



A coupled-wave analysis framework for inhomogeneous elastic gratings and underwater metasurfaces customization

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ABSTRACT

Underwater acoustic metasurfaces are pivotal for marine applications due to their compact sub-wavelength profiles and powerful wave-control capabilities. However, conventional design strategies face a dilemma: generalized Snell's law (GSL)-based approaches neglect periodic diffraction and strong fluid–solid coupling, while full-wave finite-element simulations (FEM) suffer from prohibitive computational costs. To address this, we present a Coupled-Wave Analysis (CWA) model for strongly coupled acoustic-elastic systems. In particular, we derive Fourier-domain governing equations for laterally inhomogeneous elastic media, explicitly retaining the effects of spatially varying material properties on harmonic coupling. The resulting formulation efficiently solves periodic elastic gratings and agrees well with FEM. Within this framework, we investigate material-gradient effects, high-spatial-frequency components, and inter-unit coupling, providing further physical insight into periodic gradient structures. We further identify a simplified material profile featuring a single spatial-frequency component for beam deflection and sound insulation. Owing to the computational efficiency of CWA, we also establish a general customization strategy for metasurface material distributions, capable of steering incident waves to prescribed angles. Overall, our approach provides a practical, physics-informed tool for designing underwater metasurfaces, with potential extensions to three-dimensional geometries for complex underwater acoustic engineering.

1. Introduction

Underwater acoustic devices are increasingly important in applications such as sonar stealth, underwater communication, acoustic imaging, and vibration control (Croëne et al., 2025; Dong et al., 2023). Due to the long-wavelength feature of underwater environments, underwater structures tend to be bulky and heavy. Recent advancements in acoustic metamaterials have opened new avenues for sound control at subwavelength scales (Fang et al., 2006; Liu et al., 2000). Among existing techniques, acoustic metasurfaces have garnered significant attention as planar structures capable of reshaping wavefronts, owing to their ability to achieve effective wave control within compact configurations (Chen et al., 2022; Liang et al., 2018). Early metasurface designs largely followed the Generalized Snell's Law (GSL): by introducing spatially varying phase shifts across a period, one can realize beam steering (Li et al., 2013; Xie et al., 2014), focusing (Qi et al., 2017), self-

bending (Li et al., 2014b), mode conversion (Cao et al., 2020), and other functions in both airborne and waterborne settings (Chu et al., 2020; Fang et al., 2018). However, GSL-based designs typically treat each unit cell as locally reacting and neglect inter-unit coupling. This approximation is often acceptable in airborne scenarios, where the large impedance contrast between the solid and air and the absence of strong structural resonances impair inter-unit coupling. This is not the case in underwater environments due to the close impedances of the structure and the surrounding heavy medium. This induces strong acoustic-solid coupling, which in turn amplifies inter-unit interactions. Such interactions are, in most cases, more difficult to predict than the acoustic-solid coupling itself; they complicate the resulting wave behaviors, challenge the existing meta-surface design paradigm and may substantially degrade device performance.

To address the inter-unit coupling issue, researchers have pursued several strategies. Some studies structurally minimize the coupling

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between units by introducing air gaps (Zhou et al., 2022) or using weak mechanical connections (Li et al., 2024; Mai et al., 2026), thereby allowing unit designs to be treated independently and obviating the need to consider coupling details. Alternatively, some researchers have adopted the global strategy to design the entire structure through parameter optimization (Zhou et al., 2021), topology optimization (Cheng et al., 2025; Dong et al., 2022; Noguchi and Yamada, 2021; Zhou et al., 2023) and machine-learning-based inverse design (Zhang and Wang, 2025; Zhou et al., 2023), which implicitly capture the full coupling effect. These “nonlocal” approaches extend the design space beyond local GSL prescriptions and have produced high-performance devices, although the required repeated numerical evaluations can lead to substantial computational cost. However, their numerically driven nature may provide limited direct physical interpretation of the underlying coupling mechanisms. Generally speaking, the aforementioned methods either avoid discussing the impact of inter-unit interactions or account for them implicitly, providing limited physical insights into the underlying mechanisms. Moreover, they typically rely on repeated finite-element simulations, which are computationally expensive and limit exploration of the design space. Therefore, a physically transparent, computationally efficient modeling framework is needed for the design of periodic elastic structures in fluid environments, capable of capturing the inter-unit coupling, material inhomogeneity, and near-field effects while enabling rapid structural design, optimization and customization.

Several analytical and semi-analytical methods exist for modeling periodic structures. For instance, a power-flow-conformal approach was proposed to derive structural forms for highly efficient airborne acoustic steering (Díaz-Rubio et al., 2019); however, its applicability to elastic media has not yet been demonstrated. Other commonly used methods include the transfer-matrix method, which is well suited for planar structures (Parrinello and Ghiringhelli, 2016) and multilayer stacks (Shahsavari et al., 2021); the plane-wave expansion, widely used for phononic-crystal band-structure calculations (Lu et al., 2022; Xie et al., 2017); and effective-medium models for homogenized metamaterials (Li et al., 2014a; Wen et al., 2021). Another relevant frequency-domain approach is Coupled-Wave Analysis (CWA), which provides a natural framework for describing the coupling among spatial harmonics in periodic diffraction problems. It was originally developed in electromagnetics and has since been adapted to optical gratings (Hench and Strakos, 2008; Mohamad et al., 2020) and acoustic gratings (Yang et al., 2022; Yang et al., 2020). Compared with these formulations, the problem considered here involves a laterally inhomogeneous elastic region with vector displacement fields, coupling between compressional and shear responses, spatially varying material properties, and acoustic fluid loading. The corresponding formulation must therefore account simultaneously for the internal elastic harmonic coupling and the external reflected and transmitted acoustic diffraction fields. Incorporating these features into a unified Fourier-domain open-scattering framework requires additional treatment beyond conventional acoustic coupled-wave formulations.

Although gratings and metasurfaces are often described using different theoretical frameworks, with gratings analyzed in terms of multi-order diffraction and metasurfaces in terms of subwavelength wavefront control, they share a common geometric feature: a thin, periodic arrangement of engineered unit cells. This commonality naturally leads to treating metasurfaces as gratings and applying a modal framework for their analysis and design. In this context, one would expect a Fourier-modal approach that naturally captures diffraction orders, inter-unit coupling, and material inhomogeneity, thereby offering a practical and powerful framework for the analysis and design of metasurfaces.

In this work, we develop a CWA formulation for the diffraction scattering of laterally inhomogeneous elastic gratings in underwater environments. Starting from the inhomogeneous elastodynamic equations and Fourier expansion, we derive the Fourier-domain governing equations while explicitly retaining the effects of spatially varying

material properties on the coupling among spatial harmonics. The resulting elastic Fourier system is further coupled to the acoustic diffraction fields in the surrounding fluids through the fluid-solid interface conditions, allowing both the external scattering response and the internal elastic displacement field to be obtained within the same semi-analytical framework. The formulation shows good agreement with FEM simulations while substantially reducing the required computational cost.

Based on this framework, we further examine the physical effects associated with material gradients, high-spatial-frequency components, and inter-unit interactions. In addition, we identify a simplified material profile dominated by a single spatial-frequency component for beam deflection and sound insulation. We also demonstrate that higher-order spatial-harmonic coupling can be leveraged to enable high-performance device design. To exploit this, we develop an optimization strategy that takes advantage of CWA’s computational efficiency, allowing the customization of steering metasurfaces with both continuous and discrete property distributions at prescribed deflection angles. Moreover, we show that material discontinuities can generate high-spatial-frequency components that strongly influence the scattering performance. For discretely parameterized designs, the internal elastic response and inter-unit interactions are further examined through spatial-harmonic reconstruction and perturbation-based physical measures. These representative design examples demonstrate the analysis and customization capabilities enabled by the proposed formulation.

This paper is organized as follows. After the introduction, Section II develops the CWA framework for underwater elastic gratings. Section III presents representative case studies covering normal and oblique incidence for continuous and discrete gratings, and validates CWA against numerical simulations; it also discusses the influence of the derivative (gradient) terms in the elastic wave equations. Section IV describes the design of high-efficiency metasurfaces using the CWA framework and showcases the resulting optimized designs and their performance; the roles of high-spatial-frequency components, internal elastic-wave characteristics, and inter-unit interactions are further analyzed using representative configurations. Finally, Section V summarizes the main findings and outlines prospects for future work.

2. Coupled-wave analysis of underwater inhomogeneous elastic gratings

We consider a beam-steering periodic structure embedded in an infinite water medium, as illustrated in Fig. 1(a). The structure is infinitely periodic along the x_1 direction, extends infinitely along the x_2 direction, and is homogeneous along the x_3 direction. Accordingly, the schematic of the corresponding 2D configuration of the structure is depicted in Fig. 1(b). The discrete color patches in the figure are for visualization only and do not imply piecewise-constant material parameters; the formulation covers both continuous and discrete distributions. When an incident acoustic wave in water impinges on the grating, both reflection and transmission occur. Owing to the periodicity of the structure, parasitic higher-order diffraction components inevitably arise, governed by the following relationship

$$k \sin \theta_n = k \sin \theta_0 + G_n, \quad (1)$$

where $k = 2\pi/\lambda_w$ is the wavenumber in the surrounding medium (water), with λ_w denoting the corresponding wavelength, in this case; θ_n is the n -th diffraction angle relative to the grating normal; θ_0 is the incident angle; and $G_n = 2\pi n/L = nK$ represents the reciprocal lattice vector corresponding to the n -th diffraction order, where $n = 0, \pm 1, \pm 2, \dots, \pm \infty$, and L is the period of the structure, K represents the corresponding spatial frequency. For clarity of illustration, the entire space is divided into three regions: Regions I and III correspond to the reflection and transmission fields in water, respectively, while Region II represents the grating, in which elastic wave propagation takes place. The

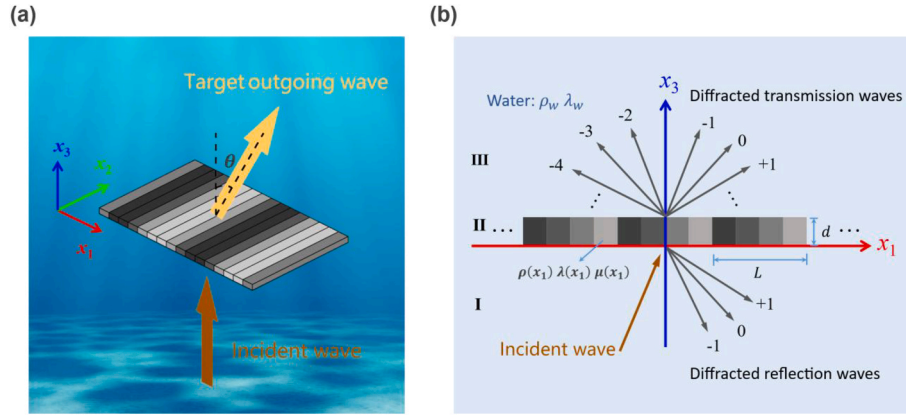


Fig. 1. (a) Schematic of the underwater metasurface. The brown arrow indicates the incident plane wave, while the yellow arrow represents the steered transmitted wave at a prescribed angle θ . (b) Illustration of elastic-wave diffraction by a periodic grating, showing the generation of multiple diffraction orders.

normalized fields in Regions I and III can be expressed as a superposition of the incident wave and all diffraction-order components, under the time-harmonic assumption $e^{j\omega t}$, which is omitted here for brevity, as given by:

$$P_I = \exp(-jk_{x10}x_1 - jk_{x30}x_3) + \sum_n P_n \exp(-jk_{x1n}x_1 + jk_{x3n}x_3), \quad (2)$$

$$P_{III} = \sum_n P_n \exp(-jk_{x1n}x_1 - jk_{x3n}(x_3 - d)), \quad (3)$$

where $k_{x10} = k \sin \theta_0$ and $k_{x30} = k \cos \theta_0$ denote the x_1 and x_3 components of the incident wavevector, respectively. For any integer n , the x_1 component of the n -th diffraction order is given by $k_{x1n} = k \sin \theta_0 + nK$.

The corresponding x_3 component is $k_{x3n} = +\sqrt{k^2 - k_{x1n}^2}$ for propagating modes (i.e., $|k_{x1n}| < k$), and $k_{x3n} = -j\sqrt{k_{x1n}^2 - k^2}$ for evanescent modes (i.e., $|k_{x1n}| > k$). The terms P_m and P_n denote normalized complex amplitudes of the n -th order reflected and transmitted waves, respectively, encapsulating both amplitude and phase information. Besides, d denotes the thickness of the structure. Since the grating is composed of elastic materials, it cannot be described by a scalar function alone. Instead, the elastic wave field within the structure is described using displacement components in two orthogonal directions u_1 and u_3 , corresponding to the x_1 - and x_3 - directions, respectively. Meanwhile, owing to the periodicity, the displacement field can be expressed as a superposition of spatial harmonics through a Fourier expansion:

$$u_1 = \sum_{n=-\infty}^{+\infty} U_{1n}(x_3) e^{-jk_{x1n}x_1}, u_3 = \sum_{n=-\infty}^{+\infty} U_{3n}(x_3) e^{-jk_{x1n}x_1}, \quad (4)$$

in which U_{1n} and U_{3n} denote the complex amplitudes of the n -th spatial harmonic of the displacement components u_1 and u_3 , respectively, as functions of x_3 . The displacements are governed by the equations of motion for two-dimensional elastic wave propagation, derived from Newton's second law, written as:

$$\rho \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{13}}{\partial x_3}, \quad \rho \frac{\partial^2 u_3}{\partial t^2} = \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{33}}{\partial x_3}, \quad (5)$$

where ρ denotes the mass density, and σ_{ij} are the components of the Cauchy stress tensor. To fully describe elastic wave propagation, the strain-displacement relations $\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$, together with the constitutive relations $\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}$, are coupled to the momentum equations. Here, λ and μ are the Lamé parameters, and the indices i, j , $k \in \{1, 3\}$. Owing to the spatial inhomogeneity of the constitutive parameters along the x_1 direction, the equation of motion for the elastic gratings is expressed as follows:

$$\begin{aligned} \rho \frac{\partial^2 u_1}{\partial t^2} &= (\lambda + 2\mu)_{,1} u_{1,1} + \lambda_{,1} u_{3,3} + (\lambda + 2\mu) u_{1,11} + \lambda u_{3,31} + \mu (u_{1,33} + u_{3,13}), \\ \rho \frac{\partial^2 u_3}{\partial t^2} &= \mu_{,1} (u_{1,3} + u_{3,1}) + \mu (u_{1,31} + u_{3,11}) + (\lambda + 2\mu) u_{3,33} + \lambda u_{1,13}. \end{aligned} \quad (6)$$

Here, the numbers following the comma in the subscripts denote directional partial derivatives. As shown in Eq. (6), the structural inhomogeneity introduces additional gradient terms into the equation of motion. Since all material properties vary periodically with respect to the position x_1 , they can likewise be represented by a Fourier series expansion, as:

$$\rho = \sum_{n=-\infty}^{+\infty} R_n e^{-jnKx_1}, \lambda = \sum_{n=-\infty}^{+\infty} \Lambda_n e^{-jnKx_1}, \mu = \sum_{n=-\infty}^{+\infty} M_n e^{-jnKx_1}, \quad (7)$$

where R_n , Λ_n and M_n are the complex coefficients of Fourier expansion of ρ , λ and μ , respectively. It is worth noting that, to ensure a real-valued property distribution after Fourier reconstruction, the expansion coefficients are typically conjugate-symmetric with respect to their negative indices. However, this symmetry can be broken when material properties are complex-valued, as in lossy or gain-bearing (active) media. By substituting Eqs. (4) and (7) into the elastic wave equation (Eq. (6)), the n -th coupled-wave equations can be obtained as:

$$\begin{aligned} -\sum_q M_{n-q} \frac{d^2 U_{1q}}{dx_3^2} + \sum_q j(M_{n-q} + \Lambda_{n-q}) k_{x1q} \frac{dU_{3q}}{dx_3} + \sum_q j(n-q) K \Lambda_{n-q} \frac{dU_{3q}}{dx_3} + \sum_q (n-q) K (\Lambda_{n-q} + 2M_{n-q}) k_{x1q} U_{1q} + \sum_q (\Lambda_{n-q} + 2M_{n-q}) k_{x1q}^2 U_{1q} - \omega^2 \sum_q R_{n-q} U_{1q} &= 0, \\ -\sum_q (\Lambda_{n-q} + 2M_{n-q}) \frac{d^2 U_{3q}}{dx_3^2} + \sum_q j(n-q) K M_{n-q} \frac{dU_{1q}}{dx_3} + \sum_q j M_{n-q} k_{x1q} \frac{dU_{1q}}{dx_3} + \sum_q j \Lambda_{n-q} k_{x1q} \frac{dU_{1q}}{dx_3} + \sum_q (n-q) K k_{x1q} M_{n-q} U_{3q} + \sum_q M_{n-q} k_{x1q}^2 U_{3q} - \omega^2 \sum_q R_{n-q} U_{3q} &= 0. \end{aligned} \quad (8)$$

In Eq. (8), $n \in \mathbb{Z}$ labels the harmonic order of the resulting coupled-wave equation, while $q \in \mathbb{Z}$ is the summation index over all displacement harmonics. The index difference $n-q$ arises from the Fourier convolution between the material parameters and the displacement field.

Assembling the CWA equations yields the following matrix form:

$$\begin{aligned} & \begin{bmatrix} -\mathbf{M} & 0 \\ 0 & -(\mathbf{A} + 2\mathbf{M}) \end{bmatrix} \begin{bmatrix} \mathbf{U}_1^+ \\ \mathbf{U}_3^+ \end{bmatrix} \\ & + \begin{bmatrix} 0 & j[(\mathbf{A} + \mathbf{M})\mathbf{K}_{x1} + \mathbf{BK}] \\ j[(\mathbf{A} + \mathbf{M})\mathbf{K}_{x1} + \mathbf{BK}] & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}_1^- \\ \mathbf{U}_3^- \end{bmatrix} \\ & + \begin{bmatrix} K(\mathbf{A} + 2\mathbf{B})\mathbf{K}_{x1} + (\mathbf{A} + 2\mathbf{M})\mathbf{K}_{x1}^2 - \omega^2\mathbf{R} & 0 \\ 0 & [\mathbf{KBK}_{x1} + \mathbf{MK}_{x1}^2 - \omega^2\mathbf{R}] \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_3 \end{bmatrix} \\ & = \mathbf{0}. \end{aligned} \quad (9)$$

Here, $\mathbf{A}$, $\mathbf{M}$, and $\mathbf{R}$ are the Fourier convolution matrices associated with the spatially varying material parameters, whereas $\mathbf{A}$ and $\mathbf{B}$ are the weighted convolution matrices arising from the corresponding material-gradient terms. These matrices possess a Toeplitz structure, with their elements defined as

$$\begin{aligned} [\mathbf{A}]_{nq} &= \Lambda_{n-q}, [\mathbf{M}]_{nq} = M_{n-q}, [\mathbf{R}]_{nq} = R_{n-q}, [\mathbf{A}]_{nq} = (n-q)\Lambda_{n-q}, [\mathbf{B}]_{nq} \\ &= (n-q)M_{n-q}. \end{aligned} \quad (10)$$

Besides, $\mathbf{K}_{x1}$ is a diagonal matrix defined by $[\mathbf{K}_{x1}]_{nn} = k_{x1n}\delta_{nn}$. The vectors $\mathbf{U}_i$ and $\mathbf{U}_i'$ are the first and second derivatives of the displacement vector $\mathbf{U}_i$ with respect to x_3 , while ω denotes the angular frequency. The dimension of the displacement vectors $\mathbf{U}_i$ depends on the truncated modes. For numerical implementation, the retained harmonic orders range from $-H$ to H , then the dimension of the vector is $N = 2H + 1$. We can simplify Eq. (9) as:

$$\mathbb{M} \begin{bmatrix} \mathbf{U}_1^+ \\ \mathbf{U}_3^+ \end{bmatrix} + \mathbb{C} \begin{bmatrix} \mathbf{U}_1^- \\ \mathbf{U}_3^- \end{bmatrix} + \mathbb{K} \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_3 \end{bmatrix} = \mathbf{0}. \quad (11)$$

For illustrative purposes, Eq. (11) is written in a form analogous to multi-degree-of-freedom damped-vibration equations; however, the terms have fundamentally different physical meanings because the variables are functions of spatial coordinates rather than time. Since this system of second-order equations is not straightforward to solve, we reformulate it into a first-order system using the state-space transformation method:

$$\mathbf{Y}' = \mathbb{S}\mathbf{Y}, \quad (12)$$

where $\mathbf{Y} = [\mathbf{U}_1 \quad \mathbf{U}_3 \quad d\mathbf{U}_1/dx_3 \quad d\mathbf{U}_3/dx_3]^T$ is the state vector of size $4N \times 1$, and $\mathbb{S}$ is the system matrix of the state-space equation, given by:

$$\mathbb{S} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}. \quad (13)$$

Equation (12) admits a direct solution of the form $\mathbf{Y} = e^{\mathbb{S}x_3}\mathbf{Y}_0$, where $\mathbf{Y}_0$ represents the initial state vector. However, preliminary calculations suggest that employing eigenvalues and eigenvectors enhances numerical stability by decoupling the system into independent modes while reducing error amplification in matrix exponential operations. Therefore, the solution is reformulated in the following form

$$\mathbf{Y}_a(x_3) = \sum_m \mathbf{w}_{a,m} e^{\gamma_m x_3} \mathbf{c}_m, \quad (14)$$

where γ_m denotes the m -th eigenvalue of matrix $\mathbb{S}$, $\mathbf{w}_{a,m}$ represents the (a, m) element of eigenvector matrix $\mathbf{W}$, and the coefficients $\mathbf{c}_m$ contain the information related to system characteristics and initial conditions,

which must be determined accordingly. Therefore, Eq. (14) can be written into matrix form as $\mathbf{Y} = \mathbf{W}e^{\Gamma x_3}\mathbf{C}$, where Γ is a diagonal matrix consisting of the eigenvalues. To obtain the vector $\mathbf{C}$, the boundary conditions at the interfaces between the structure and the surrounding fluid medium must be matched. These conditions include the continuity of normal velocity and normal stress on both upper and lower interfaces ($x_3 = 0, d$), along with the requirement of zero shear stress, since fluids cannot sustain shear—as outlined sequentially below:

$$-p|_{\text{fluid}} = \sigma_{33}|_{\text{solid}}, -\frac{1}{j\omega\rho_w} \frac{\partial p}{\partial x_3} \Big|_{\text{fluid}} = j\omega u_3|_{\text{solid}}, 0 = \sigma_{31}|_{\text{solid}}. \quad (15)$$

Substituting Eqs. (2)-(4), Eq. (7), and the solution $\mathbf{Y}(x_3)$ into Eq. (15) yields the n -th boundary condition equations as:

$$\begin{aligned} -(\delta_{n0} + P_m) &= \sum_q \left[(\Lambda_{n-q} + 2M_{n-q}) \frac{dU_{3q}(x_3)}{dx_3} - j\Lambda_{n-q} k_{x1q} U_{1q}(x_3) \right] \text{ at } x_3 = 0, \\ & -\frac{1}{j\omega\rho_w} (-jk_{x30}\delta_{n0} + jk_{x3n}P_m) = j\omega U_{3n}(x_3) \text{ at } x_3 = 0, \\ 0 &= \sum_q M_{n-q} \left(\frac{dU_{1q}(x_3)}{dx_3} - jk_{x1q} U_{3q}(x_3) \right) \text{ at } x_3 = 0, \\ -P_m &= \left[\sum_q (\Lambda_{n-q} + 2M_{n-q}) \frac{dU_{3q}(x_3)}{dx_3} - j\Lambda_{n-q} k_{x1q} U_{1q}(x_3) \right] \text{ at } x_3 = d, \\ & -\frac{1}{j\omega\rho_w} (-jk_{x3n}P_m) = j\omega U_{3n}(x_3) \text{ at } x_3 = d, \\ 0 &= \sum_q M_{n-q} \left(\frac{dU_{1q}(x_3)}{dx_3} - jk_{x1q} U_{3q}(x_3) \right) \text{ at } x_3 = d. \end{aligned} \quad (16)$$

We can then cast all the equations into the following matrix form

$$\begin{aligned} -\delta_{n0} - \mathbf{P}_r &= \begin{bmatrix} -j\mathbf{A}\mathbf{K}_{x1} & \mathbf{0} & \mathbf{0} & \mathbf{A} + 2\mathbf{M} \end{bmatrix} \mathbf{WC}, \\ -jk_{x30}\delta_{n0} + jk_{x3n}\mathbf{P}_r &= \begin{bmatrix} \mathbf{0} & \omega^2\rho_w\mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{WC}, \\ \vec{\mathbf{0}} &= \begin{bmatrix} \mathbf{0} & -j\mathbf{M}\mathbf{K}_{x1} & \mathbf{M} & \mathbf{0} \end{bmatrix} \mathbf{WC}, \\ -\mathbf{P}_t &= \begin{bmatrix} -j\mathbf{A}\mathbf{K}_{x1} & \mathbf{0} & \mathbf{0} & \mathbf{A} + 2\mathbf{M} \end{bmatrix} \mathbf{WEC}, \\ -jk_{x3n}\mathbf{P}_t &= \begin{bmatrix} \mathbf{0} & \omega^2\rho_w\mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{WEC}, \\ \vec{\mathbf{0}} &= \begin{bmatrix} \mathbf{0} & -j\mathbf{M}\mathbf{K}_{x1} & \mathbf{M} & \mathbf{0} \end{bmatrix} \mathbf{WEC} \end{aligned} \quad (17)$$

Here, $\vec{\mathbf{0}}$ and $\mathbf{0}$ represent zero vector of size $N \times 1$ and the zero matrix of size $N \times N$, respectively; $\mathbf{I}$ is the identity matrix of size $N \times N$, δ_{n0} is the Kronecker delta function representing the incident wave, which equals 1 only when $n = 0$ and 0 otherwise; $\mathbf{K}_{x3}$ is the diagonal matrix with elements k_{x3n} . Finally, $\mathbf{E}$ is the $4N \times 4N$ diagonal modal-propagation matrix across the elastic layer of thickness d , defined as $\mathbf{E} = e^{\Gamma d}$, whose m -th diagonal element is $e^{\gamma_m d}$. The Gaussian elimination method is used to eliminate the unknown vectors $\mathbf{P}_r$ and $\mathbf{P}_t$, which denote the complex amplitude vectors of the reflected and transmitted waves, respectively, as follows:

$$\begin{aligned} -jk_{x30}\delta_{n0} - jk_{x3n}\delta_{n0} &= \begin{bmatrix} \mathbf{K}_{x3}\mathbf{A}\mathbf{K}_{x1} & \omega^2\rho_w\mathbf{I} & \mathbf{0} & j\mathbf{K}_{x3}\mathbf{A} + 2j\mathbf{K}_{x3}\mathbf{M} \end{bmatrix} \mathbf{WC} \\ \vec{\mathbf{0}} &= \begin{bmatrix} \mathbf{0} & -j\mathbf{M}\mathbf{K}_{x1} & \mathbf{M} & \mathbf{0} \end{bmatrix} \mathbf{WC} \\ \vec{\mathbf{0}} &= \begin{bmatrix} \mathbf{K}_{x3}\mathbf{A}\mathbf{K}_{x1} & -\omega^2\rho_w\mathbf{I} & \mathbf{0} & j\mathbf{K}_{x3}\mathbf{A} + 2j\mathbf{K}_{x3}\mathbf{M} \end{bmatrix} \mathbf{WEC} \\ \vec{\mathbf{0}} &= \begin{bmatrix} \mathbf{0} & -j\mathbf{M}\mathbf{K}_{x1} & \mathbf{M} & \mathbf{0} \end{bmatrix} \mathbf{WEC} \end{aligned} \quad (18)$$

We isolate the coefficient vector $\mathbf{C}$ and consolidate all remaining terms on the right-hand side into a matrix $\mathbf{D}$ of size $4N \times 4N$. Additionally, we define the source vector as $\mathbf{b} = [(-jk_{x30}\delta_{n0} - jk_{x3n}\delta_{n0}) \quad \vec{\mathbf{0}} \quad \vec{\mathbf{0}} \quad \vec{\mathbf{0}}]^T$. Consequently, the solution vector $\mathbf{C}$ can be obtained by solving the linear system $\mathbf{C} = \mathbf{D}^{-1}\mathbf{b}$. Finally, by substituting $\mathbf{C}$ into the expressions $\mathbf{P}_r = [j\mathbf{A}\mathbf{K}_{x1} \quad \mathbf{0} \quad \mathbf{0} \quad -(\mathbf{A} + 2\mathbf{M})] \mathbf{WC} - \delta_{n0}$ and $\mathbf{P}_t = [j\mathbf{A}\mathbf{K}_{x1} \quad \mathbf{0} \quad \mathbf{0} \quad -(\mathbf{A} + 2\mathbf{M})] \mathbf{WEC}$, we obtain the coefficient amplitudes of all reflected and transmitted diffraction orders. The acoustic pressure field can then be reconstructed using Eqs. (2) and (3).

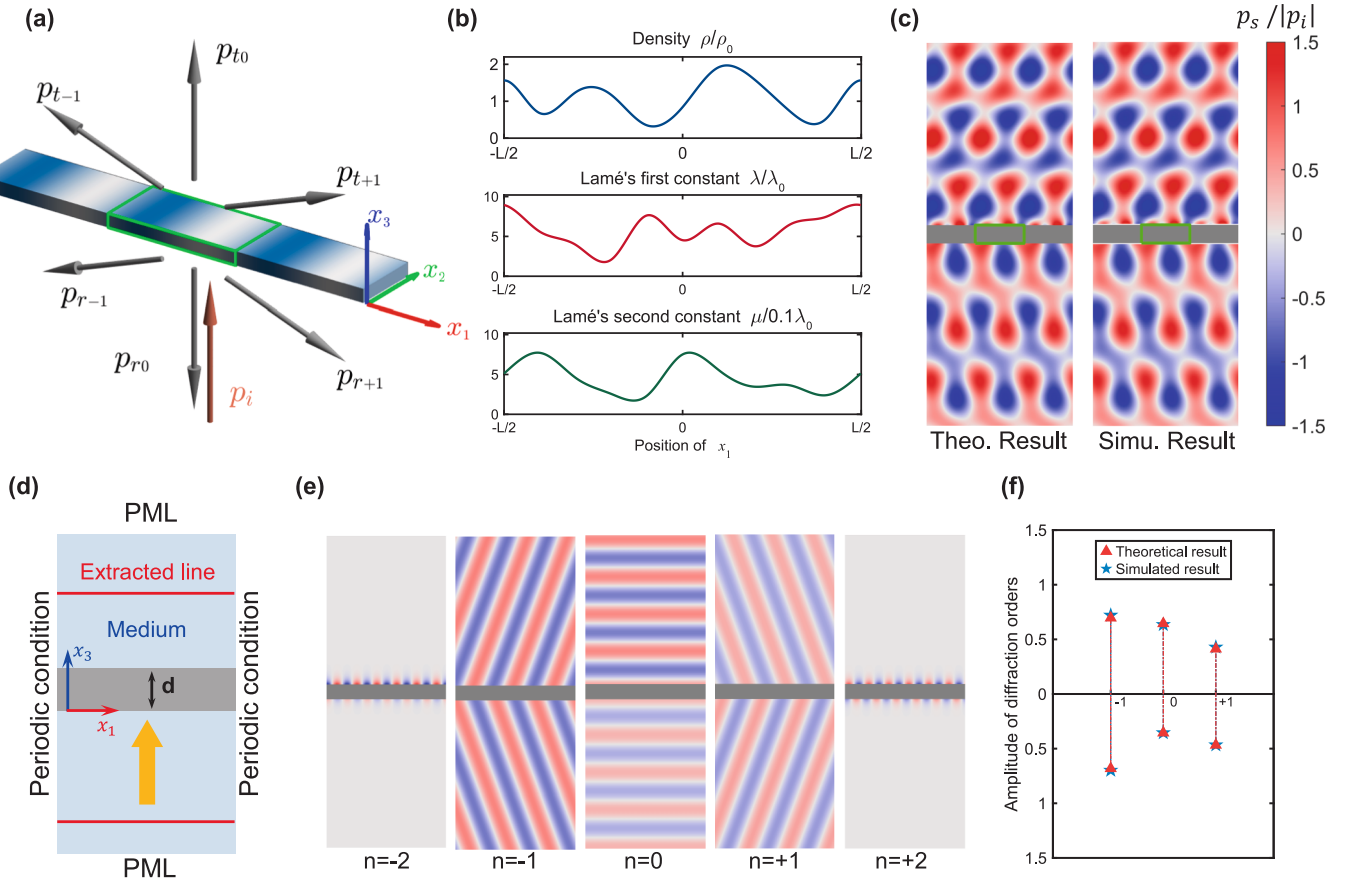


Fig. 2. Elastic gradient grating. (a) Schematic of the three-period grating structure and diffraction orders. The green box highlights a single period of the grating. (b) Example of an arbitrary material-property distribution within one period, including density, Lamé's first parameter, and Lamé's second parameter. (c) Comparison of normalized scattered acoustic pressure fields obtained by CWA and FEM. (d) Simulation model and field-extraction configuration. (e) Acoustic pressure fields of the five central diffraction orders. (f) Comparison of propagating-mode amplitudes obtained by theory and simulation. In (b), the density and Lamé's first parameter are normalized by the water density and bulk modulus, respectively, while Lamé's second parameter is normalized by one-tenth of the water bulk modulus. In (f), transmitted and reflected diffraction orders are plotted above and below the x -axis, respectively.

3. Inhomogeneous elastic wave gratings

3.1. Diffraction characteristics of the grating under normal incidence

To verify the proposed formulation, we consider a grating embedded in water with material properties that vary periodically along x_1 . The structure, having a finite thickness along x_3 , is infinite along x_2 , and is illuminated at normal incidence ($\theta_0 = 0^\circ$), as illustrated in Fig. 2(a). As previously discussed, the elastic structure can be fully characterized by three constitutive parameters: the density $\rho(x_1)$, Lamé's first parameter $\lambda(x_1)$ and Lamé's second parameter $\mu(x_1)$. For a more general treatment, we consider a continuously varying property distribution, encompassing the discrete case as a special instance, which will be discussed later. An example featuring an arbitrary, continuously differentiable distribution of material properties along x_1 is presented in Fig. 2(b), representing a structure referred to as a continuous elastic wave grating. This distribution is generated by selecting ten equally spaced sampling values within one period and reconstructing the continuous profile using periodic cubic spline interpolation.

Here, the frequency f of the incident wave is set to 3000 Hz, and the thickness of structure is chosen as $d = 2\lambda_w/5$, where $\lambda_w = c_w/f$ is the acoustic wavelength in water. The acoustic velocity in water, c_w , is set to 1500 m/s. The period of grating L is set to $\lambda_w/\sin 70^\circ$. In this situation, there are only three orders of modes (-1 st, 0 th, $+1$ st) propagating modes in both reflected and transmitted regions under the normal incidence, as determined by the wavenumbers of k_{x1n} and k_{x3n} . The

corresponding propagation angles relative to the normal are -70° , 0° , and $+70^\circ$, in both the transmitted and reflected region.

Using the CWA formulation described above, we obtain the coefficient amplitudes and reconstruct the scattered field by superimposing all diffraction orders, as shown on the left panel in Fig. 2(c). For comparison, the simulation result obtained using the finite element software COMSOL Multiphysics is presented on the right panel in Fig. 2(c). The finite element model applies periodic boundary conditions on the lateral sides, with perfectly matched layers (PMLs) implemented at the top and bottom boundaries of the fluid domain to simulate an infinite medium, as illustrated in Fig. 2(d). The corresponding mesh convergence of the FEM model is verified in Appendix A.2. It is evident that the CWA method proposed here shows good agreement with the finite element simulation, as demonstrated by the highly similar scattered field patterns. This confirms the correctness and the efficacy of the method for accurately predicting the performance of underwater elastic grating structures. Fig. 2(e) displays the corresponding five central diffraction modes, including the three propagating modes ($0, \pm 1$) and two evanescent modes (± 2), computed using the CWA proposed herein. For a more accurate comparison, the scattered pressure field is extracted along a line located at an appropriate distance from and parallel to the grating, in both the reflection and transmission regions. The extracted lines, depicted in red, are shown in Fig. 2(d). The distances of the extracted lines are selected to minimize the influence of strong near-field effects. A Fourier Transform (FT) is then applied to the scattered pressure distribution along the lines to extract the coefficients corresponding

to each diffraction order, given by $|P_m| = \frac{1}{L} \left| \int_{-L/2}^{L/2} P_s e^{ik_{x1n}x_1} dx_1 \right|$; reflection amplitudes are obtained similarly. Fig. 2(f) shows the comparison in terms of the amplitudes of the propagating modes obtained by the CWA proposed here and by the FE simulation, which shows a good consistency between them. Taking the FE simulation as a reference, the maximum relative error in the amplitudes of the propagating modes is 2.2%, calculated as $100\% \times \frac{|P_t^C| - |P_t^S|}{|P_t^S|}$, where the superscripts C and S denote the CWA method and simulation, respectively.

It is worth noting that determining the total number of diffraction orders involves a trade-off between the computational efficiency and accuracy to be achieved. In our case, the Fourier expansion is truncated at the maximum harmonic order $H = 15$, such that the orders $n = -15, \dots, 15$ are retained, corresponding to a total of $N = 2H + 1 = 31$ Fourier harmonics. This truncation achieves the reported accuracy while maintaining computational efficiency. The corresponding convergence analysis, as presented in Appendix A.1, shows that the results do not change beyond this order. In addition to the truncation order, numerical stability is also affected by the evaluation of the modal exponential terms. In the numerical implementation, these terms are re-referenced to the appropriate interfaces according to their decay directions, thereby avoiding the direct evaluation of exponentially growing factors.

3.2. Influence of constitutive parameter gradients

As shown in Eq. (6), the spatial variation of the Lamé parameters introduces additional terms containing the spatial derivatives of the material parameters and the first derivatives of displacement. These terms arise directly from the product rule when the divergence-form elastodynamic equations are expanded and are therefore part of the complete governing equations for an inhomogeneous elastic medium.

To examine the influence of the material-gradient terms, the result obtained from the complete CWA formulation in Fig. 2 is used as the reference. For comparison, an additional calculation is performed using the same material-property distribution and calculation parameters, but with the material-gradient terms omitted from the governing operator. Specifically, matrices **A** and **B**, which are the Fourier-domain representations of the terms involving the spatial derivatives of λ and μ , respectively, are set to zero.

Fig. 3(a) compares the normalized scattered pressure fields obtained from the two formulations. The left panel shows the result obtained from

the complete formulation, while the right panel shows the result obtained after matrices **A** and **B** are set to zero. A substantial change in the scattering pattern can be observed after the material-gradient terms are omitted. The corresponding amplitudes of the propagating diffraction orders are compared in Fig. 3(b), which further shows that the two formulations predict different distributions of the scattered waves among the propagating orders. However, the order amplitudes alone do not provide a direct measure of the total scattered power because the power carried by each order also depends on its propagation angle. We therefore evaluate the reflected and transmitted powers by accounting for the contributions of all propagating orders.

The normal component of the time-averaged acoustic intensity carried by the n -th propagating order is calculated as $I_n = |P_n|^2 \cos \theta_n / 2\rho_w c_w$, where P_n and θ_n are the pressure amplitude and propagation angle of the n -th diffraction order, respectively, and ρ_w and c_w are the density and sound speed of water. The total reflected and transmitted powers are obtained by summing the contributions of all propagating reflected and transmitted orders. For the complete CWA formulation, the normalized reflected and transmitted powers are $R = 0.360591$ and $T = 0.639409$, respectively, giving $R + T = 1.000000$, with a power-balance residual of 9.17×10^{-13} . After matrices **A** and **B** are omitted, R and T become 0.788564 and 1.139287, respectively, resulting in $R + T = 1.927851$. The loss of power balance results from the change in the governing operator. This is because the material-gradient terms arise naturally from applying the product rule to the spatially varying Lamé parameters. Omitting these terms while retaining the same material distribution makes the governing equation incomplete. As a result, the calculated reflected and transmitted powers no longer satisfy the energy balance expected for a lossless system. This comparison shows that the material-gradient terms must be retained to accurately describe wave propagation in inhomogeneous elastic media.

3.3. Additional validation of the generality of the proposed CWA method

To demonstrate the general applicability of the proposed CWA method, additional examples under different scenarios are presented in Fig. 4. First, we examine an oblique-incidence case (Fig. 4(a-c)), where the incident wave impinges on the grating at -40° relative to the normal while all other settings remain unchanged. Wavenumber analysis shows that, under these conditions, only the -1 st and 0 th diffraction orders are propagating in both reflection and transmission. The material profile

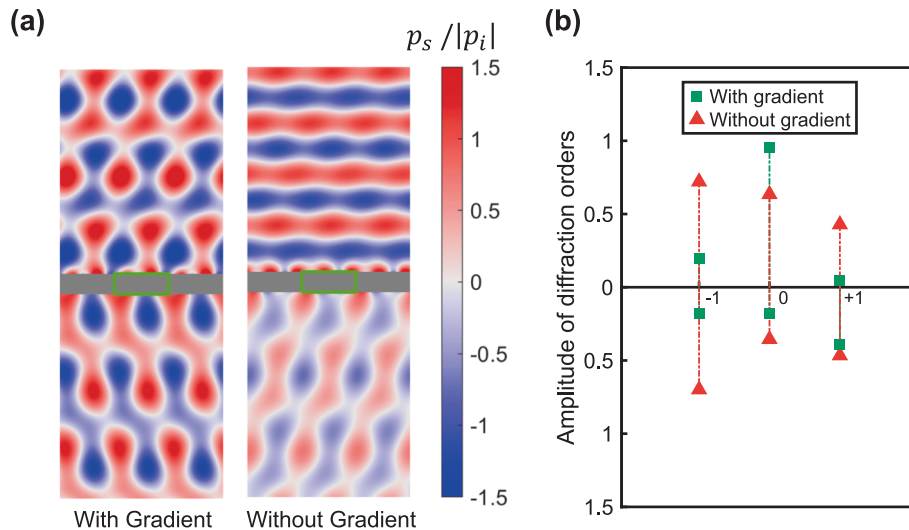


Fig. 3. Comparison between the complete CWA formulation and an artificially modified formulation in which the material-gradient terms are omitted. (a) Normalized scattered acoustic pressure fields. The left panel shows the result obtained using the complete formulation, while the right panel shows the result obtained after setting the matrix terms **A** and **B** to zero. Green boxes indicate one period of the grating. (b) Comparison of the propagating-order amplitudes obtained from the two formulations.

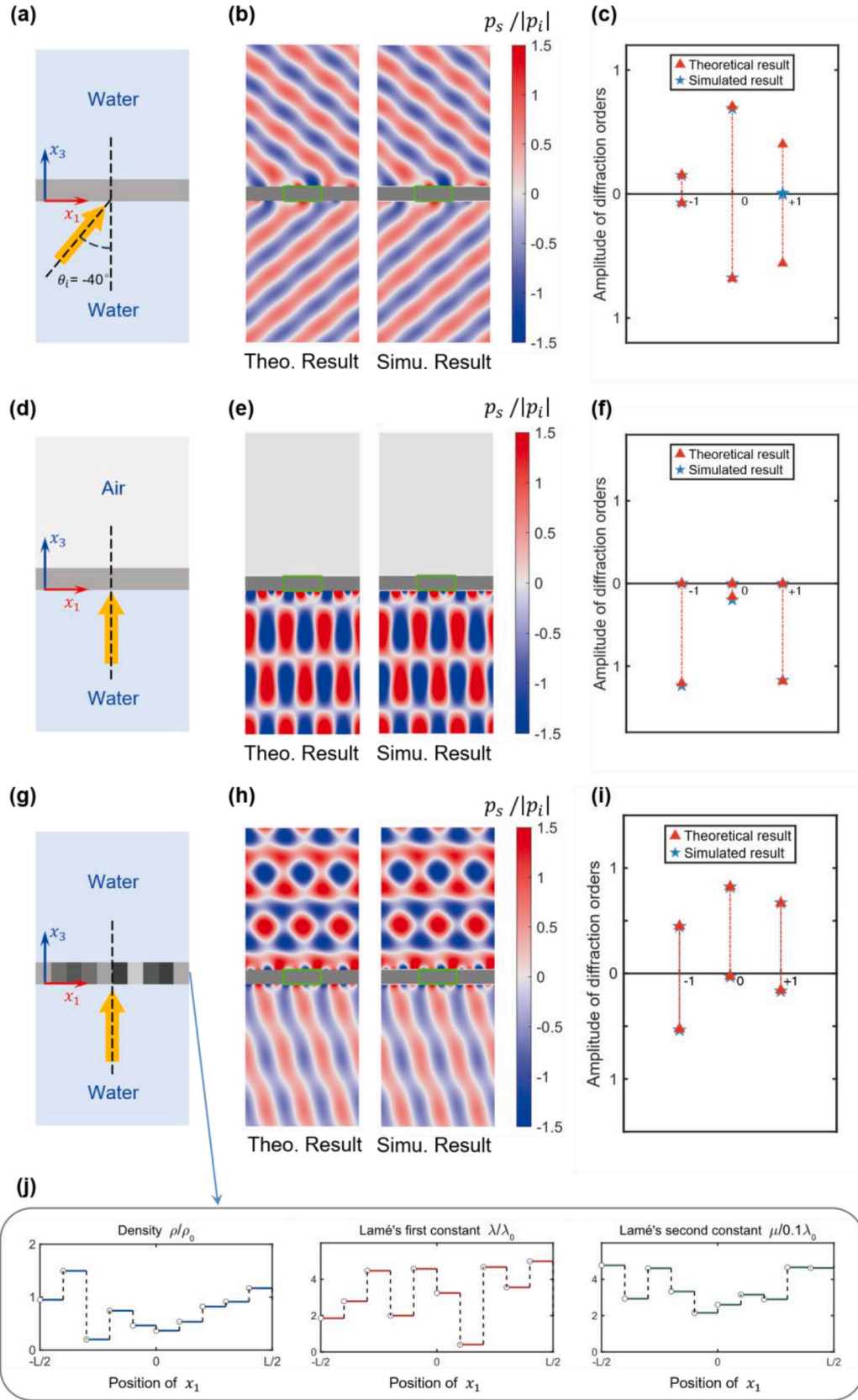


Fig. 4. Elastic wave gratings under various scenarios. (a–c) Oblique-incidence case (incident angle $\theta_i = -40^\circ$ relative to the grating normal): (a) schematic diagram; (b) scattered fields from CWA and simulation (yellow arrows indicate incidence, green boxes indicate one period); (c) amplitude comparison. (d–f) Reflection-grating case (reflection region: water; transmitted region: air): (d) schematic diagram; (e) scattered fields; (f) amplitude comparison. (g–i) Discretely parameterized grating with piecewise-constant material properties: (g) schematic diagram: discrete blocks denote the units with differing properties that compose the structure; (h)

scattered fields; (i) amplitude comparison; (j) property distribution: hollow circles mark discontinuities, solid lines indicate discrete profiles, and dashed lines show step changes.

is identical to that of the first example. The sole difference is the incident angle; the model is shown in Fig. 4 (a). The scattered field (Fig. 4(b)) and the amplitude comparison (Fig. 4(c)) confirm that the CWA predictions remain highly accurate. Note that the Fourier Transform of the pressure field extracted from the simulation cannot recover the amplitude of the +1 st order because that order is evanescent here; consequently, Fig. 4 (c) shows a large discrepancy at the +1 st order, as CWA predicts a strong localized intensity, whereas the simulation-extracted amplitude is nearly zero.

Next, we consider a total-reflection scenario (Fig. 4(d–f)). This occurs when the characteristic impedance of the transmitted medium is much lower than that of the reflected region and the grating. For example, if the transmitted region is air while the reflected region is water, the upper interface behaves like a pressure-release boundary (Fig. 4 (d)). Here, the physical parameters of air are taken as $\rho_{\text{air}} = 1.225 \text{ kg/m}^3$ and

$c_{\text{air}} = 343 \text{ m/s}$. For comparison, water is characterized by $\rho_{\text{water}} = 1000 \text{ kg/m}^3$ and $c_{\text{water}} = 1500 \text{ m/s}$. Using the same property distribution, the scattered field is shown in Fig. 4 (e); no pressure disturbance appears in the transmitted region despite persistent particle displacement. The amplitude comparison in Fig. 4(f) again demonstrates good agreement between CWA and the numerical simulation.

Finally, we treat a discretely parameterized grating (Fig. 4(g–j)), a configuration commonly employed for targeted acoustic functionalities; the model is represented by Fig. 4 (g). By “discrete grating” we mean a periodic structure whose material properties are piecewise constant within each period: here each period comprises units of identical length but ten distinct material-parameter values selected randomly within a prescribed range (Fig. 4 (j)). The proposed CWA method is also applied to this discontinuous profile, with the material-property discontinuities treated using inverse-rule Fourier factorization based on Li’s general

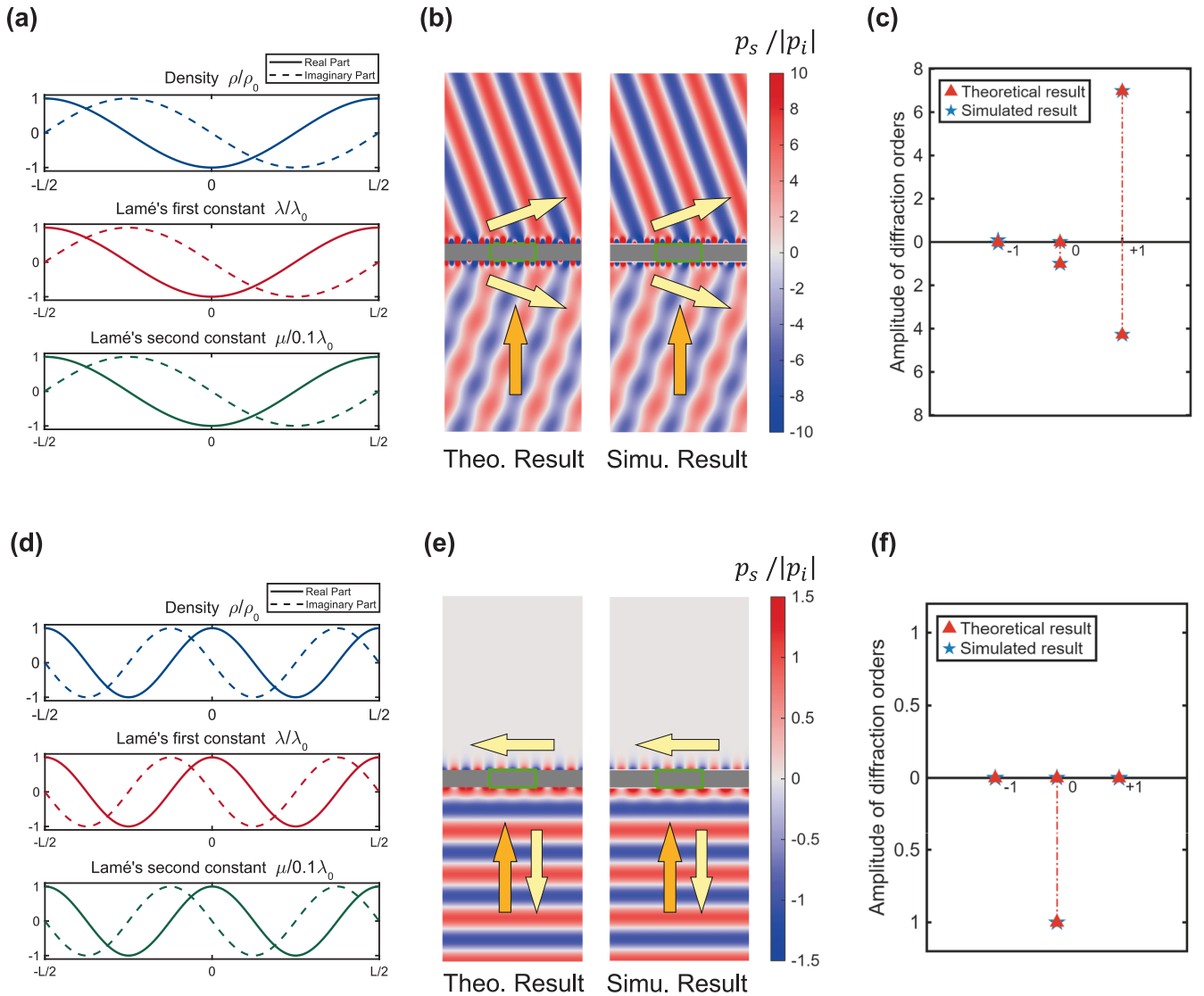


Fig. 5. Three-period elastic gratings with a single spatial-frequency component. (a–c) Grating with only the +1 component: (a) property distribution; (b) comparison of the scattered fields obtained by CWA and FEM; (c) extracted propagating-wave amplitude spectrum. (d–f) Grating with only the –2 component: (d) property distribution; (e) comparison of the scattered fields obtained by CWA and FEM; (f) extracted propagating-wave amplitude spectrum. In (a,d), solid and dashed lines denote the real and imaginary parts, respectively. In (b,e), dark-yellow arrows indicate the incident wave, light-yellow arrows indicate the primary transmitted and reflected waves, and green boxes highlight the central period.

theory and its subsequent applications to elastic-wave plane-wave-expansion formulations (Cao et al., 2004; Li, 1996). The same established factorization principle is adopted in the subsequent discrete-structure optimization, and its standard derivation is not repeated here for brevity. Fig. 4(h) compares the reconstructed scattered field from CWA with the corresponding finite-element result and shows strong agreement. Following the same procedure used elsewhere, we extract the scattered pressure along a line at a fixed distance from the grating and obtain the diffraction amplitudes via the Fourier Transform; the amplitude comparison in Fig. 4(i) confirms the accuracy of the CWA solution. This example further demonstrates that the proposed CWA formulation remains applicable to piecewise-constant material profiles, with the effects of abrupt material variations consistently retained in the Fourier-domain formulation.

The above examples demonstrate the general applicability of the proposed method. In the following section, we present the design of functional elastic gratings, hereafter referred to as metasurfaces because of their subwavelength dimensions and strong acoustic-wave manipulation capabilities, using the CWA method developed herein.

4. High-efficiency gradient elastic metasurfaces for underwater wave steering

4.1. Elastic metasurfaces with single spatial frequency components

By reviewing the CWA method derived above, it becomes evident that all diffraction orders are mutually coupled through the spatial frequency components of the material properties. Consequently, if the system contains only a single spatial frequency component, it will primarily excite the corresponding diffraction mode. To confirm this inference, we study the same geometric grating as Fig. 2 under two illustrative cases, each containing only a single nonzero Fourier component, as plotted in Fig. 5. For the exact single-spatial-frequency case, the conventional CWA formulation using the same Fourier truncation range for all field variables leads to rank-deficient convolution matrices. This special case can be treated by factoring out the common spatial harmonic and using appropriately shifted Fourier bases, while the FEM results are shown here for direct numerical verification. The corresponding formulation and treatment are described in detail in Appendix C. Fig. 5(a-c) presents the elastic grating with material properties containing only the $n=+1$ spatial frequency components, meaning all other Fourier components are set to zero. The corresponding property distribution is shown in Fig. 5(a), where the real part follows a cosine distribution and the imaginary part follows a sine function within one period. The scattered field pattern shown in Fig. 5(b) demonstrates that the incident wave is predominantly redirected into the $+1$ st order in both transmission and reflection region, as a result of the system containing only $+1$ st-order material parameter components. A small residual 0 th-order component remains in the reflected field, which can be attributed to the presence of the incident wave on the right-hand side of the 0th-order equation, preventing the equation from being identically zero. This observation is further confirmed by the amplitudes of the propagating modes shown in Fig. 5(c). It is worth noting that high diffraction orders can still emerge, as evidenced by the presence of strong evanescent waves, even when the system lacks corresponding high-order components. This phenomenon arises from energy transfer from lower to higher orders through mode coupling.

We also investigate the case where the system contains only -2 nd-order material components, as shown in Fig. 5(d-f). The results indicate that under this condition, no propagating modes emerge in the transmitted region, while only the 0 th-order propagating mode remains in the reflected region. The incident wave is completely blocked by the grating, resulting in total sound insulation. This suggests that the second-order material component cannot efficiently couple the incident field to the 1 st- or 0 th-order propagating diffraction modes, indicating that the coupling toward lower diffraction orders is significantly

suppressed in this single-component system. However, when the system contains dual Fourier components, such as the simultaneous presence of the $+1$ and -1 components, additional diffraction orders can be more readily excited through inter-order coupling. For real-valued material distributions, these paired Fourier components are generally difficult to avoid because of the conjugate symmetry of the Fourier coefficients. As a result, suppressing undesired diffraction orders and achieving highly efficient angular steering becomes more challenging, beyond the conventional explanation based solely on impedance mismatch (Díaz-Rubio and Tretyakov, 2017).

The single-harmonic gratings shown in Fig. 5 not only involve negative real parts of the effective density and elastic moduli, phenomena already reported in the open literature (Cheng et al., 2008; Fang et al., 2006; Liu et al., 2000), but also incorporate nonzero imaginary parts of the material parameters. The presence of complex-valued material parameters, such as density and elastic moduli, represent non-conservative effective material responses associated with loss or gain. For instance, consider the complex density $\rho = \rho_r + j\rho_i$, where the real part ρ_r characterizes the effective inertial response, while the imaginary part ρ_i contributes to energy dissipation or amplification. Under the $e^{j\omega t}$ convention adopted here, a negative imaginary part of the effective density corresponds to a dissipative contribution, whereas a positive imaginary part represents an amplifying contribution. Complex-valued elastic moduli similarly describe dissipative or active constitutive responses, although the corresponding sign of the imaginary part depends on the constitutive parameter and the adopted time-harmonic convention. Therefore, our results indicate that a carefully engineered spatial distribution of gain and loss can facilitate the redistribution of energy among diffraction orders, providing a potential route toward the design of high-efficiency acoustic devices, in agreement with previous studies (An et al., 2021; Rivet et al., 2018).

4.2. Optimization of continuous elastic metasurfaces

The analysis above has shown that a constitutive parameter distribution containing only the 1st-order spatial harmonic can generate a single corresponding propagating diffraction order, while a 2nd-order distribution can completely insulate the incident sound wave. However, such distributions typically involve positive, negative, and even complex values, placing stringent requirements on material design. In contrast, a distribution composed of purely real-valued material properties is more practical to implement. Since maintaining a real-valued distribution requires that Fourier coefficients appear in complex-conjugate pairs, it becomes inherently challenging to suppress the resulting parasitic symmetric diffraction modes when designing high-efficiency meta-devices for steering acoustic energy in a desired direction, particularly for large-angle deflection. The CWA framework reveals that the total acoustic field arises from the coupling of all diffraction orders; therefore, higher-order modes may play a significant role in achieving high-efficiency asymmetric transmission. Due to the complexity of accounting for the coupling effects among all diffraction modes, we adopt a Genetic Algorithm (GA) to perform the optimization and design the structures.

We continue to consider the case of normal incidence, where both the reflected and transmitted regions support three propagating modes. The objective is to maximize the acoustic energy in the $+1$ st transmitted order. By varying the period length L , the $+1$ st transmitted order can be steered into the desired direction. For normal incidence, the period is chosen as $L = \lambda_w / \sin\theta_{\text{target}}$, where λ_w is the acoustic wavelength in water and θ_{target} is the target transmission angle of the $+1$ st order measured from the structure normal. We consider two specific target angles: $+40^\circ$ and $+70^\circ$. At $f = 3000\text{Hz}$ with $c_w = 1500\text{m/s}$ (so $\lambda_w = 0.5\text{m}$), the corresponding periods are $L = 0.7779\text{m}$ and 0.5321m , respectively. The structural thickness is kept the same as in the previous cases, with a value of $2\lambda_w/5$. Thus, the structure has a subwavelength

thickness of $0.4\lambda_w$. Together with its ability to achieve the prescribed wavefront manipulation within a thin periodic layer, this sub-wavelength thickness supports its classification as an acoustic metasurface. The computation of the objective function is carried out based on the proposed CWA framework. In contrast to the finite element method, the CWA framework proposed here achieves high computational efficiency due to its mesh-free formulation, enabling rapid evaluation at reduced computational cost. The mathematical formulation of the optimization problem is written as

$$\text{Find : } \rho(x_1), \lambda(x_1), \mu(x_1) \quad (19)$$

$$\text{Subject to : } \begin{cases} \rho_{\min} \leq \rho(x_1) \leq \rho_{\max} \\ \lambda_{\min} \leq \lambda(x_1) \leq \lambda_{\max} \\ \mu_{\min} \leq \mu(x_1) \leq \mu_{\max} \end{cases},$$

$$\text{Maximizing : Fitness} = |P_{m=+1}|,$$

where $\rho(x_1)$, $\lambda(x_1)$ and $\mu(x_1)$ denote the three constitutive parameters. All parameters are real-valued and constrained within physically reasonable bounds to avoid the extreme values. $|P_{m=+1}|$ denotes the amplitude of the +1st transmitted diffraction order. For simplicity, we define 10 equally spaced sampling points within one period, reconstruct the continuous distribution via periodic cubic spline interpolation, and

optimize the values at these points to maximize the objective function.

The optimization results are presented in Fig. 6. Panels (a) and (d) show the corresponding optimized spatial distributions of material properties for target transmission angles of $+40^\circ$ and $+70^\circ$, respectively, indicating that relatively low values of density and elastic moduli are favorable for achieving high-efficiency transmission. Fig. 6(b) and (e) compare the scattered fields computed by the CWA method and by numerical simulation; a small discrepancy is apparent in the reflection region across all cases. Nevertheless, the transmission regions exhibit strong consistency, confirming the effectiveness of the proposed method. The amplitude comparisons in Fig. 6(c) and (f) corroborate this agreement. The discrepancy arises from the presence of small elastic moduli, which are associated with short wavelengths within the structure. As a result, a finer computational grid is required to improve numerical accuracy, particularly in regions where material properties approach near-zero values and exhibit rapid spatial variations. The mesh density in this context is determined by balancing computational accuracy and cost; consequently, some degree of precision may be sacrificed.

As previously mentioned, when the constitutive parameters are strictly real-valued, it becomes difficult to eliminate parasitic symmetric diffraction orders due to the inherent complex conjugate symmetry of the Fourier coefficients. Therefore, higher-order spatial-frequency components may play an important role in achieving the desired

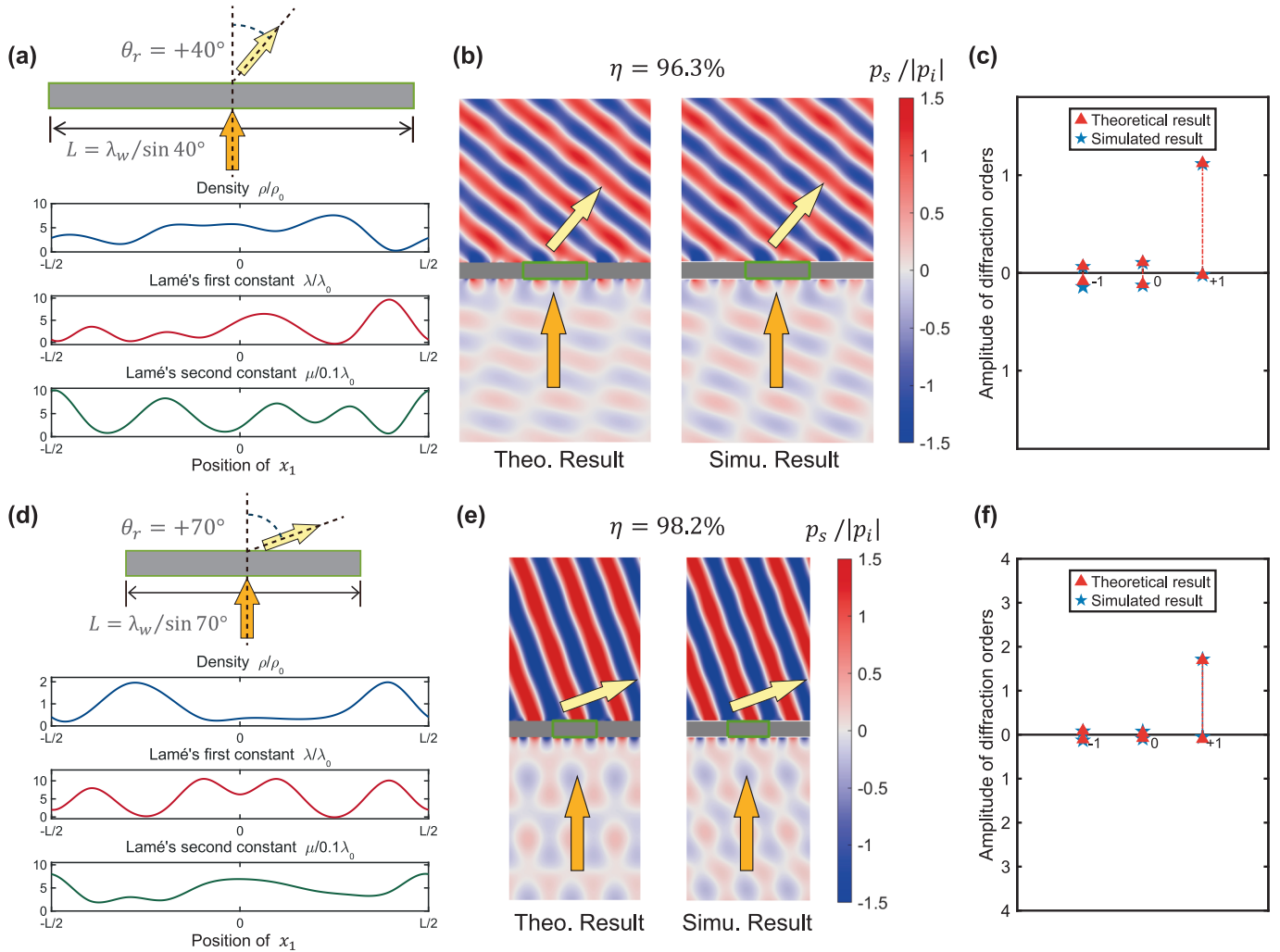


Fig. 6. Optimization results for highly efficient asymmetric transmission metasurfaces with continuous property distribution at deflection angles $+40^\circ$ and $+70^\circ$. Panels are grouped by angle: $+40^\circ$ (a–c) and $+70^\circ$ (d–f). (a, d) Model schematics and spatial distribution of optimized material properties for each target transmission angle. (b, e) Comparison of scattered acoustic pressure fields obtained from CWA and finite-element simulation. Dark yellow arrows indicate the incident wave; light yellow arrows show the targeted transmitted direction. (c, f) Comparison of diffraction-order amplitudes between CWA and simulation.

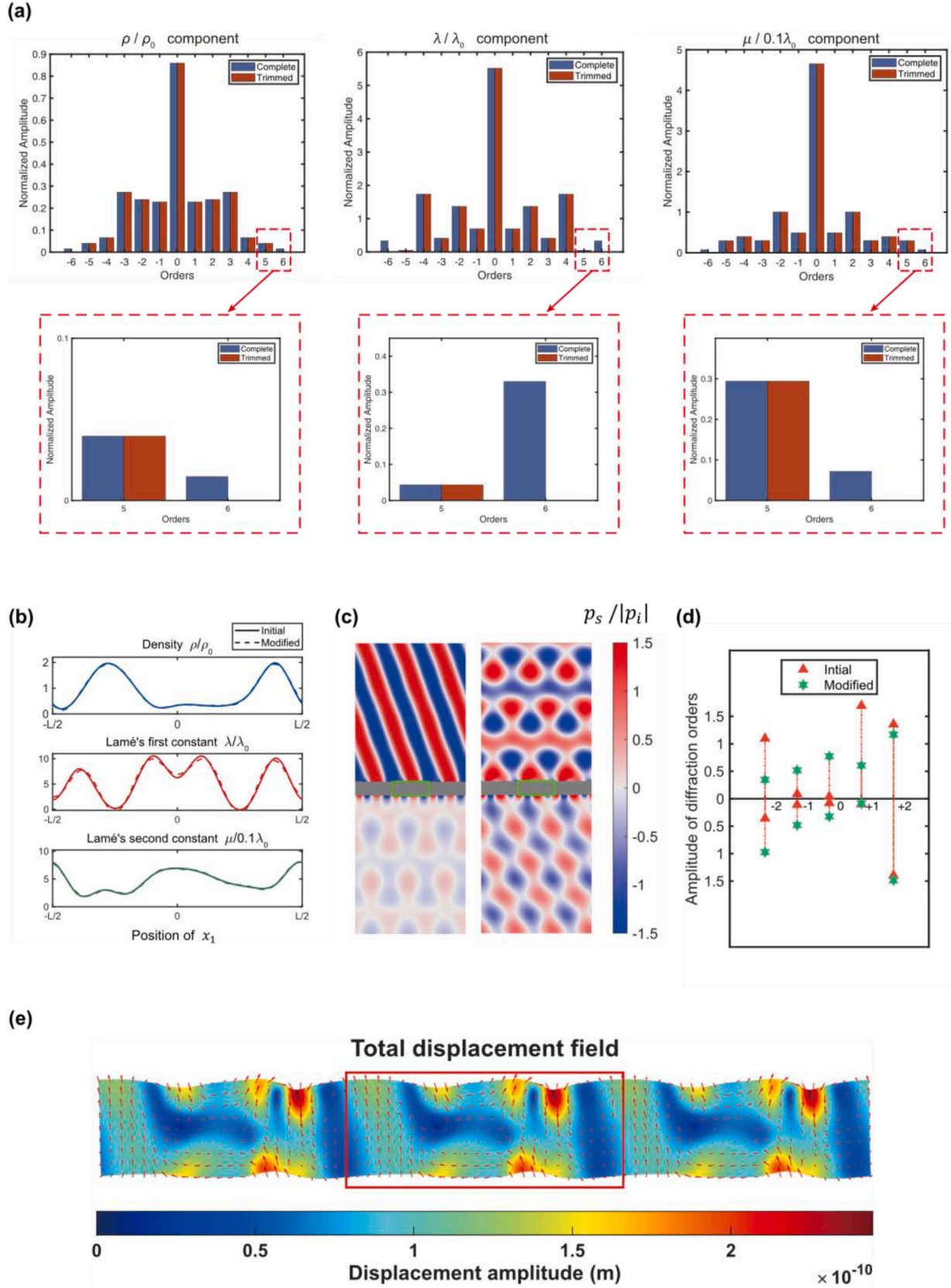


Fig. 7. Effect of high-order spatial-frequency components on the continuously optimized design for the target angle. (a) Spatial-frequency spectra of the optimized material properties before and after high-order truncation, with enlarged views of the truncated components. (b) Original optimized property distributions and their inverse-Fourier reconstructed profiles after truncation. (c) Scattered acoustic pressure fields of the original and truncated structures. (d) Comparison of the corresponding diffraction-order amplitudes. (e) Total displacement field of the optimized metasurface, with the red box highlighting one period and the arrows indicating displacement vectors.

asymmetric diffraction response with real-valued material parameters. To investigate the influence of higher-order components, we take the optimized $+70^\circ$ case as an example to selectively remove high-order terms from the Fourier coefficients, specifically those of order ± 6 and above (as shown in Fig. 7(a)), in order to observe their impact on the performance. The property distribution is reconstructed via inverse Fourier transform, as shown in Fig. 7(b). Compared with the original distribution, the differences are minimal, with the most noticeable deviation observed in the Lamé's first parameter. However, the scattered field shown in Fig. 7(c) reveals a significant difference between the original and modified structures. The grating is no longer able to efficiently steer acoustic energy toward the $+70^\circ$ direction, which corresponds to the $+1$ st diffraction order. To examine the energy distribution across diffraction orders, the amplitude components of both the propagating modes and the two nearest evanescent modes are presented by CWA, as shown in Fig. 7(d). It is observed that, after removing the high spatial-frequency components of the material properties, most of the energy is redistributed among the three propagating modes in both the transmission and reflection regions, while the evanescent components in the transmission region are also significantly weakened. Therefore, the high-order spatial-frequency components make an important contribution to maintaining high-efficiency acoustic steering. We also perform a robustness analysis by perturbing the control-point values, with the corresponding results provided in Appendix B. The optimized design maintains high steering efficiency under small perturbations, while larger perturbations lead to a gradual reduction in efficiency and increased performance variation. We also reconstruct the total displacement field inside the structure, as shown in Fig. 7(e), to provide a direct view of the elastic-wave response. The displacement amplitude

exhibits a strongly nonuniform distribution, with localized enhancements appearing near both the upper and lower interfaces. The field pattern repeats periodically along the metasurface, consistent with the periodic material distribution. Such a full-field reconstruction is directly obtained from the Fourier components solved within the CWA framework.

Although the total displacement field provides an overall picture of the structural response, it does not distinguish the different wave components contributing to the observed field pattern. Further decomposition is therefore required to reveal the underlying characteristics of the elastic-wave response. The optimized configuration for the $+70^\circ$ deflection case is adopted again for the following analysis. We first examine the P-like and S-like components, representing the compressional and shear characteristics of the wave field, respectively, to characterize the dominant wave behavior within the structure. The P-like and S-like responses are characterized by $|\nabla \cdot \mathbf{u}|$ and $|\nabla \times \mathbf{u}|$, respectively, and are normalized using a common reference for visualization, as shown in Fig. 8(a). To quantify their overall contributions within the structure, $|\nabla \cdot \mathbf{u}|^2$ and $|\nabla \times \mathbf{u}|^2$ are integrated over the entire structural domain. The relative proportion of each component is then obtained by normalizing the corresponding integral by the sum of the two integrals. The P-like and S-like contributions are found to be 2.37% and 97.63%, respectively, indicating that the wave field inside the optimized metasurface is strongly dominated by shear-like motion. This result indicates that the internal elastic response of the optimized structure is predominantly shear-like, suggesting substantial coupling from the incident compressional excitation to shear-like motion within the structure. Such mode conversion provides an additional mechanism for redistributing the wave field among the spatial harmonics, thereby

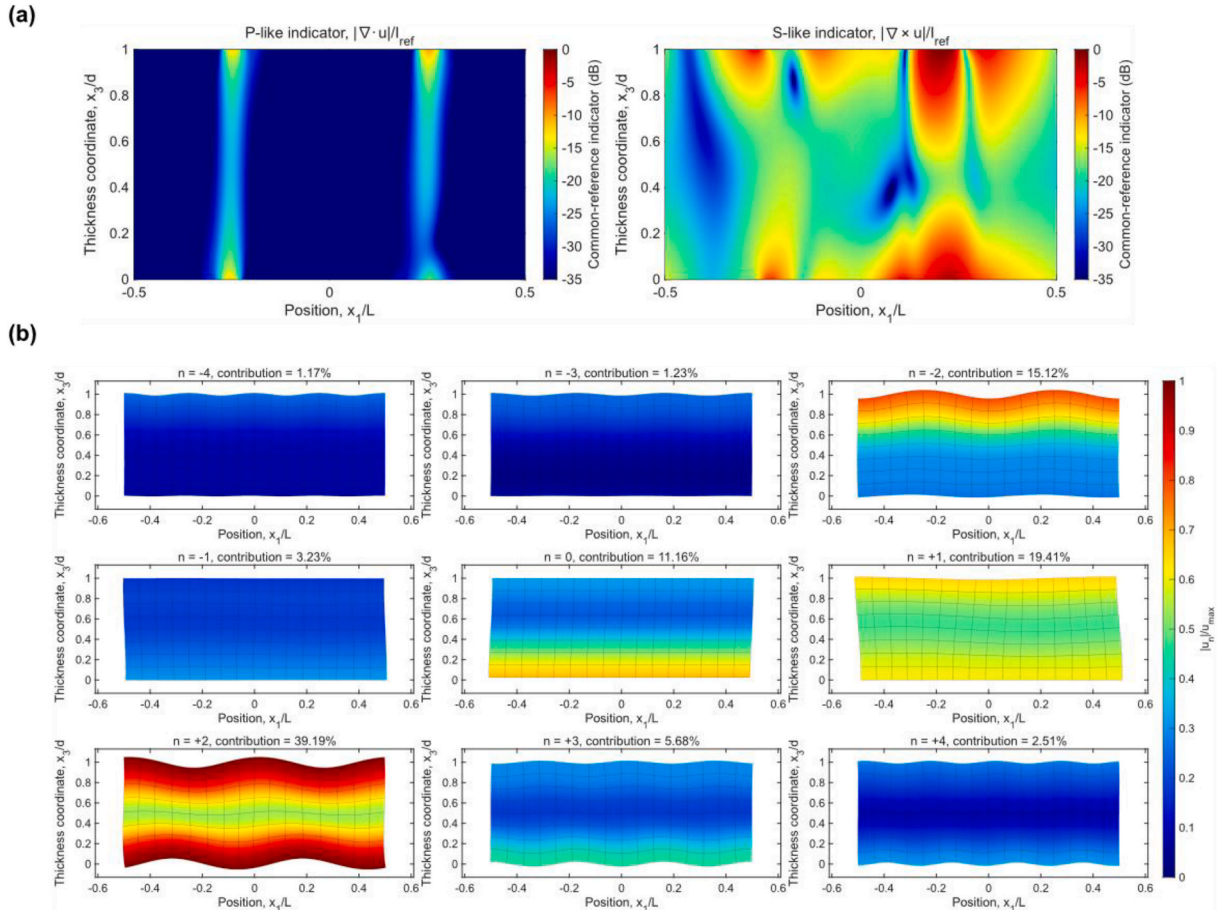


Fig. 8. Wave-component decomposition of the continuously optimized elastic metasurface. (a) Spatial distributions of the P-like and S-like components. (b) Displacement contributions associated with spatial-harmonic orders.

facilitating the transfer of energy toward the targeted diffraction channel and contributing to the high-efficiency wave steering. It should be noted that the terms P-like and S-like are used because pure P- and S-wave separation is difficult in the inhomogeneous structure. They represent only the compressional and shear characteristics of the displacement field, rather than actual P- and S-wave components. The corresponding proportions should also not be interpreted as energy fractions, since the elastic energy additionally depends on the spatially varying material properties. These indicators are therefore used only as a reference for characterizing the relative compressional and shear features of the structural response.

In addition, to further examine the displacement characteristics of the optimized structure, we take advantage of the Fourier representation inherent in the CWA formulation and reconstruct the displacement fields associated with the central spatial-harmonic orders from $n = -4$ to $+4$. This order-resolved reconstruction allows the contribution and spatial distribution of individual Fourier components to the total displacement field to be examined separately. For quantitative comparison, the relative contribution of each spatial-harmonic order is evaluated from the integrated squared displacement amplitude through the structural thickness and normalized by the sum over all retained Fourier harmonics, as

$$\text{Relative contribution of order } n = \frac{\int_0^d \left[|U_{1n}(x_3)|^2 + |U_{3n}(x_3)|^2 \right] dx_3}{\sum_q \int_0^d \left[|U_{1q}(x_3)|^2 + |U_{3q}(x_3)|^2 \right] dx_3} \times 100\% \quad (20)$$

The order-resolved displacement fields are shown in Fig. 8(b). The positive-order components are generally stronger than the corresponding negative-order components, which is consistent with the targeted positive steering direction of the optimized metasurface. Among these components, the $n = +2$ order gives the largest contribution, indicating that the structural displacement response is mainly concentrated in this spatial harmonic.

The above P-/S-like and Fourier-order analyses provide a clearer physical picture of the wave behavior inside the optimized metasurface. The P-/S-like results reveal a predominantly shear-like internal response, while the Fourier-order decomposition shows a preferential contribution from positive spatial harmonics consistent with the targeted steering direction. These observations further clarify the internal wave characteristics associated with the optimized steering performance.

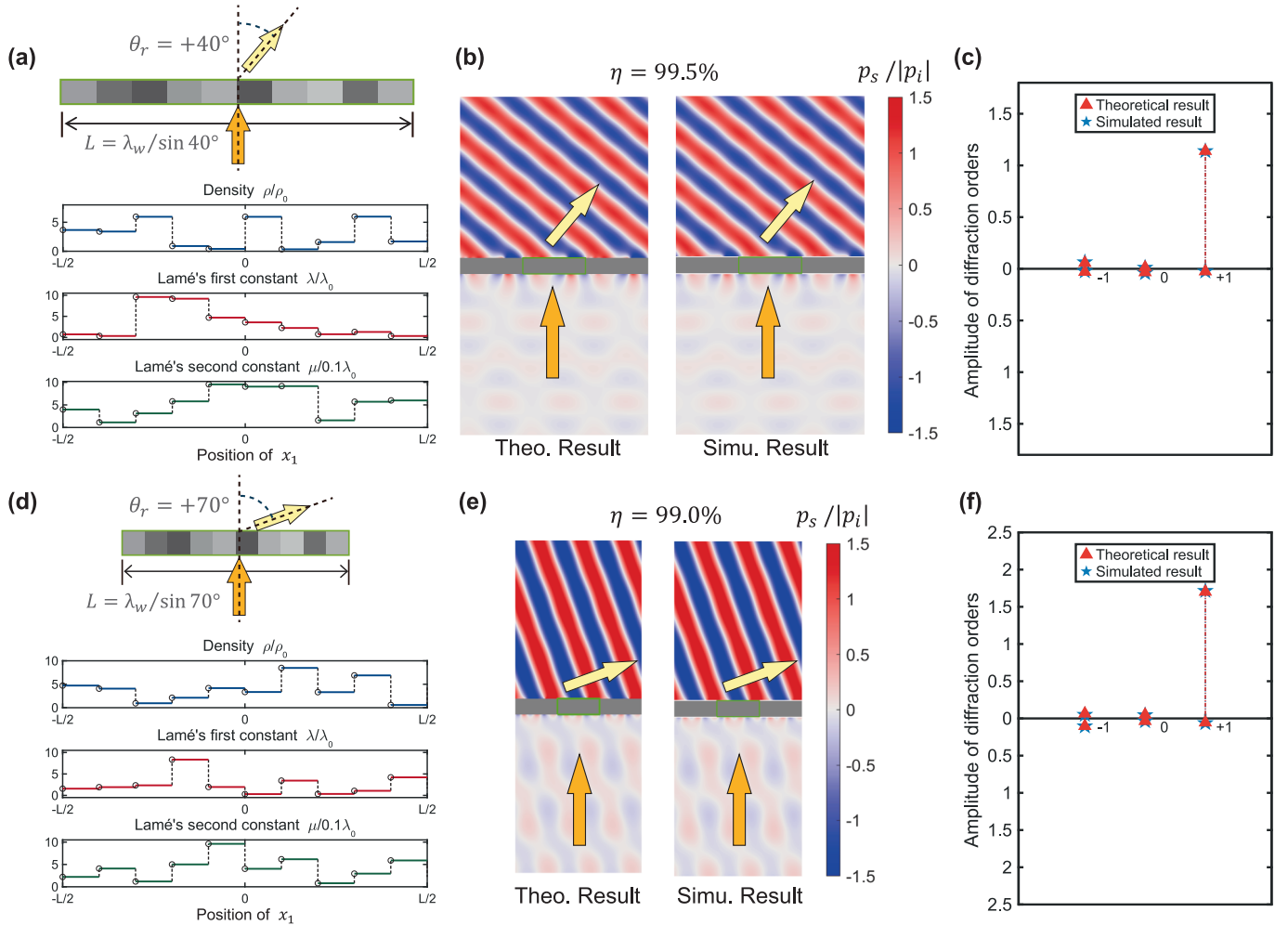


Fig. 9. Optimization results for highly efficient asymmetric transmission gratings with discrete property distribution at deflection angles $+40^\circ$ and $+70^\circ$. Panels are grouped by angle: $+40^\circ$ (a–c) and $+70^\circ$ (d–f). (a, d) Model schematics and spatial distribution of optimized material properties for each target angle. Hollow circles indicate points of discontinuity, and dashed lines represent step changes. (b, e) Comparison of scattered pressure fields obtained from CWA and finite-element simulation. Dark yellow arrows indicate the incident wave; light yellow arrows show the targeted transmitted direction. (c, f) Comparison of diffraction-order amplitudes between CWA and simulation.

4.3. Optimization of discrete elastic metasurfaces

In the previous subsection, we showed that the proposed CWA method effectively handles gratings composed of discrete units, a class of structures widely studied in the literature. Discrete configurations, involving fewer variables than continuous ones, offer advantages in design and fabrication, which deserve further examination. Here we present two discrete designs using the same overall procedure as in the continuous case. Each period is divided into ten equal-length units whose material properties are held constant within prescribed ranges, while all other geometric parameters are identical to the continuous-structure examples. Such an arrangement produces piecewise-constant material profiles with abrupt discontinuities at the interfaces between adjacent units. The discrete-structure calculations reported below use $H = 31$, whose adequacy is verified a posteriori by comparing the convergence behavior of the direct Fourier factorization and the inverse-rule Fourier factorization in Appendix A.4. Our objective is to determine the property values of these ten units to achieve a highly efficient acoustic function, specifically maximizing the +1st transmitted order, as in the continuous cases. For illustration, we still select two target transmission angles of $+40^\circ$ and $+70^\circ$. The corresponding unit lengths are $L/10 = 0.07779\text{m}$ ($0.156\lambda_0$) and 0.05321m ($0.106\lambda_0$), respectively. Using the GA, the optimized results are shown in Fig. 9. Panels (a) and (d) depict the optimized spatial distributions of material properties for the two target angles, respectively. Panels (b) and (e) present the scattered fields obtained from the proposed CWA method and corresponding FEM simulations for both scenarios. The steering objectives are achieved by our method, as evidenced by the amplitude comparisons shown in Panels (c) and (f).

We further examine the discrete optimized result for the $+70^\circ$ deflection case. The optimized material parameters exhibit piecewise-constant distributions, and the sharp interfaces naturally contain rich high-spatial-frequency components. To quantify the role of these components, the high-order Fourier components of all three constitutive parameters are progressively set to zero while the computational truncation order is kept unchanged, following the same procedure used for

the continuous optimized structure. The resulting scattering responses are then compared to assess the influence of the removed high-spatial-frequency components. Here, h denotes the Fourier order of the material-property spectra; components satisfying the specified range of $|h|$ are retained, while all higher-order components are set to zero. Retaining $|h| \leq 17$ largely preserves the steering performance, with the efficiency decreasing only slightly from 99.02% for the original optimized structure to 97.39%, whereas further truncation to $|h| \leq 16$ reduces the efficiency markedly to 53.08%; this deterioration is evident in the scattered fields (Fig. 10(b)) and the amplitude comparison (Fig. 10(c)). Fig. 10(a) shows the original distributions of the three constitutive parameters and their inverse-FFT reconstructions after retaining different spatial-frequency ranges. Compared with continuous profiles, discrete gratings contain substantially more high-spatial-frequency content, a feature that can be exploited in metasurface design. The reconstructed displacement field of the optimized structure is shown in Fig. 10(d) to facilitate interpretation. Pronounced displacement concentrations appear near the interfaces between the medium and the structure.

In the CWA formulation, a local variation in the material properties modifies the corresponding Fourier coefficients and gradient-related terms, thereby altering the coupling among spatial harmonics. This spectral representation provides the basis for examining how the influence of a local perturbation extends across the discrete metasurface. We further investigate the interaction between structural units using the discrete optimized metasurface, in which the individual units are explicitly defined. The ten units within one period are numbered from 1 to 10 from left to right. To examine these interactions quantitatively, we introduce small perturbations to the material parameters of selected units and evaluate their effects in terms of the material-property dependence, the local mechanical-energy response within the structure, the resulting variation in the far-field diffraction response, and the non-additive interaction between pairs of units.

To evaluate the material-property dependence of the inter-unit response, each material parameter $\chi \in \{\rho, \lambda, \mu\}$ in unit α is independently increased by 2%, while all other parameters are kept unchanged.

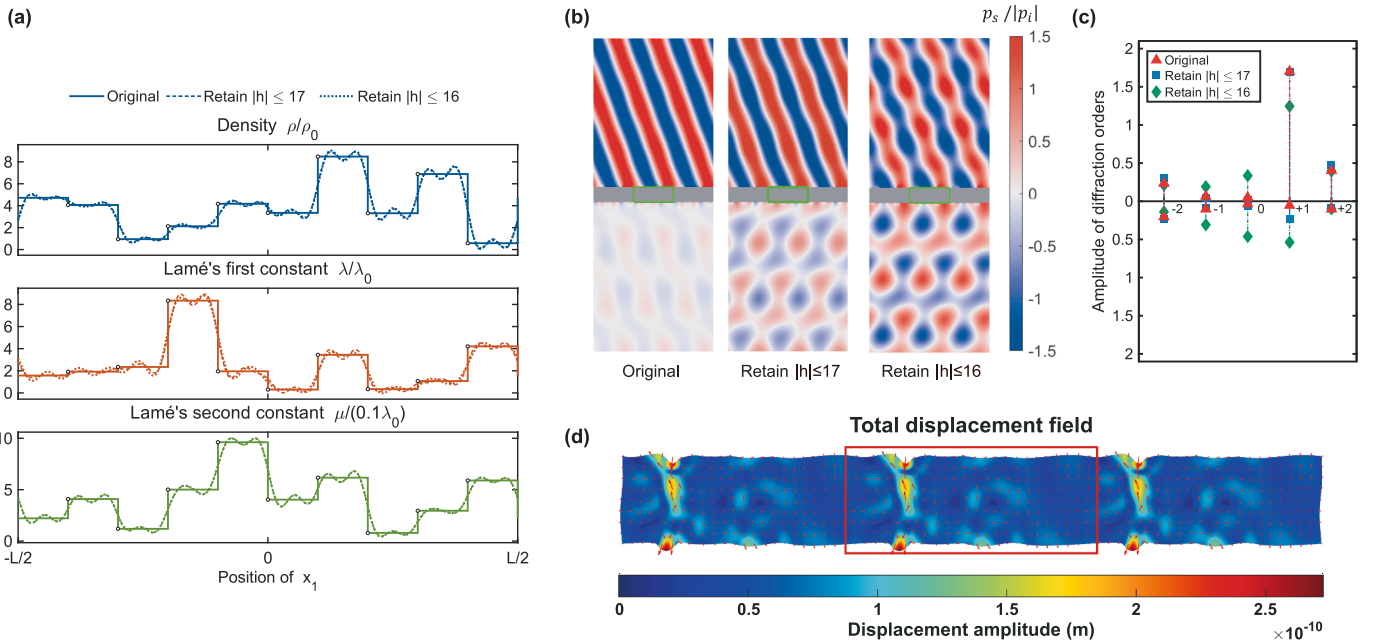


Fig. 10. Investigation of the effect of high-order spatial-frequency components on the discrete optimized design targeting $+70^\circ$. (a) Comparison of the original material-property distributions and their Fourier reconstructions after retaining different spatial-frequency ranges, including density, Lamé's first parameter, and Lamé's second parameter. (b) Normalized scattered acoustic pressure fields of the original optimized structure and the structures reconstructed with truncated spatial-frequency components. (c) Comparison of the amplitudes of the central diffraction orders for the original and spectrally truncated structures. (d) Total displacement field of the optimized discrete structure, with the red box highlighting one grating period.

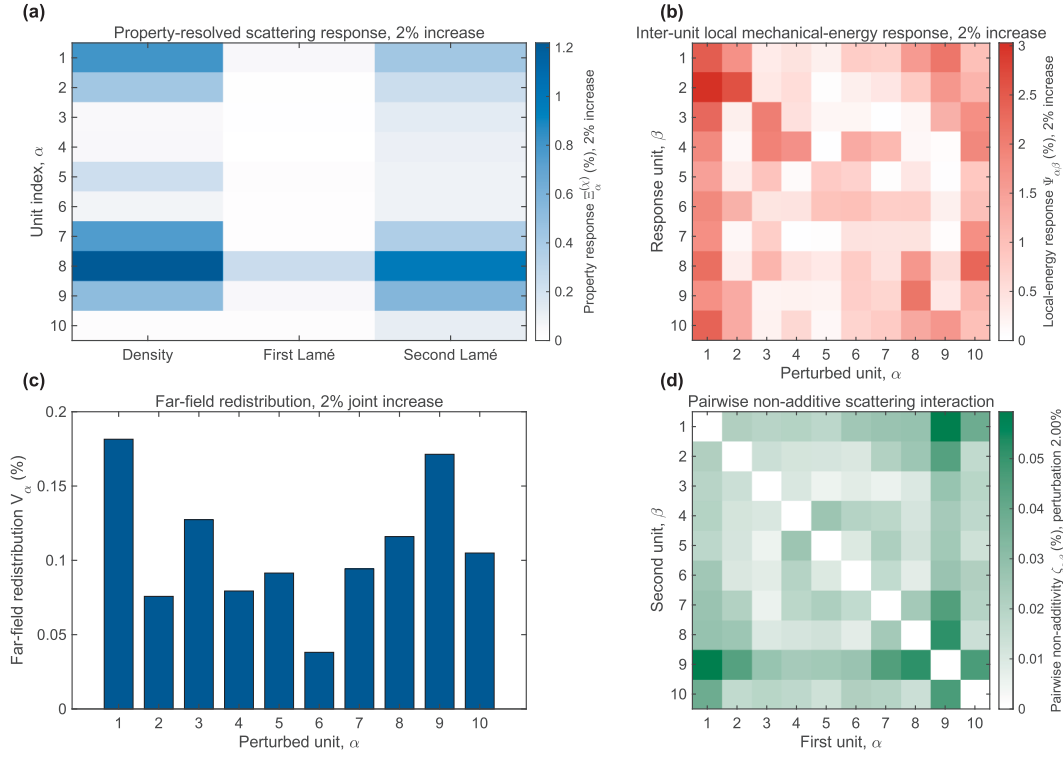


Fig. 11. Perturbation-based characterization of inter-unit interactions in the optimized $+70^\circ$ discrete metasurface. (a) Property-resolved far-field scattering response. (b) Inter-unit local mechanical-energy response. (c) Single-unit-induced far-field energy redistribution. (d) Pairwise non-additive scattering interaction.

The resulting change in the scattering response is quantified as

$$\Xi_{\alpha}^{(\chi)} = \|\Phi^{(\alpha, \chi)} - \Phi^{(0)}\|_2, \quad (21)$$

where Φ denotes the normalized acoustic-intensity vector of all propagating diffraction orders, obtained by normalizing the previously defined acoustic intensities by the incident intensity. The superscripts (α, χ) and (0) denote the perturbed and original structures, respectively. The resulting $\Xi_{\alpha}^{(\chi)}$ for the three material parameters and all ten units are shown in Fig. 11(a). For the present optimized design, density generally produces the largest change in the scattering response, followed by the second Lamé constant, whereas the influence of the first Lamé constant is relatively weak. Considerable variation is also observed among different units under the same perturbation, indicating that the influence of a local material perturbation strongly depends on the unit in which it is introduced.

To further examine how such a perturbation affects the mechanical response of other units within the metasurface, all three material parameters of unit α are simultaneously increased by 2%, and the local mechanical energy of each unit is evaluated. The mechanical energy of unit β is calculated as $\mathcal{E}_{\beta}^{(\alpha)} = \int_{\Omega_{\beta}} (e_{\text{kin}} + e_{\text{strain}}) d\Omega$, where $e_{\text{kin}} = \rho \omega^2 \mathbf{u}^* \cdot \mathbf{u} / 4$ and $e_{\text{strain}} = \text{Re}(\boldsymbol{\sigma} : \boldsymbol{\epsilon}^*) / 4$ are the time-averaged kinetic- and strain-energy densities, respectively. The corresponding relative energy response is defined as

$$\Psi_{\alpha\beta} = \frac{|\mathcal{E}_{\beta}^{(\alpha)} - \mathcal{E}_{\beta}^{(0)}|}{\mathcal{E}_{\beta}^{(0)}}, \quad (22)$$

where α and β denote the perturbed and response units, respectively. A larger $\Psi_{\alpha\beta}$ indicates that the perturbation introduced in unit α produces a stronger change in the local mechanical energy of unit β , corresponding to a stronger energetic influence between the two units. The resulting $\Psi_{\alpha\beta}$ is shown in Fig. 11(b). Pronounced responses are observed beyond neighboring units, and the response strength does not decrease

monotonically with unit separation. This indicates that the influence of a local perturbation can extend across multiple units rather than being restricted to nearest-neighbor interactions.

The previous measure evaluates the scattering variation caused by perturbing each material parameter separately and is therefore used to compare their relative contributions. To quantify the overall redistribution of propagating energy when a unit is perturbed as a whole, we further define

$$V_{\alpha} = \frac{1}{2} \|\Phi^{(\alpha)} - \Phi^{(0)}\|_1, \quad (23)$$

where $\Phi^{(\alpha)}$ denotes the response when the three material parameters of unit α are simultaneously increased by 2%. A larger V_{α} indicates a greater redistribution of energy among the propagating diffraction channels. Fig. 11(c) shows that the far-field energy redistribution varies markedly among the perturbed units. For the present optimized design, perturbations in units 1 and 9 produce the largest V_{α} , whereas unit 6 gives the weakest response. This further indicates that different units contribute unequally to the redistribution of energy among the propagating diffraction channels.

The preceding measures reveal how local perturbations modify the mechanical-energy distribution within the metasurface and redistribute energy among the far-field diffraction channels. However, these responses alone do not distinguish whether the effects of two units are simply additive or involve a genuine pairwise interaction. To quantify this non-additive contribution, we compare the joint response of two perturbed units with the superposition of their individual responses. The pairwise interaction is then defined as

$$\zeta_{\alpha\beta} = \frac{1}{2} \|\Phi^{(\alpha, \beta)} - \Phi^{(\alpha)} - \Phi^{(\beta)} + \Phi^{(0)}\|_1, \alpha \neq \beta, \quad (24)$$

where $\Phi^{(\alpha, \beta)}$ denotes the scattering response when the three material parameters of units α and β are simultaneously increased by 2%. If the two units act independently, their joint response can be approximated

by the superposition of their individual responses, giving $\zeta_{ap} \approx 0$. A larger ζ_{ap} therefore indicates a stronger non-additive interaction between the two units. The results in Fig. 11(d) show clear off-diagonal responses, with particularly strong interactions involving unit 9, indicating that the interaction strength varies considerably among different unit pairs.

Overall, these results show that the inter-unit interaction in the optimized metasurface is strongly dependent on both the material parameter and the unit location. Local perturbations can influence the mechanical-energy response of non-neighboring units and lead to different degrees of far-field energy redistribution, while the non-additive responses further confirm that different unit pairs exhibit distinct interaction strengths. These observations indicate that the scattering behavior of the discrete metasurface arises from the collective response of multiple units rather than from independent local contributions.

5. Conclusions

In this paper, we develop a Coupled-Wave Analysis (CWA) framework for elastic grating design and analysis in underwater environments. The Fourier-domain governing equations are derived from the inhomogeneous elastodynamic equations, with the effects of spatially varying material properties retained explicitly in the coupling among spatial harmonics. By combining the Fourier displacement representation with the governing equations and enforcing continuity at the fluid-structure interfaces, we obtain a unified coupled-wave system for the elastic and acoustic fields and solve for the complex amplitudes of all diffraction orders. The scattered pressure and internal displacement fields are then reconstructed, and the formulation is validated against FEM simulations.

We first demonstrate the efficacy of the proposed method using a representative example. Compared with full-scale numerical simulations, our approach achieves high accuracy with reduced computational cost by operating on relatively small matrices. Next, we assess the role of gradient terms in the inhomogeneous elastic wave equation. Removing these terms from the calculations not only alters the coupling among spatial harmonics but also violates the energy-conservation principle, resulting in unphysical energy growth. The results demonstrate that the gradient terms are essential for accurately modeling inhomogeneous elastic systems. Then, we apply the proposed method to several additional case studies, including oblique-incidence gratings, reflective gratings, and discretely parameterized gratings, to further verify the applicability of the formulation.

After validating the CWA formulation, we further use the Fourier-domain equations to explore directly derived material distributions for acoustic wave manipulation. It is shown that the spatial-frequency components of the material variation profile strongly influence the coupling among spatial harmonics. Based on this relation, we derive a simplified elastic material profile dominated by a single spatial-frequency component, which can realize efficient large-angle beam steering and total sound insulation. These examples further show that judicious arrangements of gain and loss can enhance device performance and demonstrate how the CWA formulation can provide direct physical guidance for material-profile design.

Moreover, owing to the low computational cost of the proposed CWA, we develop an optimization-based customization strategy for high-efficiency steering metasurfaces. Using this strategy, continuous and discrete material distributions are optimized for prescribed deflec-

tion angles of $+40^\circ$ and $+70^\circ$. The optimized designs achieve the target performance and contain rich high-spatial-frequency components. To examine the role of these high-spatial-frequency components, we remove them from the optimized grating and compare the modified structure to the original one; the results show that the high-spatial-frequency components can strongly influence the optimized steering performance. The internal elastic response of a representative optimized structure is further examined using P-like and S-like indicators together with order-resolved spatial-harmonic displacement fields, providing complementary information on the wave characteristics inside the structure.

Using one of the discrete optimized structures, we further investigate the interactions among structural units through perturbation-based measures of material-parameter sensitivity, local mechanical-energy response, far-field energy redistribution, and pairwise non-additivity. The results show that local perturbations can influence non-neighboring units and produce distinct far-field and pairwise responses. These observations indicate that the scattering behavior of the discrete metasurface arises from the collective response of multiple structural units rather than from independent local contributions.

As a final remark, the proposed CWA framework may be extended to three-dimensional elastic functional structures by introducing the corresponding additional field components while retaining the Fourier-domain formulation. Combined with the transfer-matrix method, the approach may also be extended to multilayer gratings, thereby providing additional flexibility for structural design. Although specific discrete microstructures were not explicitly constructed in this study, the proposed solver provides a theoretical framework for evaluating and optimizing complex material distributions. The optimized distributions can further serve as reference targets for practical realization using suitable metamaterial concepts. This formulation, when combined with metamaterial concepts such as pentamode materials (Dong et al., 2021; Liang et al., 2023), may facilitate the realization of underwater acoustic functional structures.

CRediT authorship contribution statement

Tianbao Liang: Writing – original draft, Visualization, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Mu He:** Writing – review & editing, Formal analysis. **Liang Xia:** Writing – review & editing. **Tao Yang:** Writing – review & editing. **Li Cheng:** Writing – review & editing, Supervision, Methodology, Formal analysis, Conceptualization.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this manuscript, the first author used ChatGPT in order to improve the clarity and readability of the language. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the published article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Numerical convergence and validation of the CWA and FEM methods

A.1. Convergence of the CWA method

To examine the convergence of the proposed CWA method with respect to the Fourier truncation, the maximum retained harmonic order H is gradually increased, corresponding to a total of $N = 2H + 1$ Fourier harmonics. The convergence analysis presented here is conducted for the continuous material-profile case. Fig. A1(a) and (b) show the reflected and transmitted amplitudes of all propagating diffraction orders, respectively. The amplitudes approach stable values as N increases, and only small variations are observed after $N = 31$.

The convergence is further quantified in Fig. A1(c) using the relative error with respect to the reference solution at $H = 31$ ($N = 63$), the variation between successive truncation orders, and the energy-balance residual. The energy-balance residual is calculated as $|1 - R - T|$, where R and T are obtained by summing the normalized acoustic intensities of all propagating reflected and transmitted orders, respectively. Both convergence errors decrease with increasing N , while the energy-balance residual remains small throughout the calculation. At $N = 31$, the scattering amplitudes are already close to their converged values, and further increasing the number of retained harmonics produces only minor changes. Therefore, $N = 31$ is adopted for the continuous-profile calculations in the main text to balance numerical accuracy and computational efficiency.

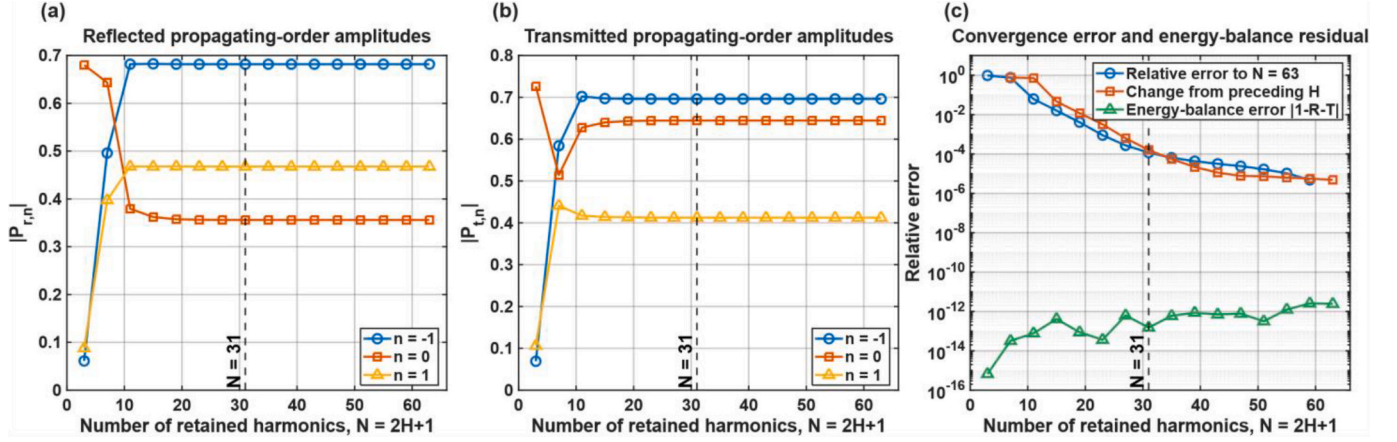


Fig. A1. Convergence analysis of the proposed CWA method. (a) Reflected propagating-order amplitudes. (b) Transmitted propagating-order amplitudes. (c) Convergence error and energy-balance residual.

A.2. Mesh convergence of the FEM model

Following the convergence analysis of the CWA method, the mesh convergence of the FEM model is also examined. Five successively refined meshes are considered, with total degrees of freedom of 8,040, 21,816, 69,040, 134,890, and 416,422. Fig. A2(a) and (b) show the reflected and transmitted amplitudes of the propagating diffraction orders, respectively. The results gradually approach stable values with mesh refinement and are nearly converged at 69,040 degrees of freedom. This mesh is therefore adopted for the FEM calculations in the main text.

The convergence is further quantified in Fig. A2(c) using the finest mesh with 416,422 degrees of freedom as the reference. For the adopted mesh, the relative errors of the complex reflection- and transmission-coefficient vectors are 0.0180% and 0.0073%, respectively, giving a relative error of 0.0130% for the combined complex scattering-coefficient vector. The change between the two finest meshes is further reduced to 0.0041%, confirming that the adopted FEM mesh provides sufficient accuracy for comparison with the CWA results.

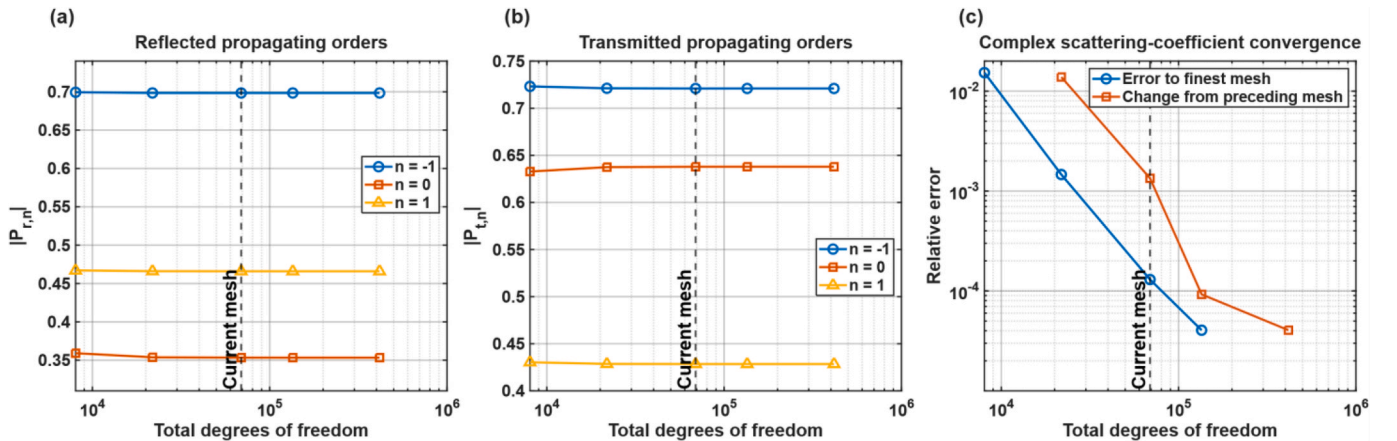


Fig. A2. FEM mesh-convergence analysis with respect to the total number of degrees of freedom. (a) Reflected propagating-order amplitudes. (b) Transmitted propagating-order amplitudes. (c) Relative convergence errors of the complex scattering coefficients. The dashed line indicates the mesh adopted in the main calculations.

A.3. Parametric validation of the CWA method

To further examine the numerical performance of the proposed CWA method under different physical conditions, a parametric validation is conducted using the configuration in Fig. 2 as the reference case. Four representative parameters are considered, including the material contrast, frequency, layer thickness, and incidence angle. Each parameter is varied independently while the remaining parameters are kept unchanged.

For the material contrast, the spatial variation of each material parameter is scaled relative to its period-averaged value according to

$$\xi_s(\mathbf{x}_1) = \langle \xi \rangle + s[\xi_0(\mathbf{x}_1) - \langle \xi \rangle], \quad \xi \in \{\rho, \lambda, \mu\}. \quad (\text{A1})$$

where $\xi_0(\mathbf{x}_1)$ denotes the reference material profile and $\langle \xi \rangle$ is its period-averaged value. The parameter s controls the strength of the spatial material modulation. Specifically, $s = 0$ corresponds to a homogeneous layer with the same mean material properties, $s = 1$ recovers the reference profile, and $s > 1$ increases the material contrast. The same parameter definition is adopted in both the CWA and FEM calculations.

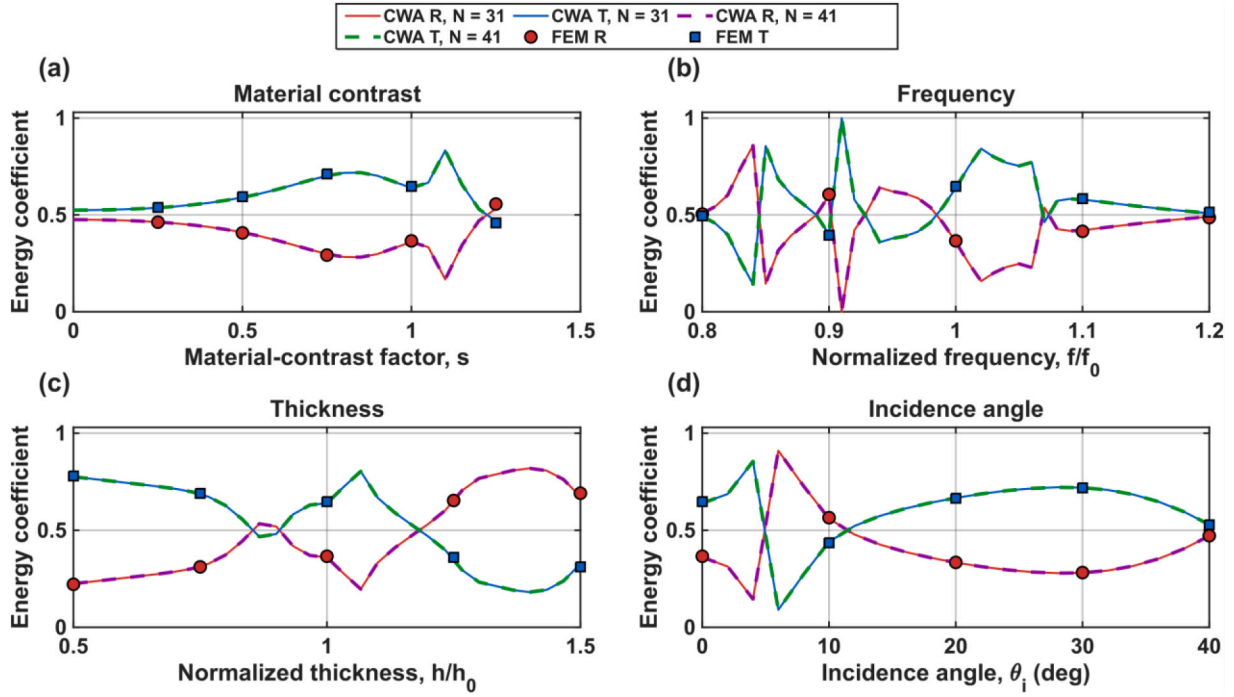


Fig. A3. Parametric validation of the CWA method against FEM. Reflection and transmission energy coefficients obtained using CWA with and are compared with the FEM results under variations in (a) material-contrast factor, (b) normalized frequency, (c) normalized thickness, and (d) incidence angle.

For the remaining parameters, the variations are defined directly relative to the reference configuration. The frequency and layer thickness are expressed in normalized forms as f/f_0 and d/d_0 , respectively, where f_0 and d_0 denote the values used in Fig. 2. The incidence angle θ is varied directly. In each case, only the parameter under consideration is changed, while all other geometrical and material parameters are kept at their reference values.

Fig. A3 compares the reflection and transmission energy coefficients obtained from CWA and FEM as the four physical parameters are varied. The solid and dashed lines correspond to the CWA results calculated using $N = 31$ and $N = 41$, respectively, while the symbols represent the FEM results. The scattering response changes noticeably with material contrast, frequency, thickness, and incidence angle, indicating that these parameters have different influences on the reflection and transmission characteristics. Meanwhile, the CWA results obtained using the two truncation orders nearly overlap and remain in good agreement with the FEM results throughout the investigated parameter ranges. The maximum relative difference between the CWA and FEM reflection and transmission energy coefficients over all sampled cases is 3.59%. These results further demonstrate the convergence, numerical stability, and accuracy of the proposed CWA method under different physical conditions.

A.4. Convergence verification for discontinuous material profiles

The discrete calculations in the main text employ inverse-rule Fourier factorization with a maximum retained harmonic order of $H = 31$, corresponding to $N = 2H + 1 = 63$ Fourier harmonics. To verify whether this adopted truncation provides sufficient accuracy for piecewise-constant material profiles, the ten-unit $+70^\circ$ design was selected as a representative discontinuous case. Its material-property distribution, geometry, frequency, and incidence conditions were held fixed, and the same structure was re-evaluated at successively higher truncation orders without re-optimization. For comparison, the fixed structure was also evaluated using the direct Fourier-product formulation, while the FEM result was used as an independent reference.

Fig. A4(a) shows the target transmitted amplitude as a function of H . Fig. A4(b) presents the flux-weighted amplitude difference from FEM considering all six propagating components, namely the reflected and transmitted orders $i = -1, 0, +1$. The squared amplitude difference of each order is weighted by $\cos\theta_i$ to account for its contribution to the normal acoustic-energy flux. With inverse-rule factorization, the target transmitted amplitude is 1.64827 at the adopted truncation $H = 31$, compared with 1.64570 at $H = 63$ and 1.64512 at $H = 400$. The difference between the $H = 31$ result and the high-order result is therefore approximately 0.19%.

At $H = 31$, the flux-weighted difference from FEM over all propagating orders is approximately 1.10%, and only minor changes occur upon further increasing H . In contrast, the direct Fourier-product formulation gives a substantially larger difference at the same truncation and approaches the FEM result much more slowly. These results confirm that inverse-rule Fourier factorization provides sufficiently converged results at the currently adopted truncation of $H = 31$. As an additional check of the final optimized structure, its target-order transmission efficiencies obtained at $H = 31, 41$, and 63 are 99.0221%, 99.0196%, and 99.0165%, respectively, further confirming that the reported optimized performance is insensitive to an increase in the Fourier truncation order.

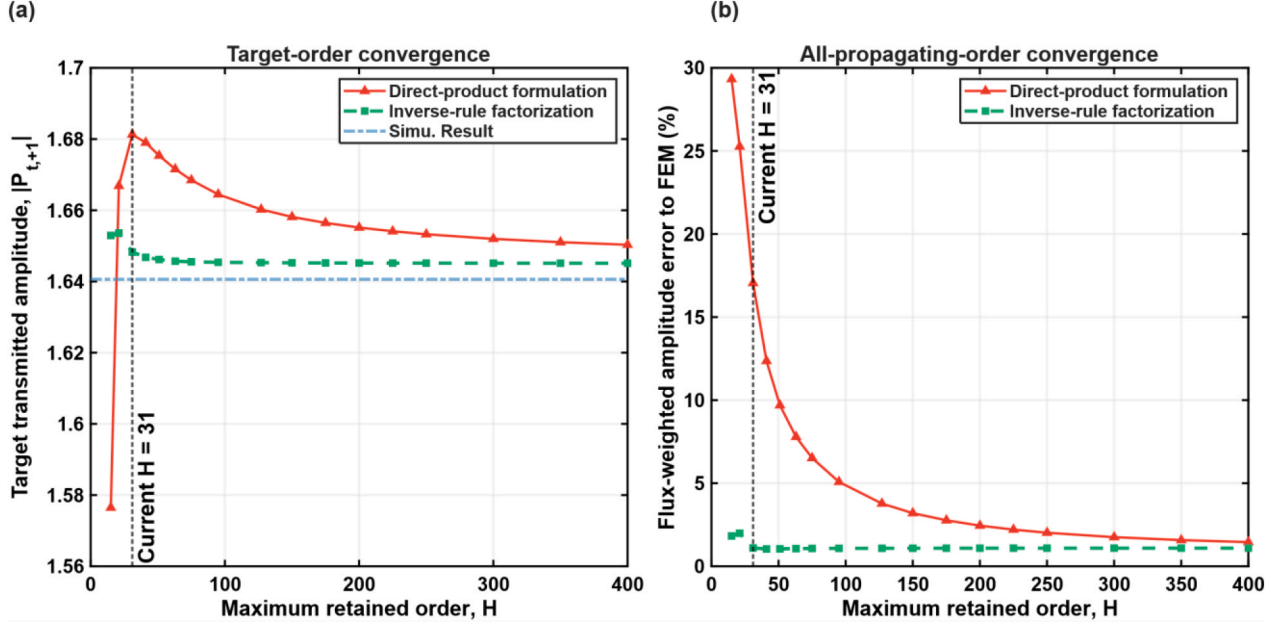


Fig. A4. Convergence verification for the ten-unit piecewise-constant design. (a) Target transmitted amplitude obtained using the direct Fourier-product formulation and inverse-rule Fourier factorization, together with the FEM reference. (b) Flux-weighted amplitude difference from FEM considering all propagating reflected and transmitted orders. The vertical dashed lines indicate the currently adopted truncation, 0.

Appendix B. Computational efficiency and optimization robustness

All calculations were performed on the same workstation equipped with an AMD Ryzen 9 9950X processor and 32 GB of RAM using MATLAB R2025b Update 4 under 64-bit Windows. Based on the convergence analyses in Appendix A, $N = 31$ is adopted for the CWA calculation, giving a relative error of 0.0117% with respect to the $N = 63$ reference solution. For FEM, the adopted mesh contains 69,040 degrees of freedom and gives a relative error of 0.0130% with respect to the finest mesh with 416,422 degrees of freedom. The computational costs of the two methods are therefore compared at similar numerical accuracies, as summarized in Table 1.

Table 1

Computational cost comparison between CWA and FEM.

Method	Computational size	Convergence error	Time per calculation	Peak working set	Incremental peak memory during solution
CWA	$N = 31$, matrix dimension 124×124	0.0117%	0.0306 s	1414 MB	3.72 MB
FEM	69,040 degrees of freedom	0.0130%	1.514 s	2850 MB	113.95 MB

For this benchmark case, CWA is approximately 49.5 times faster than FEM at comparable convergence accuracy. The incremental peak memory during the solution is also substantially lower for CWA. The total peak working set includes the memory occupied by the software environment and loaded models and is therefore provided only as an additional reference. The difference between the CWA and FEM reflection and transmission energy coefficients is 1.32%.

The repeatability of the optimization is further examined using five independent runs with different random initial populations. Each optimization contains 30 design variables and is performed using $N = 31$. As shown in Fig. B1(a), all five runs gradually converge to target amplitudes close to the optimized value reported in the main text. The runs terminate after 280–361 generations, with an average computational time of 98.91 s (approximately 1.65 min) per optimization and a total time of 8.24 min for the five runs. When the optimized designs are re-evaluated using $N = 63$, the target-order transmission efficiencies range from 98.10% to 98.21%, with a mean of 98.145% and a standard deviation of 0.040 percentage points. The similar final efficiencies obtained from different initial populations indicate good repeatability of the optimized performance.

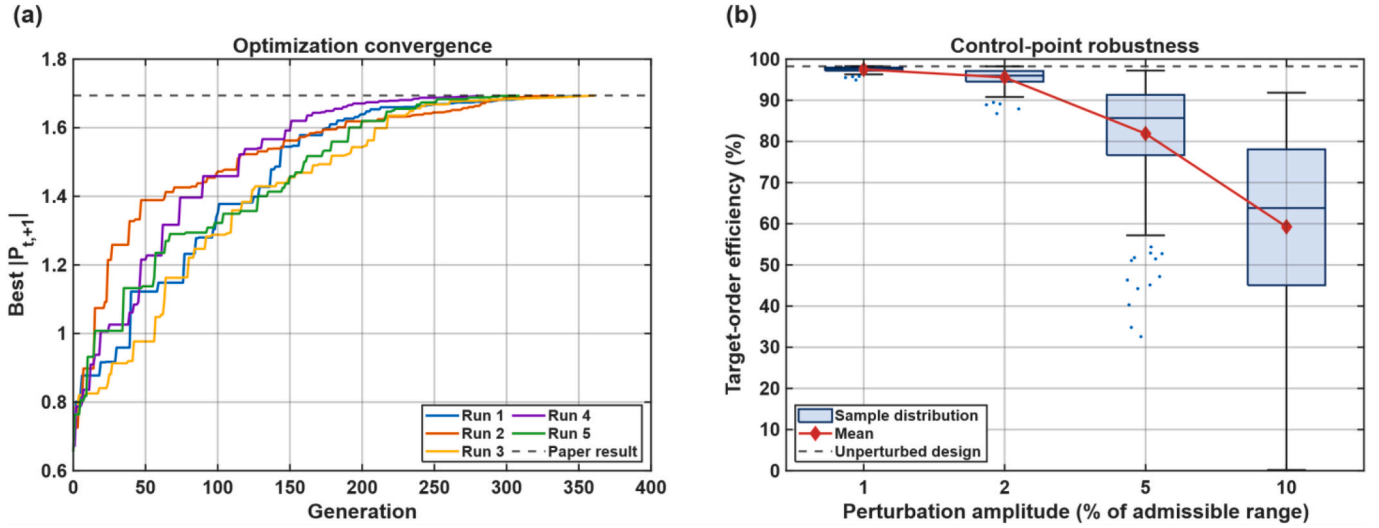


Fig. B1. Optimization repeatability and robustness of the optimized metasurface. (a) Optimization repeatability. (b) Control-point robustness.

The robustness of the optimized design is also examined by applying independent random perturbations to the 30 control variables. Perturbation levels of $\pm 1\%$, $\pm 2\%$, $\pm 5\%$, and $\pm 10\%$ of the corresponding admissible material-property ranges are considered, with 200 random samples evaluated at each level using $N = 63$. As shown in Fig. B1(b), the mean target-order transmission efficiency decreases from 98.23% for the unperturbed design to 97.50%, 95.55%, 81.90%, and 59.26% for the four perturbation levels, respectively. The optimized structure therefore maintains high and relatively stable performance under small perturbations, while larger perturbations lead to increasingly pronounced performance degradation and variability.

Appendix C. Treatment of the single-spatial-frequency case

This appendix describes the treatment of the exact single-spatial-frequency case in the CWA formulation. Suppose that each material parameter contains only one nonzero Fourier component,

$$\rho(x_1) = \rho_s e^{-jsGx_1}, \lambda(x_1) = \lambda_s e^{-jsGx_1}, \mu(x_1) = \mu_s e^{-jsGx_1}, \quad (C1)$$

with $N = 2H + 1$ retained Fourier orders $q = -H, \dots, H$. Multiplication by the material harmonic shifts a field component from order q to $q + s$. The corresponding convolution matrices can therefore be written as

$$\mathbf{M} = \mu_s \mathbf{S}_s, \mathbf{\Lambda} = \lambda_s \mathbf{S}_s, \mathbf{R} = \rho_s \mathbf{S}_s, \quad (C2)$$

where $(\mathbf{S}_s)_{nq} = \delta_{n,q+s}$. If the same finite Fourier range is used before and after the shift, some shifted orders fall outside the retained range, giving $\text{rank}(\mathbf{S}_s) = N - |s| < N$,

$$(C3)$$

and hence rank-deficient material matrices.

To avoid this truncation-induced rank deficiency, the Fourier bases are shifted consistently with the material harmonic. The displacement basis $\{-H, \dots, H\}$ is paired with the corresponding shifted output basis $\{-H + s, \dots, H + s\}$ for the constitutive, inertial, and balance-equation terms, so that the mapping $q \rightarrow q + s$ becomes one-to-one. Under this basis alignment, the material matrices reduce to

$$\mathbf{M} = \mu_s \mathbf{I}_N, \mathbf{\Lambda} = \lambda_s \mathbf{I}_N, \mathbf{R} = \rho_s \mathbf{I}_N, \quad (C4)$$

The derivative contribution associated with the extracted spatial harmonic is retained through the corresponding $-jsG$ term. This treatment removes the rank deficiency arising from the single-spatial-frequency case in the CWA formulation and allows the resulting system to be solved without regularizing the original singular convolution matrices.

Data availability

No data was used for the research described in the article.

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