A new stochastic equilibrium problem for two stage games

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Motivation:
What is impact of risk on investment of a firm?
Think of a market, for example wholesale electricity.

- Investment, e.g., in generation capacity, occurs ahead of production and trading.
- Production and sales may go on for decades. But financial instruments — like options to hedge against low prices (low revenue) in future — are often limited to 1 or 2 years at most. → market is incomplete in that not all risks are traded

How can we model this from point of view of firm?

- A firm, even a bank, is necessarily risk averse.
- Corporate finance theory models the entire market (assuming liquid, complete markets etc) → risk neutral probabilities. Whatever an individual’s risk profile, it cannot beat the market; the market is efficient.
- Risk neutral prob. are exogenous— “read” from market data.
Investment and risk

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D Ralph A new two stage stochastic equilibrium model
Risk neutral probabilities are endogenous

Our approach cannot look like standard corporate finance
- Use game with $N$ players rather than amorphous “market”
- Use coherent risk measures, e.g. cV@R, to model risk aversion, not risk neutral expectations
- Players’ interactions in market determine their risk neutral probabilities ... which are endogenous.

Our model is rather simple, only two stages:
- Stage one is investment, stage two is production and sales.
- Main task is to establish existence of equilibrium
- Give some theory behind computational model & analysis of European electricity market [Ehrenmann-Smeers-07]
- Future: How do changes in financial products (completeness) affect equilibrium?
What is impact of risk on investment of a firm?
Notation motivation: deterministic two stage game
Coherent risk measures
Stochastic two stage game
Full stochastic equilibrium model

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Section 2

Notation motivation:
A deterministic two stage investment game
Deterministic spot market (game)

Simple **spot market** of $N$ players & 1 commodity (electricity):
- Player $i$ chooses production level $z_i \geq 0$ to max profit, given:
  - production cost function $c_i(z_i)$
  - production capacity $x_i \geq 0$
  - spot market price $\lambda \geq 0$:

$$\max_{z_i} \lambda z_i - c_i(z_i) \quad \text{subject to} \quad 0 \leq z_i \leq x_i. \quad (1)$$

In **perfect competition**, $\lambda$ comes out of market clearing condition, given total demand $d > 0$:

$$0 \leq \sum_i z_i - d \quad \perp \quad \lambda \geq 0. \quad (2)$$

**Note.** Rewrite as min-max over $\lambda \geq 0$ and $z = (z_1, \ldots, z_N)$ to get unique equilibrium quantities $z$ and price $\lambda$ if
- each $c_i$ is smooth & strictly convex with $c_i(0) = 0$
- total capacity exceeds demand: $\sum_i x_i > d$ ... Slater CQ
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- total capacity exceeds demand: \( \sum_i x_i > d \) ... *Slater CQ*
First stage investment in capacity

The spot market is the second stage of the game. In the first stage Player \(i\) invests in production capacity \(x_i\):

- investment cost function \(k_i(x_i)\)
- capacity payment \(\nu \geq 0\) ... to encourage investment

**Switch from “max profit” to “min cost”**: Over both stages, Player \(i\) solves

\[
\min_{x_i, z_i} k_i(x_i) - \nu x_i + c_i(z_i) - \lambda z_i \\
\text{subject to } 0 \leq x_i \\
0 \leq z_i \leq x_i. \tag{3}
\]

Capacity payment \(\nu\) must clear the capacity market:

\[
0 \leq \sum_i x_i - d \perp \nu \geq 0. \tag{4}
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The **two stage investment game** is described by (2)–(4).
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Reduction to single stage game

Player $i$’s optimization problem is equivalent to

$$\min_{x_i \geq 0} \ k_i(x_i) - \nu x_i + h_i(x_i, \lambda)$$

(5)

where cost of capital in scenario $\omega$ is

$$h_i(x_i, \lambda) = \inf_{z_i \in [0, x_i]} \{ c_i(z_i) - \lambda z_i \}.$$

Assume $\exists$ unique spot market equilibrium $z(x)$, $\lambda(x)$ if $\sum_i x_i \geq d$. Then

$$h_i(x_i, \lambda(x)) = c_i(z_i(x)) - \lambda(x)z_i(x).$$

But perfect competition $\Rightarrow$ Player $i$ does not anticipate its effect on spot price $\lambda$ ... or other players’ actions.

$\Rightarrow$ Reduced (single stage) game given by clearing capacity market (4), players’ problems (5), and $\lambda = \lambda(x)$.

Notes. 1. (5) is not an MPEC ... two stage game not an EPEC!

2. In general, $h_i$ may be convex and smooth or nonsmooth in $x_i$.
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Section 3

Coherent risk measures
Cost of future (random) outcomes

Think of $Y \in L_p(\Omega)$ as vector $(Y_\omega)_{\omega \in \Omega}$ of future outcomes or costs, each $Y_\omega \in \mathbb{R}$. Here $p \in (1, \infty]$, 

$$\|Y\| = \left(\int_\Omega |Y_\omega|^p d\omega\right)^{1/p}$$

... where $d\omega$ means $dP(\omega)$ 

So $1_{\Omega} = (1)_{\omega \in \Omega} \in L_p(\Omega)$, in fact $\int_\Omega d\omega = 1$.

What is cost (or value) of $Y$ at the time of investment?

Simplest answer: use expectation under probability measure, defined as dual element $\Pi \in L_q(\Omega)$, where $1/p + 1/q = 1$, s.t.

$$\Pi_\omega \geq 0 \text{ a.e.} \quad \text{and} \quad \Pi 1_{\Omega} = \int_\Omega \Pi_\omega dP(\omega) = 1.$$ 

Denote $\Pi Y = \int_\Omega \Pi_\omega Y_\omega dP(\omega)$ as expectation $\mathbb{E}_\Pi[Y]$.

$\rightarrow$ investment game over two stage stochastic programs.

Alternative: coherent risk measure — risk function for short — $\rho : L_p(\Omega) \rightarrow \mathbb{R} \cup \infty$ is more general, recommended in finance, see [Artzner-et-al-99]. Risk functions include straight expectations.

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Coherent risk measures

In general, $\rho$ is risk function if for all $Y, Y' \in L_p(\Omega)$ and $\alpha \in \mathbb{R}$:

- **proper**: $\rho(Y) > -\infty$
- **sublinear**: $\rho(Y + Y') \leq \rho(Y) + \rho(Y')$
- **positive homogeneous**: $\rho(\alpha Y) = \alpha \rho(Y)$ for $\alpha > 0$
- **monotone**: $\rho(Y) \leq \rho(Y')$ if $Y_\omega \leq Y'_\omega$ a.e.
- **translation invariant**: $\rho(Y + \alpha \mathbb{1}) = \rho(Y) + \alpha$

To these we add the property of being lower semicontinuous (lsc).

**Classical result** [Hörmander 54]: $\rho$ is proper, sublinear, positive homogeneous and lsc $\iff \rho = \sigma_D$, the support function of a nonempty, closed, convex set $D$ in $L_q(\Omega)$, where

$$\sigma_D(Y) = \sup_{\zeta \in D} \zeta Y.$$ 

**Refinement** [Shapiro-Ruczinski-06]: $\rho$ is lsc risk function $\iff \rho = \sigma_D$ for nonempty, closed, convex set $D$ of probability meas.
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Risk aversion vs risk neutrality

Example

**Conditional value at risk**, $cV@R$, is one of best known risk functions [Rockafellar-Uryasev-02].
$cV@R_{\Pi,\beta}(Y)$ is (roughly) $\Pi$-expectation of all but $\beta\%$ of lowest outcomes of $Y$.
Thus $cV@R_{\Pi,\beta}(Y) \geq E_{\Pi}[Y] \rhd cV@R$ is risk averse.

Generally say risk measure $\rho = \sigma_D$ is **risk neutral** if $D$ is a singleton $\{\Pi\}$, and **risk averse** otherwise.

[Rucz.-Shap.-06, Example 7] shows that $cV@R_{\Pi,\beta} = \sigma_D$ where $D$ is set of probability measures $\Pi'$ such that $\Pi'_\omega \leq \Pi_\omega/(1 - \beta)$ a.e.

Note also that risk measures combine robust optimization with stochastic programming (c.f. Nemirovsky et al).
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Section 4

Stochastic two stage game

Subsection 4.1
Costing an uncertain future
Costing uncertain spot market outcomes

For each spot scenario $\omega$, Player $i$’s spot market cost is

$$G_{i\omega}(z_i, \lambda) = c_{i\omega}(z_i) - \lambda z_i.$$  

We may choose a different $z_i = Z_{i\omega}$ in each scenario, and the spot price may vary: $\lambda = \Lambda_{\omega}$.  

Future outcomes are listed as $G_i(Z_i, \Lambda) = (G_{i\omega}(Z_{i\omega}, \Lambda_{\omega}))_{\omega \in \Omega}$.  

Player $i$ minimizes investment & spot market cost:

$$\min_{x_i, Z_i} k_i(x_i) - \nu x_i + \rho_i(G_i(Z_i, \Lambda))$$

subject to

$$0 \leq x_i$$

$$0 \leq Z_i \leq x_i$$

(6)

Spot market clearing condition is given pointwise over scenarios,

$$0 \leq \sum_i Z_{i\omega} - D_\omega \perp \Lambda_{\omega} \geq 0 \text{ a.e.}$$

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where $D = (D_\omega)_\omega$ is vector of nonnegative demands.
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where $D = (D_\omega)_\omega$ is vector of nonnegative demands.
To be sure of meeting spot demand, assume $D$ is bounded and let $d = \sup D$ ignoring zero measure sets

$$d = \text{ess sup } D_\omega < \infty.$$ 

Clearing capacity market is same as before,

$$0 \leq \sum_{i=1}^{N} x_i - d \perp \nu \geq 0. \tag{8}$$

**Two stage stochastic game** given by (6), (7) & (8)
Reduction to single stage stochastic game

Cost of capital (or investment) in spot scenario $\omega$ is

$$H_{i\omega}(x_i, \lambda) = \inf_{z_i \in [0, x_i]} G_{i\omega}(z_i, \lambda)$$  \hspace{1cm} (9)

Define vector of spot capital cost $H_i(x_i, \Lambda) = (H_{i\omega}(x_i, \Lambda_\omega))_\omega$.

Assume (1), when $\sum_i x_i \geq d$, that each spot scenario $\omega$ has unique equilibrium $Z_{\omega}(x) \in \mathbb{R}^N$, $\Lambda_{\omega}(x) \geq 0$, with $Z(x), \Lambda(x) \in L_\infty(\Omega)$.

Proposition (Pointwise decomposition)

$Z_i(x)$ is a global solution of $\min_{0 \leq Z_i \leq x_i} \rho_i(G_i(Z_i, \Lambda(x)))$.

This is based on an interchangeability principle, see [Rucz.-Shap.-06, Proposition 4], c.f. [Rock.-Wets-88].
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Recall the game where each player has a two stage stochastic problem (6), plus clearing of spot & capacity markets (7) & (8).

This reduces to the **single stage stochastic game**

$$\min_{x_i \geq 0} k_i(x_i) - \nu x_i + \rho_i(H_i(x_i, \Lambda))$$

with capacity market condition (8) and $\Lambda = \Lambda(x)$.

(As in simple deterministic game, Player $i$ doesn’t anticipate its effect on price or other players.)

**Main point:** have reduced infinite dimensional game to finite dimensions.
Single stage stochastic game

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Section 4

Stochastic two stage game

Subsection 4.2

Hedging: A financial market in risk
Options trading to hedge against low prices

Players want to **hedge** against a low spot price. So financial traders offer a list of **strike prices** $\Lambda^O = (\Lambda^O_\theta)_{\theta \in \Theta}$ where $\Theta$ is another measure space.

- Anyone can buy any amount of any **option** $\theta$.
- If the spot price is $\lambda$, one unit of the option pays $(\Lambda^O_\theta - \lambda)_+ = \max\{\Lambda^O_\theta - \lambda, 0\}$
- This is a hedge against price falling below $\Lambda^O_\theta$.

Player $i$ buys/sells $W_{i\theta}$ of option $\theta$ where $W_i = (W_{i\theta})_{\theta \in \Theta} \in L_{p'}(\Theta)$ and $p' \in (1, \infty]$.

In spot scenario $\omega$, given investments $\sum_i x_i \geq d$, Player $i$ is paid

$$P^O_\omega(x)W_i = \int_{\Theta} (\Lambda^O_\theta - \Lambda_\omega(x))_+ W_{i\theta} d\theta$$

This defines linear mapping $P^O(x) : L_{p'}(\Theta) \rightarrow L_\infty(\Omega)$. 

D Ralph A new two stage stochastic equilibrium model
Options trading to hedge against low prices

Players want to **hedge** against a low spot price. So financial traders offer a list of **strike prices** \( \Lambda^O = (\Lambda^O_\theta)_{\theta \in \Theta} \) where \( \Theta \) is another measure space.

- Anyone can buy any amount of any **option** \( \theta \).
- If the spot price is \( \lambda \), one unit of the option pays 
  \[ (\Lambda^O_\theta - \lambda)_+ = \max \{ \Lambda^O_\theta - \lambda, 0 \} \]
- This is a hedge against price falling below \( \Lambda^O_\theta \).

Player \( i \) buys/sells \( W_{i\theta} \) of option \( \theta \) where \( W_i = (W_{i\theta})_{\theta \in L_{p'}(\Theta)} \) and \( p' \in (1, \infty] \).

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Pricing options

The cost (market price) for options is a nonnegative vector $C^O = (C^O_\theta)_\theta \in L_{q'}(\Theta)$ where $1/p' + 1/q' = 1$.

What determines the cost of options? Develop equilibrium conditions to find out!

First, Player $i$’s (reduced) problem now looks like

$$
\min_{x_i \geq 0, W_i \in L_{p'}(\Theta)} k_i(x_i) - \nu x_i + C^O W_i \\
+ \rho_i(H_i(x_i, \Lambda) - P^O W_i)
$$

where $\Lambda = \Lambda(x)$ and $P^O = P^O(x)$.

We also have conservation of cash:

$$
\sum_{i=1}^{N} W_i = 0.
$$
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\begin{align*}
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A new two stage stochastic equilibrium model
Optimality conditions for Player $i$

**Assume (II)** smoothness of $k_i$ and $H_i$ with respect to $x_i$, Player $i$’s optimality condition for (10) is

$$\Pi_i \in \partial \rho_i(H_i(x_i, \Lambda) - P^O W_i)$$

(12)

$$0 \leq \nabla x_i k_i(x_i) - \nu + \Pi_i \nabla x_i H(x_i, \Lambda) \perp x_i \geq 0$$

(13)

$$C^O = \Pi_i P^O.$$  

(14)

Note each $\Pi_i$ is a probability measure . . . a risk neutral probability for Player $i$.

The full equilibrium conditions also require: clearing capacity market (8), $\Lambda = \Lambda(x)$, $P^O = P^O(x)$, and cash conservation (11).
Assume (II) smoothness of \( k_i \) and \( H_i \) with respect to \( x_i \), Player \( i \)'s optimality condition for (10) is

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\]

\[
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The full equilibrium conditions also require: clearing capacity market (8), \( \Lambda = \Lambda(x) \), \( P^O = P^O(x) \), and cash conservation (11).
Full stochastic equilibrium model
Existence of equilibrium

Assume (III) that

1. (a) all player’s risk functions have form \( cV@R_{\Pi_i, \beta_i} \) for same \( \Pi \) but possibly different \( \beta_i \in (0, 1) \)
   (b) \( \Pi_\omega \geq \epsilon \) a.e. for some \( \epsilon > 0 \)

2. some technical conditions, e.g., \( PO(x) : L_{p'}(\Theta) \to L_\infty(\Omega) \)
   has closed range in \( L_1(\Omega) \) whenever \( \sum_i x_i \geq d \).

Assumption (III) part 1 requires overlap in sets \( D_i \) where \( \rho_i = \sigma_{D_i} \): players’ risk profiles are similar enough to avoid **arbitrage**.
(The conjecture holds if there are only finitely many options.)

Let \( E(x) \) be set of

\[
(\Pi_1 \nabla_{x_1} H_1(x_1, \Lambda), \ldots, \Pi_N \nabla_{x_N} H_N(x_N, \Lambda)) \in \mathbb{R}^N
\]

where \( \Lambda = \Lambda(x) \), \( PO = PO(x) \), and ((11)) & (12) hold.
Existence of equilibrium

Assume (III) that

1. (a) all player’s risk functions have form $cV@R_{II_i},\beta_i$ for same $\Pi$ but possibly different $\beta_i \in (0, 1)$
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2. some technical conditions, e.g., $P^O(x) : L_{p'}(\Theta) \to L_{\infty}(\Omega)$ has closed range in $L_1(\Omega)$ whenever $\sum_i x_i \geq d$.

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$$\left( \Pi_1 \nabla x_1 H_1(x_1, \Lambda), \ldots, \Pi_N \nabla x_N H_N(x_N, \Lambda) \right) \in \mathbb{R}^N$$

where $\Lambda = \Lambda(x)$, $P^O = P^O(x)$, and ((11)) & (12) hold.
Existence of equilibria

Conjecture

The set mapping $\mathcal{E} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is upper semicontinuous and has nonempty, compact, convex values.

This allows us to convert the optimality conditions (12)-(14) and capacity market condition (8) into a standard fixed point setting. Main result becomes application of Kakutani’s fixed point theorem.

Theorem

Let Assumptions (I), (II) and (III) hold. If each $k_i$ is strongly convex then there exists a solution $x^* = (x_1^*, \ldots, x_N^*) \in \mathbb{R}^N$ of the reduced stochastic investment game with options trading.

Note this provides a risk neutral probability $\Pi_i$ for each Player $i$. 
What is impact of risk on investment of a firm?
Notation motivation: deterministic two stage game
Coherent risk measures
Stochastic two stage game
Full stochastic equilibrium model

Existence of equilibria

Conjecture

The set mapping $\mathcal{E} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is upper semicontinuous and has nonempty, compact, convex values.

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Extensions

Many technical/practical extensions possible

- **Multiple time periods**, c.f. [Ehrenmann-Smeers-07]
- several commodities: \( x_i \) lies in \( \mathbb{R}^n \) not \( \mathbb{R} \)
- Player \( i \)’s problem may directly depend on other players decisions \( x_{-i} \) or \( Z_{-i} \)
- nonsmooth (but still convex) objective functions and constraints
- More complex constraints under suitable constraint qualification

**Real interest**: interplay between financial products and firms’ investments
- whether few options (incomplete market)
- or many options (nearly complete)
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**Real interest**: interplay between financial products and firms’ investments

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Second, writing $W = (W_1, \ldots, W_N)$, cost $C^O$ is dual multiplier of above constraint in the **hedging problem**

$$
\min_{W} \sum_{i=1}^{N} \rho_i \left( H_i(x_i, \Lambda) - P^O(x)W_i \right)
$$

subject to (11)
Most technical part of analysis occurs here.