



Delayed resonator for displacement-excited vibration control: combining with a high-static-low-dynamic-stiffness primary structure

Yifan Liu¹ · Ge Yan² · Li Cheng¹

Received: 22 April 2026 / Revised: 19 July 2026 / Accepted: 2 September 2026
© The Author(s) 2026

Abstract

Delayed resonators (DRs) refer to a class of active vibration absorbers driven by the control forces following past (delayed) system states. By tuning the delay, DRs can enable complete vibration suppression at a given frequency. Different from the existing DRs, which mainly focus on force-excited vibrations, this work considers a displacement-excited case where the primary is mounted on a vibrating frame (base). Particularly, the high-static-low-dynamic-stiffness (HSLDS) feature is injected between the primary and the frame, given its widely known benefits in vibration isolation, finally yielding a combined system with a DR and an HSLDS primary structure. Targeting complete vibration suppression, with additional efforts made to seek performance enhancement in the HSLDS case over the classical constant-stiffness (CS) case, the investigation includes dynamical analysis, control parameter tuning, and stiffness optimization. Meanwhile, special attention is paid to evaluating the nonlinear effect of the HSLDS feature, and the shown negligible effects constitute the basis for using limited-order models without compromising accuracy. Finally, experiments showcase a simple yet effective HSLDS mechanism based on magnetic springs. This work not only extends the application scope of DRs but also exploits the mechanical advantage of the HSLDS feature to achieve complete vibration suppression.

Keywords Active vibration control · Delayed resonator · High-static-low-dynamic stiffness · Dynamical control

1 Introduction

In 1994, Olgac and Holm-Hansen [1] proposed the concept of the delayed resonator (DR), a scheme that employs a feedback force based on the system's past (delayed) state to drive the vibration absorber. By properly tuning the delay, DRs can enable complete vibration suppression at a given frequency. Compared with the nondelayed control logic [2], the DR scheme can model the inevitable loop delay to achieve more accurate dynamical prediction and thus more complete vibration suppression. Besides, one can directly alter the force phase by manipulating the delay, so the DR scheme can be implemented by only incorporating displacement [1], velocity [3], or acceleration [4] to simplify sensor deployment

and reduce the number of control terms. Over the 30 years' development, DRs have become a typical counter-intuitive example that increasing loop delay can be beneficial.

Among the extensive research on DRs, Sun and Xu [5, 6] identified the hardware delay to guide delay tuning. Pilbauer et al. [7] and Kučera et al. [8] proposed a distributed delayed DR to reduce the effect of the measurement noise by calculating the actuation force as per the dynamical sum of all absorber states over a past time period. Liu et al. [9] considered the effect of the loop delay on the distributed delayed logic. Eris et al. [10] actuated a DR with both delayed and nondelayed control terms to reduce the transient process. Zhang et al. [11] evaluated the friction effect on the DR tuning. Karama et al. [12] and Kammer et al. [13] studied the energy harvesting efficiency of DRs. Šika et al. [14–16] achieved multi-dimensional vibration suppression in robotic applications. The DR was also used to address robotic force tracking and jitter issues [17]. Wang et al. [18] designed a DR with an adjustable stiffness to fit the variable stiffness of a primary structure. Vyhldal et al. [19] compared the tuning mechanisms of the DRs incorporating different system states. Cai et al. [20] summarized the tuning mechanisms using

✉ Li Cheng
li.cheng@polyu.edu.hk

¹ Department of Mechanical Engineering, The Hong Kong Polytechnic University, Hong Kong, China

² State Key Laboratory of Ocean Engineering, Department of Engineering Mechanics, School of Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai, China

the fractional-order calculator. Kuře et al. [21] enhanced the robustness of the DRs by extending the antiresonance valley, see also [22]. By deploying multiple-degree-of-freedom DRs [23–25], the location of the active force can differ from the location where the vibration is suppressed. Particularly, time-delayed systems have infinite characteristic roots, conveniently enabling the DR-based scheme to suppress multi-frequency vibrations by driving a single-mass absorber [26–28]. One can also refer to [29–36] for using the delayed control to enhance (incomplete) vibration reduction.

Note that the existing DRs are mainly used to settle a force-excited primary structure mounted on a fixed frame or base. However, many practical scenarios involve a vibrating frame serving as the excitation, and the task is to reduce the displacement transferred to the primary (vibration isolation). Hence, this work aims to explore the applicability of the DR's handling of such displacement-excited vibrations, with the main focus still on complete vibration suppression at a given frequency. On the other hand, vibration isolation problems frequently call for high-static-low-dynamic-stiffness (HSLDS) design, a nonlinear stiffness configuration that can significantly extend the isolation bandwidth without affecting the isolator's static load capacity [37–39]. Recent design examples include the isolators by Zhang et al. [40], Ma et al. [41], Xu et al. [42], Jiao et al. [43], Wang et al. [44], Pan et al. [45], and Li et al. [46]. Besides, Yan et al. [47] established a stiffness compensation method to achieve the HSLDS feature. The HSLDS feature can also be implemented by material deformation to achieve a lightweight and compact isolator design [48–51]. More advances on the HSLDS isolators can be found in the review articles [52–55]. Meanwhile, the idea of combining vibration absorption and HSLDS isolation has also been reported [56–61]. In this regard, the enhanced system performances usually include improved isolation efficacy and an extended isolation band. Given the benefits of the HSLDS feature, the displacement-excited primary structure under investigation is designed to be HSLDS. Particularly, deploying a DR to completely settle a force-excited HSLDS primary structure has been considered in our previous work [62]. The benefits of the 'low-dynamic' stiffness were, however, not fully utilized since the completely settled primary structure has no relative motion to the vibration source. To our best knowledge, both the DR's handling of displacement-excited vibration and the combination of the HSLDS isolator with an active vibration absorber remain a research gap.

First, this work adopts a feedforward control logic for the DR design, which drives the DR as per the real-time displacement excitation rather than the system response as in the classical feedback-controlled cases. This is motivated by the fact that the feedforward control does not induce instability [27, 32]. With this simplification, we can concentrate more on the mechanical advantages of the HSLDS feature.

Then, we explore three problems or challenges for the resulting combined system, which is constituted by a linear DR and an HSLDS primary structure mounted on a vibrating frame.

- (i) How would the combination of the HSLDS feature with delayed control affect system dynamics?
- (ii) How to achieve the so-called complete vibration suppression at a given frequency by tuning a linear DR in the HSLDS case?
- (iii) If complete vibration suppression is achievable, what benefits can be drawn from the HSLDS feature compared with the original constant-stiffness (CS) case?

Particularly, for (ii), since a linear DR (driven by linear control forces) can only exert linear forces on the primary structure, achieving complete vibration suppression, which means all dynamical forces on the primary structure should be ideally neutralized, requires that the HSLDS feature causes negligible nonlinear effects. We show that this can be achieved by attaching a linear absorber to the HSLDS isolator, as also observed in [57], and this is unaffected by the DR's driving force. For (iii), for the considered system, even when the displacement-excited primary structure is completely settled, the low-dynamic stiffness is still effective, signifying the possible benefits of the HSLDS feature for the DR subsystem, in addition to enhancing the settling of the primary structure. This was not considered in [62]. In addition, justified by the experimental tests, we apply a simpler yet effective way to implement the HSLDS feature using magnets, which simplifies the multiple-spring mechanisms in [57, 62] to facilitate practical applications.

In what follows, Sect. 2 formulates the problem. Section 3 establishes system dynamics. Section 4 evaluates the nonlinear effect of the HSLDS feature and the passive vibration control. Section 5 addresses the tuning mechanism of the DR. The benefits for vibration suppression are evaluated in Sect. 6. The absorber performance is studied in Sect. 7. Case studies are given in Sect. 8. Conclusions are drawn in Sect. 9. Italic symbols without the bar superscript ' $\bar{\cdot}$ ' are dimensionless throughout the text.

2 Preliminaries

The classical operation mode of the DR is shown in Fig. 1a, where an active force $\bar{u}$ following delayed (past) system states is introduced to the absorber, aiming to completely suppress the force-excited vibration on the primary structure at a given frequency, with $\bar{f}_e$ denoting the excitation.

In this work, we explore the applicability of the DR in handling the displacement-excited vibrations. As shown in Fig. 1b, the frame is now force-excited to generate the dis-

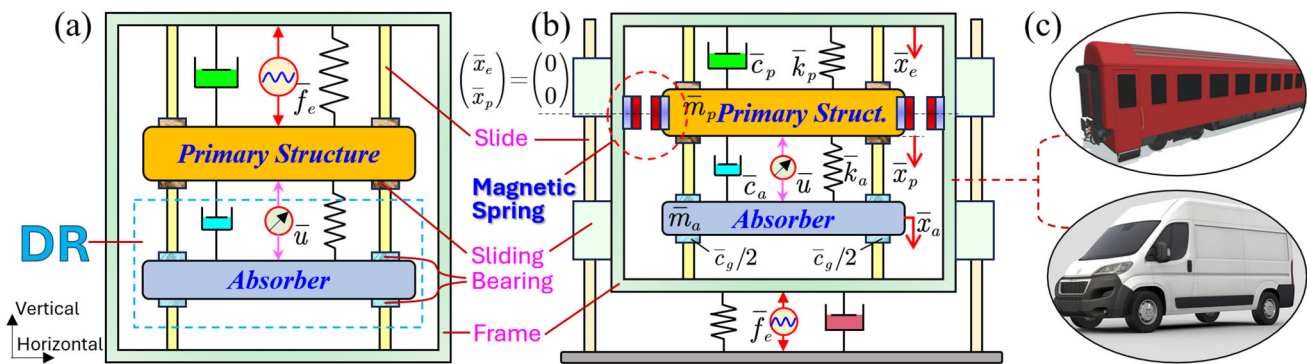


Fig. 1 **a** Classical operation mode of the DR, where the primary structure is excited by the force $\tilde{f}_e$. **b** Operation mode considered in this work, with $\bar{x}_e$ denoting the displacement excitation. **c** Two potential applications of the model in **(b)**

placement vibration $\bar{x}_e$. Given the benefits of the HSLDS feature to vibration isolation (from $\bar{x}_e$ to $\bar{x}_p$), magnets are additionally used between the frame and the primary to yield repulsive magnetic forces in the vertical direction when $\bar{x}_p \neq \bar{x}_e$, thus reducing the dynamical stiffness of the primary without affecting the static load stiffness $\bar{k}_p$. Figure 1c exemplifies two applications of the considered model, where the frame box can be a vibrating carriage of trains or vehicles. The problem is then to properly combine the nonlinear HSLDS vibration isolation and the linear active (delayed) vibration absorption.

2.1 Dynamical modeling

We only focus on the dynamics of the primary structure and the absorber under a given displacement excitation of $\bar{x}_e$. As marked in Fig. 1b, the governing equations follow

$$\begin{cases} \bar{m}_a \ddot{\bar{x}}_a + \bar{c}_a (\dot{\bar{x}}_a - \dot{\bar{x}}_p) + \bar{k}_a (\bar{x}_a - \bar{x}_p) + \bar{c}_g (\dot{\bar{x}}_a - \dot{\bar{x}}_e) = \bar{u}, \\ \bar{m}_p \ddot{\bar{x}}_p + \bar{c}_p (\dot{\bar{x}}_p - \dot{\bar{x}}_e) + \bar{k}_p (\bar{x}_p - \bar{x}_e) + \bar{f}_{NL} \\ + \bar{c}_a (\dot{\bar{x}}_p - \dot{\bar{x}}_a) + \bar{k}_a (\bar{x}_p - \bar{x}_a) = -\bar{u}, \end{cases} \quad (1)$$

where $\bar{m}(\cdot)$, $\bar{c}(\cdot)$, $\bar{k}(\cdot)$, and $\bar{x}(\cdot)$ represent mass, damping, stiffness, and absolute displacement, respectively; $\bar{f}_{NL}$ is a nonlinear restoring force arising from the magnetic springs to achieve the HSLDS feature, and is expanded as

$$\begin{aligned} \bar{f}_{NL}(\bar{r}_p) = & n_0 \bar{k}_p + \hat{n}_1 \bar{k}_p \bar{r}_p + n_2 \bar{k}_p \bar{r}_p^2 \\ & + n_3 \bar{k}_p \bar{r}_p^3 + \dots + n_n \bar{k}_p \bar{r}_p^n, \end{aligned} \quad (2)$$

where $(n_0, \hat{n}_1, n_2, \dots, n_n) \in \mathbb{R}^n$, and $\bar{r}_p = \bar{x}_p - \bar{x}_e$ is the relative displacement of the primary with respect to $\bar{x}_e$. Particularly, $n_{2k} = 0, k = 0, 1, 2, \dots$ since the repulsive magnetic forces are made symmetrical about $\bar{r}_p = 0$. $\bar{x}(\cdot)$, $\bar{r}_p$, and $\bar{u}$ are functions of the time $\bar{t}$. Performing the variable transformations

$$\begin{aligned} \bar{\omega}_p &= \sqrt{\frac{\bar{k}_p}{\bar{m}_p}}, \bar{\omega}_a = \sqrt{\frac{\bar{k}_a}{\bar{m}_a}}, \\ \zeta_p &= \frac{\bar{c}_p}{2\bar{m}_p\bar{\omega}_p}, \zeta_a = \frac{\bar{c}_a}{2\bar{m}_a\bar{\omega}_a}, \zeta_g = \frac{\bar{c}_g}{2\bar{m}_a\bar{\omega}_a}, \\ \mu &= \frac{\bar{m}_a}{\bar{m}_p}, \nu = \frac{\bar{\omega}_a}{\bar{\omega}_p}, x_{(\cdot)} = \frac{\bar{x}(\cdot)}{\bar{l}}, r_p = \frac{\bar{r}_p}{\bar{l}}, t = \bar{t}\bar{\omega}_p, \\ u &= \frac{\bar{u}}{\bar{k}_p\bar{l}}, f_{NL} = \frac{\bar{f}_{NL}}{\bar{k}_p\bar{l}}. \end{aligned} \quad (3)$$

where $\bar{l}$ is the unit length ($\bar{l} = 1$ m) and $x_{(\cdot)}, r_p \ll 1$, Eq. (1) can be cast into the dimensionless form as

$$\begin{cases} \mu [\ddot{x}_a + 2\zeta_a \nu \dot{r}_a + \nu^2 r_a + 2\zeta_g \nu (\dot{x}_a - \dot{x}_e)] = u, \\ \ddot{r}_p + \ddot{x}_e + 2\zeta_p \dot{r}_p + f_{total}(r_p) - \mu (2\zeta_a \nu \dot{r}_a + \nu^2 r_a) = -u, \end{cases} \quad (4)$$

where $r_a = x_a - x_p$, and f_{total} denotes the total restoring forces of the primary, following

$$f_{total}(r_p) = r_p + f_{NL}(r_p) = n_1 r_p + n_3 r_p^3 + n_5 r_p^5 + \dots, \quad (5)$$

in which $n_1 = \hat{n}_1 + 1$, incorporating the dynamical effects of both the load stiffness and the linear stiffness term in (2). In the static case $|x_e| = 0$, the primary structure should be settled at $x_p = 0$, which requires positive total stiffness at $r_p = 0$, i.e.,

$$\begin{aligned} \left[k_{total}(r_p = 0) = \frac{\partial f_{total}(r_p)}{\partial r_p} \Big|_{r_p=0} > 0 \right] \\ \iff [n_1 > 0 \text{ or } \hat{n}_1 > -1]. \end{aligned} \quad (6)$$

Note that the constant-stiffness (CS) case corresponds to $k_{total} \equiv 1$, or $n_1 = 1$ while $(n_3, n_5, \dots) = (0, 0, \dots)$. In the HSLDS case, $n_1 < 1$ given the low-dynamic stiffness, and we let $(n_3, n_5, \dots) > 0$ to model a hardening stiffness [37]

given the weakening magnetism as $|r_p|$ increases. Clearly, vibration suppression expects smaller values of $|x_p|/|x_e|$.

2.2 Control law design and stiffness truncation

When designing the active delayed force u to achieve the DR scheme following the classical feedback control laws, i.e.,

$$u_{ap}(t) = u_a(t, x_a, \dot{x}_a, \ddot{x}_a) + u_p(t, x_p, \dot{x}_p, \ddot{x}_p), \quad (7)$$

where u_a and u_p are the control forces related to system responses (x_a, x_p) and their time derivatives, the operable bandwidth is restricted to system stability [20]. Besides, a complete stability analysis can be involved due to the coupling between the delay and the nonlinear dynamics. To simplify analysis, following [27, 32], we alternatively design u as

$$u(t) = u_e(t, x_e, \dot{x}_e, \ddot{x}_e), \quad (8)$$

which is based on the displacement excitation. Note that u_e is a feedforward control law and guarantees stability since the system input is independent of the responses (x_a, x_p) . That is, u_e can be regarded as an artificially made external (independent) excitation, thus leading to no destabilization. In this work, we simply specify u_e based on the displacement x_e as

$$u_e(t) = g x_e(t - \tau), \quad (9)$$

where g is the gain, and τ is the time delay, which are the two dimensionless control parameters and have the dimensional forms as $(\bar{g}, \bar{\tau}) = (g\bar{k}_p, \tau/\bar{\omega}_p)$.

Remark 1 While adopting (9) rather than (7) avoids instability, the pure feedforward control may lack robustness against measurement noises and unmeasured external disturbances and may increase the sensor installation difficulty. Given the robustness issue, the feedforward control usually works together with the feedback control in practice [27, 32]. However, this combination is not considered in this paper to simplify stability analysis, given that the main focus is on exploiting the mechanical advantage of the HSLDS feature for the DR's vibration control performance.

On the other hand, considering that the vibration absorber can confine the response amplitude $|x_p|$, the nonlinear total restoring force $f_{total}(r_p)$ is truncated at the third order to simplify analyses, assuming that the effect of the higher-order terms on system dynamics is negligible. Hence, the primary has the so-called Duffing-type restoring force as

$$f_{total}(r_p) \approx n_1 r_p + n_3 r_p^3, \quad (10)$$

the effectiveness of which can be justified by either showing that the errors for predicting the dynamics of the exact model (5) are sufficiently small or proving that the effect of n_3 is much smaller than that of n_1 . The latter aspect is tested later due to the lack of the exact values of $(n_0, n_1, n_2, \dots, n_n)$. The effects of (n_1, n_3) are graphically demonstrated in Appendix A. Substituting Eqs. (9) and (10) into (4) yields

$$\begin{cases} \mu[\ddot{x}_a + 2\zeta_a v \dot{r}_a + v^2 r_a + 2\zeta_g v(\dot{x}_a - \dot{x}_e)] = g x_e(t - \tau), \\ \ddot{r}_p + \ddot{x}_e + 2\zeta_p \dot{r}_p + n_1 r_p + n_3 r_p^3 - \mu(2\zeta_a v \dot{r}_a + v^2 r_a) \\ = -g x_e(t - \tau). \end{cases} \quad (11)$$

Then, the HSLDS feature and the delayed control affect system performance via (n_1, n_3, g, τ) . As an extension of the DR research, (g, τ) are later tuned to achieve the complete vibration suppression at a given frequency. Besides, the effect of the HSLDS feature with $n_1 < 1$ and $n_3 > 0$ are compared with the original CS case with $(n_1, n_3) = (1, 0)$.

3 Frequency response analysis

We first address the frequency responses of the system. Due to the existing nonlinearity signified by $n_3 > 0$, the commonly used Laplace transformation-based methods for analyzing the DR performance in linear systems are inoperable. Following [57], the harmonic balance method is adopted instead.

3.1 Harmonic balance analysis

We let the harmonic displacement excitation take the form

$$x_e(\omega, t) = R_e \cos(\omega t), \quad (12)$$

where $R_e \in \mathbb{R}^+$, and $\omega \in \mathbb{R}^+$ is the angular frequency. The dimensional values of (R_e, ω) are $(\bar{R}_e, \bar{\omega}) = (R_e \bar{l}, \omega \bar{\omega}_p)$. Revisiting (11), system responses are governed by (x_a, r_a, r_p) and their time derivatives. Note that when x_e is given, (r_a, r_p) are governed by (x_a, x_p) and vice versa. Since (r_a, r_p) appear more in (11) and since the nonlinear term r_p^3 exists, using (r_a, r_p) to depict steady-state dynamics can simplify equations. Then, we assume that (r_a, r_p) in the steady state satisfy

$$\begin{cases} r_a(\omega, t) = A_{ra} \cos(\omega t) + B_{ra} \sin(\omega t) \\ \quad = R_{ra} \cos(\omega t + \phi_{ra}), \\ r_p(\omega, t) = A_{rp} \cos(\omega t) + B_{rp} \sin(\omega t) \\ \quad = R_{rp} \cos(\omega t + \phi_{rp}), \end{cases} \quad (13)$$

where $(A_{ra}, B_{ra}, A_{rp}, B_{rp}) \in \mathbb{R}^4$, $R_{ra,rp} = \sqrt{A_{ra,rp}^2 + B_{ra,rp}^2}$, and (ϕ_{ra}, ϕ_{rp}) are the phase differences of (r_a, r_p) from x_e . With (12) and (13), the absolute displacements (x_a, x_p) can be expressed as

$$\begin{cases} x_a(\omega, t) = (x_a - x_p) + (x_p - x_e) + x_e = r_a + r_p + x_e \\ \quad = (A_{ra} + A_{rp} + R_e) \cos(\omega t) \\ \quad \quad + (B_{ra} + B_{rp}) \sin(\omega t), \\ x_p(\omega, t) = r_p + x_e \\ \quad = (A_{rp} + R_e) \cos(\omega t) + B_{rp} \sin(\omega t). \end{cases} \quad (14)$$

The steady-state amplitudes of (x_a, x_p) are then

$$\begin{cases} R_a(\omega) = |x_a(\omega, t)| \\ \quad = \sqrt{(A_{ra} + A_{rp} + R_e)^2 + (B_{ra} + B_{rp})^2}, \\ R_p(\omega) = |x_p(\omega, t)| = \sqrt{(A_{rp} + R_e)^2 + B_{rp}^2}. \end{cases} \quad (15)$$

Substituting the expressions of (r_a, r_p, x_a) in (13) and (14) into (11) and balancing the coefficients of $(\sin(\omega t), \cos(\omega t))$ on both sides lead to

$$\begin{cases} G_1(\omega, \Theta) = \mu [v^2 B_{ra} - (B_{ra} + B_{rp}) \omega^2 - 2((A_{ra} + A_{rp}) \zeta_g + \zeta_a A_{ra}) v \omega] - g R_e S = 0, \\ G_2(\omega, \Theta) = \mu [v^2 A_{ra} - (R_e + A_{ra} + A_{rp}) \omega^2 + 2((B_{ra} + B_{rp}) \zeta_g + \zeta_a B_{ra}) v \omega] - g R_e C = 0, \\ G_3(\omega, \Theta) = \frac{1}{4} \left[3n_3 B_{rp}^3 + \lambda(\omega, A_{rp}) B_{rp} \right] + 2\omega (\mu v \zeta_a A_{ra} - A_{rp} \zeta_p) - \mu v^2 B_{ra} + g R_e S = 0, \\ G_4(\omega, \Theta) = \frac{1}{4} \left[3n_3 A_{rp}^3 + \lambda(\omega, B_{rp}) A_{rp} \right] - 2\omega (\mu \zeta_a v B_{ra} - B_{rp} \zeta_p) - \mu v^2 A_{ra} + g R_e C - R_e \omega^2 = 0, \end{cases} \quad (16)$$

where $\Theta = [A_{ra}, B_{ra}, A_{rp}, B_{rp}]$, $(C, S) = (\cos(\omega\tau), \sin(\omega\tau))$, and $\lambda(\omega, \gamma) = 3\gamma^2 n_3 - 4\omega^2 + 4n_1$, with $\gamma \in [A_{rp}, B_{rp}]$. The frequency responses are then governed by the common solutions (ω, Θ) of (16). To reduce computational complexity, we first solve (A_{ra}, B_{ra}) since they have the lowest degree (first order). From $G_1(A_{ra}, B_{ra}) = 0$ and $G_2(A_{ra}, B_{ra}) = 0$, one has

$$\begin{cases} A_{ra}(\omega, A_{rp}, B_{rp}) = \frac{1}{\mu [\omega^4 + (4(\zeta_a + \zeta_g)^2 - 2)v^2 \omega^2 + v^4]} \sum_{i=0}^4 \alpha_i^{[1]}(A_{rp}, B_{rp}) \omega^i, \\ B_{ra}(\omega, A_{rp}, B_{rp}) = \frac{1}{\mu [\omega^4 + (4(\zeta_a + \zeta_g)^2 - 2)v^2 \omega^2 + v^4]} \sum_{i=0}^4 \beta_i^{[1]}(A_{rp}, B_{rp}) \omega^i, \end{cases} \quad (17)$$

where $i \in \mathbb{Z}$, and $(\alpha_i^{[1]}, \beta_i^{[1]})$ are real polynomial coefficients in (A_{rp}, B_{rp}) , see Appendix B. Substituting (17) into $G_3(A_{ra}, B_{ra}) = 0$ and $G_4(A_{ra}, B_{ra}) = 0$, the numerators of the resulting two equations lead to

$$\begin{cases} E_1(\omega, A_{rp}, B_{rp}) = \sum_{i=0}^6 \alpha_i^{[2]}(A_{rp}, B_{rp}) \omega^i = 0, \\ E_2(\omega, A_{rp}, B_{rp}) = \sum_{i=0}^6 \beta_i^{[2]}(A_{rp}, B_{rp}) \omega^i = 0, \end{cases} \quad (18)$$

where $(\alpha_i^{[2]}, \beta_i^{[2]}) \in \mathbb{R}^2$ are given in Appendix B. By sweeping ω , Eq. (18) at each ω grid corresponds to two polynomial equations in (A_{rp}, B_{rp}) , and thus the exhaustive solutions are available. Back-substituting the solution composition (ω, A_{rp}, B_{rp}) of (18) into (17) finally yields the desired common solutions (ω, Θ) of (16).

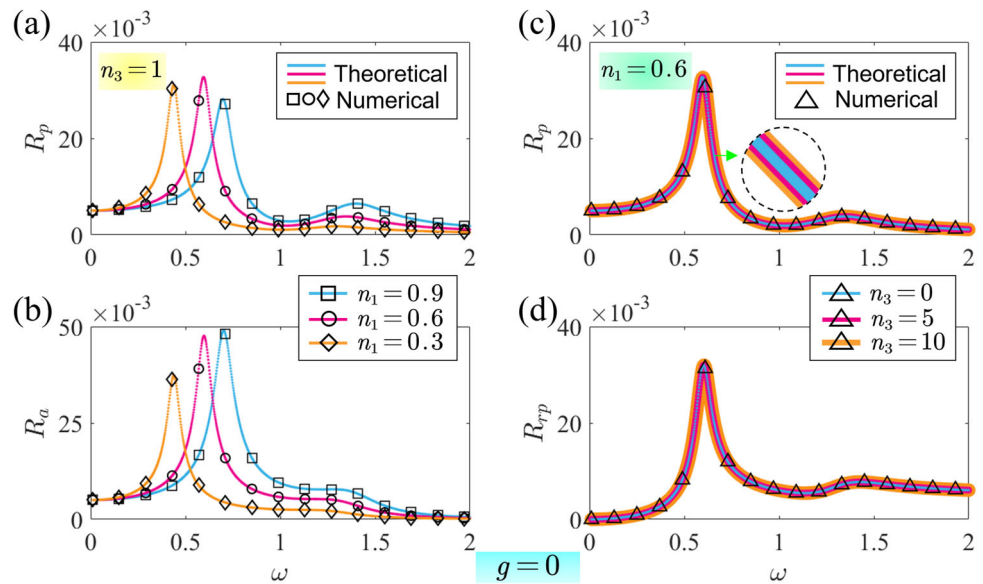
Remark 2 The simplification process based on (17) and (18) for solving Eq. (16) is independent of the truncation order of the nonlinear total restoring force f_{total} in (10). The reason is that $G_1 = 0$ and $G_2 = 0$ are always linear equations in (A_{ra}, B_{ra}) as the nonlinear terms do not appear in the first equation of (11). Besides, $G_3 = 0$ and $G_4 = 0$ are always polynomial equations in $(A_{ra}, B_{ra}, A_{rp}, B_{rp})$. Hence, the third-order truncation in (10) does not affect the generality of the adopted calculation simplification for the frequency response analysis.

3.2 Numerical verifications for response prediction

We here test the solution accuracy of (16) for predicting the response of (11). The coupled system to be considered has the dimensionless structural parameters as

$$\mu = 0.454, v = 1.03, \zeta_p = 0.027, \zeta_a = 0.101, \zeta_g = 0.038. \quad (19)$$

Fig. 2 Comparisons between the theoretical and numerical frequency responses for $R_e = 0.005$ and $g = 0$ (passive case). **a, b** Absolute displacements of the primary (R_p) and the absorber (R_a) versus ω for $n_1 \in [0.9, 0.6, 0.3]$ and $n_3 = 1$. **c, d** R_p versus ω and the amplitude R_{rp} of the relative displacement r_p versus ω for $n_3 \in [0, 5, 10]$ and $n_1 = 0.6$. Curves with different thicknesses correspond to three n_3 values



The corresponding dimensional parameters and the experimental setup are to be introduced in Sect. 8.1. Starting with the passive case $g = 0$ with $R_e = 0.005$, the theoretical frequency responses governed by the absolute displacement amplitudes (R_p , R_a) in (15) are shown as the solid curves in Fig. 2a, b.

For comparisons, the numerical results in Fig. 2a, b at each frequency ω are obtained by directly solving (16) with respect to t and then getting the response amplitudes in the steady states. Clearly, the numerical results agree well with the theoretical ones, thus verifying the effectiveness of the solutions of (16). Note that three cases with different n_1 values and a given n_3 are considered. It shows that reducing n_1 shifts the resonance peaks to the lower-frequency band, which agrees with the lower dynamic stiffness due to the stronger negative stiffness of the magnetic springs, i.e., $\hat{n}_1 \rightarrow -1$.

On the other hand, Fig. 2c compares the frequency responses of the primary when varying n_3 and fixing n_1 , showing that the responses are insensitive to the variations of the nonlinear stiffness n_3 . Figure 2d further verifies this by showing that the relative displacement r_p of the primary with respect to x_e is insensitive to n_3 . Hence, combining the HSLDS vibration isolation with the linear vibration absorption achieves the beneficial low dynamic stiffness while limiting the nonlinear effects, as to be further tested. The limited effect of n_3 justifies the limited-order truncation of the nonlinear restoring force in (10).

The active case with an example pair of control parameters $(g, \tau) = (0.5, 3)$ is considered in Fig. 3, where the theoretical solutions again well predict the numerical responses as expected. Note from Fig. 3c, d that system responses are still insensitive to n_3 . Comparing Figs. 2 and 3, the active force can significantly affect the system responses. Particularly,

from Fig. 3a, the given control parameter pair can achieve a sufficiently low antiresonance peak of R_p around $\omega = 0.8$. The inverse problem to determine the values of (g, τ) that yield an ideal zero antiresonance peak $R_p = 0$ at a given frequency ω , which is the so-called complete vibration suppression in the classical DR studies, is to be addressed in Sect. 5.

3.3 Frequency spectrum

Since the theoretical solutions are based on the harmonic balance at the fundamental frequency, i.e., Eq. (13), the good match between the theoretical and numerical solutions in Figs. 2 and 3 implies that the super- and sub-harmonic vibrations arising from the nonlinearity, if any, are sufficiently small. Ensuring that the vibration energy is concentrated at the fundamental frequency is critical to the DR tuning later. To this end, the numerical responses with $(n_1, n_3) = (0.6, 10)$, which corresponds to the strongest effect of the nonlinear stiffness in Figs. 2 and 3, are further checked. With fast Fourier transform, Fig. 4 extracts the first ten dominant frequency components of the primary structure's response x_p within the steady state and calculates the ratio of their total energy to the total response energy.

Figure 4a, b correspond to the passive case with $g = 0$. From Fig. 4a, the frequency components are distributed very closely to the excitation frequency. The multiple-component feature is attributed to the limited resolution of the fast Fourier transform and the non-simple harmonic response caused by the inherent nonlinear excitation. Combined with Fig. 4b, these frequency components account for over 96% of the total response energy. The same can be observed in the active case in Fig. 4c, d, which exhibit almost the same spec-

Fig. 3 Comparisons between the theoretical and numerical frequency responses for $R_e = 0.005$ and $(g, \tau) = (0.5, 3)$ (active case). **a, b** Absolute displacements of the primary (R_p) and the absorber (R_a) versus ω for $n_1 \in [0.9, 0.6, 0.3]$ and $n_3 = 1$. **c, d** R_p versus ω and the amplitude R_{rp} of the relative displacement r_p versus ω for $n_3 \in [0, 5, 10]$ and $n_1 = 0.6$. Curves with different thicknesses correspond to three n_3 values

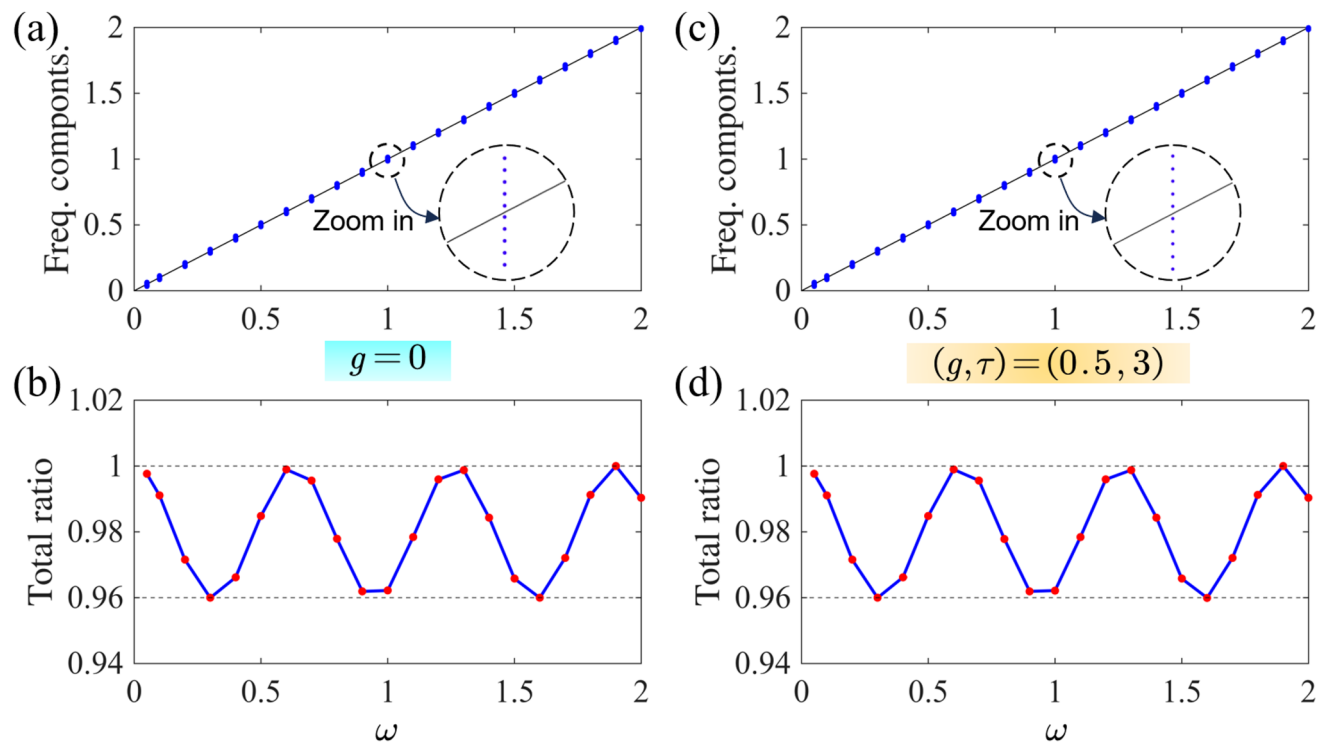
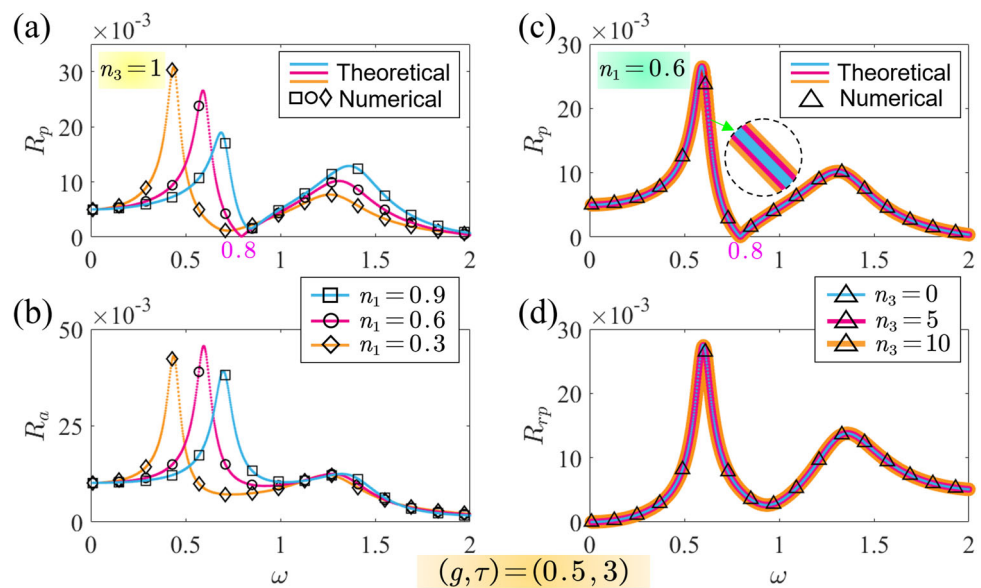


Fig. 4 Frequency spectrum of the steady-state displacement response x_p corresponding to $(n_1, n_3) = (0.6, 10)$ in Figs. 2 and 3a, c. The distributions of the first ten dominant frequency components. **b, d** The ratios

of the total energy of the first ten frequency components to the total response energy. **a, b** correspond to Fig. 2 for $g = 0$. **c, d** correspond to Fig. 3 for $(g, \tau) = (0.5, 3)$

trum as Fig. 4a, b. The above justifies the negligible sub- and super-harmonics and the accuracy of the harmonic balance at the fundamental frequency. For further discussions on the existence conditions of sub- and super-harmonics, one can refer to [63, 64].

4 Suppressed nonlinear effects and passive vibration control

From Figs. 2 and 3, the HSLDS springs can reduce the dynamical stiffness while posing very limited nonlinear effects (i.e., the effect of $n_3 \neq 0$ is negligible). Before tuning

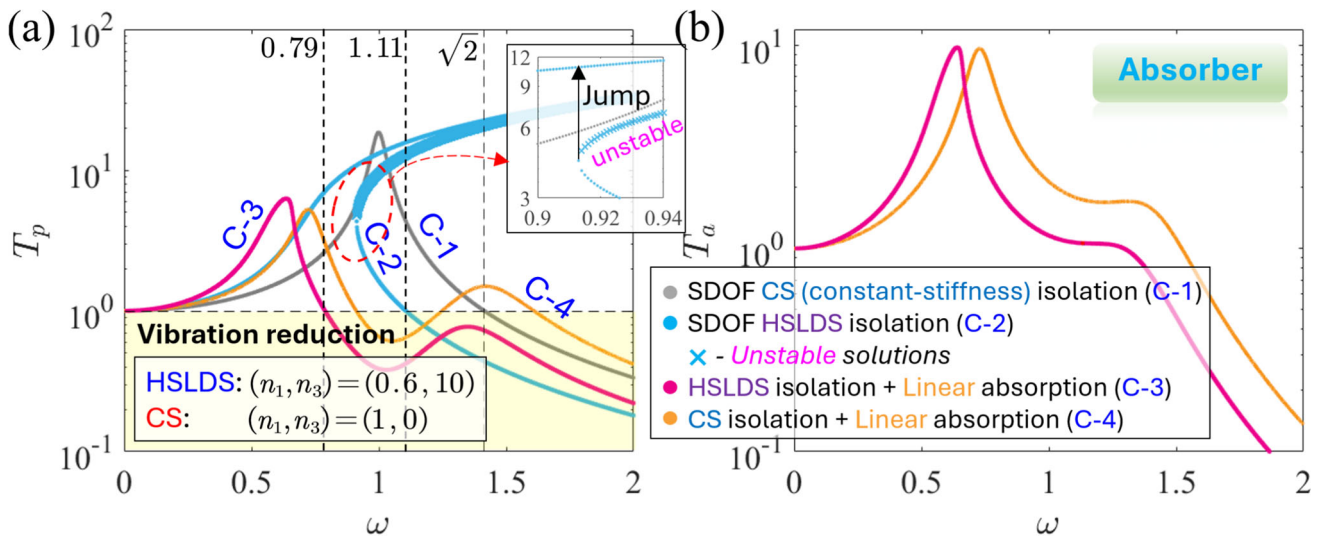


Fig. 5 Displacement transmissibility (T_p , T_a) in different vibration control models. **a** T_p versus ω . Effective vibration reduction corresponds to $T_p < 1$. Only C-1 and C-4 correspond to the CS case. **b** T_a versus ω . SDOF refers to single-degree-of-freedom

the active force u , we briefly demonstrate how this benefit can be attributed to the additional vibration absorber. To this end, three basic passive vibration control models are compared with the present one (i.e., Fig. 1b). The results also show the benefits of combining the HSLDS isolation and the linear vibration absorption for the passive vibration control, an important prerequisite to explore the advantage of the HSLDS feature to the active control.

4.1 Basic vibration control models

The considered model in Fig. 1b can be seen as a modification of (1) the single-degree-of-freedom CS vibration isolation, (2) the single-degree-of-freedom HSLDS vibration isolation, and (3) the two-degree-of-freedom linear vibration absorption. The single-degree-of-freedom vibration isolation refers to

$$\ddot{r}_p + 2\zeta_p \dot{r}_p + n_1 r_p + n_3 r_p^3 = -\ddot{x}_e, \quad (20)$$

where the CS case corresponds to $(n_1, n_3) = (1, 0)$, and the adopted hardening HSLDS feature corresponds to $0 < n_1 < 1$ and $n_3 > 0$, as aforementioned. For the 2DOF linear vibration absorption, it follows the reduced form of (11),

$$\begin{cases} \mu [\ddot{x}_a + 2\zeta_a v \dot{r}_a + v^2 r_a + 2\zeta_g v (\dot{x}_a - \dot{x}_e)] = 0, \\ \ddot{r}_p + 2\zeta_p \dot{r}_p + r_p - \mu (2\zeta_a v \dot{r}_a + v^2 r_a) = -\ddot{x}_e. \end{cases} \quad (21)$$

Following a similar harmonic balance analysis as Eqs. (12)–(18), we can obtain the frequency responses governed by the absolute displacement amplitudes (R_p, R_a) . To facilitate discussions, we convert (R_p, R_a) into the displacement

transmissibility as

$$(T_p, T_a) = \left(\frac{R_p}{R_e}, \frac{R_a}{R_e} \right). \quad (22)$$

The vibration reduction then requires $T_p < 1$. Note that (T_p, T_a) depend on the excitation amplitude R_e in the HSLDS case since R_e affects the solutions of (16), while (T_p, T_a) and R_e are independent of each other in the CS case including (21).

4.2 Comparisons of passive vibration control

To demonstrate the nonlinear effects of the HSLDS vibration isolation with (20), we consider $(n_1, n_3) = (0.6, 10)$ with a stronger excitation of $R_e = 0.02$ ($\bar{R}_e = 20$ mm). Figure 5 shows the variations of (T_p, T_a) to the four vibration control models.

From Curve 1 (C-1) in Fig. 5a, the isolation band where $T_p < 1$ of the single-degree-of-freedom CS isolator is lower-bounded at $\omega = \sqrt{2}$, a determinative value independent of ζ_p [37]. Introducing the nonlinear HSLDS feature leads to C-2, which shows that the isolation band can be significantly extended, with the lower bound being reduced by 21.5% to $\omega = 1.11$. However, the nonlinearity can lead to multiple solutions and unstable solutions at a given frequency ω , which can result in impacts and thus should be avoided [38]. To this end, we can additionally introduce a linear vibration absorber, thus yielding C-3. In this case, the undesirable solution instability and multiplicity (known as the typical feature of the nonlinearity) in vibration control disappear. Meanwhile, the lower bound of the isolation band is further reduced to $\omega = 0.79$. Similar observations that vibration

absorbers can suppress the unfavorable nonlinear effects can also be found in [57], despite a different system under force excitation being studied.

To show that the observed advantage of C-3 is owing to both HSLDS vibration isolation and linear vibration absorption, the results of the purely linear vibration absorption governed by (21) are superposed as C-4 for comparison. Clearly, C-3, despite lacking the nonlinear feature (solution multiplicity), yields a broader isolation band and a higher vibration reduction efficacy. Besides, Fig. 5b compares the T_a values in the two cases, showing that the HSLDS vibration isolation can also help settle the absorber response.

From the above, on the one hand, the HSLDS feature can enhance passive vibration control, paving the way for exploring its advantage in the active case, i.e., the HSLDS feature is expected to benefit both cases. On the other hand, it qualitatively justifies Fig. 2, Fig. 3, Fig. 4 that an additional linear vibration absorber can confine the nonlinear behaviors of the HSLDS isolation without confining the excitation intensity.

5.1 Tuning mechanism of the control parameters

Achieving $R_p = 0$ at a given frequency ω by tuning (g, τ) requires that Eq. (16) with this pair of (g, τ) yields a fixed solution $(A_p, B_p) = (0, 0)$. In this case, (r_a, r_p) defined in (13) reduce to

$$\begin{cases} r'_a(\omega, t) = x_a(\omega, t) = A_a \cos(\omega t) + B_a \sin(\omega t), \\ r'_p(\omega, t) = -x_e(\omega, t) = -R_e \cos(\omega t), \end{cases} \quad (23)$$

where the superscript $(\cdot)'$ highlights $R_p = 0$. Then, comparing (13) and (23), $R_p = 0$ is equivalent to

$$\begin{cases} (A'_{ra}, B'_{ra}) = (A_a, B_a), \\ (A'_{rp}, B'_{rp}) = (-R_e, 0). \end{cases} \quad (24)$$

Substituting (24) into (16) gives

$$\begin{cases} G'_1(\omega, A_a, B_a, g, \tau) = \mu [B_a(v^2 - \omega^2) + 2(R_e \zeta_g - A_a(\zeta_g + \zeta_a))v\omega] - gR_e S = 0, \\ G'_2(\omega, A_a, B_a, g, \tau) = \mu [A_a(v^2 - \omega^2) + 2B_a(\zeta_g + \zeta_a)v\omega] - gR_e C = 0, \\ G'_3(\omega, A_a, B_a, g, \tau) = \mu (2A_a\omega v\zeta_a - B_a v^2) + 2\zeta_p R_e \omega + gR_e S = 0, \\ G'_4(\omega, A_a, B_a, g, \tau) = \mu (2B_a\omega v\zeta_a + A_a v^2) + n_1 R_e + 3n_3 R_e^3/4 - gR_e C = 0, \end{cases} \quad (25)$$

Remark 3 The HSLDS magnetic springs can be generalized to any mechanism satisfying (5) and (6), see various negative-stiffness structures in [52–55]. The benefits of using magnetic springs include that the two magnetic components are contactless and do not introduce additional degrees of motion freedom, thus simplifying modeling and practical installation.

5 Control tuning for complete vibration suppression at a given frequency

Built upon the energy concentration at the fundamental frequency, as shown in Fig. 4, we next address the inverse problem aforementioned in Fig. 3 to determine the values of (g, τ) that achieve an ideal zero antiresonance peak $R_p = 0$ at a given frequency ω , i.e., the so-called tuned pair. Meanwhile, the behaviors of the tuned pair as (ω, n_1, n_3) vary are investigated. The suppressed nonlinear effects of the HSLDS feature mean that the n_3 value should not significantly affect the tuned pair, as to be demonstrated.

where τ is implied in (C, S) . To solve (g, τ) for a given ω , we first solve (A_a, B_a) , which can be obtained from the two equations $G'_3(A_a, B_a) = 0$ and $G'_4(A_a, B_a) = 0$ as

$$\begin{cases} A'_a(\omega, g, \tau) = \frac{R_e [(-3n_3 R_e^2 + 4gC - 4n_1)v - 8\zeta_a(2\omega\zeta_p + gS)\omega]}{4\mu v(4\omega^2\zeta_a^2 + v^2)}, \\ B'_a(\omega, g, \tau) = \frac{R_e [((-3n_3 R_e^2 + 4gC - 4n_1)\zeta_a + 4\zeta_p v)\omega + 2gSv]}{2\mu v(4\omega^2\zeta_a^2 + v^2)}. \end{cases} \quad (26)$$

Back substituting (26) into $G'_1(A_a, B_a) = 0$ and $G'_2(A_a, B_a) = 0$, getting the numerators of the resulting equations yields

$$\begin{cases} G'_1(\omega, g, \tau) = \alpha_1^{[3]}(\omega, C, S)g + \alpha_0^{[3]}(\omega) = 0, \\ G'_2(\omega, g, \tau) = \beta_1^{[3]}(\omega, C, S)g + \beta_0^{[3]}(\omega) = 0, \end{cases} \quad (27)$$

which are two linear equations in g , where $(\alpha_i^{[3]}, \beta_i^{[3]})$, $i = 0, 1$ are real coefficients. Leaving the expressions of $(\alpha_0^{[3]}(\omega), \beta_0^{[3]}(\omega))$ to (55) of Appendix B, we point out that

$$\begin{cases} \alpha_1^{[3]}(\omega, C, S) = -[4(\zeta_g v^2 + \omega^2 \zeta_a)C + 2(1 - 4\zeta_a \zeta_g)vS]\omega, \\ \beta_1^{[3]}(\omega, C, S) = 4[2(\zeta_a \omega^2 + \zeta_g v^2)S + (4\zeta_a \zeta_g - 1)v\omega C]\omega, \end{cases} \quad (28)$$

each of which is a linear combination of (C, S) without residual terms, agreeing with the form of (25), where (C, S) are always the linear multipliers of g owing to the linear control law (9). From $G'_1(\omega, g, \tau) = 0$, one can solve g as

$$g(\omega, C, S) = -\frac{\alpha_0^{[3]}(\omega)}{\alpha_1^{[3]}(\omega, C, S)}. \quad (29)$$

Further substituting (29) into $G'_2(\omega, g, \tau) = 0$ leads to

$$-\beta_1^{[3]}(\omega, C, S)\alpha_0^{[3]}(\omega) + \beta_0^{[3]}(\omega)\alpha_1^{[3]}(\omega, C, S) = 0. \quad (30)$$

Given the forms of $(\alpha_1^{[3]}, \beta_1^{[3]})$ in (28), Eq. (30) can be combined into

$$\alpha^{[4]}(\omega)C + \beta^{[4]}(\omega)S = 0, \quad (31)$$

where $(\alpha^{[4]}, \beta^{[4]}) \in \mathbb{R}^2$ are independent of τ , as given in (56) of Appendix B. Hence, one can obtain

$$\tan(\omega\tau) = -\frac{\beta^{[4]}(\omega)}{\alpha^{[4]}(\omega)}. \quad (32)$$

Solving τ from (32) and then substituting the result into (29) lead to the tuned pair for achieving $R_p = 0$ at a given ω ,

$$u_t \Leftrightarrow \begin{cases} \tau_t(\omega) = \frac{1}{\omega} \left[a \tan\left(-\frac{\beta^{[4]}(\omega)}{\alpha^{[4]}(\omega)}\right) + (b-1)\pi \right], \\ g_t(\omega) = -\frac{\alpha_0^{[3]}(\omega)}{\alpha_1^{[3]}(\omega, \tau_t(\omega))}, \end{cases} \quad (33)$$

where the subscript $(\cdot)_t$ denotes ‘tuned’, u_t is called the tuned force, and $b = 1, 2, \dots$, which stems from the periodicity of the function $\tan(\cdot)$, is called the branch number [19]. Note that the values of the tuned delay τ_t can be multiple-valued due to the optional b values. Besides, the gain polarity $\text{sgn}(g_t)$ can reverse when b changes from odd to even, and vice versa, which is attributed to the fact that $\alpha_1^{[3]}$ is an odd function of τ_t (see (28)) and that $b = 2, 4, \dots$ correspond to odd multiples of the half-cycle delay shifts. Particularly, if the nonlinear effect is negligible, the tuned pair reduces to

$$\begin{cases} \tau_{t,n3=0}(\omega) = \frac{1}{\omega} \left[\text{atan}\left(-\frac{\beta_{n3=0}^{[4]}(\omega)}{\alpha_{n3=0}^{[4]}(\omega)}\right) + (b-1)\pi \right], \\ g_{t,n3=0}(\omega) = -\frac{\alpha_{0,n3=0}^{[3]}(\omega)}{\alpha_{1,n3=0}^{[3]}(\omega, \tau_{t,n3=0}(\omega))}. \end{cases} \quad (34)$$

where $(\alpha_{n3=0}^{[4]}, \beta_{n3=0}^{[4]}, \alpha_{0,n3=0}^{[3]})$ represent the corresponding coefficients with $n_3 = 0$ substituted in.

Remark 4 Eq. (34) with $n_1 = 1$ corresponds to the tuned pair in the original linear CS case (without magnetic springs), which holds in both the force- (Fig. 1a) and the displacement-excited cases.

5.2 Analysis of the tuned pair (g_t, τ_t)

Taking $(n_1, n_3, R_e) = (0.6, 1, 0.005)$ as an example, the variations of the tuned pair (g_t, τ_t) with respect to the vibration frequency ω for $b = 1, 2, 3$ are shown in Fig. 6.

Figure 6a depicts the variations of the tuned delay τ_t while excluding $\tau_t < 0$. It shows that increasing the branch number b can prolong τ_t due to the delay shift $b\pi/\omega$ in (33). Besides, for a given branch b , the variations of τ_t jump at $\omega_c = 1.0028$, which is caused by the sign reversal of $\beta^{[4]}/\alpha^{[4]}$ as ω crosses ω_c , see (32). Since $(\beta^{[4]}, \alpha^{[4]})$ are continuous to ω , the jump does not affect the implementation of τ_t at ω_c by switching b . On the other hand, τ_t reduces as ω increases. For the classical feedback-controlled DRs [19], it can severely limit the operable frequency band since prolonging τ_t by increasing b can destabilize the system, whereas the stable values of τ_t within the high-frequency band can be too small to be practically implemented. In comparison, manipulating b to alter τ_t is no longer an issue here since the feedforward control (9) does not induce instability.

Figure 6b shows the variations of the tuned gain g_t . Clearly, the values of b can affect the gain polarity, as aforementioned. The feature that $|g_t| \rightarrow \infty$ as $\omega \rightarrow 0$ can have an intuitive physical explanation. Consider the limit case $\omega = 0$, i.e., x_e is constant, it requires a unidirectional force $u > 0$ to achieve $x_p \equiv 0$. However, this force makes $\ddot{x}_a > 0$ so it must increase with time t to balance the growing forces on the primary exerted by (k_a, c_a) . For $\tau_t \rightarrow \infty$ as $\omega \rightarrow 0$, this is due to that a larger delay τ is required when ω is small to yield the desired phase difference $\omega\tau$ between u_t and x_e , as pointed out in [27].

5.3 Effects of (n_1, n_3)

We next evaluate the effects of (n_1, n_3) on (g_t, τ_t) . To facilitate demonstration, we now focus on the tuned gain amplitude $|g_t|$ and the first two branches $b = 1, 2$ of the positive delays $\tau_t > 0$ while discarding the jumped values. Similar to Figs. 2 and 3, we first vary n_1 while fixing $(n_3, R_e) = (1, 0.005)$. The resulting effects on $(|g_t|, \tau_t)$ are shown in Fig. 7a, b, where the original CS case with $(n_1, n_3) = (1, 0)$, i.e., Eq. (34), is marked as the solid black curves for reference.

From Fig. 7a, the effect of n_1 on τ_t is limited, so the dynamic stiffness of the primary does not significantly affect the phase difference between the tuned force u_t and x_e . From Fig. 7b, the tuned gain amplitude $|g_t|$ depends on n_1 , and reducing n_1 leads to a smaller $|g_t|$, which can be explained as follows. In the steady state where $x_p \equiv 0$, a smaller dynamic

Fig. 6 Variations of the tuned pair (g_t , τ_t) with respect to frequency ω for $(n_1, n_3, R_e) = (0.6, 1, 0.005)$ and $b = 1, 2, 3$. **a** τ_t versus ω with negative cases $\tau_t < 0$ omitted. **b** g_t versus ω

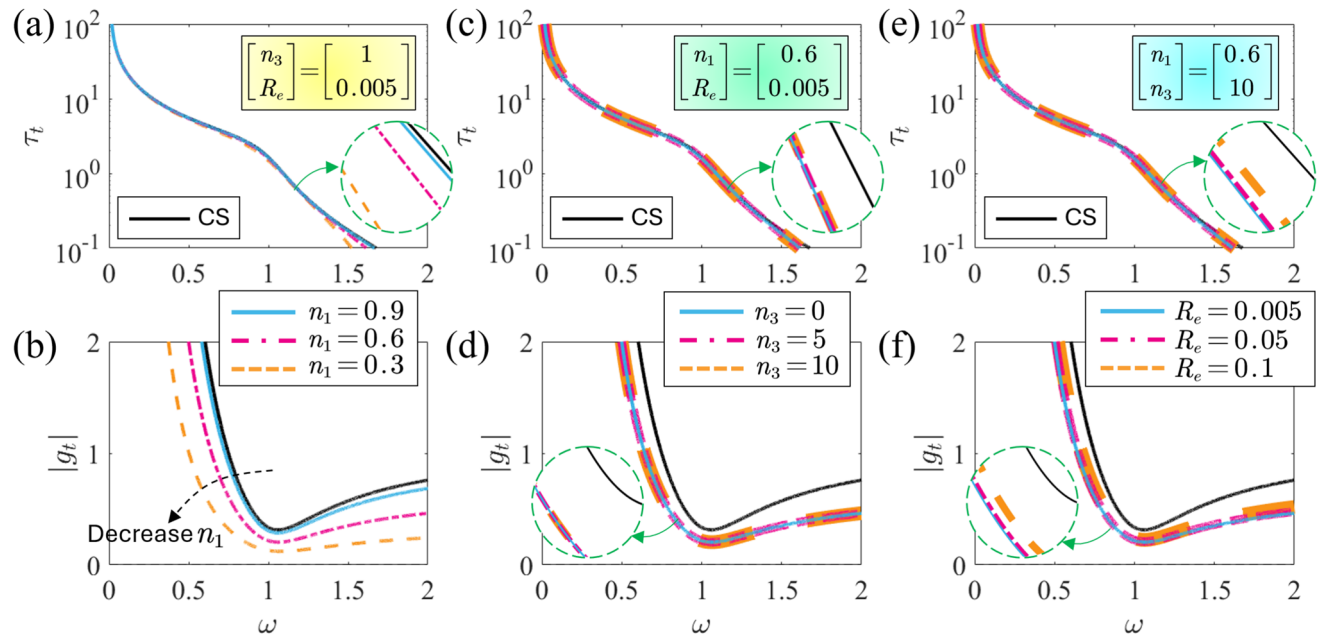
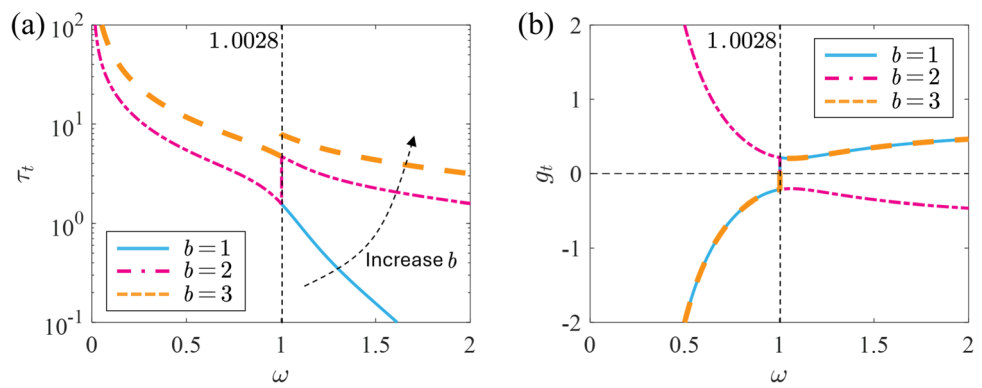


Fig. 7 Effects of (n_1, n_3) on the tuned pair $(|g_t|, \tau_t)$. **a, b** n_1 varies while fixing $(n_3, R_e) = (1, 0.005)$. **c, d** n_3 varies while fixing $(n_1, R_e) = (0.6, 0.005)$. **e, f** R_e varies while fixing $(n_1, n_3) = (0.6, 10)$ to further test the nonlinear effect when $n_3 \neq 0$. The CS case corresponds to the condition $(n_1, n_3) = (1, 0)$

stiffness results in smaller forces arising from the displacement excitation x_e exerted on the primary structure. Hence, to neutralize such forces to yield $x_p \equiv 0$, the total forces resulting from the tuned DR exerted on the primary structure should also reduce as n_1 reduces. Since the relationship between the forces of the DR exerted on the primary structure via (k_a, c_a) and u follows a linear transfer function, the required smaller total forces of the tuned DR necessitate smaller values of $|g_t|$, as $|g_t|$ governs the tuned force amplitude (i.e., $|u_t|$ for a given R_e). Then, a smaller n_1 signifies possible lower energy consumption of u_t , as to be discussed in Sect. 7.2.

Figure 7c, d show the results when varying n_3 and fixing n_1 . Clearly, $(|g_t|, \tau_t)$ both are almost independent of the values of n_3 . Considering that the nonlinear force resulting from $n_3 > 0$ is also related to the motion amplitude $|r_p|$, Fig. 7(e, f) further test the effect of n_3 while increas-

ing the excitation amplitude R_e . It shows that even with an excitation with $R_e = 0.1$ (i.e., $\bar{R}_e = |\bar{x}_e| = 0.1$ m, which is a very exaggerated intensity for our experimental setup that yields the dimensionless system (19)), n_3 still poses limited effects on $(|g_t|, \tau_t)$ over a broad frequency band, indicating that the residual vibrations arising from the nonlinearity (super- and sub-harmonics) are limited. This provides a cross-verification of the results in Sect. 3.3.

Remark 5 The limited effects of n_3 with respect to the relative decrease in n_1 and the increase in R_e in both the non-tuned and tuned cases show that the linear absorber can suppress the nonlinearity of the HSLDS feature within a broad range of the excitation intensity, regardless of whether the primary structure is settled or not (i.e., whether $R_p = 0$ or not). Owing to this, the HSLDS feature can now be regarded as mainly

providing linear negative stiffness to reduce the dynamic stiffness of the primary structure.

Remark 6 A linear absorber can suppress the nonlinearity of the HSLDS feature but does not guarantee that the suppressed nonlinearity is definitely negligible. However, even if a sufficiently strong excitation intensity leads to notable nonlinearity with super- and sub-harmonics, causing efficacy deterioration of a tuned DR with significant residual vibrations, it should not invalidate the absorber's benefit. At least, it can increase the lower threshold of the excitation intensity causing such deterioration, as qualitatively discussed in Fig. 5 by comparing C-2 and C-3. Moreover, a small or moderate excitation intensity is also common in practice, which justifies the applicability of the considered model tested with a limited excitation intensity. The control problem with insuppressible strong nonlinearity deserves a special investigation.

6 Vibration control by the tuned DR and enhancement

This section examines the tuned DR from the perspective of frequency responses and seeks the benefits in the HSLDS case compared with the CS case. Given the negligible nonlinear effects, we only consider the effect of $n_1 < 1$ while fixing $n_3 = 0$ to avoid redundancy. Then, R_e does not affect the tuned pair (g_t, τ_t) and the displacement transmissibility (T_p, T_a) due to the linearization, which agrees with the fact that R_e does not affect the common solution Θ of Eq. (16) when $n_3 = 0$.

6.1 Vibration suppression efficacy

We start with a sample frequency $\omega = 0.8$ to test the effectiveness of the tuned pair. Similar to Fig. 7, the tuned pair corresponding to the smallest positive tuned delay τ_t is considered hereafter. One then has $(g_t, \tau_t) = (0.784, 3.063)$ in the CS case. Particularly, $(g_t, \tau_t) = (0.481, 2.983)$ in the HSLDS case with $n_1 = 0.6$, which verifies the corresponding antiresonance peak in Fig. 3a, c. The variations of (T_p, T_a) with respect to ω for $(g, \tau) = (g_t, \tau_t)$ in various CS and HSLDS cases are compared in Fig. 8.

The variations of T_p are shown in Fig. 8a, where the passive case is also included. It shows that a passive (damped) absorber can only achieve a non-zero antiresonance peak around its natural frequency, signified by the dimensionless value of $\omega = v$, see the definition of v in (3). Clearly, the vibration suppression is far from complete. In the active case, the tuned DR achieves the expected complete vibration suppression by creating $T_p = 0$ at the designated frequency $\omega = 0.8$, regardless of whether the primary structure is CS or HSLDS. Figure 8b reflects the absorber's response ampli-

tude, showing that the HSLDS feature can also confine the DR's working space, as to be further discussed in Sect. 7.1, see also Fig. 5b for similar results in the passive case.

Revisiting Fig. 8a, although the tuned DR achieves $T_p = 0$ at the designated frequency (next called 'tuning frequency') in different cases, the non-zero values of T_p differ within the entire frequency band. The benefits of the HSLDS feature for the vibration control of the primary structure are then sought in terms of T_p from the perspective of the antiresonance valley at the tuning frequency, aiming to enhance the control robustness of the DR to better achieve complete vibration suppression.

6.2 Extended antiresonance valley

The antiresonance valley for the tuning frequency at $\omega = 0.8$ is circled in Fig. 8a. Clearly, the HSLDS feature can extend the valley in the CS case, and reducing n_1 , which yields a lower dynamic stiffness, is favorable to the extension. An extended antiresonance valley means lower sensitivity of the primary structure with respect to the mismatch between the detected vibration frequency (i.e., the tuning frequency) and the actual one. Since the consequence of the frequency mismatch is causing deviations in the tuned pair (g_t, τ_t) and since T_p for a continuous system should not significantly vary under small parametric variations, the conclusion of lower sensitivity can be generalized from the frequency mismatch to the case where the parameters involved in the tuning mechanism of (g_t, τ_t) are inaccurate. Hence, an extended antiresonance valley can suppress the residual vibrations caused by the inaccurate (g_t, τ_t) to enhance control robustness, which complements the analysis in [22].

To evaluate the antiresonance valley width, we first extract the T_p values around the tuning frequency from Fig. 8a, thus arriving at Fig. 9a, where the CS and the HSLDS cases with $n_1 = 0.6$ are considered.

In [22], the width of the antiresonance valley was evaluated by the difference between the two frequencies that achieve the same T_p value on the two sides of the tuning frequency. As marked in Fig. 9a, if selecting $T_p = 0.2$ to evaluate the valley width, say W_{AV} , then $W_{AV} = 0.021$ and $W_{AV} = 0.063$ in the CS and the HSLDS cases, respectively. However, this evaluation can omit the possible fluctuations within the valley. As a further step, we here use the area enclosed by the T_p curve and the ω axis within a given frequency interval around the tuning frequency for evaluation, i.e.,

$$A_{AV}(\omega_t) = \int_{\omega_t - \delta_{low}}^{\omega_t + \delta_{up}} T_p(\omega, \omega_t) d\omega, \quad (35)$$

where $(\delta_{low}, \delta_{up}) \in \mathbb{R}^{2+}$ and ω_t represents the tuning frequency to discriminate it from ω , which serves as a variable

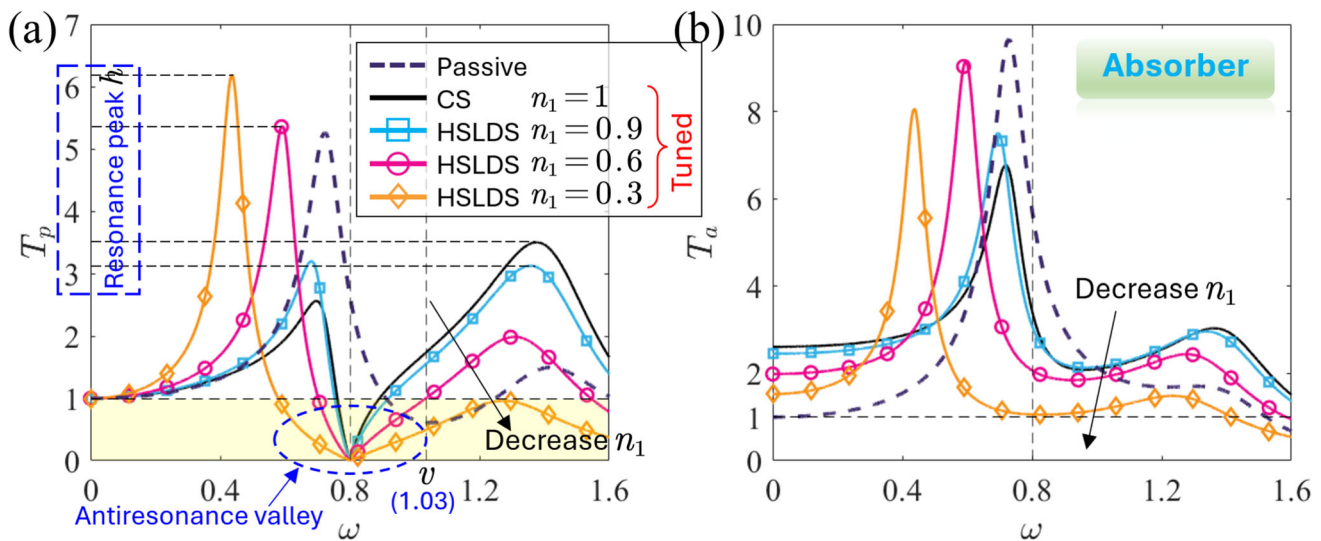


Fig. 8 Demonstration of the complete vibration suppression at $\omega = 0.8$ by tuning the DR in different CS and HSLDS cases. **a** T_p versus ω . Vibration reduction corresponds to $T_p < 1$. **b** T_a versus ω

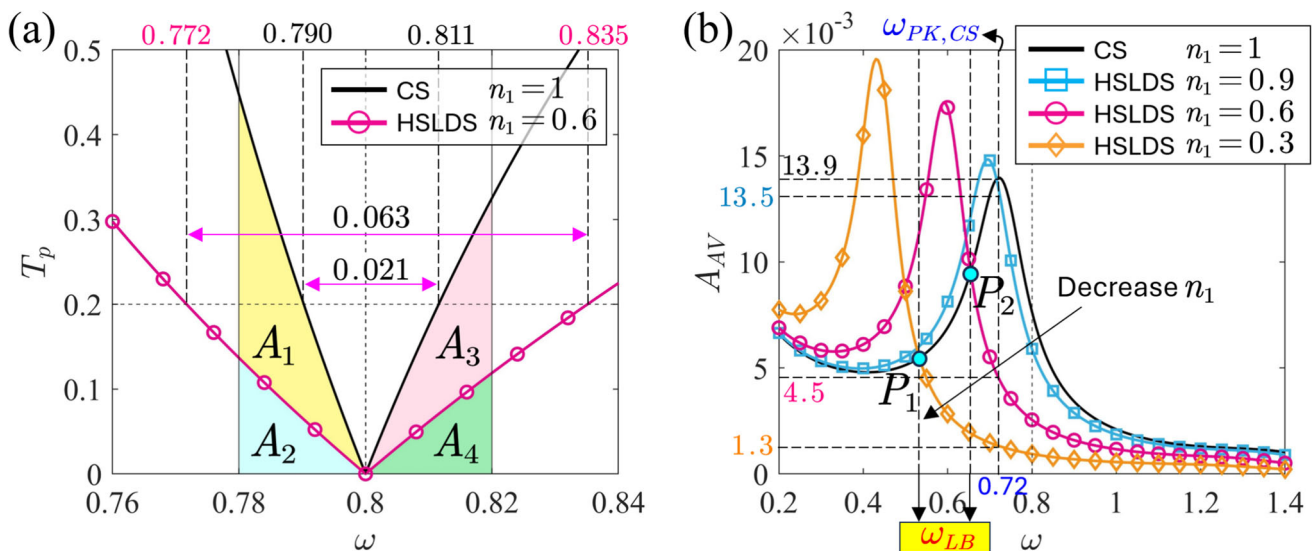


Fig. 9 **a** Zoomed plot of Fig. 8a for the antiresonance valley in the CS case and the HSLDS case with $n_1 = 0.6$. **b** A_{AV} versus ω in different cases, where ω represents the frequency where to achieve $T_p = 0$

in (35). For instance, if $\omega_t = 0.8$ and $\delta_{low} = \delta_{up} = 0.02$, then $A_{AV} = A_1 + A_2 + A_3 + A_4$ in the CS case, and $A_{AV} = A_2 + A_4$ in the HSLDS case, where the areas (A_1, A_2, A_3, A_4) are defined in Fig. 9a. Clearly, a smaller A_{AV} value yields a wider valley and thus is preferred. Compared with the method in [22], evaluating the antiresonance valley via A_{AV} rather than W_{AV} actively defines the frequency interval of the antiresonance valley and takes all frequency grids within the valley interval of interest into account, thus yielding a more comprehensive quantitative evaluation of the vibration control robustness against the inaccuracies in (g_t, τ_t) .

Since the T_p value for each ω is known, as shown in Fig. 9a, one can use the trapezoidal rule to numerically calculate (35). Sweeping ω_t , the obtained values of A_{AV} for $\delta_{low} = \delta_{up} = 0.02$ in different cases are shown in Fig. 9b, where the abscissa is denoted by the excitation frequency ω assuming that $\omega_t = \omega$ to streamline with the values of (g_t, τ_t) in Figs. 6 and 7. A_{AV} reflected in Fig. 9a is then the special case $\omega = 0.8$ of Fig. 9b. We can find that the HSLDS can extend the antiresonance valley within a broadband range, and reducing n_1 can improve the extension while achieving the extension within a lower-frequency band. However, it can compromise the raising of A_{AV} within an even lower band.

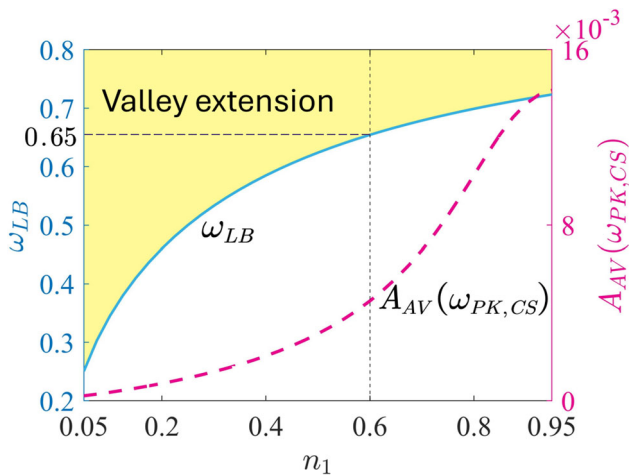


Fig. 10 Variations of $(\omega_{LB}, A_{AV}(\omega_{PK,CS}))$ versus n_1 , a result generalized from Fig. 9b

To guide the design of n_1 , the frequency corresponding to the intersection point between the A_{AV} curve in the HSLDS case and that in the CS case signifies the lower-frequency bound, denoted as ω_{LB} , of the valley extension, see (P_1, P_2) in Fig. 9b. Besides, the A_{AV} value at the peak frequency, say $\omega_{PK,CS}$, of the A_{AV} curve in the CS case is considered as a supplementary indicator, e.g., $\omega_{PK,CS} = 0.72$ and $A_{AV}(\omega_{PK,CS}) = 13.9 \times 10^{-3}$ in Fig. 9b. Clearly, the valley extension expects smaller values of both ω_{LB} and $A_{AV}(\omega_{PK,CS})$, the variations of which with respect to n_1 are shown in Fig. 10.

From Fig. 10, reducing n_1 simultaneously yields smaller values of ω_{LB} and $A_{AV}(\omega_{PK,CS})$. To utilize the extended valley to enhance the vibration control performance, one should design n_1 to meet $\omega > \omega_{LB}$. Besides, considering that a smaller value of n_1 is desired to maximize the benefits of the valley extension, the primary structure should have a certain degree of flexibility to simplify the practical implementation of the negative stiffness $n_1 < 1$. That is, one should deploy an appropriately soft stiffness $\bar{k}_p$ if allowed. Note also that the extended antiresonance valley resulting from $n_1 < 1$ with $n_3 = 0$ can also be obtained by directly reducing $\bar{k}_p$, which, however, reduces the static load capacity of the primary.

Remark 7 Using the area enclosed by the T_p curve (i.e., Fig. 10) as the metric for optimizing the parameter, i.e., n_1 here, is known as H_2 optimization [65]. Another widely known design metric for the vibration absorber is H_∞ optimization, which focuses on the resonance suppression of the T_p curve, see also our previous work [22] for using it to guide the DR control. However, additional issues need to be considered if one aims to achieve resonance control when introducing the HSLDS feature, i.e., reducing n_1 . The trade-off is briefly discussed in Appendix C.

Remark 8 The extended antiresonance valley was also achieved in [21, 22] via special control laws. In comparison, we achieve the extension with purely mechanical strength, based on a simple structural modification (by installing magnets). In addition to control simplification, it reduces the actuator in [21] for exerting stabilizing forces on the primary and the monitoring sensor in [22] for the primary.

7 Absorber behaviors of the tuned DR

This section considers (steady-state) behaviors of the DR subsystem when achieving complete vibration suppression $T_p = 0$ at a given ω , aiming to establish the design guideline of the HSLDS parameters to match the hardware constraint of the DR's practical application. The analysis is twofold: i). the absorber vibration, and ii). the amplitude and the energy consumption of the tuned force u_t .

7.1 Absorber vibration

Given the condition $T_p = 0$, i.e., $R_p = 0$, we proceed with the analysis from Eq. (25), which leads to

$$\begin{cases} G'_5(\omega, A_a, B_a) = G'_1(\omega, A_a, B_a, g, \tau) + G'_3(\omega, A_a, B_a, g, \tau) \\ \quad = [-\mu(2(A_a - R_e)v\zeta_g + B_a\omega) + 2R_e\zeta_p] \\ \quad \quad \omega = 0, \\ G'_6(\omega, A_a, B_a) = G'_2(\omega, A_a, B_a, g, \tau) - G'_4(\omega, A_a, B_a, g, \tau) \\ \quad = \mu(2B_a\omega v\zeta_g - A_a\omega^2) \\ \quad \quad - n_1R_e - 3n_3R_e^3/4 = 0, \end{cases} \quad (36)$$

which are two linear equations in (A_a, B_a) and can be solved as

$$\begin{cases} A'_a(\omega) = \frac{R_e[16v\zeta_g(\zeta_p + \mu v\zeta_g) - 4n_1 - 3n_3R_e^2]}{4\mu(4v^2\zeta_g^2 + \omega^2)}, \\ B'_a(\omega) = \frac{R_e[v(4n_1 + 4\mu\omega^2 + 3n_3R_e^2)\zeta_g + 4\zeta_p\omega^2]}{2\mu\omega(4v^2\zeta_g^2 + \omega^2)}. \end{cases} \quad (37)$$

Note that Eq. (37) is equivalent to (26) with $(g, \tau) = (g_t, \tau_t)$. The amplitude of the absorber vibration when the tuned DR achieves $T_p = 0$ at ω is then governed by

$$R'_a(\omega) = \sqrt{A'_a(\omega) + B'_a(\omega)} = R_e T'_a(\omega), \quad (38)$$

where T'_a denotes the displacement transmissibility from x_e to x_a when $T_p = 0$ and is a special case of T_a in (22), with

$$T'_a(\omega) = \frac{R'_a(\omega)}{R_e}$$

$$= \sqrt{\frac{64(\mu v \zeta_g + \zeta_p)^2 \omega^2 + 9(n_3 R_e^2 + \frac{4n_1}{3})^2}{16\mu^2 \omega^2 (4v^2 \zeta_g^2 + \omega^2)}}. \quad (39)$$

Clearly, T'_a determines R'_a for a given R_e . Furthermore, T'_a only depends on the structural parameters in addition to the vibration frequency ω and is independent of the tuned control parameters (g_t , τ_t).

Remark 9 In fact, T'_a holds regardless of the form of the control laws. The reason is that the total force exerted by the tuned DR absorber upon the primary structure, including the tuned force u_t plus the forces related to $(\bar{c}_a, \bar{k}_a)$, for complete vibration suppression at a given frequency ω is definite since the force exerted by the displacement excitation x_e when $R_p = 0$ is definite. Hence, the corresponding dynamics of the tuned absorber, which are governed by the reaction force, should also be definite. Similar conclusions can also be found in the cases where the DR is feedback-controlled [19].

Then, revisiting (39), n_3 appears in the form of $n_3 R_e^2$ so its effect on T'_a must be very limited compared with n_1 given that $R_e \ll 1 < 4/3$. This also justifies the conclusions with $n_3 = 0$ in Sect. 6. On the other hand, one has

$$\frac{\partial T'_a}{\partial n_1} = \frac{1}{2T'_a} \frac{\partial (T'_a)^2}{\partial n_1} = \frac{1}{2T'_a} \frac{3n_3 R_e^2 + 4n_1}{2\omega^2 \mu^2 (4v^2 \zeta_g^2 + \omega^2)}. \quad (40)$$

Considering that $\partial T'_a / \partial n_1 > 0$ requires $3n_3 R_e^2 + 4n_1 > 0$, or equivalently, $n_1 > -3n_3 R_e^2 / 4$, T'_a can always be reduced in the HSLDS case $0 < n_1 < 1$ compared with the CS case $n_1 = 1$, regardless of ω , which verifies Fig. 8b. The variations of T'_a with respect to n_1 when the DR is tuned as per different ω with $n_3 = 0$ are shown in Fig. 11.

In Fig. 11, T'_a reduces as n_1 reduces regardless of ω , and the result in the case $\omega = 0.8$ covers that in Fig. 8b at the same frequency, both of which are expected. Note that T'_a in the CS case can be unduly large when handling the low-frequency vibrations, so the absorber may need a large working space in practice, which can limit the DR's applications.

On the other hand, a known issue of the classical feedback-controlled DRs is that their low-frequency operability can be severely restricted by system instability [20]. This work avoids instability by adopting a feedforward-based control logic (9), which can then extend the operable bandwidth, as aforementioned in Sect. 2.2. Hence, the confined working space of the DR absorber, signified by the reduction of T'_a when $n_1 < 1$ compared with T'_a when $n_1 = 1$, indicates that the HSLDS feature can better leverage the strength of the feedforward control.

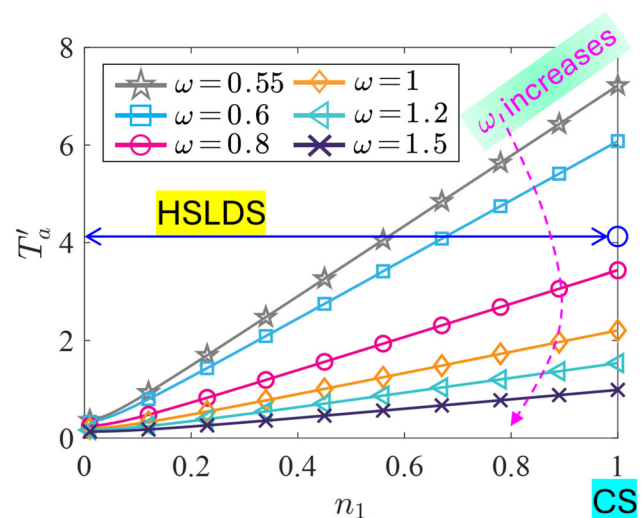


Fig. 11 Variations of T'_a versus n_1 when the tuned DR achieves $T_p = 0$ at different frequencies ω . The analysis is based on $n_3 = 0$ so $n_1 = 1$ and $n_1 < 1$ correspond to the CS and the HSLDS cases, respectively

7.2 Amplitude and energy consumption of the tuned force

From the first equation of (4), when the complete vibration suppression $T_p = 0$ or $R_p = 0$ is achieved, u_t satisfies

$$\mu [\ddot{x}_a + 2(\zeta_a + \zeta_g) v \dot{x}_a + v^2 x_a - 2v \zeta_g \dot{x}_e] = u_t. \quad (41)$$

Substituting $x_e(t) = R_e \cos(\omega t)$ and $x_a = x'_a(t)$, with $x'_a(t) = A'_a \cos(\omega t) + B'_a \sin(\omega t)$, into (41) yields

$$u_t(\omega, t) = A'_u(\omega) \cos(\omega t) + B'_u(\omega) \sin(\omega t), \quad (42)$$

which is the general solution of the tuned force u_t to achieve $R_p = 0$, regardless of the form of control laws, with

$$\begin{cases} A'_u(\omega) = \mu [A'_a(v^2 - \omega^2) + 2B'_a(\zeta_a + \zeta_g)v\omega], \\ B'_u(\omega) = \mu [B'_a(v^2 - \omega^2) + 2(\zeta_g R_e - (\zeta_a + \zeta_g)A'_a)v\omega]. \end{cases} \quad (43)$$

One can then obtain the general form of the amplitude of u_t as

$$R_{ut}(\omega) = |u_t(\omega, t)| = \sqrt{(A'_u(\omega))^2 + (B'_u(\omega))^2}. \quad (44)$$

In light of [19], the instantaneous power of u_t in the steady state $R_p = 0$ is governed by

$$p_{ut}(\omega, t) = u_t(\omega, t) \dot{x}'_a(t), \quad (45)$$

where $\dot{x}'_a(t) = \omega (B'_a \cos(\omega t) - A'_a \sin(\omega t))$. Letting $(C_n, S_n) = (\cos(n\omega t), \sin(n\omega t))$, which should be discrim-

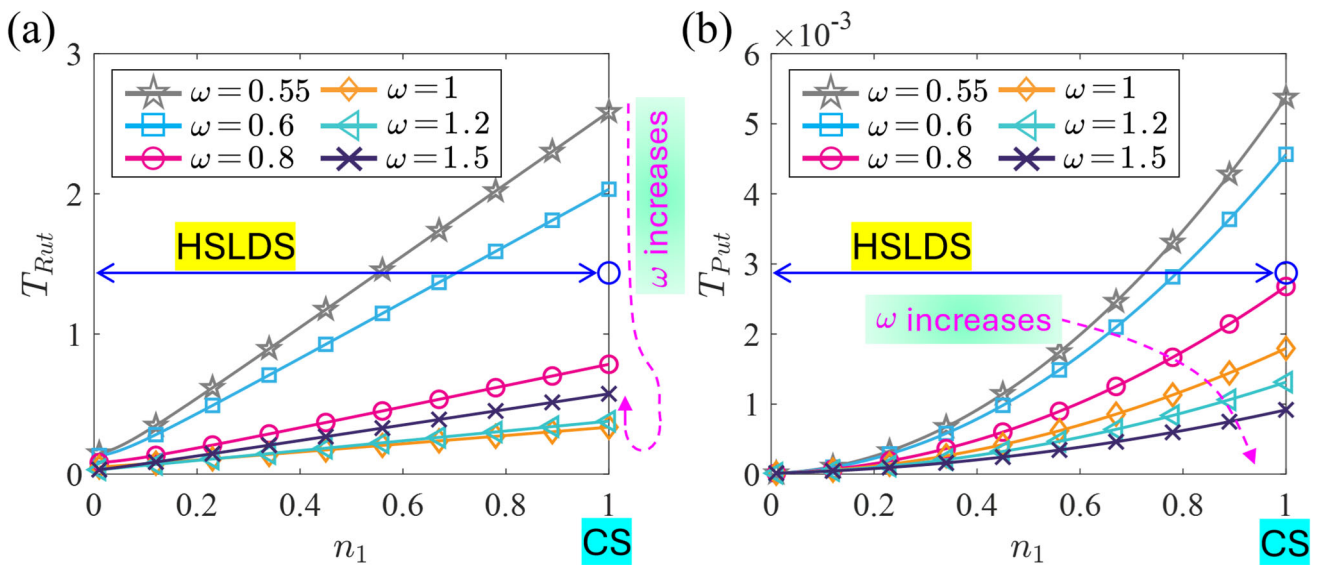


Fig. 12 Variations of (T_{Rut}, T_{Put}) versus n_1 when the tuned DR achieves $T_p = 0$ at different frequencies ω . **a** T_{Rut} versus n_1 . **b** T_{Put} versus n_1

inated from (C, S) in (16), $p_{ut}(\omega, t)$ can be expanded as

$$\begin{aligned} p_{ut}(\omega, t) &= (A'_u C_1 + B'_u S_1) \times \omega (B'_a C_1 - A'_a S_1) \\ &= \frac{\omega}{2} [(B'_u B'_a - A'_u A'_a) S_2 + (B'_u A'_a + A'_u B'_a) C_2 \\ &\quad + A'_u B'_a - B'_u A'_a]. \end{aligned} \quad (46)$$

The average power of the energy consumption of u_t over a cycle $T = 2\pi/\omega$ is then

$$\begin{aligned} P_{ut,av}(\omega) &= \frac{1}{T} \int_0^T p_{ut}(t) dt = \frac{1}{T} \int_0^T p_{ut}(t) dt \\ &= \frac{1}{T} \int_0^T \frac{\omega}{2} (A'_u B'_a - B'_u A'_a) dt \\ &= \frac{\omega}{2} (A'_u(\omega) B'_a(\omega) - B'_u(\omega) A'_a(\omega)). \end{aligned} \quad (47)$$

From the forms of (37) and (43), the excitation amplitude R_e acts as a scaler of $(R_{ut}, P_{ut,av})$. Similar to (22), we define

$$(T_{Rut}(\omega), T_{Put}(\omega)) = \left(\frac{R_{ut}(\omega)}{R_e}, \frac{P_{ut,av}(\omega)}{R_e} \right). \quad (48)$$

Note that even if $\zeta_g R_e$ is small enough to be negligible, (T_{Rut}, T_{Put}) still depend on R_e via the term $n_3 R_e^2$ in (A'_a, B'_a) as per (37), and n_3 takes effect via the same term. However, the effects of (n_3, R_e) on (T_{Rut}, T_{Put}) are negligible, as aforementioned. Following Fig. 11, we fix $n_3 = 0$ to omit such negligible effects. The variations of (T_{Rut}, T_{Put}) with respect to n_1 when tuning the DR at different frequencies ω are shown in Fig. 12. Clearly, a smaller tuned force u_t and a lower energy consumption require smaller values of (T_{Rut}, T_{Put}) .

Figure 12a reflects the variations of the tuned force amplitude R_{ut} . For a given n_1 , R_{ut} first reduces and then increases as ω increases, which agrees with the variations of the tuned gain amplitude $|g_t|$ shown in Figs. 6b and 7b, d, f. Note that R_{ut} is minimized around $\omega = 1$ regardless of n_1 , which indicates that a smaller active force is needed to resonate the absorber to achieve $T_p = 0$. The reason is simply that the (dimensionless) natural frequency of the adopted absorber lies around $\omega = 1$, i.e., $\omega_a/\omega_p = v \approx 1.03$, see also Fig. 8a, so the absorber structure itself can provide sufficient inherent resonance. On the other hand, for all the frequencies ω , R_{ut} reduces as n_1 reduces, and thus the HSLDS feature $n_1 < 1$ can reduce the rated force output of the actuator, again simplifying the DR's implementation.

Different from R_{ut} , Fig. 12b shows that for a given n_1 , the energy consumption of the tuned DR monotonically reduces as ω increases. It means that the reduced absorber motion amplitude due to the increase in ω , as reflected in Fig. 11, takes a more significant effect than the variations of R_{ut} on p_{ut} . As for the effect of n_1 , T_{Put} reduces as n_1 reduces regardless of ω . Hence, the HSLDS feature can also reduce the energy consumption compared with the CS case $n_1 = 1$.

Remark 10 Figures 11 and 12 can be used to design the n_1 value or examine its operability as per the practical requirements for (T'_a, T_{Rut}, T_{Put}) and the range of the vibration frequency ω . Particularly, n_1 has a closed-form upper bound depending on both the upper bound of T'_a and the lower bound of the frequency range, given the closed-form of (39) and the condition (40). If

$$T'_a < T'_{a,\max}, \omega > \omega_{\min}, \quad (49)$$

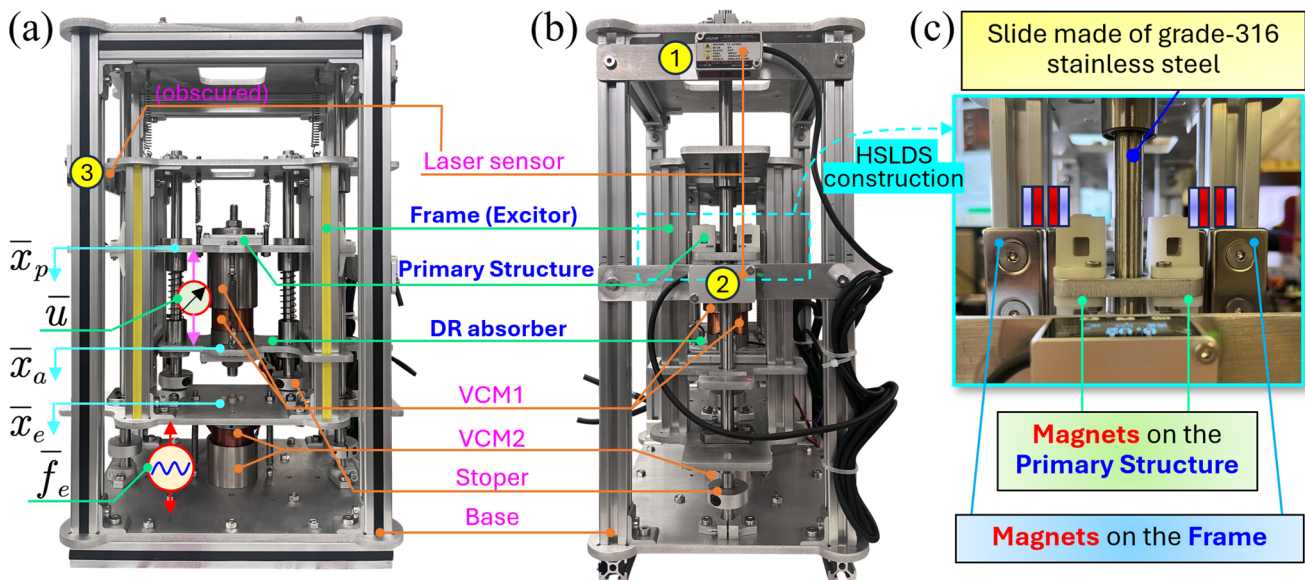


Fig. 13 Mechanical main body of the experimental setup. VCM refers to the voice coil motor. **a** Front view. **b** Right-side view. **c** Close-up of the introduction of magnetic springs. Subplots **a**, **b** correspond to the CS case, and **c** introduces HSLDS magnetic springs

where $(T'_{a,\max} \omega_{\min}) \in \mathbb{R}^{2+}$, we then have $n_1 < n_{1,\max}$, with

$$n_{1,\max} = \omega_{\min} \sqrt{\mu^2 (4v^2 \zeta_g^2 + \omega_{\min}^2) (T'_{a,\max})^2 - 4(\mu v \zeta_g + \zeta_p)^2 - \frac{3}{4} n_3 R_e^2}. \quad (50)$$

Clearly, the smaller values of $(T'_{a,\max} \omega_{\min})$ yield a smaller $n_{1,\max}$, which agrees with Fig. 11.

Remark 11 In light of Remark 5, the HSLDS feature with suppressed nonlinearity linearly reduces the dynamic stiffness of the primary structure, thus reducing the external force exerted on the system. This is the basis of the reduced tuned active force and the confined absorber's motion as n_1 reduces when achieving the complete vibration suppression at a given frequency. However, the HSLDS feature is not guaranteed to be beneficial if frequency variations are involved, e.g., the tuned force amplitude for a given n_1 can be increased as per Fig. 12a. Besides, reducing n_1 can shrink the low-frequency antiresonance valley as per to Fig. 9b, and amplify the resonance response of the primary (see Appendix C). Hence, the benefits of the HSLDS feature should not be taken for granted, despite the external force being reduced. Then, the parametric analysis of the HSLDS feature is necessary to guide practical design.

8 Case studies

This section verifies the dynamical modelling of the tuned DR and the benefits of the HSLDS feature. Both numerical and experimental tests are conducted. The corresponding

SIMULINK-based simulation models and results are available in Appendix D.

8.1 Experimental setup

Following Fig. 1b, the mechanical body in the original CS case is constructed as Fig. 13a, b. The frame is excited by a force $\bar{f}_e$ to generate the displacement excitation $\bar{x}_e$. The upper and lower platforms inside the frame constitute the primary structure and the DR absorber, respectively. Two voice coil motors are deployed to generate $\bar{f}_e$ and the active force $\bar{u}$ between the primary and the absorber to achieve the DR scheme, i.e., VCM1 for $\bar{u}$ and VCM2 for $\bar{f}_e$. The VCM control by manipulating the coil current has been detailed in our previous works [22, 27]. The displacements $(\bar{x}_e, \bar{x}_p, \bar{x}_a)$ are real-time monitored by three laser sensors fixed on the base, as numbered in yellow circles, in which laser 3 for $\bar{x}_p$ is obscured.

Figure 13c shows the implementation of the HSLDS feature by introducing magnets, arranged in such a way that the ones fixed on the frame and the primary repel each other. Note that the magnets on the primary structure are fixed, and we only introduce or remove the magnets on the frame. In addition, the slide is made of grade 316 stainless steel to avoid the magnetic coupling between the primary and the base, and most other mechanical components of the setup are made of grade 304 stainless steel and aluminum to achieve a similar effect. Based on system measurement and identification, we obtain

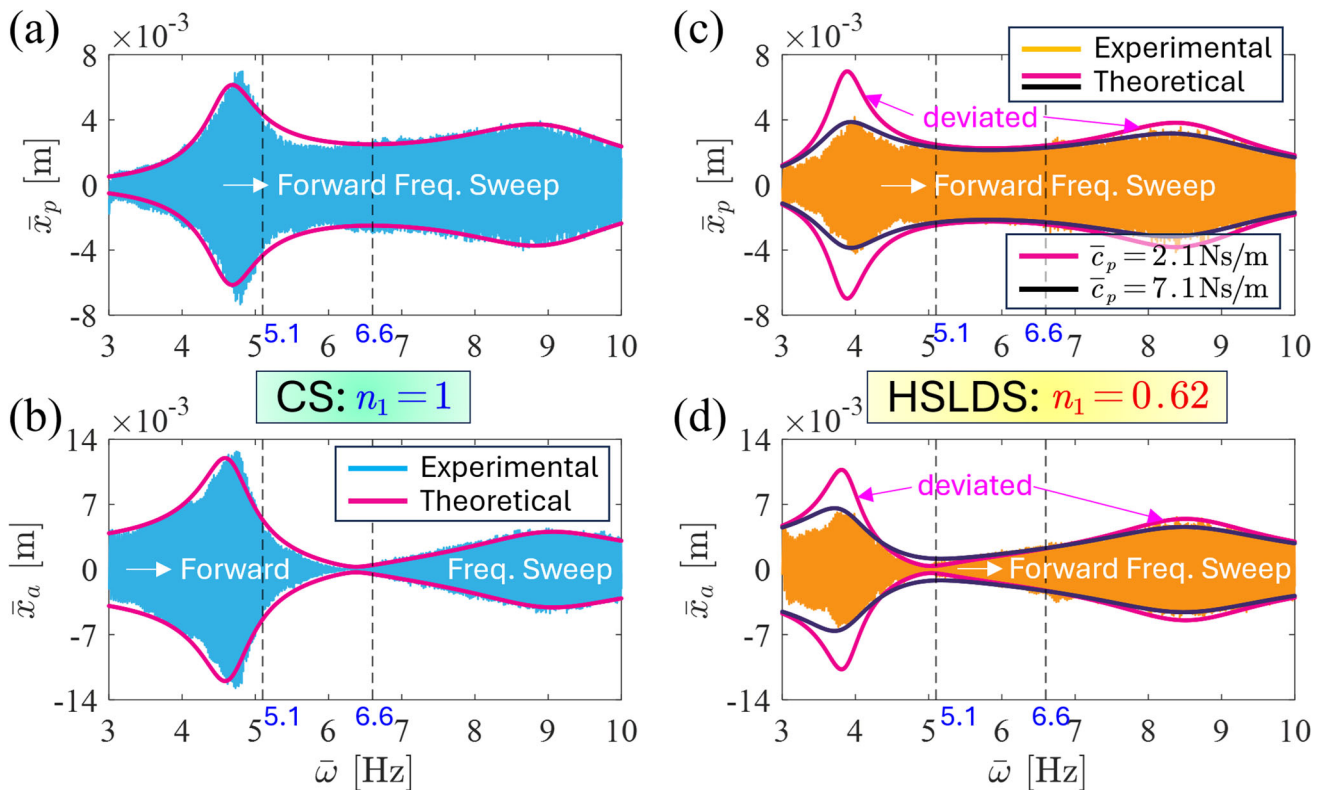


Fig. 14 Frequency sweeping test when fixing the frame under the excitation $\bar{u}$ with the amplitude $|\bar{u}| = 2$ N and the sweeping rate $+0.05$ Hz/s. **a**, **b** The CS case. **c**, **d** The HSLDS case with magnetic springs introduced

$$\begin{cases} \bar{m}_p = 0.97 \text{ kg}, \bar{c}_p = 2.1 \text{ Ns/m}, \bar{k}_p = 1553 \text{ N/m}, \\ \bar{m}_a = 0.44 \text{ kg}, \bar{c}_a = 3.7 \text{ Ns/m}, \bar{c}_g = 1.4 \text{ Ns/m}, \\ \bar{k}_a = 756 \text{ N/m}, \end{cases} \quad (51)$$

which correspond to the dimensionless variables in (19). Besides, with static measurements, the magnetic springs lead to $n_1 = 0.62$ in the HSLDS case.

8.2 Parametric accuracy and experimental issues

To verify the effectiveness of the structural parameters in (51), we first fix the frame on the base as in Appendix B of [27] and then activate $\bar{u}$ to excite $(\bar{m}_p, \bar{m}_a)$. During this process, a frequency sweeping excitation is applied, with a fixed force amplitude $|\bar{u}| = 2$ N plus a sufficiently low sweeping rate of $+0.05$ Hz/s to suppress the effect of the transient responses. The results in the CS case with $n_1 = 1$ are shown in Fig. 14a, b, where the theoretical solutions are obtained by solving the dynamical equations

$$\begin{cases} \bar{m}_a \ddot{\bar{x}}_a + \bar{c}_a (\dot{\bar{x}}_a - \dot{\bar{x}}_p) + \bar{k}_a (\bar{x}_a - \bar{x}_p) + \bar{c}_g \dot{\bar{x}}_a = \bar{u}, \\ \bar{m}_p \ddot{\bar{x}}_p + \bar{c}_p \dot{\bar{x}}_p + n_1 \bar{k}_p \bar{x}_p + \bar{c}_a (\dot{\bar{x}}_p - \dot{\bar{x}}_a) + \bar{k}_a (\bar{x}_p - \bar{x}_a) \\ = -\bar{u}, \end{cases} \quad (52)$$

which can be obtained by substituting $|\bar{x}_e| = 0$ into (1). Clearly, the theoretical solutions agree well with the experimental ones.

Then, we introduce the HSLDS mechanism (i.e., the magnets on the frame, as shown in Fig. 13c) and repeat the frequency-sweeping test, yielding Fig. 14c, d. Letting $n_1 = 0.62$, the theoretical solutions resulting from (51) and (52), which correspond to the magenta solid envelopes, however, deviate from the experimental responses. Given that the magnets do not contact the absorber-primary system (i.e., the system mass is unaffected) but introduce repulsive forces coupled between the primary and the frame, the deviation is inferred to be related to the enhanced friction between the primary and the frame on which the slide is rigidly fixed. In light of refs. [11, 66], which show that the friction can result in the viscous damping effect, $\bar{c}_p$ is updated to $\bar{c}_p = 7.1$ Ns/m, which best fits the experimental responses, see the black solid envelopes. For more on the friction mechanism and damping fitting, see Appendix E. The good accuracy of the linear model also signifies the negligible nonlinear effect of the HSLDS mechanism.

Since the theoretical analyses are based on (51), the damping value $\bar{c}_p = 2.1$ Ns/m is consistently used in the following simulation tests, regardless of whether in the CS $(n_1, n_3) = (1, 0)$ or the HSLDS $(n_1 = 0.62)$ cases, to facil-

Table 1 Tuned pairs used in the tests in the CS case at two given frequencies

$\bar{\omega}$	$(\bar{g}_t, \bar{\tau}_t) = (-N/m, -ms)$
5.1 Hz	(1212.4, 76.4)
6.6 Hz	(491.8, 34.2)

Table 2 Tuned pairs used in the tests in the HSLDS case at two given frequencies

$\bar{\omega}$	$(\bar{g}_t, \bar{\tau}_t) = (-N/m, -ms)$	$\bar{c}_p = 7.1 \text{ Ns/m}$
	$\bar{c}_p = 2.1 \text{ Ns/m}$	
5.1 Hz	(767.2, 74.6)	(795.8, 69.8)
6.6 Hz	(328.2, 33.3)	(339.4, 29.2)

itate verification. Besides, both simulations and experiments are conducted at two designated frequencies $\bar{\omega} = 6.6 \text{ Hz}$ and $\bar{\omega} = 5.1 \text{ Hz}$ for two reasons: (1) The simulation responses in the HSLDS case should not significantly deviate from the experimental ones within the steady state, since the theoretical envelopes for both $\bar{c}_p = 2.1 \text{ Ns/m}$ and $\bar{c}_p = 7.1 \text{ Ns/m}$ at the two frequencies are sufficiently accurate, see Fig. 14c, d. (2) The insignificant effect of $\bar{c}_p$ at the two frequencies is the basis to make fair experimental comparisons between the CS and the HSLDS cases.

To verify the suppressed nonlinear effect, the tuned pair in all the tests follows the reduced form (34), with the smallest positive tuned delay considered. Accordingly, with the conversion $(\bar{g}_t, \bar{\tau}_t) = (g_t \bar{k}_p, \tau_t / \bar{\omega}_p)$, Table 1 lists the dimensional tuned pair in the CS case. Table 2 lists the tuned pairs for both $\bar{c}_p = 2.1 \text{ Ns/m}$ and $\bar{c}_p = 7.1 \text{ Ns/m}$ in the HSLDS case. Note from Table 2, the small differences between the tuned pairs regarding the two $\bar{c}_p$ values indicate that the simulation results ($\bar{c}_p = 2.1 \text{ Ns/m}$) are sufficiently accurate to predict the experimental results ($\bar{c}_p = 7.1 \text{ Ns/m}$) of the tuned DR, as to be shown next. This is attributed to the fact that the damping force related to $\bar{c}_p$ reduces as $|\bar{x}_p| \rightarrow 0$.

8.3 Complete vibration suppression and tuned absorber behavior

To test the vibration suppression performance of the tuned DR, we consider the case $\bar{\omega} = 6.6 \text{ Hz}$ (i.e., $\bar{\omega} = \bar{\omega}_a$ or $\omega = \nu$), which corresponds to the antiresonance frequency of the primary structure in the passive absorber case, see Fig. 8a. In the simulation, the excitation has the amplitude $|\bar{x}_e| = \bar{R}_e = 5 \text{ mm}$, and $n_3 = 10 \gg n_1$ is considered in the HSLDS case. The simulation results are shown in Fig. 15.

Figure 15a shows the responses of the primary. The time interval $\bar{t} \in [0, 2s]$ corresponds to the passive absorber. One can find that although the slightly damped absorber with $\zeta_a = 0.101$ achieves obvious vibration suppression signified by

$|\bar{x}_p| = \bar{R}_p < \bar{R}_e$, the suppression is far from complete. From $\bar{t} = 2 \text{ s}$ onwards, we activate the tuned DR, which then suppresses the vibration to a sufficiently low level in both the CS and the HSLDS cases, as expected. More intuitively, Fig. 15b dynamically evaluates the root mean square of $\bar{x}_p$ as per 500 data sampled within a past duration of 0.5 s (the sampling time is 1 ms). The rather small residual vibration in steady state is attributed to the limited simulation accuracy rather than nonlinearity since the same appears in the CS case, see the close-up in Fig. 15a. In addition, the HSLDS feature can enhance the vibration control in the passive case, i.e., $\bar{t} \in [0, 2s]$, which agrees with Fig. 5a.

Figure 15c, d show the time histories of the absorber response $\bar{x}_a$ and the active force $\bar{u}$, respectively. As per (39) and (48), $(T'_a, T'_{Rut}) = (2.05, 491.98)$ in the CS case, and $(T'_a, T'_{Rut}) = (1.28, 328.17)$ in the HSLDS case. The corresponding amplitude predictions are superposed as the black dashed lines, which agree well with the numerical results. Clearly, the HSLDS feature can confine both the absorber's working space and the force output of the actuator and thus can facilitate the practical implementation of the DR scheme, which verifies the analysis in Sect. 7. The experimental results are shown in Fig. 16.

Let us start with Fig. 16c, which depicts the time history of the displacement excitation $\bar{x}_e$. Note that the excitation amplitude $\bar{R}_e$ is not constant since the frame is excited by an open-loop controlled force $\bar{f}_e$, see the VCM2 in Fig. 13a, i.e., the output $\bar{x}_e$ is affected by the responses $(\bar{x}_p, \bar{x}_a)$. However, by manipulating $\bar{f}_e$, a constant amplitude $\bar{R}_e = 5 \text{ mm}$ is achieved in the steady state (e.g., $\bar{t} > 4 \text{ s}$) in both the CS and the HSLDS cases to match the test condition in Fig. 15c. To avoid impacts in the experiments, the corresponding two amplitudes of $\bar{f}_e$ in both the CS and the HSLDS cases are unchanged in the passive case (i.e., $\bar{t} < 2 \text{ s}$), which finally results in the corresponding differences of $\bar{R}_e$, i.e., $\bar{R}_e = 3.3 \text{ mm}$ and $\bar{R}_e = 2.3 \text{ mm}$ in the CS and HSLDS cases, respectively, as marked in Fig. 16c.

Then, from Fig. 16a, b, the tuned DR activated from $\bar{t} = 2 \text{ s}$ onwards settles the primary structure as expected, and the inevitable residual vibration can be caused by the measurement noise, the actuator uncertainty, the inaccuracy in the tuned pair due to inaccurate structural parameters or models (e.g., the asymmetric magnetic forces in the HSLDS case, as mentioned in Appendix E), etc. The theoretical amplitude $\bar{R}_a = 6.4 \text{ mm}$ of the tuned DR obtained in Fig. 15c is also superposed in Fig. 16d in the black dashed lines, which well predicts the real amplitude $\bar{R}_a = 6.9 \text{ mm}$ ($T'_a = 6.9/5 = 1.38$), despite the difference in $\bar{c}_p$, agreeing with Sect. 8.2. As for the previously observed enhanced passive vibration control, we focus on the time interval $\bar{t} \in [0, 2s]$ of Fig. 16b, c. One can find that the excitation amplitude $\bar{R}_e$ is reduced by $1 - 2.3/3.3 = 30.3\%$ in the HSLDS case, but it corresponds to a much higher reduction of $\bar{R}_p$, i.e., $50\% > 30.3\%$, in

Fig. 15 Simulation tests of the complete vibration suppression in the CS and the HSLDS $((n_1, n_3) = (0.62, 10))$ cases, with $(\bar{\omega}, \bar{R}_e) = (6.6 \text{ Hz}, 5 \text{ mm})$. **a** Responses of the primary structure. **b** Dynamical evaluation of the root mean square of $\bar{x}_p$. **c** Absorber responses. **d** Time history of the active force. The tuned DR is activated from $\bar{t} = 2 \text{ s}$ onwards

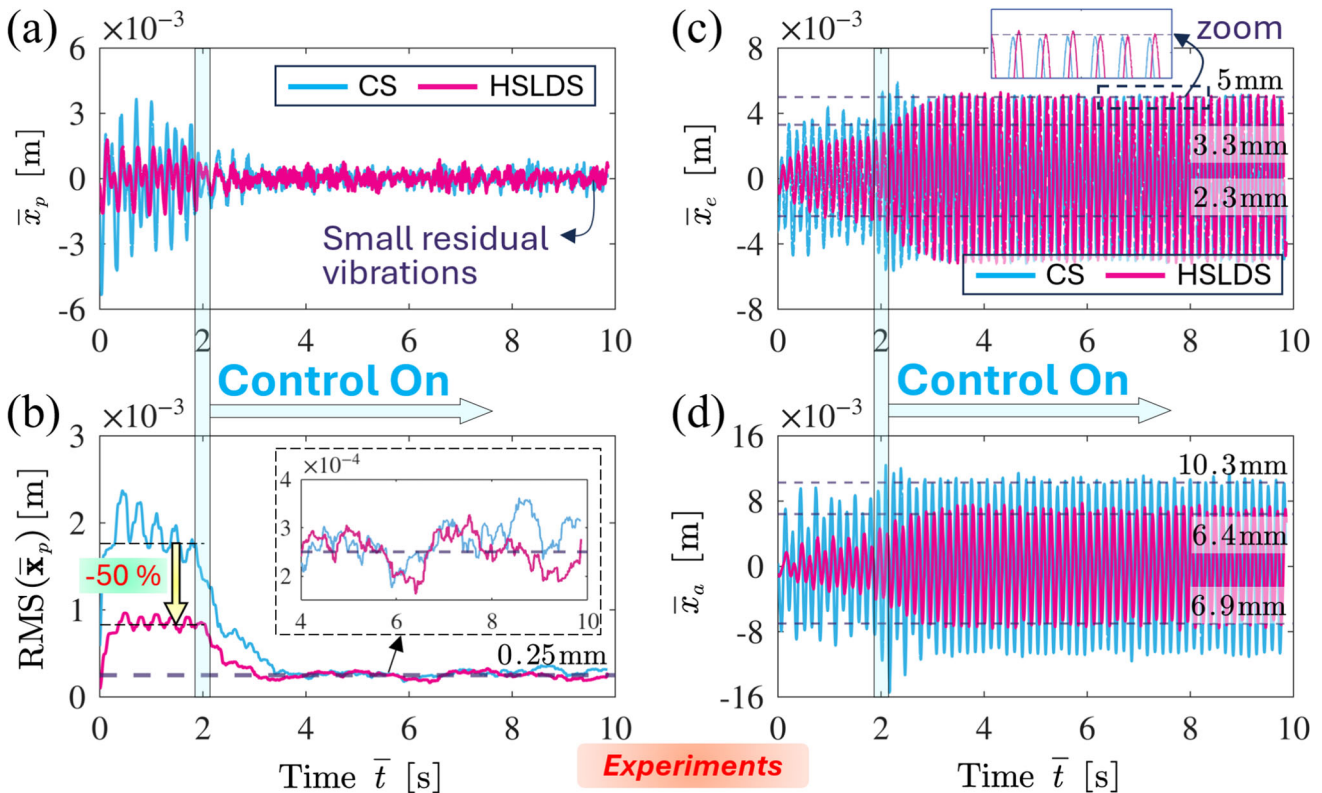
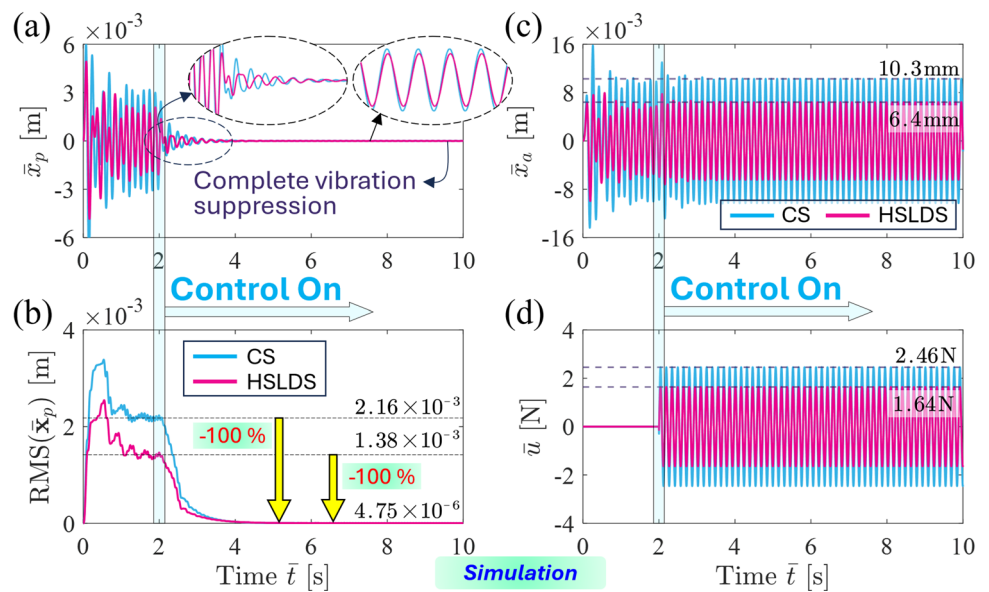


Fig. 16 Experimental tests of the complete vibration suppression of the tuned DR in the CS and the HSLDS cases at $\bar{\omega} = 6.6 \text{ Hz}$. **a** Responses of the primary structure. **b** Dynamical evaluation of the root mean square

of $\bar{x}_p$. **c** Time history of the displacement excitation $\bar{x}_e$. **d** Responses of the absorber. The tuned DR is activated from $\bar{t} = 2 \text{ s}$ onwards

the CS case. Note that we have experimentally shown that the HSLDS feature can be realized by the magnetic springs and effectively modelled by Eq. (10) with negligible non-linear effects, in addition to testing the complete vibration suppression, see also Remark 3.

8.4 Extended antiresonance valley in the HSLDS case

We next test the extended antiresonance valley in the HSLDS case, which is expected to enhance the control robustness against the deviations of the control parameters, as discussed

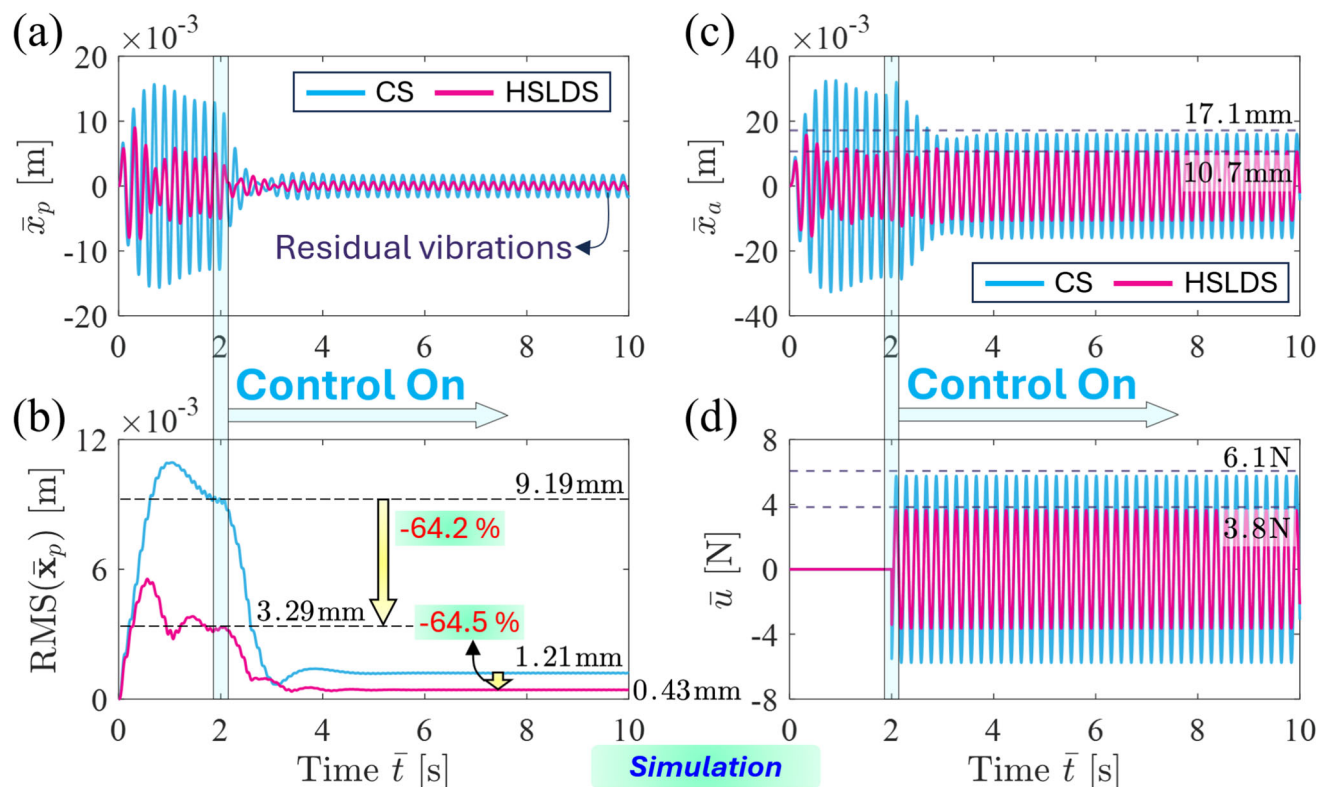


Fig. 17 Simulation test of the enhanced control robustness in the HSLDS case against the deviated control parameters, with $(\bar{\omega}, \bar{R}_e) = (5.1 \text{ Hz}, 5 \text{ mm})$. **a** Responses of the primary structure. **b** Dynamical root

mean square evaluation of $\bar{x}_p$. **c** Absorber responses. **d** Time history of the active force. The tuned DR is activated from $\bar{t} = 2$ s onwards

in Sect. 6.2. Following Fig. 9a, we consider the case $\omega = 0.8$ (i.e., $\bar{\omega} = 5.1 \text{ Hz}$), which satisfies $\omega > \omega_{LB}$ as per Fig. 10. Similar to Fig. 15, $(n_3, \bar{R}_e) = (10, 5 \text{ mm})$ are considered in the simulation. To simulate the deviations, the DR is driven by $(\bar{g}, \bar{\tau}) = (0.95\bar{g}_t, 1.05\bar{\tau}_t)$. The simulation results are shown in Fig. 17.

We first observe from the time interval $\bar{t} \in [0, 2 \text{ s}]$ of Fig. 17a, b that $\bar{R}_p$ is significantly increased compared with the previous case in Fig. 15, signifying the performance degradation of a passive absorber not operating around its natural frequency. Besides, the HSLDS feature once again enhances the passive vibration control. Both observations agree with Fig. 5a. After activating the tuned DR at $\bar{t} = 2$ s, the primary structure is significantly settled, and the persisting residual vibration reflects the deviated or inaccurate tuned pair. Particularly, the residual vibration level in the HSLDS case is much lower (64.5% reduction) than that in the CS case, signifying the stronger control robustness thanks to the extended antiresonance valley. Figure 17c, d show the time histories of $(\bar{x}_a, \bar{u})$, where the mismatch between the theoretical amplitude marked in the black dashed lines and the simulation is attributed to the fact that the complete vibration suppression $|\bar{x}_p| = 0$ is not achieved at last. Note, however,

that the confinement of the HSLDS feature to the amplitudes of $(\bar{x}_a, \bar{u})$ is still effective.

The experimental results at the same frequency with the same deviations in the tuned pair are shown in Fig. 18. Given that $\bar{f}_e$ is upper-bounded due to the hardware constraint of the corresponding voice coil motor (i.e., VCM2 in Fig. 13), a smaller excitation amplitude $\bar{R}_e = 3 \text{ mm}$ is achieved in the steady state after activating the DR, as shown in Fig. 18c. The suppressed residual vibrations of the primary structure in the HSLDS case, as well as the enhanced passive vibration control, can be clearly found in Fig. 18a, b. At last, Fig. 18d once again experimentally verifies the effectiveness of the theoretical amplitude $\bar{R}_a$ (marked in the black dashed lines) in both the CS and the HSLDS cases, alongside the confined absorber vibration in the HSLDS case.

9 Conclusions

In this work, DR is used to suppress the displacement-excited vibration. Particularly, we additionally introduce the HSLDS feature between the primary structure and the displacement excitation, given its benefits to vibration isolation, finally

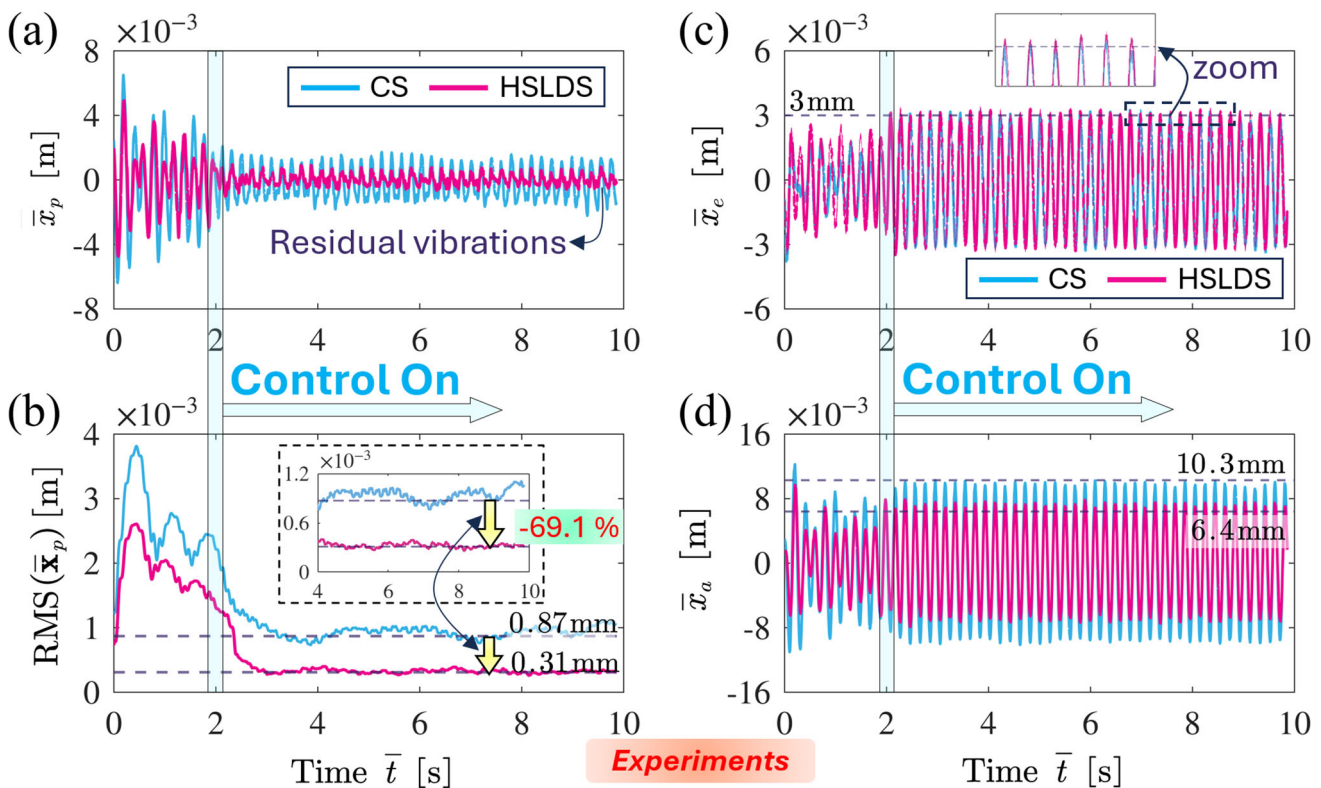


Fig. 18 Experimental tests of the enhanced control robustness of the tuned DR in the HSLDS cases at $\bar{\omega} = 5.1$ Hz. **a** Responses of the primary structure. **b** Dynamical root mean square evaluation of $\bar{x}_p$. **c** Time

history of the displacement excitation $\bar{x}_e$. **d** Responses of the absorber. The tuned DR is activated from $\bar{t} = 2$ s onwards

yielding a coupled system combined with HSLDS vibration isolation and linear active DR-based vibration absorption. The main results are summarized as follows.

- (i) System dynamics are established with the harmonic balance method. It is shown that a linear absorber can significantly suppress the nonlinear effect of the HSLDS feature. This determines the efficacy of a linear DR, which needs to neutralize the dynamical forces on the primary. Moreover, it can simplify the dynamical modelling by reducing the truncation order, e.g., the 3rd-order stiffness truncation in (10) and the harmonic balance at the fundamental frequency in (13).
- (ii) The combined system improves the passive vibration control performance of both the pure HSLDS vibration isolation and the combination of linear isolation and linear absorption (i.e., the CS case). Therefore, the benefits of the HSLDS feature to the DR do not degrade passive control.
- (iii) The tuned pair (g_t, τ_t) for complete vibration suppression at a given frequency is analytically obtained. Due to the negligible nonlinear effect, the HSLDS feature mainly affects (g_t, τ_t) by the linear dynamical stiffness

n_1 of the primary structure. Furthermore, reducing n_1 reduces the dynamical excitation forces exerted on the primary structure, thus leading to a smaller g_t .

- (iv) Compared with the CS case, the benefits of the HSLDS feature to the DR are twofold: (1) For the primary structure, we can extend the antiresonance valley at the tuning frequency to enhance the robustness of the tuned DR against the deviations or inaccuracies in the tuned pair (g_t, τ_t) . (2) For the absorber, thanks to the reduced dynamical forces exerted on the primary structure, we can reduce the absorber vibration and the force output of the actuator to realize a more compact absorber's working space and reduce the requirements for the actuator performance.

Overall, this work extends the application scope of the DR technique and enhances the vibration control performance via a mechanical approach, i.e., altering the dynamic stiffness of the primary structure without affecting the load capacity and inducing significant nonlinearity.

However, many issues deserve further exploration. Our future work would design advanced control logic to suppress the residual vibrations in the HSLDS case, as observed

in Fig. 16a, b. In this regard, the feedback control loop as (7) will be activated. Meanwhile, the issue discussed in Appendix C and the effect of the asymmetric stiffness (omitted in this work) when implementing the HSLDS feature on the vibration control need to be addressed. On the other hand, the nonlinear behaviors under extremely strong excitations deserve a specialized study on nonlinear dynamics.

A Graphic demonstration of the nonlinear restoring force

The variations of the nonlinear total restoring force f_{total} following (10) with respect to r_p , as well as the associated variations of the stiffness k_{total} , which can be calculated as per (6), for different pairs of (n_1, n_3) are shown in Fig. 19.

Figure 19a, b show the cases where n_1 varies while fixing n_3 . Clearly, the variations of n_1 affect the stiffness k_{total} at the static equilibrium point $r_p = 0$. The low-dynamic stiffness related to $n_1 < 1$ is reflected by $k_{total} < 1$ around $r_p = 0$. Besides, k_{total} increases as $|r_p|$ increases, signifying the nonlinear hardening characteristic with $n_3 > 0$. Figure 19c, d show the cases where n_3 varies while fixing n_1 . This time, k_{total} at $r_p = 0$ is unaffected, and the nonlinear hardening characteristics appear again due to $n_3 > 0$. Note

that $(n_1, n_3) = (1, 0)$ corresponds to the CS case $k_{total} \equiv 1$, as marked in Fig. 19d.

B Coefficients or expressions omitted in the main text

The coefficients omitted in (17) write

$$\begin{cases} \alpha_4^{[1]} = -\mu (R_e + A_{rp}), \\ \alpha_3^{[1]} = -2\mu B_{rp} \zeta_a v, \\ \alpha_2^{[1]} = -\left[\mu (4\zeta_a \zeta_g A_{rp} + 4\zeta_g^2 A_{rp} - R_e - A_{rp}) v^2 - g R_e C \right], \\ \alpha_1^{[1]} = -2v [\mu \zeta_g B_{rp} v^2 + g (\zeta_a + \zeta_g) R_e S], \\ \alpha_0^{[1]} = g R_e C v^2, \\ \beta_4^{[1]} = -\mu B_{rp}, \\ \beta_3^{[1]} = 2\mu v (\zeta_g R_e + (R_e + A_{rp}) \zeta_a), \\ \beta_2^{[1]} = -\mu (4\zeta_a \zeta_g + 4\zeta_g^2 - 1) B_{rp} v^2 - g R_e S, \\ \beta_1^{[1]} = 2v (\mu \zeta_g A_{rp} v^2 + g (\zeta_a + \zeta_g) R_e C), \\ \beta_0^{[1]} = g R_e S v^2. \end{cases} \quad (53)$$

The full expressions of the two polynomial equations in (18) write

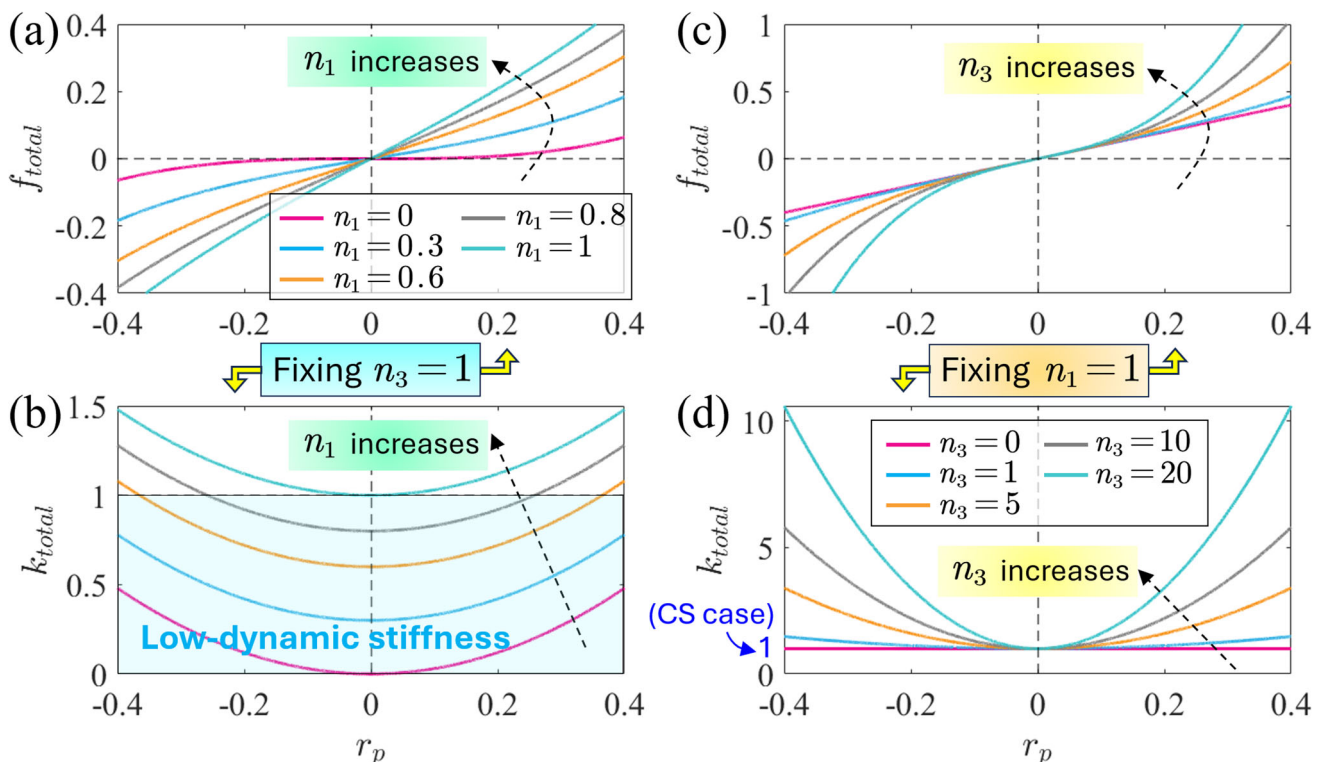


Fig. 19 Graphic demonstration of the nonlinear total restoring force f_{total} defined in (10) and the associated stiffness k_{total} . **a, b** Varying n_1 while fixing n_3 . **c, d** Varying n_3 while fixing n_1

$$\begin{aligned}
& \left\{ \begin{aligned} E_1(\omega, A_{rp}, B_{rp}) &= -4B_{rp}\omega^6 - 8[(\mu v\zeta_a + \zeta_p)A_{rp} + R_e\mu v\zeta_a]\omega^5 \\ &+ \left\{ \begin{aligned} 3n_3B_{rp}(A_{rp}^2 + B_{rp}^2) + 4n_1B_{rp} + 4R_egS \end{aligned} \right\} \omega^4 \\ &- 8v\{4v[\mu v\zeta_a\zeta_g(\zeta_a + \zeta_g) + \Phi_1\zeta_p]A_{rp} + R_e(\mu v^2\zeta_g + Cg\zeta_a)\}\omega^3 \\ &+ 4v^2\left\{ \begin{aligned} (3n_3(A_{rp}^2 + B_{rp}^2) + 4n_1)\Phi_1B_{rp} + B_{rp}v^2(4\mu\zeta_g^2 - \mu - 1) \\ + R_egS[4\zeta_g(\zeta_a + \zeta_g) - 1] \end{aligned} \right\} \omega^2 \\ &- 8v^3[v(\mu v\zeta_g + \zeta_p)A_{rp} + C R_eg\zeta_g]\omega + B_{rp}v^4[3n_3(A_{rp}^2 + B_{rp}^2) + 4n_1], \end{aligned} \right. \\
& \left\{ \begin{aligned} E_2(\omega, A_{rp}, B_{rp}) &= -4(A_{rp} + R_e)\omega^6 + 8B_{rp}(\mu v\zeta_a + \zeta_p)\omega^5 \\ &+ \left\{ \begin{aligned} 3n_3A_{rp}(A_{rp}^2 + B_{rp}^2) + 4n_1A_{rp} + 4R_egC \\ - 16v^2[A_{rp}(\Phi + \mu\zeta_a^2 - \frac{\mu}{4}) + R_e(\Phi_1 + \mu\zeta_a(\zeta_a + \zeta_g) - \frac{\mu}{4})] \end{aligned} \right\} \omega^4 \\ &+ 8v[4B_{rp}v^2\mu\zeta_a\zeta_g(\zeta_a + \zeta_g) + 4B_{rp}v\zeta_p\Phi_1 + R_egS\zeta_a]\omega^3 \\ &+ 4v^2\left\{ \begin{aligned} (3n_3(A_{rp}^2 + B_{rp}^2) + 4n_1)\Phi_1A_{rp} + v^2A_{rp}(4\mu\zeta_g^2 - \mu - 1) \\ - R_e(\mu + 1)v^2 + CgR_e[4\zeta_g(\zeta_a + \zeta_g) - 1] \end{aligned} \right\} \omega^2 \\ &+ 8v^3[B_{rp}v(\mu v\zeta_g + \zeta_p) + R_egS\zeta_g]\omega + A_{rp}v^4[3n_3(A_{rp}^2 + B_{rp}^2) + 4n_1], \end{aligned} \right. \end{aligned} \quad (54)$$

where $\Phi_1 = (\zeta_a + \zeta_g)^2 - 1/2$. The coefficients $(\alpha_0^{[3]}, \beta_0^{[3]})$ in (28) write

Accordingly, Fig. 20b shows the variations of the peak frequency ω_h , i.e., the frequency where h appears.

$$\begin{aligned}
& \left\{ \begin{aligned} \alpha_0^{[3]}(\omega) &= \{16v[\mu v\zeta_a^2\zeta_g + \zeta_p(\zeta_a^2 + \zeta_a\zeta_g - \frac{1}{4})] + 4\zeta_a\Phi_2\}\omega^3 + 4v^2(\mu v^2\zeta_g + v\zeta_p + \zeta_g\Phi_2)\omega, \\ \beta_0^{[3]}(\omega) &= 16\zeta_a\zeta_p\omega^4 + 16v(\zeta_g\zeta_p v - \Phi_2\Phi_3)\omega^2 - 4v^3\Phi_2, \end{aligned} \right. \end{aligned} \quad (55)$$

where $\Phi_2 = 3R_e^2n_3/4 + n_1$, and $\Phi_3 = \zeta_a^2 + \zeta_a\zeta_g - 1/4$. The coefficients $(\alpha^{[4]}, \beta^{[4]})$ in (31) write

$$\begin{aligned}
& \left\{ \begin{aligned} \alpha^{[4]}(\omega) &= 8\zeta_p\omega^4 + 8v\{4\mu\zeta_gv^2(\zeta_a\zeta_g - \frac{1}{4}) + 4\zeta_p v[\zeta_g(\zeta_a + \zeta_g) - \frac{1}{4}] - \zeta_a\Phi_2\}\omega^2 - 8\Phi_2\zeta_gv^3, \\ \beta^{[4]}(\omega) &= [16\zeta_a v(\mu\zeta_g v + \zeta_p) + 4\Phi_2]\omega^3 + 16v^2[\zeta_g v(\mu\zeta_g v + \zeta_p) + \Phi_2\Phi_3]\omega. \end{aligned} \right. \end{aligned} \quad (56)$$

C Issues with the HSLDS feature on resonance response

We here discuss the effect of the HSLDS feature $n_1 < 1$ on the resonance peak, which corresponds to the maximum of T_p , so we define $h = \max\{T_p\}$. From Fig. 8a, n_1 can affect h when achieving the complete vibration suppression $T_p = 0$ at $\omega = 0.8$. Clearly, a smaller h is desired to confine the maximum vibration caused by (non-random) perturbations. To evaluate the effect of n_1 on h , we consider six cases where the tuned DR separately achieves $T_p = 0$ at $\omega \in [0.55, 0.6, 0.8, 1, 1.2, 1.5]$. The associated variations of h with respect to n_1 in different cases are shown in Fig. 20a.

From Fig. 20a, in the CS case $n_1 = 1$, h can significantly rise when achieving $T_p = 0$ within the low-frequency band, exemplified by the two cases $\omega = 0.55$ and $\omega = 0.6$. This can be avoided by having $n_1 < 1$. Note that although h is minimized as $n_1 \rightarrow 0$ regardless of ω , introducing the HSLDS feature can increase h if the resulting reduction of n_1 from $n_1 = 1$ is limited. On the other hand, although having $n_1 < 1$ can effectively confine h when achieving $T_p = 0$ at a small ω , it can compromise the performance degradation in the passive case. More specifically, revisiting C-3 and C-4 in Fig. 5a, T_p within the low-frequency band can be increased in the HSLDS case, and the same conclusion applies to the absorber vibration signified by T_a as per Fig. 5b. A possible solution for limiting h when a limited reduction of n_1 is available is to additionally introduce the state of the primary

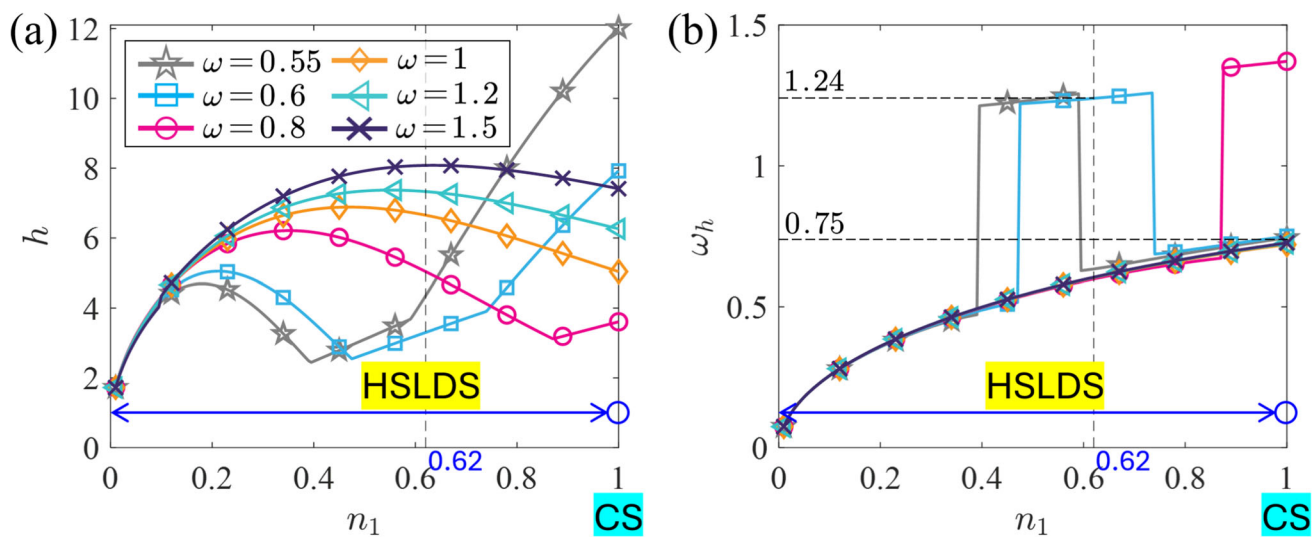


Fig. 20 **a** Variations of $h = \max\{T_p\}$ versus n_1 when achieving complete vibration suppression $T_p = 0$ at different frequencies ω , a result generalized from Fig. 8a. **b** Peak frequency ω_h related to h versus n_1

structure into the control loop [22]. Further discussions are left to another ongoing work.

D Open-source materials

The SIMULINK-based simulation models and simulation results related to Figs. 15 and 17 are available at <https://tinyurl.com/26zrdqsl>.

E Issues with magnet installation and damping fitting

In Fig. 14c, d, the system parameters identified in the CS case are ineffective in response prediction after introducing the magnets on the frame. However, good agreement between the theoretical envelopes and the experimental responses is achieved by increasing the damping $\bar{c}_p$. Here are some explanations.

Figure 21a shows the schematic diagram for the expected installation relationship between the magnets and the primary structure, which refers to two ideal conditions: (1) The repulsive force on either side of the primary is symmetric about the y axis, i.e., Eq. (5), and (2) The two repulsive forces on the two sides of the primary are symmetric to each other about the x axis. Then, introducing or removing the magnets on the frame does not affect the contact between the slide bearing and the slide since the components of the magnetic repulsive forces along the y axis are always collinearly neutralized by each other even when the primary structure vibrates. Hence, the friction between the primary structure and the frame is unaffected.

However, such two ideal conditions are difficult to guarantee in practice (at least in our case). Figure 21b, c show the possible installation errors of the magnet positions on the frame. The asymmetry of the repulsive forces about the x and y axes causes a torque to the slide bearing as well as the pressures exerted on the slide (the frame) by the bearing (the primary structure), as marked in Fig. 21b, c. Note that the pressures can also result from the case where the two repulsive forces (even if they are collinear along the y axis) are not equal due to the differences in the mechanical properties of each magnet, the schematic diagram of which is omitted here. The resulting pressures increase the friction between the primary and the frame. In light of [11, 66], the friction force between the bearing and the slide can be modelled as the Stribeck friction,

$$\bar{f}_r(\bar{t}) = \bar{f}_{r,1} + \bar{f}_{r,2}, \quad (57)$$

where

$$\begin{cases} \bar{f}_{r,1} = \bar{F}_c \left[1 + \frac{\sigma_s - 1}{1 + \left(\frac{\dot{\bar{r}}_p(t)}{\bar{V}_s} \right)^2} \right] \text{sgn}(\dot{\bar{r}}_p(\bar{t})), \\ \bar{f}_{r,2} = \bar{c}_r \dot{\bar{r}}_p(\bar{t}), \end{cases} \quad (58)$$

in which $\bar{r}_p$ is the relative displacement of the primary, as defined in (3), $\bar{c}_r$ is the viscous damping coefficient, $\bar{F}_c$ denotes Coulomb (static) friction force, σ_s represents the Stribeck ratio of static friction to Coulomb friction, $\bar{V}_s$ is the Stribeck velocity. With good lubrication (our case), $\bar{V}_s \rightarrow 0$ so $\bar{f}_r \rightarrow \bar{f}_{r,2}$ when $\dot{\bar{r}}_p \neq 0$, see also [11, 66]. Given the viscosity-pressure effect [67–69], we conclude that the

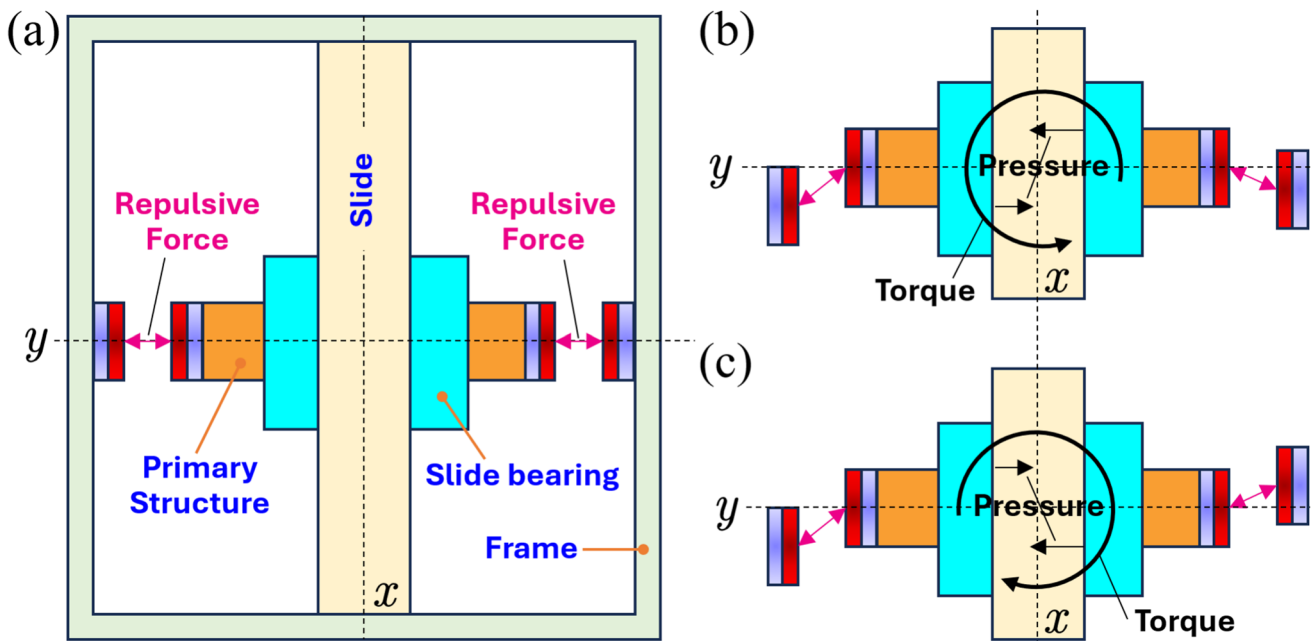


Fig. 21 **a** Schematic ideal relationship between the magnets and the primary structure. **b, c** Non-ideal cases where the repulsive forces are asymmetric about the y axis

increased pressures by the magnets raise $\bar{c}_r$, which finally leads to the increase in $\bar{c}_p$.

Since the magnetic repulsive forces exerted on the primary structure along the y axis vary as the primary structure vibrates, the pressures should vary, yielding a time-varying value of $\bar{c}_r$. However, exactly modeling the $\bar{c}_r$ is far beyond the scope of this work. In light of [70–72], within a limited frequency band and a limited motion amplitude, linearizing $\bar{c}_r$ with a constant value can be sufficiently accurate for dynamic modelling. This explains the increased accuracy of the theoretical envelopes after a constant increase in $\bar{c}_p$. Note that, strictly speaking, all damping parameters should be time-varying, especially given that no specific damper component is used, while such variations are less considered in the open literature, regardless of whether linear or non-linear vibration control systems with a similar structure to our experimental setup. From this perspective, the constant modelling of $\bar{c}_p$ is used.

Particularly, the increase from $\bar{c}_p = 2.1$ Ns/m in the CS case to $\bar{c}_p = 7.1$ Ns/m in the HSLDS case, i.e., 238% increase, may seem over-large. However, this can be reasonable given that: (1) The primary structure is installed on the frame parallel to the gravity, so the pressure inherent in the assembly exerted on the slide can be small. Note that $\bar{c}_a$ incorporates the damping effect inside the voice coil motor so $\bar{c}_a > 2.1$ Ns/m. (2) $\bar{c}_p = 2.1$ Ns/m corresponds to a small damping ratio of $\zeta_p = 0.027$, which is susceptible to changes in external conditions, i.e., a small condition change can cause a significant relative fluctuation. Note that even

$\bar{c}_p = 7.1$ Ns/m corresponds to a small damping ratio of $\zeta_p = 0.091$.

Author contributions Y. Liu conceived the idea and wrote the draft. L. Cheng supervised the work. G. Yan and L. Cheng reviewed and revised the manuscript. All authors contributed to the study and approved the final manuscript.

Funding Open access funding provided by The Hong Kong Polytechnic University. The authors have not disclosed any funding.

Data availability The datasets used and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors declare no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- Olgac, N., Holm-Hansen, B.T.: A novel active vibration absorption technique: delayed resonator. *J. Sound Vib.* **176**(1), 93–104 (1994)
- Elmali, H., Renzulli, M., Olgac, N.: Experimental comparison of delayed resonator and PD controlled vibration absorbers using electromagnetic actuators. *J. Dyn. Sys., Meas., Control* **122**(3), 514–520 (2000)
- Filipovic, D., Olgac, N.: Torsional delayed resonator with velocity feedback. *IEEE/ASME Trans. Mechatron.* **3**(1), 67–72 (1998)
- Vyhlídal, T., Olgac, N., Kučera, V.: Delayed resonator with acceleration feedback—complete stability analysis by spectral methods and vibration absorber design. *J. Sound Vib.* **333**(25), 6781–6795 (2014)
- Sun, Y., Xu, J.: Experiments and analysis for a controlled mechanical absorber considering delay effect. *J. Sound Vib.* **339**, 25–37 (2015)
- Xu, J., Sun, Y.: Experimental studies on active control of a dynamic system via a time-delayed absorber. *Acta. Mech. Sin.* **31**, 229–247 (2015)
- Pilbauer, D., Vyhlídal, T., Olgac, N.: Delayed resonator with distributed delay in acceleration feedback—design and experimental verification. *IEEE/ASME Trans. Mechatron.* **21**(4), 2120–2131 (2016)
- Kučera, V., Pilbauer, D., Vyhlídal, T., Olgac, N.: Extended delayed resonators—design and experimental verification. *Mechatronics* **41**, 29–44 (2017)
- Liu, Y., Olgac, N., Cheng, L.: Delayed resonator with multiple distributed delays—Considering and optimizing the inherent loop delay. *J. Sound Vib.* **576**, 118290 (2024)
- Eris, O., Alikoc, B., Ergenc, A.F.: A new delayed resonator design approach for extended operable frequency range. *J. Vib. Acoust.* **140**(4), 041003 (2018)
- Zhang, X., Xu, J., Ji, J.: Modelling and tuning for a time-delayed vibration absorber with friction. *J. Sound Vib.* **424**, 137–157 (2018)
- Karama, M., Hamdi, M., Habbad, M.: Energy harvesting in a nonlinear energy sink absorber using delayed resonators. *Nonlinear Dyn.* **105**(1), 113–129 (2021)
- Kammer, A.S., Olgac, N.: Delayed-feedback vibration absorbers to enhance energy harvesting. *J. Sound Vib.* **363**, 54–67 (2016)
- Šika, Z., Vyhlídal, T., Neusser, Z.: Two-dimensional delayed resonator for entire vibration absorption. *J. Sound Vib.* **500**, 116010 (2021)
- Šika, Z., Krivošej, J., Vyhlídal, T.: Three dimensional delayed resonator of Stewart platform type for entire absorption of fully spatial vibration. *J. Sound Vib.* **571**, 118154 (2024)
- Vyhlídal, T., Michiels, W., Neusser, Z., Bušek, J., Šika, Z.: Analysis and optimized design of an actively controlled two-dimensional delayed resonator. *Mech. Syst. Signal Process.* **178**, 109195 (2022)
- Wang, G., Xie, N., Che, H., Zhang, Q.: Enhancing constant force tracking in uncertain contact surfaces: an admittance controller utilizing virtual delayed resonator. *Robot. Auton. Syst.* **192**, 105008 (2025)
- Wang, F., Sun, X., Meng, H., Xu, J.: Time-delayed feedback control design and its application for vibration absorption. *IEEE Trans. Industr. Electron.* **68**(9), 8593–8602 (2020)
- Vyhlídal, T., Pilbauer, D., Alikoc, B., Michiels, W.: Analysis and design aspects of delayed resonator absorber with position, velocity or acceleration feedback. *J. Sound Vib.* **459**, 114831 (2019)
- Cai, J., Gao, Q., Liu, Y., Olgac, N.: Control design, analysis, and optimization of fractional-order delayed resonator for complete vibration absorption. *J. Sound Vib.* **571**, 118083 (2024)
- Kuře, M., Bušek, J., Boussaada, I., Michiels, W., Niculescu, S.-I., Vyhlídal, T.: Robust delayed resonator with acceleration feedback—design by double root assignment and experimental validation. *J. Sound Vib.* **576**, 118261 (2024)
- Liu, Y., Yan, B., Cheng, L.: H_∞ optimization of a hybrid multiple-delayed delayed resonator vibration absorber. *Int. J. Mech. Sci.* **300**, 110381 (2025)
- Olgac, N., Jenkins, R.: Actively tuned noncollocated vibration absorption: an unexplored venue in vibration science and a benchmark problem. *IEEE Trans. Control Syst. Technol.* **29**, 294–304 (2020)
- Saldanha, A., Michiels, W., Kuře, M., Bušek, J., Vyhlídal, T.: Stability optimization of time-delay systems with zero-location constraints applied to non-collocated vibration suppression. *Mech. Syst. Signal Process.* **208**, 110886 (2024)
- Silm, H., Kuře, M., Bušek, J., Michiels, W., Vyhlídal, T.: Spectral design and experimental validation of noncollocated vibration suppression by a delayed resonator and time-delay controller. *IEEE Trans. Control Syst. Technol.* **32**, 73–85 (2024)
- Valášek, M., Olgac, N., Neusser, Z.: Real-time tunable single-degree of freedom, multiple-frequency vibration absorber. *Mech. Syst. Signal Process.* **133**, 106244 (2019)
- Liu, Y., Yan, B., Shao, J., Cheng, L.: Enhanced delayed resonator for complete single- and multiple-frequency vibration suppression: a hybrid feedforward-feedback control approach. *Mech. Syst. Signal Process.* **240**, 113317 (2025)
- Timorian, S., Valášek, M., Neusser, Z.: Real-time tuning of an SDOF dual frequency vibration absorber with fully guaranteed stability. *Mech. Syst. Signal Process.* **237**, 113129 (2025)
- Peng, J., Xia, H., Lenci, S., Xie, X., Wang, L.: Nonlinear time-delay feedback control of a suspended cable under temperature effect. *Int. J. Non-Linear Mech.* **170**, 104975 (2025)
- Du, J., Liu, X., Dai, H., Long, X.: Robust combined time delay control for milling chatter suppression of flexible workpieces. *Int. J. Mech. Sci.* **274**, 109257 (2024)
- Cai, J., Gao, Q., Zhang, X.: Proportional-retarded control of a quasi-zero-stiffness vibration isolator. *J. Sound Vib.* **578**, 118309 (2024)
- Yan, B., Wang, X., Ma, H., Lu, W., Li, Q.: Hybrid time-delayed feedforward and feedback control of lever-type quasi-zero-stiffness vibration isolators. *IEEE Trans. Ind. Electron.* **71**(3), 2810–2819 (2024)
- Wang, F., Sun, X., Meng, H., Xu, J.: Tunable broadband low-frequency band gap of multiple-layer metastructure induced by time-delayed vibration absorbers. *Nonlinear Dyn.* **107**(3), 1903–1918 (2022)
- Peng, J., Li, Y., Li, L., Lenci, S., Sun, H.: Time-delay feedback control of a suspended cable driven by subharmonic and superharmonic resonance. *Chaos, Solitons & Fractals* **181**, 114646 (2024)
- Liu, Z., Mao, X.: Dynamical analysis of a stay cable with a nonlinear energy sink and time-delayed feedback control. *Eng. Struct.* **310**, 118074 (2024)
- Cai, J., Zhu, S., Xu, H., Gao, Q.: Stability robustness and performance optimization of a semi-active/active hybrid quasi-zero-stiffness isolator with multiple-delayed feedback. *Mech. Syst. Signal Process.* **240**, 113335 (2025)
- Carrella, A., Brennan, M.J., Waters, T.P., Lopes, V., Jr.: Force and displacement transmissibility of a nonlinear isolator with high-static-low-dynamic-stiffness. *Int. J. Mech. Sci.* **55**(1), 22–29 (2012)
- Kovacic, I., Brennan, M.J.: The Duffing equation: nonlinear oscillators and their behaviour. John Wiley & Sons, (2011)
- Carrella, A., Brennan, M., Waters, T., Shin, K.: On the design of a high-static-low-dynamic stiffness isolator using linear mechanical springs and magnets. *J. Sound Vib.* **315**, 712–720 (2008)
- Zhang, Y., Lei, Y., Cao, J., Liu, Q., Liao, W.-H.: Nonlinear wire rope isolator with magnetic negative stiffness. *Mech. Syst. Signal Process.* **222**, 111791 (2025)
- Ma, H., Zhu, J., Zheng, W., Wang, K., Yan, B.: Buckling-induced high-static-low-dynamic stiffness vibration isolator with semiac-

- tive nonlinear damping. *IEEE/ASME Trans. Mechatron.* **31**(2), 2124–2135 (2026)
42. Xu, J., Su, X., Chen, X., Lai, Z., Zhu, B., Leng, Y.: Multi-magnet cluster levitation structure (MMCLS) for low-frequency vibration isolation with gravity compensation. *J. Sound Vib.* **623**, 119534 (2026)
 43. Jiao, G., Zeng, J., Wang, S.: A compact magnetic-curved-spring QZS isolator for supporting uncertain loads. *Nonlinear Dyn.* **113**(9), 9217–9238 (2025)
 44. Wang, S., Ling, P., Yan, G., Ma, H., Yan, B.: Multi-stage quasi-zero stiffness vibration isolation with linear-to-torsion magnetic negative stiffness mechanism for variable load. *Mech. Syst. Signal Process.* **249**, 114055 (2026)
 45. Pan, G., Jiao, X., Lin, C., Guan, Z., Wu, J., Liu, H., Zhao, H., Huang, H.: High load-bearing quasi-zero stiffness metamaterials for vibration isolation. *Int. J. Mech. Sci.* **293**, 110225 (2025)
 46. Li, X., Ding, B., Dzwaairo, D., Dong, X.: A magnetorheological damping-based quasi-zero-stiffness isolator for large-amplitude vibration isolation. *Int. J. Mech. Sci.* **304**, 110702 (2025)
 47. Yan, G., Lu, J.-J., Qi, W.-H., Liu, F.-R., Yan, H., Zhao, L.-C., Wu, Z.-Y., Zhang, W.-M.: Linear and nonlinear stiffness compensation for low-frequency vibration isolation: a comparative study. *Nonlinear Dyn.* **112**(8), 5955–5973 (2024)
 48. Chen, Y., Niu, M.-Q., Fan, Y., Chen, L.-Q.: Quasi-zero-stiffness vibration isolation via a double-layer compliant mechanism. *Int. J. Mech. Sci.* **304**, 110670 (2025)
 49. Zhang, Z.-Q., Niu, M.-Q., Chen, L.-Q.: Analysis and optimization of a quasi-zero-stiffness vibration isolator based on partially rigidized compliant mechanisms. *Eng. Struct.* **343**, 121048 (2025)
 50. Liu, J., Wang, A., Wang, L., Sun, Y., Wang, X., Wang, L., Gao, H., Zhang, Z., Han, H., Ding, Q.: Ultra-low and ultra-broad-band nonlinear quasi-zero-stiffness metastructure enabled by suspended oscillators. *Thin-Walled Struct.* **228**, 115053 (2026)
 51. Qiu, X., Lan, Y., Sun, Y., Xia, Y., Kadir, M.R.A., Lyu, Y.: Design of a quasi-zero-stiffness metastructure for vibration isolation using bistable inclined beam. *European J. Mechanics-A/Solids* **118**, 106086 (2026)
 52. Hamzehei, R., Bodaghi, M., Wu, N.: Mastering the art of designing mechanical metamaterials with quasi-zero stiffness for passive vibration isolation: a review. *Smart Mater. Struct.* **33**, 083001 (2024)
 53. Liu, C., Zhang, W., Yu, K., Liu, T., Zheng, Y.: Quasi-zero-stiffness vibration isolation: designs, improvements and applications. *Eng. Struct.* **301**, 117282 (2024)
 54. Li, H., Bi, K., Han, Q., Ma, R.: A state-of-the-art review on negative stiffness-based structural vibration control. *Eng. Struct.* **323**, 119247 (2025)
 55. Li, H., Li, Y., Li, J.: Negative stiffness devices for vibration isolation applications: a review. *Adv. Struct. Eng.* **23**, 1739–1755 (2020)
 56. Zhang, Z., Zhang, Y.-W., Ding, H.: Vibration control combining nonlinear isolation and nonlinear absorption. *Nonlinear Dyn.* **100**, 2121–2139 (2020)
 57. Xing, Z.-Y., Yang, X.-D.: A combined vibration isolation system capable of isolating large amplitude excitation. *Nonlinear Dyn.* **112**(4), 2523–2544 (2024)
 58. Sun, X., Xu, J.: Vibration control of nonlinear absorber–isolator-combined structure with time-delayed coupling. *Int. J. Non-Linear Mech.* **83**, 48–58 (2016)
 59. Zhou, S., Hou, B., Zheng, L., Xu, P., Yu, T., Ren, Z.: Nonlinear property and dynamic stability analysis of a novel bio-inspired vibration isolation–absorption structure. *Nonlinear Dyn.* **112**, 887–902 (2024)
 60. Xing, Z.-Y., Yang, X.-D.: A combined vibration isolation system with quasi-zero stiffness and dynamic vibration absorber. *Int. J. Mech. Sci.* **256**, 108508 (2023)
 61. Liu, C., Yu, K.: A high-static-low-dynamic-stiffness vibration isolator with the auxiliary system. *Nonlinear Dyn.* **94**, 1549–1567 (2018)
 62. Liu, Y., Cheng, L.: Delayed resonator for complete vibration suppression of primary structures with nonlinear stiffness. *Int. J. Non-Linear Mech.* **161**, 104689 (2024)
 63. Liu, L., Thomas, J.P., Dowell, E.H., Attar, P., Hall, K.C.: A comparison of classical and high dimensional harmonic balance approaches for a Duffing oscillator. *J. Comput. Phys.* **215**(1), 298–320 (2006)
 64. Dai, H.-H., Schnoor, M., Atluri, S.N.: A simple collocation scheme for obtaining the periodic solutions of the duffing equation, and its equivalence to the high dimensional harmonic balance method: subharmonic oscillations. *Comput. Model. Eng. Sci.* **84**(5), 459 (2012)
 65. Wang, J., Zhang, Y., Looi, D.T.W.: Analytical H_{∞} and H_2 optimization for negative-stiffness inerter-based systems. *Int. J. Mech. Sci.* **249**, 108261 (2023)
 66. Marton, L., Lantos, B.: Modeling, identification, and compensation of stick-slip friction. *IEEE Trans. Industr. Electron.* **54**(1), 511–521 (2007)
 67. Needs, S.: Viscosity-pressure effect on friction and temperature in a journal bearing. *Trans. Am. Soc. Mech. Eng.* **80**, 1099–1102 (1958)
 68. Lu, J., Wang, Q.J., Ren, N., Lockwood, F.E.: Correlation between pressure-viscosity coefficient and traction coefficient of the base stocks in traction lubricants: a molecular dynamic approach. *Tribol. Int.* **134**, 328–334 (2019)
 69. Prach, A., Kouba, O., Bernstein, D.S.: How slippery is viscous friction? [Lecture Notes]. *IEEE Control Syst. Mag.* **39**, 73–82 (2019)
 70. Liang, J.-W., Feeny, B.F.: Balancing energy to estimate damping parameters in forced oscillators. *J. Sound Vib.* **295**, 988–998 (2006)
 71. Detroux, T., Noël, J.-P., Virgin, L.N., Kerschen, G.: Experimental study of isolas in nonlinear systems featuring modal interactions. *PLoS ONE* **13**, e0194452 (2018)
 72. Friis, T., Tarpø, M., Katsanos, E.I., Brincker, R.: Equivalent linear systems of nonlinear systems. *J. Sound Vib.* **469**, 115126 (2020)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.