

Periodic Oscillations to Chaos: Nonlinear Dynamics of a Tristable Piezoelectric Flutter System

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<https://doi.org/10.2514/1.J065758>

Piezoelectric flutter systems exhibit rich nonlinear dynamics under aerodynamic excitation. Alongside their beneficial effects in terms of energy harvesting, nonlinearities may also induce chaotic responses, potentially compromising system performance. Unlike bistable energy harvesters in which chaotic behaviors have not been reported, tristable flutter systems exhibit experimentally observed aperiodic oscillations; yet, the underlying chaos mechanisms remain theoretically unexplored. This paper investigates the nonlinear dynamics of a tristable piezoelectric flutter system, elucidating the transition from periodic oscillations to chaos for the first time. Using a harmonic balance method, Lyapunov exponent analysis, and bifurcation mapping, the complete evolutionary pathway is revealed. Originating from the central potential well, the system undergoes Hopf bifurcations followed by a symmetry-breaking bifurcation that triggers period-doubling cascades, which are fundamentally different from classical flutter instabilities. The chaotic regime features positive Lyapunov exponents and fractal basin boundaries, representing unique dynamical behavior. Analyses reveal that chaotic responses reduce power output by up to 80%. Parametric studies show that reducing cubic pitching stiffness substantially lowers the critical wind speed for interwell limit cycles, effectively suppressing pure chaotic regions. This work provides the theoretical foundation for chaos phenomena in tristable piezoelectric flutter systems and actionable design guidelines for performance enhancement.

Nomenclature

$A_{\text{exp,max}}$	=	experimental peak amplitudes
A_i, B_i	=	harmonic coefficients
A_n	=	noise amplitude used to control the noise intensity
$A_{\text{sim,max}}$	=	simulated peak amplitudes
$\mathbf{B}_Z$	=	nonlinear state vector
b	=	half-chord length of the airfoil, m
$\mathbf{C}$	=	damping matrix
c_h	=	plunging damping coefficient, (N · s)/m
c_p	=	piezoelectric capacitance, F
c_a	=	torsional damping coefficient, (N · m · s)/rad
$\bar{c}_h$	=	normalized plunging damping coefficient
$\bar{c}_1, \dots, \bar{c}_9$	=	normalized coefficients
$\bar{k}_1, \dots, \bar{k}_6$	=	Lyapunov dimension
D_L	=	aerodynamic coefficients
$d_0, \dots, d_4$	=	dimensionless elastic potential energy
E	=	plunging displacement, m
h	=	airfoil moment of inertia, kg · m ²
I_α	=	dimensionless moment of inertia ratio
$\bar{I}_1$	=	stiffness matrix
$\mathbf{K}$	=	linearized stiffness matrix
$\mathbf{K}_l$	=	plunging stiffness, N/m

k_{h1}, k_{h3}, k_{h5}	=	plunging stiffness polynomial coefficients, N/m, N/m ³ , N/m ⁵ , respectively
k_α	=	pitching stiffness, (N · m)/rad
$\bar{k}_{a3}$	=	nonlinear pitching stiffness, (N · m)/rad ³
$\bar{k}_{h1}, \bar{k}_{h3}, \bar{k}_{h5}$	=	dimensionless linear, cubic, and quintic plunging stiffnesses, respectively
$\bar{k}_\alpha, \bar{k}_{a3}$	=	dimensionless linear and cubic pitching stiffnesses, respectively
L	=	aerodynamic lift force, N
$\mathbf{M}$	=	mass matrix
M_α	=	aerodynamic moment, N · m
m_f	=	airfoil mass, kg
m_T	=	equivalent total mass, kg
$\bar{m}_2$	=	dimensionless mass unbalance parameter
$\mathbf{N}(z)$	=	nonlinear term vector
R	=	load resistance, Ω
$\bar{R}$	=	dimensionless load resistance
U_r	=	reduced wind speed
$\bar{v}$	=	dimensionless output voltage
x_α	=	dimensionless distance between the center of the gravity position and the hinge axis
y	=	dimensionless plunging displacement
y^*	=	equilibrium plunging displacement
$\mathbf{z}$	=	state vector; $[y \ \alpha \ x \ y' \ \alpha' \ x' \ \bar{v}]^T$
α	=	pitching rotation angle, rad
δ	=	Feigenbaum constant
θ	=	electromechanical coupling coefficient, C/N
$\xi(t)$	=	Gaussian white noise with zero mean and unit standard deviation
ρ	=	air density, kg/m ³
τ	=	dimensionless time
ω	=	system response frequency, rad/s
ω_α	=	natural torsional frequency, rad/s

Subscripts

h	=	plunging motion
p	=	piezoelectric
α	=	pitching motion

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I. Introduction

AEROELASTIC flutter, a self-excited vibration phenomenon caused by fluid–structure interaction [1,2], commonly occurs in aerospace systems, wind turbine blades, and long-span bridges. This phenomenon arises from the dynamic coupling-induced instability among aerodynamic forces, structural elastic forces, and inertial forces. When airflow velocity exceeds a critical threshold, the system's energy input surpasses dissipation, leading to exponentially growing response amplitudes, and eventually resulting in catastrophic structural failure. As such, flutter boundary analysis for aircraft wings remains a critical safety issue in aerospace structural design [3–6].

Recent advances in piezoelectric materials have shown great promise for flutter control and energy harvesting [7–10]. In active control strategies, researchers have leveraged multiphysics coupling mechanisms, such as piezoelectric actuation, adaptive circuit design, and real-time feedback to enable active or semiactive regulation of aeroelastic systems [11–14]. Representative approaches include self-powered semipassive control via synchronized switch damping [15], simultaneous damage sensing and active control in cracked nanocomposite beams [16], and nonlinear energy sinks (NESs) for hypersonic airfoil vibration suppression [17]. A critical trend emerges: nonlinear mechanisms outperform linear counterparts. A critical insight emerges from these studies: nonlinear mechanisms offer distinct advantages over linear approaches in certain regimes, particularly for postflutter stabilization and broadband operation. De Sousa et al. [18] demonstrated that shape memory alloys' pseudoelastic hysteresis combined with nonlinear piezoelectric circuits stabilizes postflutter limit cycle oscillations (LCOs), whereas Zhou et al. [19] showed that macrofiber composites can modify supersonic aeroelastic stability boundaries by adjusting prestress fields. Luongo and Zulli [20] revealed that NESs suppress amplitude growth in the postcritical regime through targeted energy transfer. These studies converge on a key insight: nonlinear stiffness modification expands control bandwidth and robustness, enabling multifunctional integration in smart structures.

For energy harvesting [21], key objectives include improving output power, environmental adaptability, robustness, and reliability. For instance, Zhao et al. [22,23] developed a novel mechanically intelligent energy harvesting approach to accomplish these objectives. Wang et al. [24] improved reliability by designing a U-shaped base to limit the amplitude of a galloping energy harvester, thereby preventing structural damage caused by excessive amplitudes at high wind speeds. Nonlinear structural design also provides a critical pathway: Theoretical analyses reveal that cubic nonlinear stiffness elevates cut-in speeds; yet, when coupled with nonlinear piezoelectric effects, it synergistically improves overall performance [25]. Advanced designs incorporating magnetoelectroelastic materials [26] and freeplay-induced subcritical bifurcations [27] further demonstrate that strategic nonlinearity enables stable LCOs below linear flutter speeds, optimizing energy output performance.

Among nonlinear strategies, multistable systems represent a significant advancement in performance. Bistable harvesters exploit two potential wells separated by a barrier, enabling large-amplitude interwell motion that amplifies power output while simultaneously reducing cut-in speed through bifurcation point modification. Li et al. [28] transformed supercritical flutter into subcritical flutter via magnetic coupling, achieving over 50% reduction in cut-in wind speed. Building on this success, tristable systems (featuring three potential wells and two barriers) might theoretically offer superior performance. Zhou et al. [29,30] demonstrated that optimized tristable configurations achieve high-energy interwell oscillations at lower excitation thresholds than bistable counterparts. Previous work by Li et al. [31] confirmed this potential experimentally: periodic oscillations spanning all three wells yielded significantly higher power output.

However, this performance enhancement comes at a cost. Unlike bistable systems in which chaotic responses have not been reported [28], the tristable flutter-based energy harvester (FEH) exhibits persistent aperiodic large-amplitude oscillations alongside the

desired periodic LCOs [31]. This contrast stems from the fundamental topological differences between the two systems: tristable systems possess two potential barriers creating multiple competing oscillation modes, which amplify sensitivity to parameter variations. These aperiodic responses bear the hallmarks of chaos and undermine energy harvesting performance by generating irregular voltage outputs and yielding lower time-averaged power as compared to stable periodic LCOs [32–34].

These detrimental behaviors pose a fundamental barrier to practical deployment, yet their underlying mechanisms remain unresolved. Three critical questions should be answered:

- 1) Are the aperiodic responses reported in [31] genuinely chaotic?
- 2) How do they originate and evolve from stable LCOs as parameters vary?
- 3) Can stiffness parameters be optimized to suppress chaos?

This study addresses these questions through Lyapunov exponent analysis, bifurcation mapping, and parametric optimization to establish design guidelines ensuring stable high-output operation. The paper is organized as follows: Sec. II introduces the structural layout and analytical framework. Section III examines the nonlinear dynamic behavior of the tristable FEH, including experimental results, with a focus on analyzing chaotic evolution pathways. Section IV analyzes the influence of nonlinear stiffness on bifurcation characteristics. Section V presents the discussion, and Section VI summarizes the main conclusions.

II. Structure Layout and Analytical Framework

A. Structure Layout

The tristable FEH under consideration consists of a rigid airfoil section mounted on a flexible support structure (as shown in Fig. 1) modeled with two primary degrees of freedom: pitching motion (denoted by a rotation angle α) and plunging motion (denoted by a vertical displacement h). The pitching motion around the airfoil's elastic axis is driven by the aerodynamic moment M_α and resisted by the torsional stiffness [$k_\alpha(\alpha) = k_\alpha + 3k_{\alpha 3}\alpha^2$] and damping c_α of the elastic hinge. The plunging motion, driven by the aerodynamic lift force L , is constrained by the piezoelectric cantilever's vertical stiffness and damping c_h . The stiffness pattern for a tristable system is given by $k_h(h) = k_{h1} + 3k_{h3}h^2 + 5k_{h5}h^4$. The piezoelectric material embedded in the cantilever generates an electric charge in response to mechanical strain, with the efficiency of the conversion governed by the electromechanical coupling coefficient θ . The generated electrical energy is delivered to an external load resistance R , which also influences the output power of the system. The aerodynamic interactions, as a result of the lift L and the aerodynamic moment M_α , drive the coupled plunging and pitching oscillations, forming the basis for the energy harvesting process. The coupled governing equations of the system are written as

$$m_T \ddot{h} + m_f x_\alpha \ddot{\alpha} + c_h \dot{h} + k_{h1}h + k_{h3}h^3 + k_{h5}h^5 - \theta v = -L \quad (1)$$

$$m_f x_\alpha \ddot{h} + I_\alpha \ddot{\alpha} + c_\alpha \dot{\alpha} + k_{\alpha 1}\alpha + k_{\alpha 3}\alpha^3 = M_\alpha \quad (2)$$

$$C_p \dot{v} + \frac{v}{R} + \theta \dot{h} = 0 \quad (3)$$

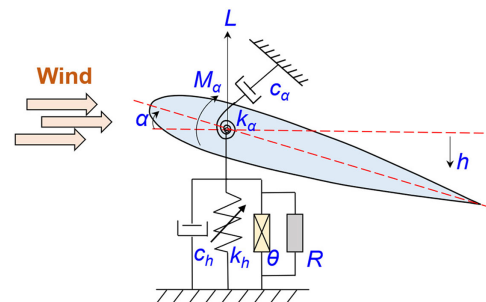


Fig. 1 Diagram of the tristable FEH.

where m_T is the total equivalent mass, m_f is the airfoil's mass, x_a is the dimensionless distance between the center of the gravity position and the hinge axis, I_a is the airfoil's moment of inertia, C_p is the capacitance of the piezoelectric material, and R is the external load resistance. These equations describe the coupling between the plunging and pitching motions with the electrical circuit, characterizing the system's dynamic responses and electrical energy output. Expressions of the aerodynamic lift force L and moment M_a are given as [35]

$$L = \pi \rho b^2 s (\ddot{h} - ba\ddot{\alpha} + U\dot{\alpha}) + 2\pi \rho Ubs(d_0 - d_1 - d_3)Q + 2\pi \rho U^3 sd_2 d_4 (d_1 + d_3)x + 2\pi \rho U^2 bs(d_1 d_2 + d_3 d_4)\dot{x} \quad (4)$$

$$M_a = \pi \rho b^2 s \left[ba\ddot{h} - Ub\left(\frac{1}{2} - a\right)\dot{\alpha} - b^2\left(\frac{1}{8} + a^2\right)\ddot{\alpha} \right] + 2\pi \rho U b^2 s \left(\frac{1}{2} + a\right)(d_0 - d_1 - d_3)Q + 2\pi \rho b U^3 s \left(\frac{1}{2} + a\right)d_2 d_4 (d_1 + d_3)x + 2\pi \rho U^2 b^2 s \left(\frac{1}{2} + a\right)(d_1 d_2 + d_3 d_4)\dot{x} \quad (5)$$

with

$$\ddot{x} + d_2 d_4 \frac{U^2}{b^2} x + (d_2 + d_4) \frac{U}{b} \dot{x} - \frac{U}{b} \alpha - \left(\frac{1}{2} - a\right)\dot{\alpha} - \frac{\dot{h}}{b} = 0 \quad (6)$$

$$Q = U\alpha + \dot{h} + \dot{\alpha}b\left(\frac{1}{2} - a\right) \quad (7)$$

The aerodynamic coefficients are given as $d_0 = 1$, $d_1 = 0.165$, $d_2 = 0.0455$, $d_3 = 0.335$, and $d_4 = 0.3$. The coupled governing equations, as given in Eqs. (1–3), have been experimentally validated for their accuracy in [31]. The nonlinear plunging force–displacement relationship $F_h = k_{h1}h + k_{h3}h^3 + k_{h5}h^5$ and the pitching moment–angle relationship $M_p = k_{\alpha1}\alpha + k_{\alpha3}\alpha^3$ adopted in this study are illustrated in Fig. 2, with the parameters obtained from [31]. The pitching direction exhibits hardening stiffness characteristics ($k_{\alpha3} > 0$), where the restoring moment increases nonlinearly with the angle of attack. From a dynamical system perspective, hardening nonlinearity typically shifts resonance peaks toward higher frequencies and can induce jump phenomena and hysteresis in frequency-response curves, creating coexisting stable and unstable solution branches that provide pathways for complex dynamics. The plunging direction displays a typical tristable force–displacement curve, characterized by five equilibrium points (zero crossings): three stable equilibria (the two outermost points and center point, where $dF_h/dh > 0$) corresponding to three potential wells, separated by two unstable equilibria (where $dF_h/dh < 0$)

forming potential barriers. This multiwell topology fundamentally predisposes the system to chaos through several mechanisms:

1) The negative stiffness regions at barriers enable rapid acceleration during interwell transitions, amplifying sensitivity to initial conditions.

2) Aerodynamic force can drive competing oscillation modes around different wells, whose nonlinear interactions generate irregular motion.

3) The symmetry breaking between well depths and barrier heights creates heteroclinic connections in phase space that facilitate chaotic trajectories.

Unlike bistable systems with a single barrier, the presence of two barriers in the tristable configurations significantly expands the homoclinic/heteroclinic tangle structure, thereby increasing the parametric regime where chaos emerges. The elastic potential energy characteristics of the system will be discussed in detail in the following section.

To facilitate the analysis, we introduce a set of nondimensional variables and parameters. The time variable is normalized as $\tau = (Ut/b)$, spatial coordinates are scaled by $y = (h/b)$, the reduced wind speed is defined as $U_r = (U/b\omega_a)$ (where $\omega_a = \sqrt{(k_a/I_a)}$), and the dimensionless output voltage is $\bar{v} = (c_p/\theta b)v$. Then, the following dimensionless parameters are introduced: $\bar{m}_2 = (m_f x_a - \pi \rho b^2 s a/m_T + \pi \rho b^2 s)$, $\bar{c}_h = (c_h/(m_T + \pi \rho b^2 s)\omega_a U_r)$, $\bar{c}_1 = (2\pi \rho b s(d_0 - d_1 - d_3)b/m_T + \pi \rho b^2 s)$, $\bar{c}_2 = [2\pi \rho b^2 s(d_0 - d_1 - d_3)((1/2) - a) + \pi \rho b^2 s/m_T + \pi \rho b^2 s]$, $\bar{c}_3 = ((2\pi \rho b s(d_1 d_2 + d_3 d_4))b\omega_a/m_T + \pi \rho b^2 s)$, $\bar{k}_{h1} = (k_{h1}/(m_T + \pi \rho b^2 s)\omega_a^2)$, $\bar{k}_1 = ((2\pi \rho b s(d_0 - d_1 - d_3))b/m_T + \pi \rho b^2 s)$, $\bar{k}_2 = ((2\pi \rho s d_2 d_4 (d_1 + d_3))b^2\omega_a/m_T + \pi \rho b^2 s)$, $\bar{k}_{h3} = (k_{h3}b^2/(m_T + \pi \rho b^2 s)\omega_a^2)$, $\bar{k}_{h5} = (k_{h5}b^4/(m_T + \pi \rho b^2 s)\omega_a^2)$, $\bar{\theta} = (\theta^2/(m_T + \pi \rho b^2 s)\omega_a^2 c_p)$, $\bar{I}_1 = [I_a + \pi \rho b^4 s((1/8) + a^2)/(m_f x_a b - \pi \rho b^3 s a)b]$, $\bar{c}_a = (c_a/(m_f x_a b - \pi \rho b^3 s a)\omega_a b)$, $\bar{c}_4 = (\pi \rho b^3 s((1/2) - a) - 2\pi \rho b^3 s((1/2) + a)(d_0 - d_1 - d_3))/m_f x_a b - \pi \rho b^3 s a$, $\bar{c}_5 = ((-2\pi \rho b^2 s((1/2) + a)(d_0 - d_1 - d_3))b/m_f x_a b - \pi \rho b^3 s a)$, $\bar{c}_6 = ((-2\pi \rho b^2 s((1/2) + a)(d_1 d_2 + d_3 d_4))b\omega_a/m_f x_a b - \pi \rho b^3 s a)$, $\bar{k}_a = (k_a/(m_f x_a b - \pi \rho b^3 s a)b\omega_a^2)$, $\bar{k}_{a3} = (k_{a3}/(m_f x_a b - \pi \rho b^3 s a)b\omega_a^2)$, $\bar{k}_3 = ((-2\pi \rho b^2 s((1/2) + a)(d_0 - d_1 - d_3))b/(m_f x_a b - \pi \rho b^3 s a))$, $\bar{k}_4 = ((-2\pi \rho b s((1/2) + a)d_2 d_4 (d_1 + d_3))\omega_a b^2/(m_f x_a b - \pi \rho b^3 s a))$, $\bar{c}_7 = \bar{k}_5 = -(1/\omega_a)$, $\bar{c}_8 = (-((1/2) - a)/\omega_a)$, $\bar{c}_9 = (d_2 + d_4)$, $\bar{k}_6 = d_2 d_4$, and $\bar{R} = (1/\omega_a R c_p)$. The dimensionless form of the system's coupled dynamic equations is derived systematically as follows:

$$y'' + \bar{m}_2 \alpha'' + \bar{c}_h \frac{1}{U_r} y' + \bar{c}_1 y' + \bar{c}_2 \alpha' + \bar{c}_3 U_r x' + \bar{k}_{h1} \frac{1}{U_r^2} y + \bar{k}_1 \alpha + \bar{k}_2 U_r x + \bar{k}_{h3} \frac{1}{U_r^2} y^3 + \bar{k}_{h5} \frac{1}{U_r^2} y^5 - \bar{\theta} \frac{1}{U_r^2} \bar{v} = 0 \quad (8)$$

$$y'' + \bar{I}_1 \alpha'' + \bar{c}_a \frac{1}{U_r} \alpha' + \bar{c}_4 \alpha' + \bar{c}_5 y' + \bar{c}_6 U_r x' + \bar{k}_a \frac{1}{U_r^2} \alpha + \bar{k}_{a3} \frac{1}{U_r^2} \alpha^3 + \bar{k}_3 \alpha + \bar{k}_4 U_r x = 0 \quad (9)$$

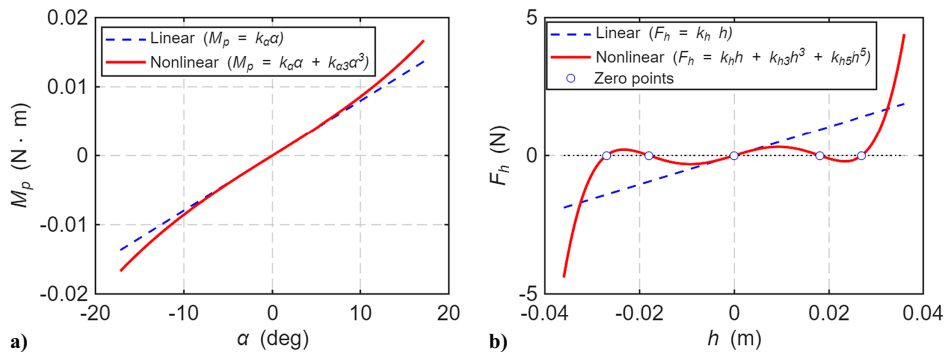


Fig. 2 Representations of a) restitution pitching moment as a function of α and b) plunge restitution force as a function of h .

$$x'' + \bar{c}_7 \frac{1}{U_r} y' + \bar{c}_8 \frac{1}{U_r} \alpha' + \bar{c}_9 x' + \bar{k}_5 \frac{1}{U_r} \alpha + \bar{k}_6 x = 0 \quad (10)$$

$$\bar{v}' + \bar{R} \frac{1}{U_r} \bar{v} + y' = 0 \quad (11)$$

For transient analysis of the tristable FEH, time-domain responses can be obtained through numerical solutions of the dimensionless coupled equations using the Runge-Kutta method. This study focuses on investigating chaotic motion characteristics and dynamic stability of the nonlinear system. Specifically, we integrate time-domain numerical results with harmonic balance (HB) analysis to reveal the evolutionary pathway from periodic motion to chaotic states. Additionally, Lyapunov's first method is applied to quantitatively analyze the stability of system equilibrium points. The subsequent sections will detail the solution process of the HB method for Eqs. (8–11).

B. HB Method for First-Order Approximate Solution

When applying the HB method to solve this dynamic system, it is first necessary to establish harmonic response assumptions for the system's state variables. Based on the fundamental characteristics of periodic responses, the first-order harmonic expansion forms of the variables are defined as

$$y = A_0 + A_1 \cos(\omega\tau) \quad (12)$$

$$\alpha = A_2 \cos(\omega\tau) + B_2 \sin(\omega\tau) \quad (13)$$

$$x = A_3 \cos(\omega\tau) + B_3 \sin(\omega\tau) \quad (14)$$

$$\bar{v} = A_4 \cos(\omega\tau) + B_4 \sin(\omega\tau) \quad (15)$$

In the application of the HB method, to model this tristable piezoelectric flutter system, the harmonic coefficients A_i and B_i , as well as the system response frequency ω , are critical parameters. The HB method typically introduces a slowly varying assumption, where the harmonic coefficients A_i , B_i , and ω are treated as functions that evolve slowly over time, whereas their second-order derivatives are neglected to simplify the governing equations. This assumption, combined with Lyapunov's first method, enables stability analysis of approximate periodic solutions. However, this study identifies the following critical issues:

1) If ω is assumed to be a time-varying parameter (retaining only first-order derivatives and neglecting higher-order terms), the time dependence embedded in the Jacobian matrix invalidates the use of Lyapunov's first method for stability assessment.

2) Although the stability of higher-order approximate periodic solutions can be analyzed via Floquet theory, the truncation errors inherent in the first-order HB approximation compromise its accuracy so that the precision requirements of Floquet theory cannot be met.

Because the focus of this study is on analyzing the bifurcation characteristics and chaotic response features of the system, the stability analysis of periodic solutions is not considered. To align with this objective, a steady-state assumption is adopted, where all harmonic coefficients A_i , B_i , and ω are treated as time-independent constants once the system reaches steady state. The first and second derivatives of Eqs. (16–22) are given by

$$y' = -A_1 \omega \sin(\omega\tau) \quad (16)$$

$$y'' = -A_1 \omega^2 \cos(\omega\tau) \quad (17)$$

$$a' = -A_2 \omega \sin(\omega\tau) + B_2 \omega \cos(\omega\tau) \quad (18)$$

$$a'' = -A_2 \omega^2 \cos(\omega\tau) - B_2 \omega^2 \sin(\omega\tau) \quad (19)$$

$$x' = -A_3 \omega \sin(\omega\tau) + B_3 \omega \cos(\omega\tau) \quad (20)$$

$$x'' = -A_3 \omega^2 \cos(\omega\tau) - B_3 \omega^2 \sin(\omega\tau) \quad (21)$$

$$\bar{v}' = -A_4 \omega \sin(\omega\tau) + B_4 \omega \cos(\omega\tau) \quad (22)$$

Substituting Eqs. (12–22) into Eqs. (8–11), neglecting higher-order harmonics, and balancing the first-order harmonics and constant terms yield the following algebraic equations:

$$\frac{\bar{k}_{h5} A_0^5}{U_r^2} + \left(\frac{5\bar{k}_{h5} A_1^2}{U_r^2} + \frac{\bar{k}_{h3}}{U_r^2} \right) A_0^3 + \left(\frac{15\bar{k}_{h5} A_1^4}{8U_r^2} + \frac{3\bar{k}_{h3} A_1^2}{2U_r^2} + \frac{\bar{k}_{h1}}{U_r^2} \right) A_0 = 0 \quad (23)$$

$$\begin{aligned} & \frac{5A_1 \bar{k}_{h5} A_0^4}{U_r^2} + \left(\frac{15\bar{k}_{h5} A_1^3}{2U_r^2} + \frac{3\bar{k}_{h3} A_1}{U_r^2} \right) A_0^2 + \frac{5\bar{k}_{h5} A_1^5}{8U_r^2} + \frac{3\bar{k}_{h3} A_1^3}{4U_r^2} \\ & + \frac{(-16U_r^2 \omega^2 + 16\bar{k}_{h1}) A_1}{16U_r^2} + \frac{(-16U_r^2 \bar{m}_2 \omega^2 + 16U_r^2 \bar{k}_1) A_2}{16U_r^2} \\ & + \bar{c}_2 \omega B_2 + \bar{k}_2 U_r A_3 + \bar{c}_3 U_r \omega B_3 - \frac{A_4 \bar{\theta}}{U_r^2} = 0 \end{aligned} \quad (24)$$

$$\begin{aligned} & \frac{(-16U_r^2 \bar{c}_1 \omega - 16U_r \bar{c}_h \omega) A_1}{16U_r^2} - \bar{c}_2 \omega A_2 + \frac{(-16U_r^2 \bar{m}_2 \omega^2 + 16U_r^2 \bar{k}_1) B_2}{16U_r^2} \\ & - \bar{c}_3 U_r \omega A_3 + \bar{k}_2 U_r B_3 - \frac{B_4 \bar{\theta}}{U_r^2} = 0 \end{aligned} \quad (25)$$

$$\begin{aligned} & A_2 \bar{I}_1 \omega^2 - B_3 U_r \bar{c}_6 \omega + A_1 \omega^2 - A_3 U_r \bar{k}_4 - B_2 \bar{c}_4 \omega \\ & - \frac{3A_2^3 \bar{k}_{a3}}{4U_r^2} - \frac{3A_2 B_2^2 \bar{k}_{a3}}{4U_r^2} - A_2 \bar{k}_3 - \frac{B_2}{U_r} \omega \bar{c}_a - \frac{A_2 \bar{k}_a}{U_r^2} = 0 \end{aligned} \quad (26)$$

$$\begin{aligned} & A_3 U_r \bar{c}_6 \omega + B_2 \bar{I}_1 \omega^2 + A_1 \bar{c}_5 \omega + A_2 \bar{c}_4 \omega - B_3 U_r \bar{k}_4 \\ & - \frac{3A_2^2 B_2 \bar{k}_{a3}}{4U_r^2} + \frac{A_2}{U_r} \omega \bar{c}_a - \frac{3B_2^2 \bar{k}_{a3}}{4U_r^2} - B_2 \bar{k}_3 - \frac{B_2 \bar{k}_a}{U_r^2} = 0 \end{aligned} \quad (27)$$

$$A_3 \omega^2 - B_3 \bar{c}_9 \omega - A_3 \bar{k}_6 - \frac{B_2 \bar{c}_8 \omega}{U_r} - \frac{A_2 \bar{k}_5}{U_r} = 0 \quad (28)$$

$$A_3 \bar{c}_9 \omega + B_3 \omega^2 + \frac{A_1 \bar{c}_7 \omega}{U_r} + \frac{A_2 \bar{c}_8 \omega}{U_r} - B_3 \bar{k}_6 - \frac{B_2 \bar{k}_5}{U_r} = 0 \quad (29)$$

$$-B_4 \omega - \frac{A_4 \bar{R}}{U_r} = 0 \quad (30)$$

$$A_1 \omega + A_4 \omega - \frac{B_4 \bar{R}}{U_r} = 0 \quad (31)$$

By simultaneously solving Eqs. (23–31), we can obtain the first-order approximate periodic solutions of the tristable FEH.

C. Stability Analysis of Equilibrium Points

The tristable FEH exhibits distinct equilibrium characteristics depending on U_r . When $U_r = 0$, the system inherently possesses three stable equilibrium points and two unstable equilibrium points, forming its basic tristable configuration. The three stable equilibrium points are, respectively,

$$y^* = 0, \pm \sqrt{\frac{-\bar{k}_{h3} + \sqrt{\bar{k}_{h3}^2 - 4\bar{k}_{h5} \bar{k}_{h1}}}{2\bar{k}_{h5}}} \quad (32)$$

Under operational conditions where $U_r \neq 0$, the stability analysis requires linearization of the system dynamics around these equilibria. To formalize the analysis, we define the state vector $\mathbf{z} = [y \ \alpha \ x \ y' \ \alpha' \ x' \ \bar{v}]^T$, and we transform Eqs. (8–11) into the state-space form as

$$\mathbf{z}' = \begin{bmatrix} y' \\ \alpha' \\ x' \\ y' \\ \alpha' \\ x' \\ \bar{v}' \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 1} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{N}_0 \\ \mathbf{0}_{1 \times 3} & -1 & \mathbf{0}_{1 \times 2} & -\bar{R}\frac{1}{U_r} \end{bmatrix} \mathbf{z} + \begin{bmatrix} \mathbf{0}_{3 \times 1} \\ -\mathbf{M}^{-1}\mathbf{N}(\mathbf{z}) \\ 0 \end{bmatrix} = \mathbf{A}\mathbf{z} + \mathbf{B}_Z(\mathbf{z}) \quad (33)$$

where the system matrices are defined as

$$\mathbf{M} = \begin{bmatrix} 1 & \bar{m}_2 & 0 \\ 1 & \bar{I}_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \bar{c}_h \frac{1}{U_r} + \bar{c}_1 & \bar{c}_2 & \bar{c}_3 U_r \\ \bar{c}_5 & \bar{c}_\alpha \frac{1}{U_r} + \bar{c}_4 & \bar{c}_6 U_r \\ \bar{c}_7 \frac{1}{U_r} & \bar{c}_8 \frac{1}{U_r} & \bar{c}_9 \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} \bar{k}_{h1} \frac{1}{U_r^2} & \bar{k}_1 & \bar{k}_2 U_r \\ 0 & \bar{k}_\alpha \frac{1}{U_r^2} + \bar{k}_3 & \bar{k}_4 U_r \\ 0 & \bar{k}_5 \frac{1}{U_r} & \bar{k}_6 \end{bmatrix}, \quad \mathbf{N}_0 = \begin{bmatrix} -\bar{\theta} \frac{1}{U_r^2} & 0 & 0 \end{bmatrix}^T,$$

$$\mathbf{N}(\mathbf{z}) = \begin{bmatrix} \bar{k}_{h3} \frac{1}{U_r^2} y^3 + \bar{k}_{h5} \frac{1}{U_r^2} y^5 \\ \bar{k}_{\alpha 3} \frac{1}{U_r^2} \alpha^3 \\ 0 \end{bmatrix}$$

For stability analysis at the equilibrium points y^* , we perform Taylor expansion and linearization, yielding

$$\mathbf{z}' \approx \begin{bmatrix} \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 1} \\ -\mathbf{M}^{-1}\mathbf{K}_l & -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{N}_0 \\ \mathbf{0}_{1 \times 3} & -1 & \mathbf{0}_{1 \times 2} & -\bar{R}\frac{1}{U_r} \end{bmatrix} \mathbf{z} = \mathbf{A}_l \mathbf{z} \quad (34)$$

where the linearized matrix $\mathbf{A}_l$ incorporates modified stiffness matrix $\mathbf{K}_l$.

When y is close to $y^* = 0$, $\mathbf{K}_l = \mathbf{K}$.

When y is close to $y^* = \pm \sqrt{(-\bar{k}_{h3} + \sqrt{\bar{k}_{h3}^2 - 4\bar{k}_{h5}\bar{k}_{h1}/2\bar{k}_{h5}})}$,

$$\mathbf{K}_l = \begin{bmatrix} \bar{k}_h \frac{1}{U_r^2} + 3\bar{k}_{h3} \frac{1}{U_r^2} y^{*2} + 5\bar{k}_{h5} \frac{1}{U_r^2} y^{*4} & \bar{k}_1 & \bar{k}_2 U_r \\ 0 & \bar{k}_\alpha \frac{1}{U_r^2} + \bar{k}_3 & \bar{k}_4 U_r \\ 0 & \bar{k}_5 \frac{1}{U_r} & \bar{k}_6 \end{bmatrix} \quad (35)$$

The eigenvalue analysis of $\mathbf{A}_l$ determines stability: eigenvalues with negative real parts indicate stable equilibria, whereas positive real parts signify instability. This approach reveals the system's two distinct cut-in wind speeds corresponding to Hopf bifurcations from the symmetric lateral equilibria and the central equilibrium, respectively.

D. Calculation of Lyapunov Exponents

For the computation of Lyapunov exponents in the tristable FEH, we employ a coevolution method in phase space and tangent space [36]. The essence of the method lies in numerically tracking the evolution of infinitesimal volume elements over long timescales. Specifically, the baseline trajectory $\mathbf{z}(\tau)$ is first generated by numerically integrating the nonlinear equations of motion [Eq. (33)] using

the Runge–Kutta method. Subsequently, an initial infinitesimal perturbation $\delta\mathbf{z}(0)$ is introduced, and its evolution is governed by the linearized equations:

$$\delta\mathbf{z}'(\tau) = \mathbf{J}(\mathbf{z}(\tau))\delta\mathbf{z}(\tau) \quad (36)$$

where the Jacobian matrix $\mathbf{J}(\mathbf{z}) = \mathbf{A} + (\partial\mathbf{B}_Z/\partial\mathbf{z})$ is computed via analytic differentiation or finite difference methods. To avoid numerical issues such as floating-point overflow caused by exponential divergence and directional degeneracy of the tangent space basis vectors $\{\delta\mathbf{z}_1, \dots, \delta\mathbf{z}_7\}$, Gram–Schmidt reorthonormalization (GSR) is performed on these vectors at fixed time intervals $\Delta\tau$:

$$\delta\mathbf{z}'_i = \delta\mathbf{z}_i - \sum_{j=1}^{i-1} \frac{\langle \delta\mathbf{z}_i, \delta\mathbf{z}'_j \rangle}{\langle \delta\mathbf{z}'_j, \delta\mathbf{z}'_j \rangle} \delta\mathbf{z}'_j \quad (37)$$

$$\delta\mathbf{z}'_i \leftarrow \frac{\delta\mathbf{z}'_i}{\|\delta\mathbf{z}'_i\|}, \quad (i = 1, 2, \dots, 7) \quad (38)$$

By monitoring the evolution of the norms of orthogonalized basis vectors and accumulating the logarithmic growth rates in each direction, the Lyapunov exponents are calculated as follows:

$$LE_i = \frac{1}{T} \sum_{k=1}^N \log_2 \frac{\|\delta\mathbf{z}_i(\tau_k)\|}{\|\delta\mathbf{z}_i(\tau_{k-1})\|} \quad (39)$$

where $T = \tau_N - \tau_0$ is the total integration time, and N is the number of GSR steps.

III. Nonlinear Dynamic Behavior of the Tristable FEH

This section analyzes the nonlinear dynamic behavior of the tristable FEH based on experimental parameters from previous work by Li et al. shown in [31]. The dimensionless parameters calculated for the tristable FEH are listed in Table 1. Based on the nondimensionalized

Table 1 Dimensionless parameters of the tristable FEH

Parameter	Value
$\bar{m}_2$	0.203
$\bar{c}_h$	0.198
$\bar{c}_1$	0.015
$\bar{c}_2$	0.0318
$\bar{c}_3$	0.134
$\bar{k}_1$	0.015
$\bar{R}$	0.5419
$\bar{k}_2$	0.0085
$\bar{I}_1$	0.6934
$\bar{\theta}$	5.62×10^{-5}
$\bar{c}_\alpha$	0.0946
$\bar{c}_4$	0.0902
$\bar{c}_5$	0.0075
$\bar{c}_6$	0.0661
$\bar{k}_\alpha$	0.6575
$\bar{k}_{\alpha 3}$	1.6227
$\bar{k}_3$	0.0075
$\bar{k}_4$	0.0042
$\bar{c}_7$	−0.0244
$\bar{c}_8$	−0.0268
$\bar{c}_9$	0.3455
$\bar{k}_5$	−0.0244
$\bar{k}_6$	0.0136
$\bar{k}_h$	0.383
$\bar{k}_{h3}$	−4.2746
$\bar{k}_{h5}$	10.16

model, this section reveals the nonlinear dynamic characteristics of the system by comparing simulation results with experimental data from [31] as shown in Fig. 3. The plots have been redrawn for clarity and consistency with the current study. In the experiments, a low-speed wind tunnel (HF400, Yunfeng Instrument Company, Ltd., China) with a speed range of 0–30 m/s is used, which is controlled by a variable-frequency device (JT330S2, JINTIAN Technology Company, Ltd., China). Wind-speed fluctuations are maintained within ± 0.15 m/s. Wind speed is measured using a micromanometer (GT1115, Kanomax, Japan) with a range of 0.2–60 m/s, with an accuracy of $\pm 1.5\%$ (at 10.0 m/s), and with a resolution of 0.01 m/s. Plunging displacement is recorded with a laser displacement sensor (HG-C1100, Panasonic, Japan) with 0.07 mm accuracy. An oscilloscope (DSOX3014T, Keysight, China) is used to measure the voltage across the resistance.

After nondimensionalizing the displacement and voltage, a sliding window is applied to the steady-state segment to extract local amplitudes, from which the mean, standard deviation, minimum, and maximum over all windows are computed. The wind-speed uncertainty of ± 0.15 m/s is converted to ΔU_r . Figure 3 presents three typical response regimes: intrawell periodic oscillation, interwell periodic oscillation (i.e. large-amplitude interwell LCO), and aperiodic vibration. The ranges of these responses are broadly consistent with the numerical simulation results. In both experiments and simulations, the amplitudes of the plunging displacement y and the output voltage $\bar{v}$ of the tristable FEH vary with the reduced wind speed U_r , in essentially the same way, validating the theoretical model's ability to predict the dynamic behavior of the actual system. The experimental results show that at low wind speeds ($U_r < 5.5$), the system primarily exhibits small-amplitude intrawell periodic vibrations; in the range $U_r = 5.46$ – 7.07 , aperiodic responses occur, representing a transitional stage of interwell response; subsequently,

the system steps into stable interwell periodic vibrations. The onset of interwell periodic response occurs at $U_r = 7.07$, whereas the simulation predicts 6.59. There is a certain discrepancy at the critical point of response-type transition. Notably, in the aperiodic response region, the error bars are significantly longer, indicating that the system is in a chaotic potential-well switching state, leading to increased amplitude fluctuations. In contrast, the error bars for intrawell and interwell periodic responses are shorter, indicating more stable dynamics and higher repeatability.

Figures 3c–3f illustrate the aperiodic response characteristics at $U_r = 6$. The time-domain curves of the plunging displacement (Fig. 3c) and voltage (Fig. 3e) exhibit highly irregular fluctuations, with displacement amplitudes fluctuating within ± 40 mm and voltage signals switching between ± 5 V, accompanied by localized extreme peaks. Wavelet spectral analysis (Figs. 3d and 3f) reveals broadband frequency distributions, with displacement spectra showing multiscale energy diffusion across 0.1–10 Hz. These features suggest the system is in a chaotic state.

Figure 4 shows how the simulation errors of the plunging displacement amplitude and voltage amplitude, relative to the experimental results, vary with U_r . The relative error is defined as $|A_{\text{sim,max}} - A_{\text{exp,max}}|/|A_{\text{exp,max}}|$, in which $A_{\text{sim,max}}$ and $A_{\text{exp,max}}$ denote the simulated and experimental peak amplitudes, respectively. For displacement, the relative error in the intrawell periodic region is generally below 20%, with only one point slightly higher but still within 30%. In the interwell LCO region, errors mostly fall within 10–20%. In the aperiodic region ($U_r = 5.46$ – 7.07), the displacement error rises markedly, with peaks exceeding 40%. This increase is consistent with the chaotic dynamics accompanying interwell switching. In addition, within a narrow band of U_r , the simulation already settles into a stable interwell LCO at $U_r = 6.59$, whereas the experiment remains aperiodic at $U_r = 6.59$ – 7.07 ;

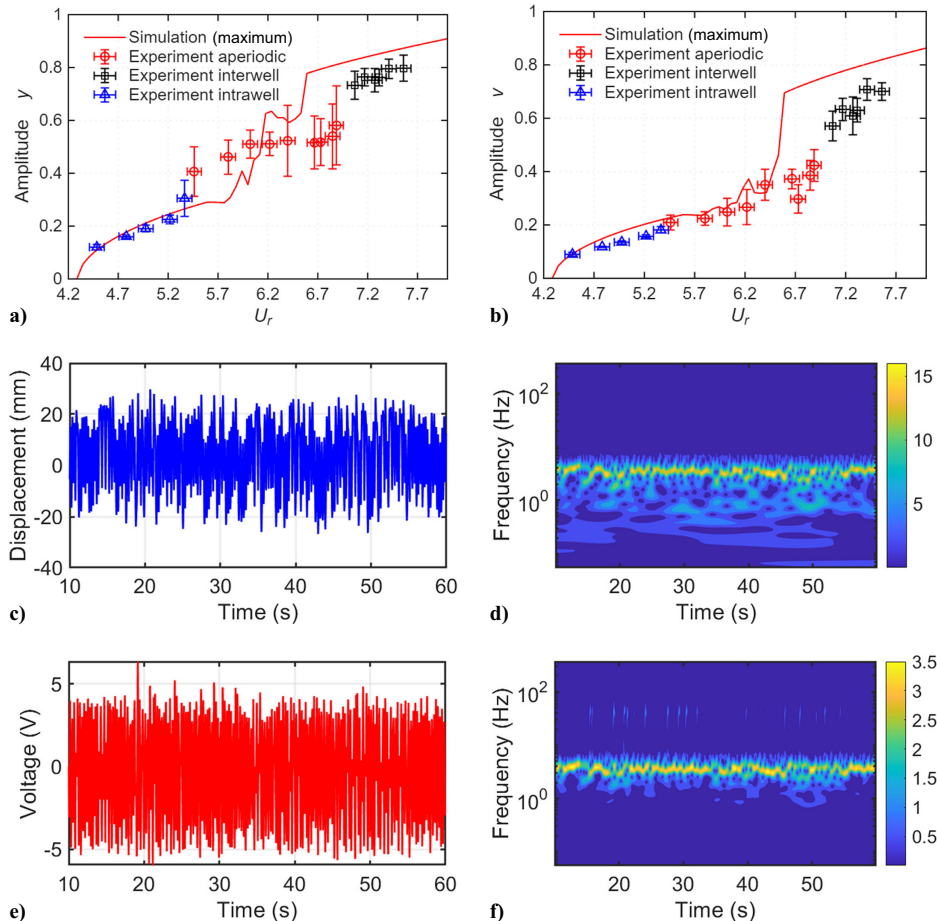


Fig. 3 Representations of a,b) experimental vs numerical comparisons (amplitudes: y , $\bar{v}$), and c–f) experimental time-domain and wavelet spectrum analysis. Experimental data are from Li et al. [31].

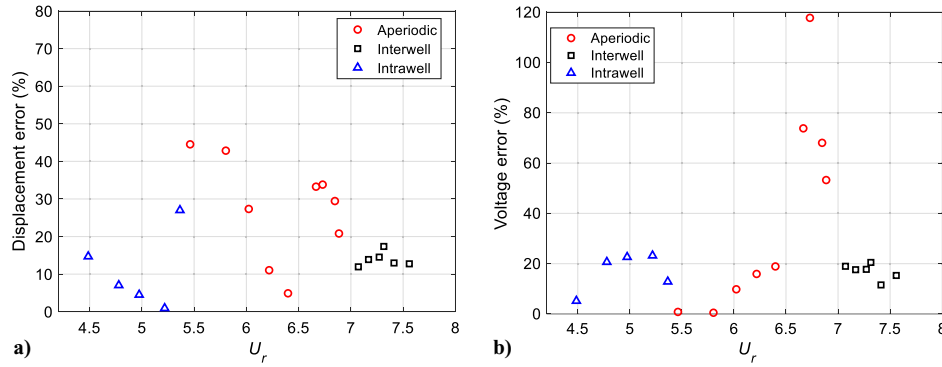


Fig. 4 Representations of a) relative error of plunging displacement amplitude versus reduced wind speed, and b) relative error of voltage amplitude versus reduced wind speed.

this mismatch in response type further enlarges the amplitude differences at the same U_r .

The relative error of voltage remains around 20% in both intrawell and interwell periodic regions but rises sharply in the aperiodic region, peaking at about 80–120% near $U_r = 6.5$ –7.0. Beyond the above response-type mismatch in this U_r band, the voltage signal is additionally affected by the electromechanical coupling and sensitivity to high-frequency content. This increases scatter during the aperiodic stage and consequently inflates the error relative to displacement.

Theoretically, a flutter system exhibits no vibration response below the cut-in speed. However, due to disturbances such as structural vibrations of the wind tunnel during operation and flow-field instabilities, small-amplitude random vibrations can still appear in experiments below the cut-in speed. Because the cut-in speed is not reached, flutter does not occur, and the amplitudes show a certain randomness (clearly distinct from intrawell periodic oscillations). Therefore, the low-wind-speed responses (below the cut-in speed) can serve as a quantitative reference for the influence of environmental disturbances on the system response. Figure 5 presents the displacement and voltage time-domain responses, power spectra, and energy contributions at $U_r = 3.1699$.

The obtained spectra show a pronounced peak in the displacement power spectral density around 5–10 Hz, close to the system's linear pitch natural frequency, indicating resonance amplification near the natural frequency under broadband environmental excitation. The spectral feature reveals a notable discrepancy between displacement and voltage responses: high-frequency components (20–100 Hz), although negligible in the displacement spectrum, contribute approximately 15% to the voltage spectrum. This difference primarily stems from the limitations in the measurement chain. The minute amplitudes of high-frequency vibrations approach the resolution limit of the laser displacement sensor, leading to an underrepresentation in the displacement data. Furthermore, when the displacement signal is converted to a voltage signal for display and storage on the oscilloscope, these already small high-frequency components are subject to further attenuation due to the limited resolution and bandwidth of the acquisition system. Consequently, their effective magnitude falls below the threshold for accurate capture. Based on this observed 15% energy contribution in the high-frequency band of the voltage spectrum, we attribute 15% of the total environmental disturbance energy to cumulative measurement system errors. This includes sensor resolution limits and signal conversion losses.

Figures 5g and 5h compare the standardized displacement and voltage with a standard Gaussian. The data are approximately symmetric with near-normal tail weight (for displacement, skewness is -0.126 and kurtosis is 2.983 ; for voltage, skewness is -0.150 and kurtosis is 2.762), and the kernel estimates largely track the Gaussian reference. Nevertheless, Kolmogorov–Smirnov (K–S) tests return $p = 0.000$ for both signals, rejecting exact normality and indicating small but detectable departures (e.g., mild shoulder/peak deviations). Given the close agreement in the first- and second-order

statistics, and for compatibility with standard estimation/control tools, we model the environmental disturbances as zero mean Gaussian noise in simulations, with intensity calibrated to the experimental standard deviations in Figs. 5g and 5h.

We repeat the sub-cut-in analysis at three additional reduced velocities without flutter. Across these cases, the response variance, spectral shape (low-frequency resonance peak and weak high-frequency tail), and skewness/kurtosis remain similar, with no significant differences. The wind-tunnel operating conditions and setup are unchanged; measured test-section wind-speed fluctuations remain within ± 0.15 m/s. These fluctuations are dominated by low-frequency content; consistent with this, the 0–5 Hz band contributes less than 2% of the response spectral energy. Hence, the disturbance magnitude can be regarded as approximately wind-speed invariant. The wind-tunnel operating conditions and setup are unchanged. Based on the aforementioned information, we use the disturbance statistics at $U_r = 3.1699$ as the reference baseline for environmental disturbances in subsequent analyses.

The wind-speed-induced amplitude uncertainty is quantified as $|(dA/dU_r)|\Delta U_r$. Here, $|dA/dU_r|$ is estimated from model simulations by perturbing U_r around the nominal value (e.g., $U_r \pm \delta U_r$) and evaluating $|A_{(U_r+\delta U_r)} - A_{(U_r-\delta U_r)}|/2\delta U_r$. All of the aforementioned factors are classified as experimental error sources. Beyond experimental errors, another major source is the model itself. Based on the aforementioned analysis, we estimate the contributions of various error sources for three distinct response states (at representative U_r values of 4.97, 6.73, and 7.17), as shown in Fig. 6.

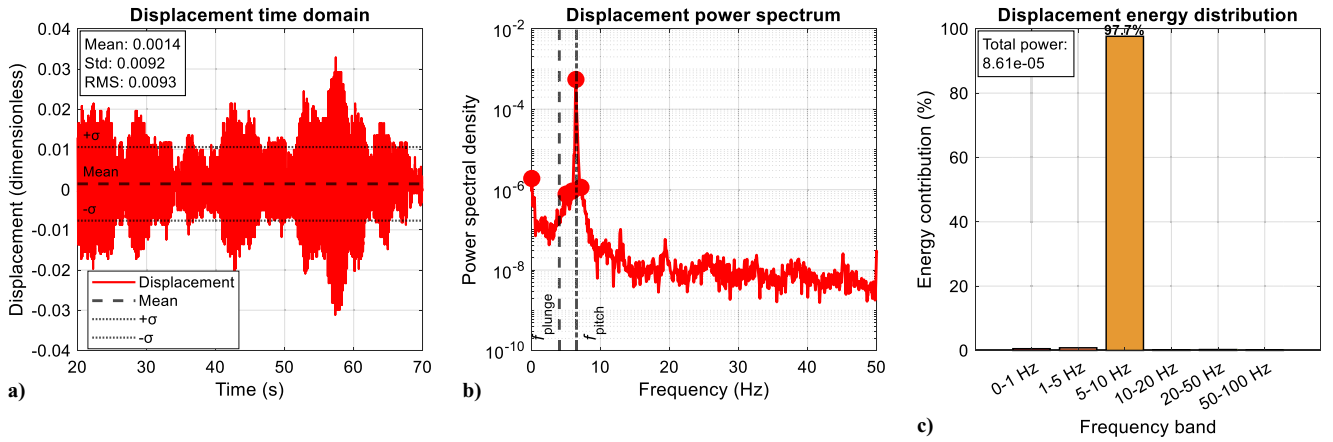
In the intrawell periodic oscillation stage, the amplitude errors exhibit a multisource superposition. The percentages indicate contributions to the total error, namely, wind-speed fluctuations of $\approx 33.4\%$, environmental disturbances of $\approx 21.3\%$, measurement error of $\approx 1\%$, and model error of $\approx 44.3\%$. Model error takes the largest share.

After entering the interwell aperiodic response stage, the error sources concentrate markedly on the model side. A typical result shows that model error is overwhelmingly dominant ($\approx 98\%$), whereas errors from wind-speed fluctuations, environmental disturbances, and measurement are each below 1%. Additional contributions are from two-dimensional simplifications (missing three-dimensional effects) and approximations in the Theodorsen-based model.

In the interwell LCO stage, the error-source distribution is similar to that of the aperiodic response, but the influence of environmental disturbances rises slightly: model error still dominates (≈ 97 – 99%), wind-speed fluctuations and measurement error each remain around 1%, and environmental disturbances can reach 1–2%. This indicates that experimental uncertainties have a slightly greater impact on large-amplitude periodic motion than on aperiodic responses; however, overall, systematic model biases in LCO amplitude remain the primary contributors to the total error.

In summary, during the small-amplitude vibration stage, experimental uncertainties contribute a noticeable portion of the total error. As wind speed increases and the system enters large-amplitude responses, the fractions of experimental uncertainties

Displacement environmental disturbance analysis ($U_r = 3.1699$)



Voltage environmental disturbance analysis ($U_r = 3.1699$)

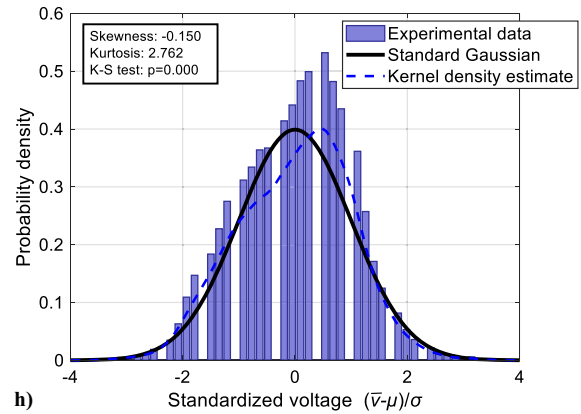
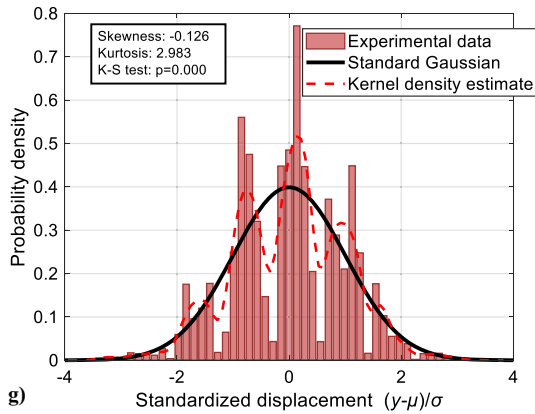
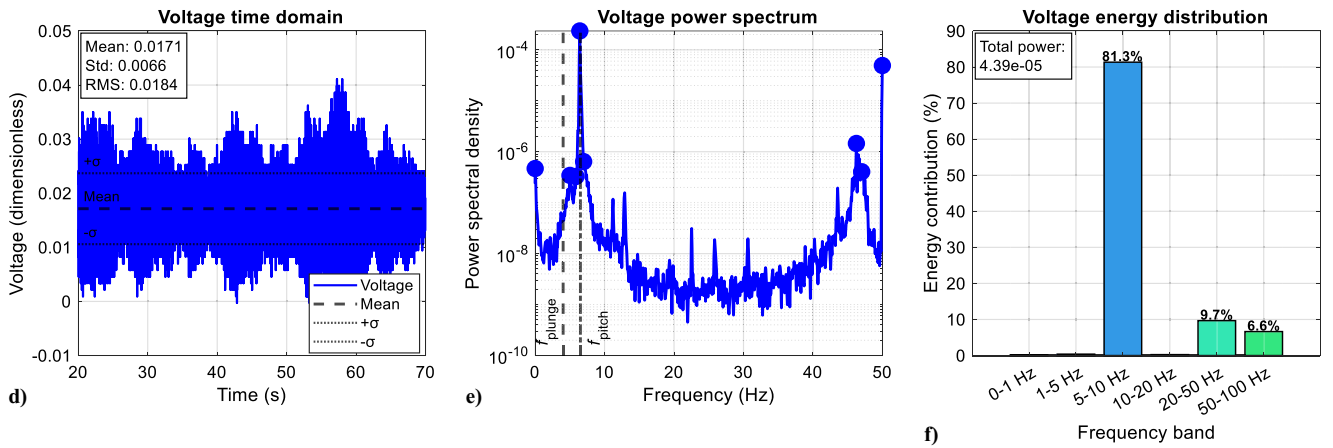


Fig. 5 At $U_r = 3.1699$ (below cut-in), a–c) displacement showing time response, power spectrum, energy contribution; d–f) voltage showing same data as Figs. 5a–5c; g) displacement; and h) voltage probability distributions vs Gaussian. Std denotes “standard deviation”.

(including environmental disturbances and measurement error) in the total error become significantly lower than that of model error. Moreover, model error is particularly prominent in the aperiodic response stage, which is closely related to its intrinsic dynamical complexity and unpredictability.

Although computational fluid dynamics (CFD) may provide more accurate results, calculating Lyapunov exponents through CFD presents significant challenges for chaos analysis and requires much more computational resources. Therefore, this study prioritizes a qualitative analysis of dynamic behavior patterns rather than striving for a precise quantitative validation.

The structural dimensionless elastic potential energy E is defined as

$$E = \frac{1}{2} \overline{k_h} \frac{1}{U_r^2} y^2 + \frac{1}{4} \overline{k_{h3}} \frac{1}{U_r^2} y^4 + \frac{1}{6} \overline{k_{hs}} \frac{1}{U_r^2} y^6 + \frac{1}{2} \overline{k_a} \frac{1}{U_r^2} \alpha^2 + \frac{1}{4} \overline{k_{a3}} \frac{1}{U_r^2} \alpha^4 \quad (40)$$

The potential energy surface at $U_r = 6$ (Fig. 7) demonstrates distinct dynamic characteristics: tristable behavior in the plunging motion y direction, with three potential energy minima (when α is fixed) and monostable behavior in the pitching motion α direction (when y is fixed). To analyze stability, the eigenvalues of the Jacobian matrix A_I near the central and lateral equilibrium points are computed. Figures 8a and 8b show the eigenvalues as a function

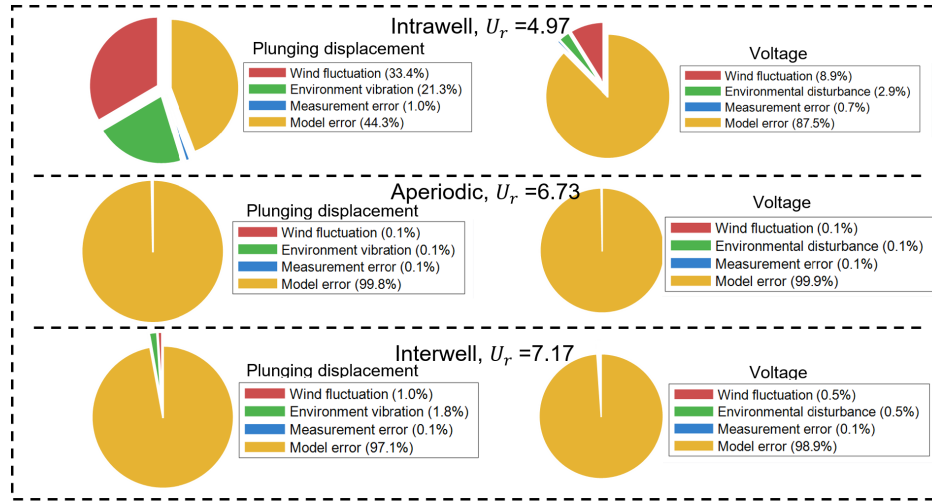


Fig. 6 Error sources and their contributions to the total error under three vibration states.

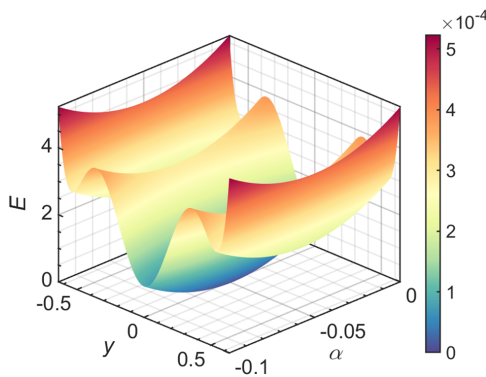


Fig. 7 Potential energy surface of the system at $U_r = 6$.

of wind speed. Both cases exhibit a pair of eigenvalues (highlighted in red and blue, respectively) crossing the zero-real-part plane, corresponding to the cut-in wind speed of $U_r = 3.96$ for the lateral equilibrium points and $U_r = 4.28$ for the central equilibrium point. These two thresholds represent two distinct Hopf bifurcation points, where the system transits from the stable equilibrium points to vibration. Figures 8c and 8d display the eigenvalue distribution when the tristable FEH initial position is near the central and lateral equilibrium points, respectively, with color mapping representing the variation of U_r . In these figures, the imaginary parts represent the normalized frequencies (dimensionless values). Three eigenvalues have imaginary parts that remain consistently zero, corresponding to a pair of eigenvalues for the aerodynamic state parameter x and one eigenvalue for the electrical state parameter $\bar{v}$. The two pairs of eigenvalues with nonzero imaginary parts correspond to the coupled structural plunging and pitching vibration modes. The reason is that the system's motion patterns, including vibration modes and frequencies, are dominated by these two states, whereas x and $\bar{v}$ serve as follower state variables. As the wind speed increases, one of the eigenvalue branches associated with the coupled plunging and pitching vibration modes crosses the imaginary axis (zero-real-part line), triggering Hopf bifurcations at $Im = 0.19$ and $Im = 0.27$ for the central and lateral equilibrium points, respectively. This crossing indicates the onset of self-induced oscillations at these specific frequencies.

As previously mentioned, the first-order approximate solution of the system's periodic response is insufficient for stability determination using Floquet theory, whereas the incremental harmonic balance (IHB) method allows for obtaining higher-order approximate solutions. In previous work [31], Li et al. specifically investigated methods for addressing multiple solution determination and

initial value selection problems of the IHB method for a tristable FEH system. To demonstrate the stability of the system's periodic solutions, we employ the method described in [31] (not detailed in this paper) to obtain the ninth-order approximate solution curve of the dimensionless plunging displacement and determine its stability using Floquet theory, as shown in Fig. 9.

It can be seen that the periodic solutions between bifurcation point 3 (B3) and bifurcation point 4 (B4) are unstable. This interval is the main region where chaos occurs. From a dynamical system perspective, these unstable periodic orbits play a crucial role in organizing the chaotic dynamics. According to the theory of chaotic attractors [37], unstable periodic orbits embedded within the chaotic attractor serve as the "skeleton" that structures the chaotic motion. When the system enters the chaotic regime, trajectories are repelled from these unstable periodic solutions but remain confined to their vicinity due to the dissipative nature of the system. This creates a complex web of unstable manifolds that guide the chaotic trajectories, leading to the sensitive dependence on initial conditions characteristic of chaos. Beyond B4, both stable and unstable periodic solutions coexist. Numerical simulation results indicate the coexistence of interwell LCO and chaos within this interval. Unlike bifurcation point 2 (B2), the branch starting from the fixed point at bifurcation point 1 (B1) first exhibits unstable periodic solutions, and then it experiences a saddle-node bifurcation where stable periodic solutions appear. This indicates that B2 is a supercritical Hopf bifurcation, whereas B1 is a subcritical Hopf bifurcation. Because this paper focuses on chaotic responses, obtaining higher-order approximate periodic solutions would significantly increase the computational time cost. Therefore, we only use Fig. 9 as the reference basis for periodic solution stability, whereas subsequent discussions on bifurcation and chaos are still based on first-order HB approximate solutions as shown in Fig. 10 and numerical solutions.

The HB solutions (as shown in Fig. 10) are categorized into three types based on their characteristics: yellow curves represent asymmetric periodic solutions around the lateral equilibrium point, red curves denote asymmetric periodic solutions around the central equilibrium point, and blue curves indicate symmetric periodic solutions around the central equilibrium point. Three key phenomena are scrutinized: Hopf bifurcations from fixed points leading to LCOs, bifurcation resulting in chaotic behavior, and saddle-node bifurcation initiating large-amplitude interwell LCOs. The four key bifurcation points are as follows:

- 1) For B1, at $U_r = 3.96$, a Hopf bifurcation occurs at the lateral equilibrium point, marking the onset of asymmetric periodic solutions around this point.
- 2) For B2, At $U_r = 4.28$, another Hopf bifurcation occurs at the central equilibrium point, triggering symmetric periodic solutions around this point.

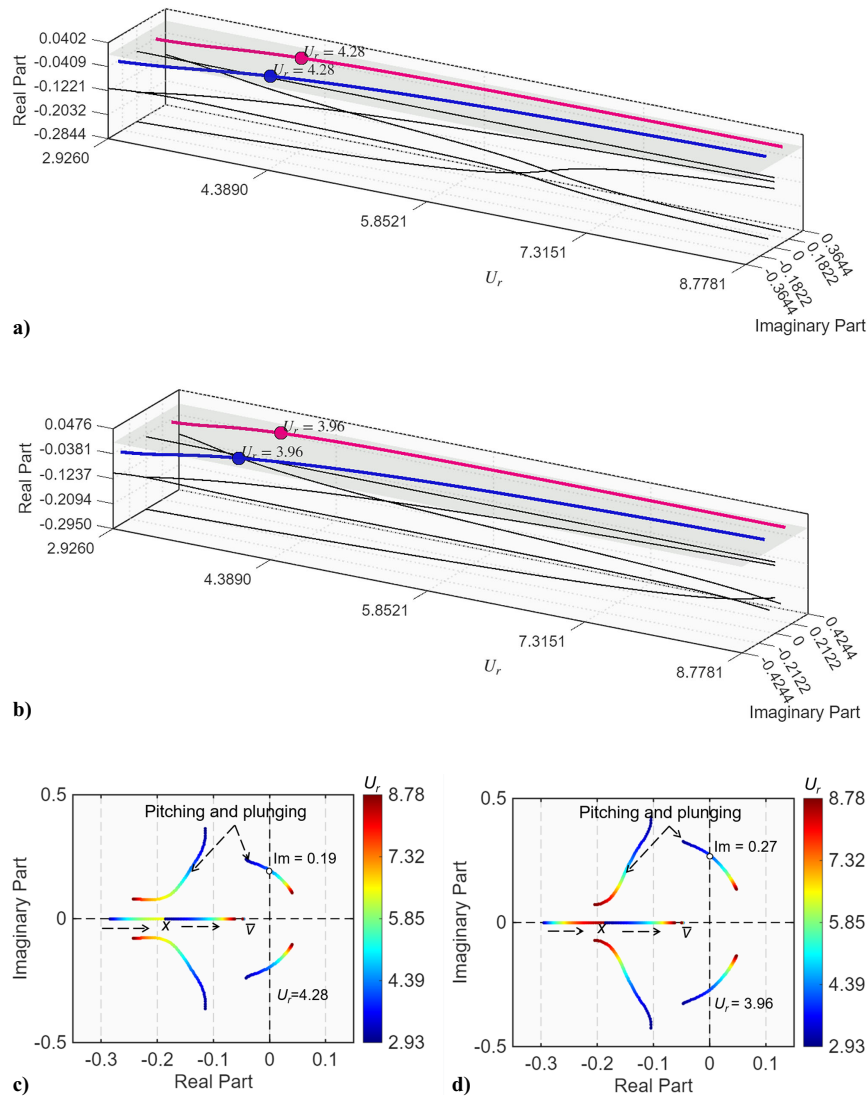


Fig. 8 Eigenvalues of Jacobian matrix A_I versus U_r projected on root-loci plane: a,c) near central and b,d) lateral equilibrium points.

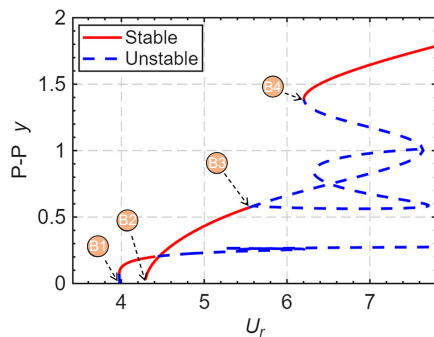


Fig. 9 Ninth-order approximate solutions of peak-to-peak (P-P) value of y and their stability.

3) For B3, at $U_r = 5.4$, a symmetry-breaking bifurcation occurs, leading to the emergence of asymmetric periodic solutions around the central equilibrium point.

4) For B4, at $U_r = 6.1$, a saddle-node bifurcation initiates large-amplitude interwell LCOs.

Based on the analysis in [31], after the symmetry-breaking bifurcation (B3), the system exhibits aperiodic solutions, indicating potential chaotic responses between B3 and B4. Stable periodic solutions exist between B1 and B3, as well as beyond B4.

To further investigate the chaotic response characteristics and dynamic behaviors of the tristable FEH, numerical solutions are

obtained using the Runge–Kutta method, with results shown in Fig. 11. The initial conditions are set as follows: displacement is $y = 0.01$, with all other state variables set to zero. The following observations can be made from the results: The system begins to generate voltage output at $U_r = 4.28$. As U_r increases, the root mean square (RMS) value of the voltage grows smoothly. When $U_r > 5.8$, the voltage output starts to fluctuate with low amplitude. Beyond $U_r = 6.5$, the system transitions to a high-output state. The bifurcation diagram reveals that the system transitions from single-periodic to aperiodic responses between $U_r = 5.8$ and $U_r = 6.5$, exhibiting chaotic characteristics. Regarding the calculation of Lyapunov exponents, a convergence analysis with respect to the simulation time and simulation step size at $U_r = 7$ is performed, with results shown in Fig. 12. In Fig. 12a, it can be observed that when the simulation time τ exceeds 2500, the Lyapunov exponents change very slowly: LE_7 changes from 0.3003 at $\tau = 2500$ to 0.3002 at $\tau = 3500$. Figure 12b shows that the time step size below 100 is required to ensure the calculation accuracy. In this study, to accommodate Lyapunov exponent calculations under different parameters, a step size of one is used with the simulation time set to 4000.

The Lyapunov exponent curves show that within this interval, there is one positive exponent, one zero exponent, and five negative exponents, confirming the presence of chaotic dynamics. To investigate the dynamic complexity of the flutter system in a chaotic state, the Lyapunov dimension of the system is calculated based on the Kaplan–Yorke conjecture [38]:

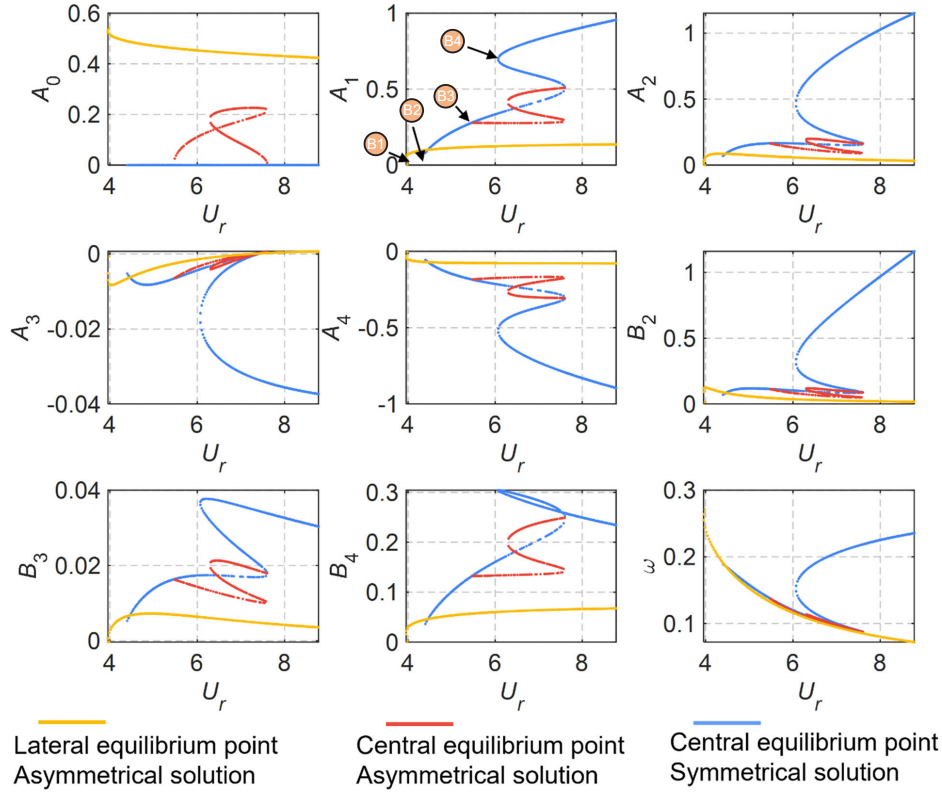


Fig. 10 Harmonic coefficients and response frequencies of the tristable FEH obtained via the HB method.

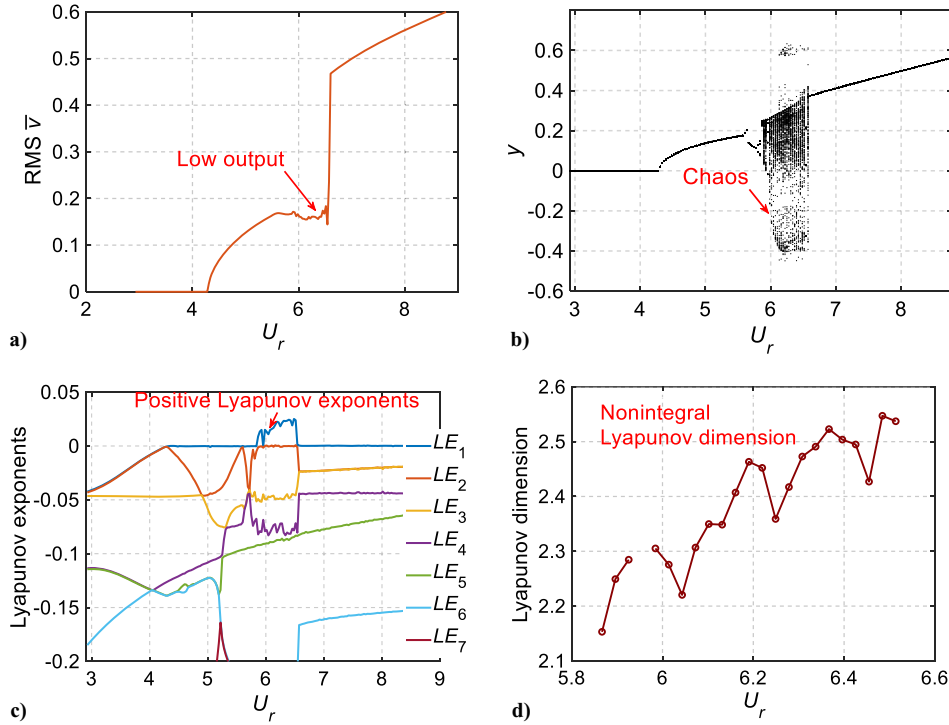


Fig. 11 Tristable FEH system numerical solutions vs U_r : a) RMS $\bar{v}$, b) bifurcation diagram of y , c) Lyapunov exponents, and d) Lyapunov dimension in chaotic regime.

$$D_L = k + \frac{\sum_{i=1}^k LE_i}{|LE_{k+1}|} \quad (41)$$

where k is the largest integer for which $\sum_{i=1}^k LE_i \geq 0$; the denominator $|LE_{k+1}|$ represents the absolute value of the $(k+1)$ th Lyapunov exponent. The calculated Lyapunov dimension is a

noninteger, indicating the presence of a noteworthy attractor: Globally, the system is stable, meaning all motions far from the attractor eventually converge to it. Locally, however, the motion within the attractor is unstable, with neighboring trajectories repelling each other and diverging exponentially. The Lyapunov dimension of the system (2.1–2.6) suggests that energy dissipation contracts the phase-space trajectories onto a low-dimensional fractal attractor

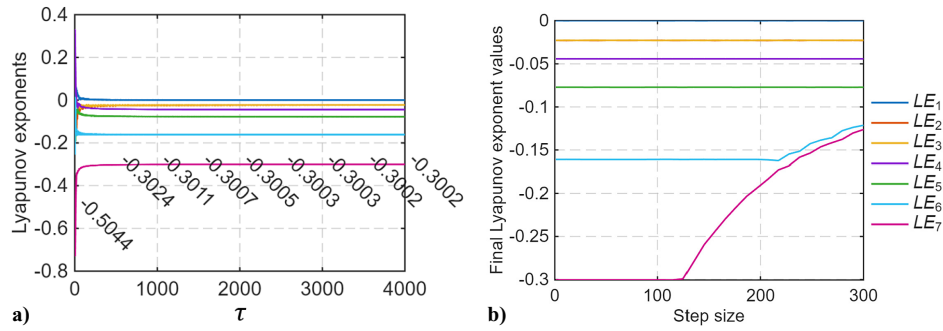


Fig. 12 Convergence analysis of Lyapunov exponents with respect to a) simulation time and b) simulation step size.

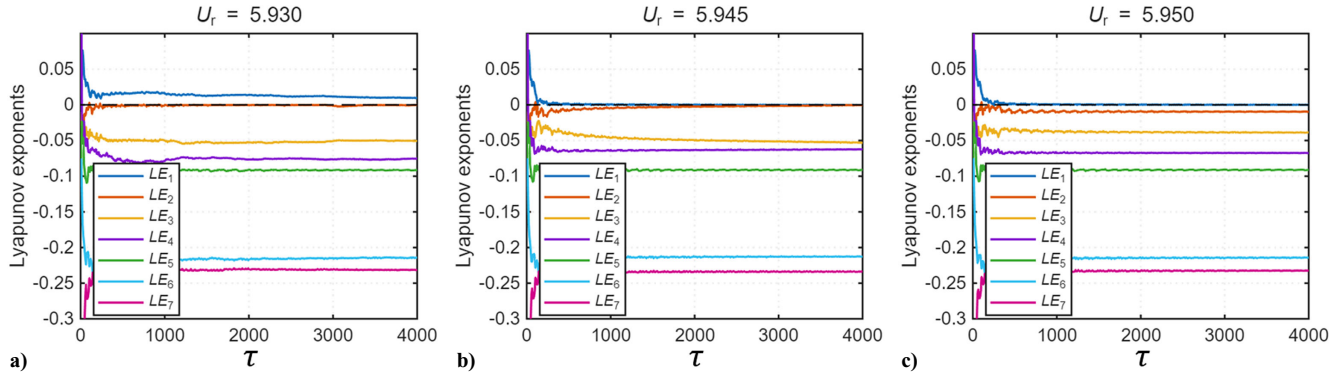


Fig. 13 Time-domain evolution of Lyapunov exponents under different wind speeds: a) $U_r = 5.930$ (chaotic state), b) $U_r = 5.945$ (quasi-periodic state), and c) $U_r = 5.950$ (periodic state).

with noninteger dimensionality. This dimensionality implies that the chaotic dynamics are primarily governed by two dominant Lyapunov modes (corresponding to the positive and neutral Lyapunov exponents, respectively), whereas the remaining modes (associated with the five negative exponents) are strongly suppressed by dissipation or synchronized to the dominant modes. When the Lyapunov dimension approaches two (e.g., 2.1), the Kaplan–Yorke formula indicates a weaker positive Lyapunov exponent LE_1 or stronger contraction in higher modes (large $|LE_3|$), leading to less pronounced chaos. As the dimension increases to three (e.g., 2.6), either a larger LE_1 (enhanced trajectory divergence) or slower contraction (smaller $|LE_3|$) entails more complex chaotic dynamics.

In Fig. 11c, it is observed that at certain specific wind-speed points, LE_1 and the second Lyapunov exponent LE_2 simultaneously approach zero. This suggests that the system may exhibit quasi-periodic oscillations. To further investigate this phenomenon and clearly distinguish quasi-periodicity from low-dimensional chaos, we examine the time-domain evolution curves of Lyapunov exponents at three typical wind speeds in Fig. 13. Figure 13a shows that at $U_r = 5.930$, LE_1 converges to a positive value, indicating that the system is in a chaotic state. Figure 13b shows that at $U_r = 5.945$, both LE_1 and LE_2 approach zero, whereas the remaining Lyapunov exponents are all negative, which is typical of quasi-periodic motion. Figure 13c shows that at $U_r = 5.950$, only LE_1 is zero, whereas the remaining exponents are all negative, corresponding to a periodic motion state. These three states clearly demonstrate that the quasi-periodic state lies in the transition region between chaos and periodicity. Further analysis indicates that the wind-speed conditions corresponding to the quasi-periodic state are extremely stringent, existing only within an extremely narrow parameter range. Because the quasi-periodic state occupies only a very small number of discrete points in the parameter space studied in this paper and is highly sensitive to changes in wind speed, a more in-depth analysis of quasi-periodic dynamics is not conducted in this study.

To verify the convergence of the maximum Lyapunov exponent calculation near the wind speed at which chaos emerges (i.e., near

the bifurcation point) and systematically evaluate its sensitivity to integration duration and initial transients, numerical experiments on a grid in the phase space defined by the initial displacement $y(0)$ and initial velocity $y'(0)$ are conducted. Figures 14 and 15, respectively, illustrate the evolution of LE_1 with respect to the dimensionless integration time τ under two typical flow velocity conditions around the bifurcation point (Fig. 14 corresponds to the stable state before bifurcation, $U_r = 5.82$, slightly below the critical value; Fig. 15 corresponds to the chaotic state after bifurcation, $U_r = 5.85$, slightly above the critical value).

In the stable parameter region before bifurcation, when the integration time is short ($\tau = 500$), LE_1 exhibits a highly nonuniform pattern on the initial condition plane, with values fluctuating violently around zero (alternating red and blue). This indicates that the system's transient response has not sufficiently decayed, and the exponent estimation is dominated by the initial transient dynamics. As the integration duration increases to $\tau = 2000$ and further to 4000, LE_1 rapidly converges to spatially uniform distribution. Most regions converge to deep blue areas corresponding to values close to zero. When $\tau > 4000$, the distribution of LE_1 almost ceases to evolve over time, demonstrating neutralized sensitivity to initial conditions, and the system converges to its stable attractor.

For the parameter region in a chaotic state after bifurcation, LE_1 holds weakly positive value (on the order of 10^{-3}). Under short integration times ($\tau = 500$ to 1000), LE_1 similarly exhibits strong dependence on initial conditions, with spatial distribution showing significant heterogeneity. However, as the integration time extends to $\tau = 4000$ and beyond, the distribution of LE_1 gradually becomes uniform. The entire initial condition plane tends toward a consistent light hue (corresponding to the weakly positive exponent), almost completely overlapping with the results for $\tau = 4000$, 6000, and 8000. This convergence behavior clearly indicates that near the bifurcation point, despite the relatively long transient process, the transient contribution become negligible after approximately 4000 dimensionless time units, and the estimation of LE_1 is no longer

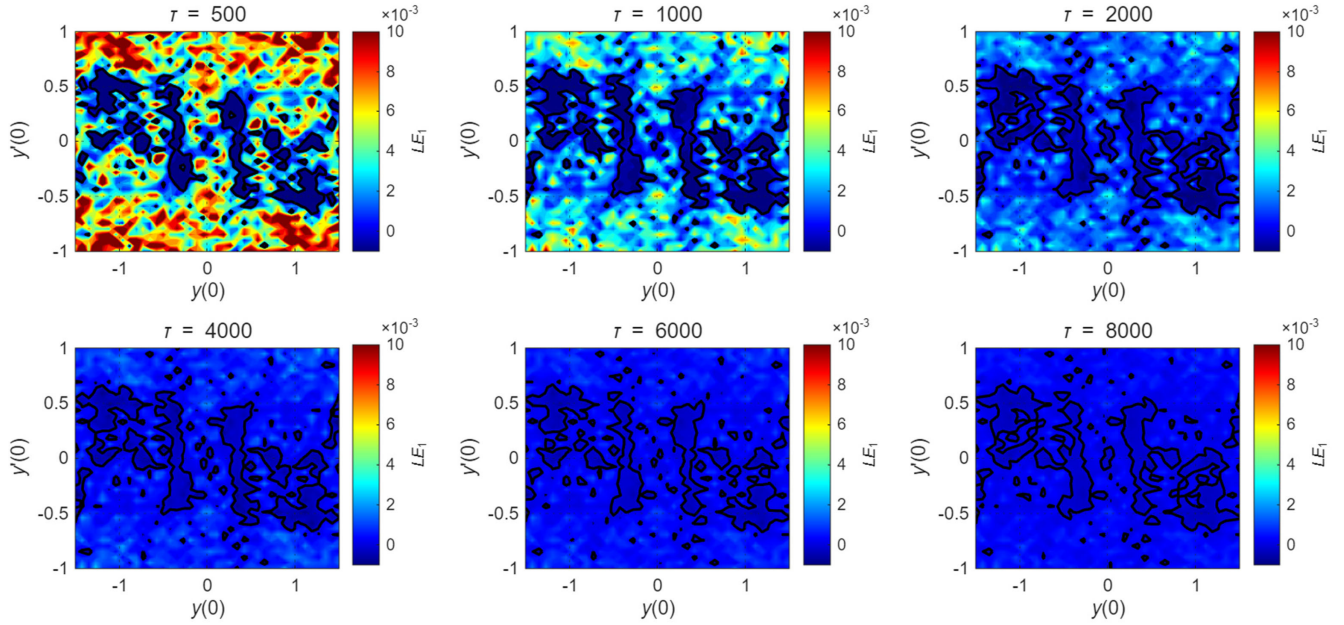


Fig. 14 Cloud maps of Lyapunov exponent convergence analysis with simulation time under different initial states ($U_r = 5.82$).

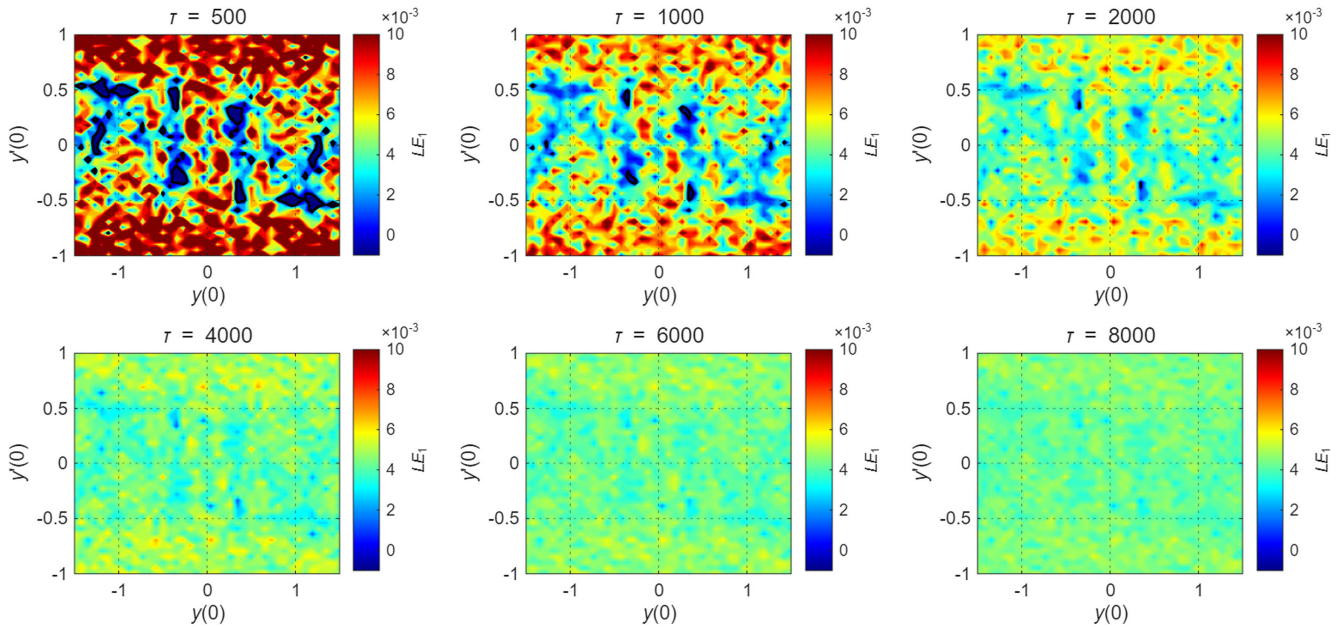


Fig. 15 Cloud maps of Lyapunov exponent convergence analysis with simulation time under different initial states ($U_r = 5.85$).

affected by the choice of initial conditions or the integration duration.

To evaluate the influence of environmental noise on Lyapunov exponent calculations, a systematic noise sensitivity analysis is conducted. According to the spectral analysis of the system's noise response below the cut-in speed, the frequency components of the noise response are primarily concentrated near the linear natural frequency in the pitching direction, which is attributed to resonance amplification at this frequency, whereas responses at other frequencies exhibit broadband distribution characteristics. This indicates that environmental disturbances possess pronounced randomness; therefore, Gaussian white noise is employed in this study to simulate environmental interference in actual measurements. The noise-contaminated plunging displacement time series can be expressed as

$$y_{\text{noise}}(t) = y(t) + A_n \cdot \xi(t) \quad (42)$$

where $\xi(t)$ denotes Gaussian white noise with zero mean and unit standard deviation, and A_n is the noise amplitude used to control the noise intensity.

The reference noise level is set to $A_n = 0.009$, which is derived from the RMS value of the dimensionless plunging displacement under environmental disturbances when the tristable FEH operates below the cut-in speed (flutter-free condition), as detailed in Fig. 5. Based on this reference, five different noise levels are selected for analysis, with noise amplitudes ranging from 0.001 to 0.04. Environmental interference in actual calculation is simulated by superimposing random noise of corresponding amplitudes onto the reconstructed time series.

Figure 16 illustrates the effects of noise on Lyapunov exponents and their associated characteristic quantities. As shown in Fig. 16a, noise does induce fluctuations in the Lyapunov exponents, but all exponent values oscillate slightly around their noise-free reference values, with the degree of deviation gradually increasing as the noise

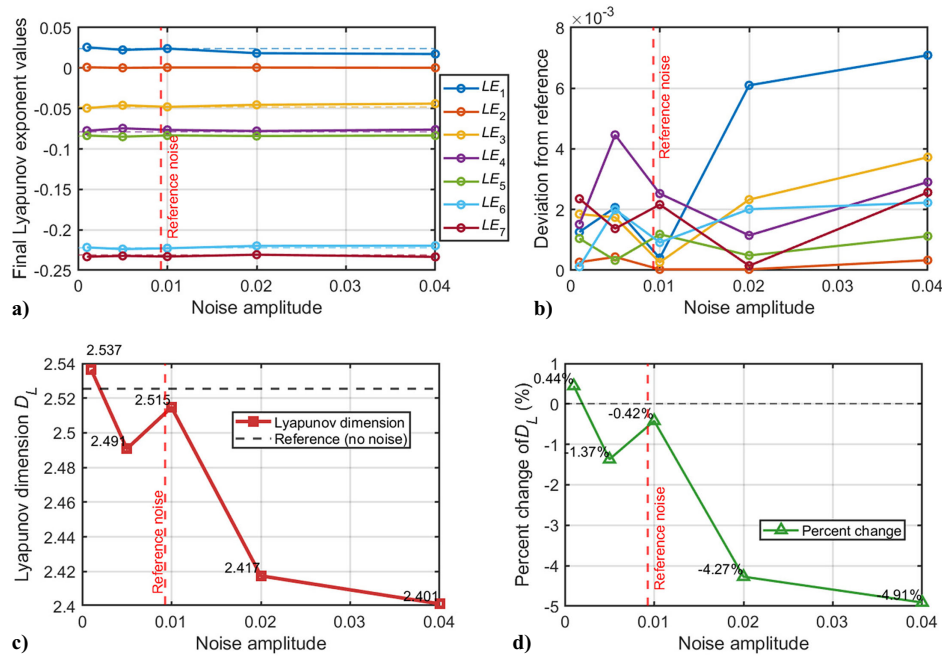


Fig. 16 Noise sensitivity analysis at $U_r = 6.5$: a) Lyapunov exponents vs noise level, b) deviation from reference, c) sensitivity, and d) percentage change of Lyapunov dimension.

level rises. Figure 16b more clearly demonstrates the magnitude relationship of these deviations: when the noise level reaches its maximum value of 0.04, the variation in the largest Lyapunov exponent is approximately 0.007. Notably, despite these deviations, the Lyapunov exponents remain on the order of 10^{-2} to 10^{-1} , whereas the deviations are merely on the order of 10^{-3} . This indicates that noise does not affect the chaos identification criterion based on positive values of the largest Lyapunov exponent.

To further quantify the impact of noise on the system's chaotic characteristics, Figs. 16c and 16d analyze the sensitivity of the Lyapunov dimension D_L to noise. The numerical results reveal that the introduction of noise causes a slight decrease in the Lyapunov

dimension. When the noise amplitude reaches 0.04, the maximum deviation of the Lyapunov dimension is -4.91% , which still falls within the acceptable range of 5%. This result demonstrates that, despite environmental noise interference, the conclusion of the existence of a strange attractor based on the Lyapunov dimension remains robust and reliable. Based on the aforementioned analysis, it can be concluded that the Lyapunov exponents' calculation method employed exhibits good robustness against environmental noise, and the computational results possess high credibility.

Figure 17 shows the evolution of the system's time-domain responses, phase-space trajectory, and frequency-domain responses as U_r increases, from before B3 to after B4. Before B3

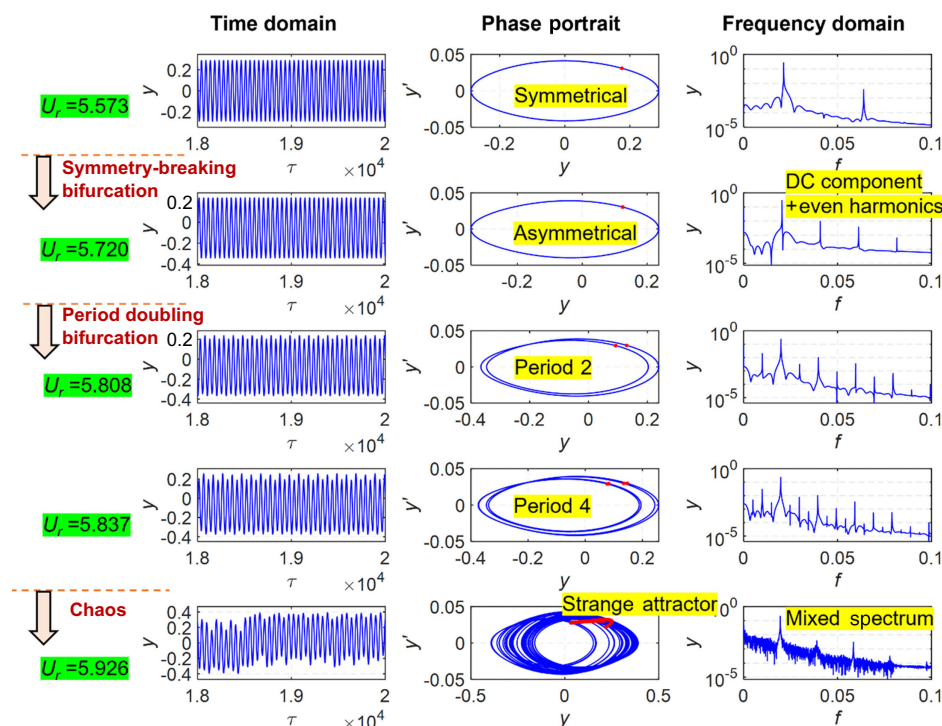


Fig. 17 Time-domain, phase-space trajectory, and frequency-domain responses of the plunging displacement y evolution as U_r increases.

(a symmetry-breaking bifurcation), the system's time-domain response exhibits a nearly sinusoidal waveform, with the phase trajectory forming a single elliptical orbit centered at the origin. The frequency-domain spectrum contains only the first and third harmonics. Upon passing through B3, the response symmetry is disrupted: the time-domain waveform shifts away from the origin, the phase trajectory no longer centers at the origin, and the frequency-domain spectrum displays not only odd harmonics but also a direct current (DC) component and even harmonics, indicating the breakdown of response symmetry. As U_r further increases, the system enters a period-doubling bifurcation pathway. At this stage, the system exhibits period-2 and period-4 motions, signifying the occurrence of period-doubling bifurcations. When U_r continues to increase, the system transitions into a chaotic state. The chaotic response frequency spectrum is dominated by a continuous component, with discrete spectral components (odd harmonics) superimposed. The discrete spectral lines correspond to periodic oscillations not entirely obscured by chaos, resulting from the system approaching a bifurcation point. As shown in Fig. 11d, the Lyapunov dimension at this location is close to two, indicating that the system's response is closer to weak chaos, thereby allowing the periodic harmonics to remain clearly discernible. As the wind speed increases, the Lyapunov dimension becomes higher, and these discrete spectral lines are gradually overwhelmed and obscured by the chaotic components.

The Poincaré section bifurcation diagram in Fig. 18 shows the cascade of period-doubling bifurcations within the reduced wind-speed range of $U_r \in [5.79, 5.849]$. When the reduced wind speed U_r exceeds the first bifurcation point $U_{r1} = 5.79215$, the system undergoes its first period-doubling bifurcation (period 1 to period 2, $1 \rightarrow 2$), where the period-1 solution loses stability and bifurcates into the period-2 solution. As U_r further increases, the system successively undergoes the second, third, and fourth period-doubling bifurcations at $U_{r2} = 5.83375$, $U_{r3} = 5.8430$, and $U_{r4} = 5.8450$, respectively ($2 \rightarrow 4 \rightarrow 8 \rightarrow 16$). Extrapolating based on the Feigenbaum constant of $\delta \approx 4.669$, the accumulation point (onset of chaos) of the period-doubling cascade is estimated at $U_{r,\infty} \approx 5.8455$. When $U_r > U_{r,\infty}$, the system enters a chaotic state, manifested as a continuous distribution of points on the Poincaré section. The computed ratios of successive bifurcation intervals are $\delta_1 = (U_{r2} - U_{r1})/(U_{r3} - U_{r2}) \approx 4.50$ and $\delta_2 = (U_{r3} - U_{r2})/(U_{r4} - U_{r3}) \approx 4.63$, which are in excellent agreement with the theoretical Feigenbaum constant of $\delta = 4.669, \dots$, and no truncation is observed. This indicates that the observed period-doubling cascade belongs to a complete Feigenbaum-type bifurcation sequence rather than an incomplete cascade truncated by global bifurcations (e.g., basin boundary crises). This result confirms that the system transitions into chaos via the period-doubling bifurcation route, which is consistent with the classic Feigenbaum scenario.

This sequence reveals the fundamental mechanism linking unstable periodic orbits to chaos emergence. The symmetry-breaking bifurcation at B3 first destabilizes the symmetric periodic solutions, creating unstable orbits within the attractor. The subsequent period-doubling cascade generates an accumulating sequence of unstable periodic orbits with periods 2^n , with each corresponding to a newly

destabilized solution. When this cascade terminates, these accumulated unstable periodic orbits become embedded within the chaotic attractor, organizing the irregular dynamics. The chaotic trajectories exhibit transient behaviors that shadow these unstable cycles before transitioning between them, creating the nonrepeating motion observed in the tristable FEH. This explains why the interval of unstable periodic solutions (B3–B4) identified through Floquet analysis corresponds precisely to the chaotic regime detected via Lyapunov exponent analysis.

To elucidate the mechanism underlying the evolution of chaos itself, Fig. 19 presents the evolution of phase trajectories at three typical wind speeds. The dynamical state at $U_r = 5.88$, shortly after the onset of chaos, is shown in Fig. 19a. The phase trajectory is confined within the central potential well (no persistent interwell motion occurs), demonstrating that chaos is localized in its initial stage, originating from the accumulation of a local period-doubling bifurcation sequence, and has not yet established a global connection with the saddle points at the potential well boundaries. As the wind speed increases to $U_r = 6.05$ (Fig. 19b), the chaotic attractor expands and begins to interact with the unstable manifolds of the potential barriers (saddle points), featuring intermittent interwell jumps. At this point, the chaotic dynamics have transitioned from local to global, suggesting the involvement of global topological mechanisms such as homoclinic tangles, but this evolution occurs after the initial onset of chaos. When the wind speed further increases to $U_r = 6.3$ (Fig. 19c), global chaos coexists with a large-amplitude LCO spanning three potential wells. At this stage, the system has fully exhibited its global dynamical properties, yet the mechanism inducing chaos must still be traced back to the earlier local bifurcation of periodic orbits.

Figure 20 provides a detailed characterization of the dynamic evolution of the basins of attraction for the tristable FEH during the aforementioned evolutionary process. When constructing the basins of attraction, only the initial y (range: -1.5 to 1.5) and initial y' (range: -1 to 1) are considered as exhaustive initial conditions, whereas the initial values of all other state variables are set to zero. The classification and color coding of the attractors in the figure are as follows: green (intrawell periodic oscillation), blue (chaos), and red (interwell LCO).

The specific evolutionary process is as follows: for $U_r = 5.70$, the system response converges to intrawell periodic oscillation (green region) for all initial conditions. For $U_r = 5.845$, chaos (blue region) appears, but it occupies only a negligible portion of the initial condition space. For $U_r = 5.97$, the system response transitions fully into chaos, with all initial conditions converging to the chaotic attractor. As U_r continues to increase, interwell LCO (red region) begins to emerge, and its basin of attraction area proportion significantly expands with increasing U_r . At $U_r = 6.70$, the basin area for interwell LCO approaches 100%. A quantitative analysis of basin area proportions shows that, after the period-doubling bifurcation, the proportion of initial conditions leading to chaotic responses rapidly increases from a small fraction to 100%. Subsequently, following the saddle-node bifurcation, a subset of initial conditions begins to converge to interwell LCOs. The basin area of interwell LCO gradually displaces the chaotic basin area, eventually dominating the entire initial condition space.

The physical implications of these basin boundary evolutions reveal fundamental mechanisms governing the tristable FEH behavior. The dramatic transformation from uniform intrawell basins (100% coverage at $U_r = 5.70$) to chaos-dominated regions occurs because increased aerodynamic forcing destabilizes symmetric oscillations through period-doubling bifurcations. The subsequent emergence of the interwell LCO basins reflects the system's enhanced ability to overcome potential energy barriers as wind speed increases. This basin competition demonstrates a critical tradeoff; whereas chaos provides broad attraction across initial conditions, interwell LCOs offer superior power output, making their gradual dominance highly beneficial for energy harvesting performance.

The geometric characteristics exhibit remarkable complexity, reflecting the underlying nonlinear dynamics. Initially, smooth

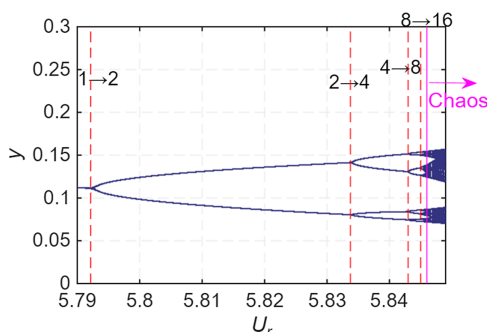


Fig. 18 Enlarged view of the period-doubling bifurcation cascade process.

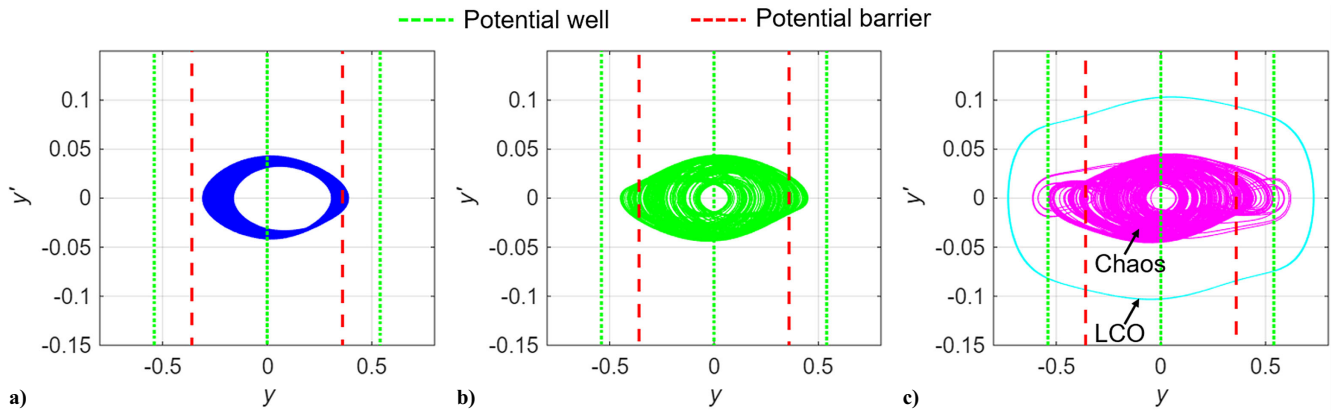


Fig. 19 Evolution of phase trajectories illustrating the transition from local to global chaotic dynamics: a) $U_r = 5.88$, b) $U_r = 6.05$, and c) $U_r = 6.3$.

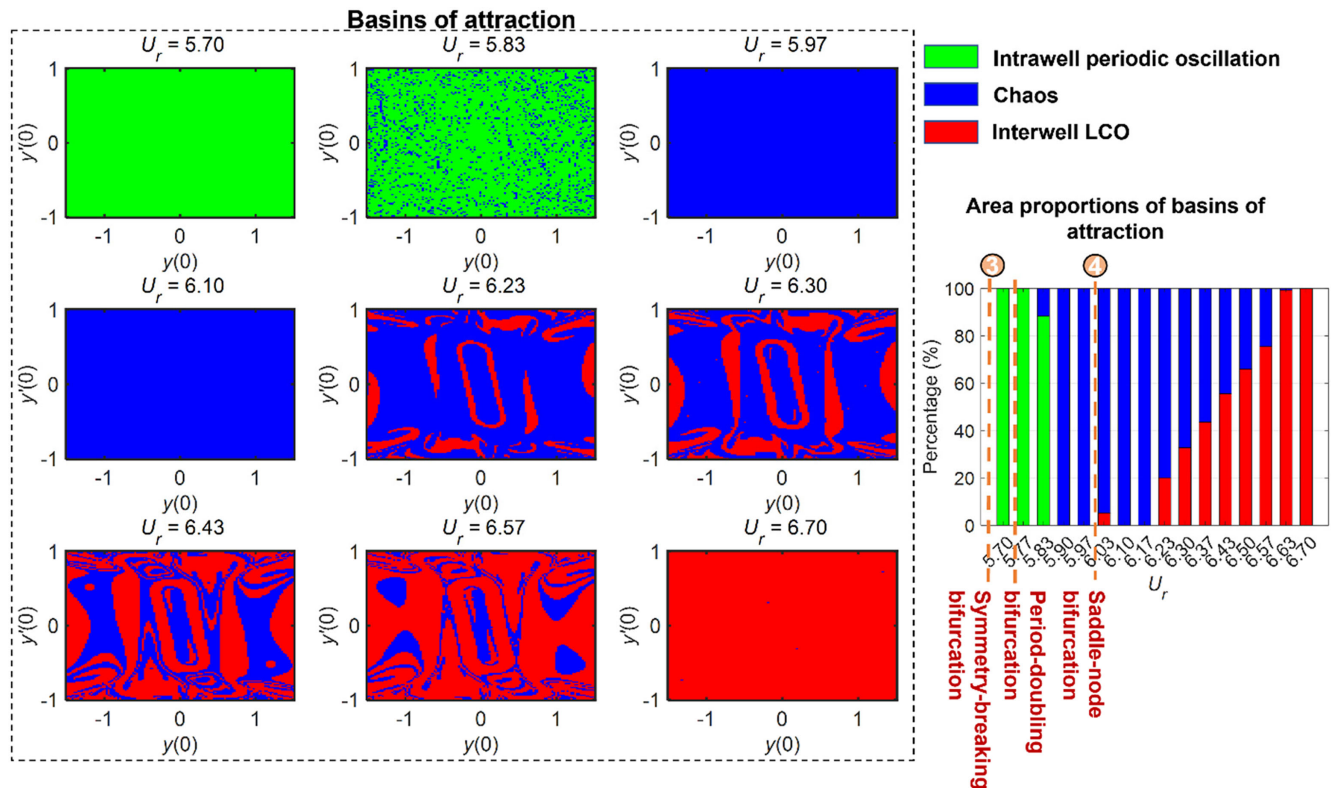


Fig. 20 Evolution of basins of attraction and their area proportions for three types of attractors in the tristable FEH.

basin boundaries evolve into fractal structures as chaos emerges ($U_r = 5.845$), with blue chaotic regions forming scattered islands possessing intricate coastlines. The coexistence regime ($U_r = 6.23$ – 6.57) reveals complex topological features where red interwell basins appear as elongated, treelike structures interweaving with blue chaotic domains. These fractal-like boundaries indicate that small changes in initial conditions can lead to dramatically different behaviors, with theoretically infinite boundary complexity at fine scales. The eventual interwell dominance occurs through a “basin invasion” process where high-energy attractors systematically conquer the phase space.

From an engineering perspective, these basin patterns have significant implications for the FEH design and operation. The fractal boundaries near critical wind speeds mean that manufacturing tolerances or environmental perturbations could unpredictably switch between high-performance interwell LCOs and lower-output chaotic responses. However, at higher wind speeds where interwell basins dominate, the system becomes robust to initialization uncertainties, achieving reliable performance, regardless of starting conditions. This suggests optimal operating strategies should target

wind ranges where beneficial attractors possess large basin coverage while avoiding regions with complex fractal boundaries that create performance sensitivity.

It should be noted that these conclusions are derived under rather ideal conditions, neglecting the effects of environmental noise and disturbances. However, environmental noise can influence the system’s response. For instance, Holness et al. [39] studied galloping energy harvesters under bounded random excitation. It was demonstrated that random excitation could lead to multiple solutions within certain parameter ranges and might cause the system to transition from a stable state to an unstable state. Consequently, random excitation and noise perturbations might alter the types of attractors, their stability, and the transitions between them. These phenomena are of significant research interest in dynamical systems theory, particularly for analyzing complex system behaviors.

In the context of this paper, the specific impact depends on the system’s characteristics and the nature of the excitation, necessitating a detailed analysis. The influence of the noise on basin stability or transitions between attractors, and the extent of such influence, should consider various factors, including the actual installation

environment and manufacturing precision. This may even require the incorporation of random excitation analysis [40]. Various research methods would suit the purpose. For example, using the stochastic averaging method to analyze the system's response under random excitation and perturbations would allow for the observation of the types and stability of attractors; employing the moment method and Monte Carlo simulation to numerically solve the system's steady-state probability density function helps verify theoretical results and explore dynamic behaviors under complex random excitations and perturbations; and calculating Lyapunov exponents would enable analysis of the stability of the system's periodic responses under random excitation and perturbations. This represents an important direction for future in-depth research.

To accurately evaluate and compare the energy harvesting performance under different response states, it is first necessary to determine the reliability of the average power calculation. Figure 21 analyzes the sensitivity of the average power calculation to the choice of time window at $U_r = 6.3$. Figure 21a displays the instantaneous power time series, characterized by chaotic fluctuations within a bounded range. Although sporadic high-power peaks occur (e.g., near $\tau = 3600$ and 4400), their contribution to the long-term average is minimal, as evidenced by the converging mean power shown in Fig. 21b. This convergence is quantitatively supported by Fig. 21c. For a given window size, different window positions yield varying average power values due to the chaotic fluctuations; the coefficient of variation quantifies these positional variations. The coefficient of variation is calculated by dividing the long-term steady-state data into multiple equal-length time windows, computing the average power for each window, and then determining the

ratio of the standard deviation of these window averages to their mean. A smaller coefficient of variation indicates a reduced sensitivity of the average power to the choice of time window, and therefore a more stable estimation. The coefficient of variation curve shows that once the time window exceeds 1000 dimensionless time units, estimation uncertainty drops sharply from 44.6% to below 4%. Collectively, these results confirm that for this chaotic response, an averaging window exceeding 1000 time units provides a stable and reliable power estimate, forming a solid basis for subsequent performance comparisons.

Based on the preceding analysis, Fig. 22 further compares the average output power (five sets of data) for two coexisting response states at the same wind speed: chaotic response and interwell limit cycle oscillation (interwell LCO). In this wind-speed range, interwell LCO can generate around $100 \mu\text{W}$, which is sufficient to operate many low-power sensors such as nanowatt-level temperature sensors, Hall effect sensors, infrared sensors, etc. Potential applications include geological hazard monitoring and meteorological data collection in mountainous regions, soil temperature and moisture monitoring, pest and disease warning, and irrigation system status detection. However, in the chaotic state, power output drops to approximately $15 \mu\text{W}$. Figure 22b clearly shows that under the same wind speed, the power output in chaotic state reaches at most 20% of that in interwell LCO, thus failing to meet the basic power requirements for the aforementioned applications.

One of the primary reasons for the reduction in power output lies in the intermittent and irregular nature of the interwell motion in chaotic response. As illustrated by the phase trajectory comparison in Fig. 19c, under the same wind speeds, the interwell LCO forms a

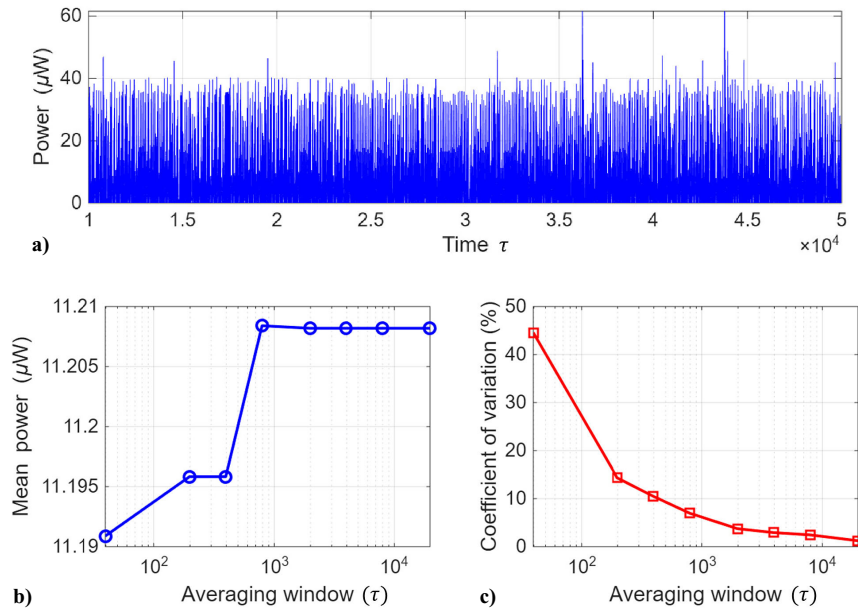


Fig. 21 Sensitivity of average power to averaging window: a) instantaneous power, b) average power, and c) coefficient of variation.

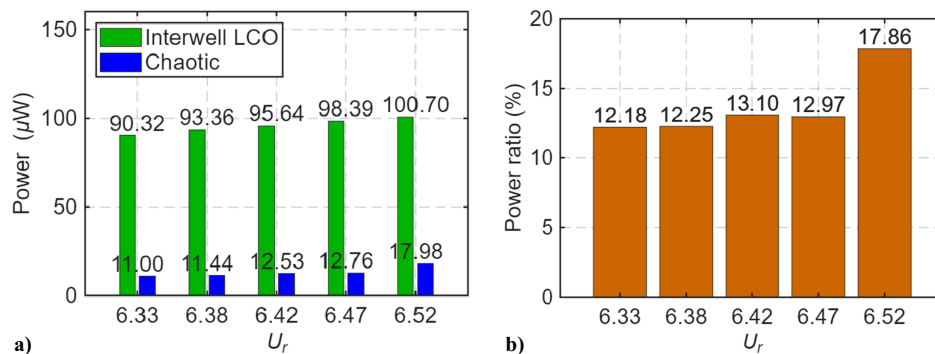


Fig. 22 Power comparison with different response states at same U_r : a) output power, and b) chaos-to-interwell LCO power ratio.

stable large-amplitude closed orbit that can continuously and periodically traverse the three potential wells. In contrast, although the chaotic response can also cross the three wells, its motion trajectory follows a randomly distributed pattern and cannot sustain a stable state of maximum amplitude. This intermittent interwell behavior prevents the system from continuously capturing the energy input provided by the aerodynamic negative damping. Furthermore, the maximum displacement amplitudes of the chaotic response are slightly smaller than those of the limit cycle oscillation. Additionally, the interwell LCO constitutes a high-frequency stable output motion, and its velocity amplitude is significantly higher than that of the chaotic response. The reduction in velocity amplitude further diminishes the power output. In the chaotic state, the phase relationship between the lift force and the vibration displacement is disordered, unlike the LCO, which can maintain a constant aerodynamic negative damping to effectively counteract the system's dissipation. The combination of these factors leads to a significant decrease in energy transfer efficiency.

For energy harvesting, if the periodic solution becomes unstable and transitions to chaos, the system may exhibit large-amplitude responses crossing potential wells. However, compared to interwell LCOs, chaotic responses yield significantly lower average power output (detailed in Fig. 22, showing at least 80% lower power) while posing similar risks of structural damage due to large amplitudes. Furthermore, the fluctuating nature of the chaos presents challenges for circuit design and system reliability. Consequently, chaos should be avoided due to its poor power output and adverse effects on system reliability. Although interwell LCOs also carry a structural risk (potentially leading to system instability upon failure due to large amplitude), their substantially higher power output offers clear energy benefits. Whereas current theoretical research prioritizes performance enhancement, practical implementation should consider other practical issues such as ensuring stable large-amplitude responses while avoiding structural damage. Strategies for achieving this balance would include incorporating magnetic limiters to constrain oscillation amplitudes or targeting lower wind speeds (where amplitude increases with wind). This requires a thorough consideration of the tristable FEH's operating environment and load requirements.

Triggering stable interwell LCOs in the wind-speed ranges where both interwell LCOs and chaotic responses coexist remains a significant challenge. In this study, interwell LCO represents a high-energy output state as compared to the chaotic response. Common approaches to address such issues include the deployment of impact mechanisms [41], which inject impulsive energy into the system to drive it toward a high-energy state. Additionally, control methods have been employed to suppress chaotic responses. For instance, Huynh et al. [42] used an Ott–Grebogi–Yorke (OGY) controller to manage chaotic responses in a vortex-induced vibration energy harvester, achieving a transition from chaotic to desired periodic vibrational responses. They demonstrated that the OGY control algorithm only requires minor perturbations in a structural parameter to implement the control effort. Furthermore, adaptive sliding mode control [43] is also a potential design to shift the system's motion trajectory from a low-energy orbit to a high-energy orbit. However, given the strong nonlinearity of the tristable FEH and the complexity of the fluid dynamics in this study, the applicability of these methods to the present system requires further in-depth investigation.

Before addressing the issue of triggering high-branch oscillations, a key question needs to be answered: whether there exist design parameters for tristable FEH systems that can eliminate wind-speed ranges where only chaos exists, thus ensuring at least the coexistence of high-output states with chaotic states as a prerequisite for triggering high-output performance. We address this issue in the following section.

IV. Effect of Nonlinear Stiffness on Bifurcation

Based on the analyses and discussions in the previous section, we can directly predict the emergence of chaotic attractors by observing

the location of B3 and interwell LCO attractors by observing B4. B1 and B2 mark the onset of vibration in the system. In this section, we will use the HB method again to perform a parametric analysis to determine the locations of B3 and B4, thereby identifying parameter regimes with smaller or negligible wind-speed ranges for purely chaotic responses. Furthermore, we must ensure that the wind speeds corresponding to B1 and B2 (the onset of vibration) are not excessively high in order to guarantee the output at low wind speeds. These two bifurcation points will be located by directly analyzing the eigenvalues of the Jacobian matrix. We first discuss the influence of nonlinear stiffness coefficients in the plunging direction. Note that $\bar{k}_{h1}$ is the linear stiffness coefficient, which primarily affects the effective stiffness near $y \approx 0$, with this influence being extensively investigated in academic research [44]. Also, $\bar{k}_{h5}$ has a stronger impact on stiffness at larger y values, whereas changes in $\bar{k}_{h3}$ significantly influence the dynamic characteristics near potential wells and potential barriers. This study will focus on analyzing the effects of $\bar{k}_{h3}$.

Figure 23 illustrates the influence of $\bar{k}_{h3}$ on the tristable FEH. From Figs. 23a and 23b, it is evident that an increase in the absolute value of $\bar{k}_{h3}$ deepens the three potential wells. Additionally, it reduces the distance between the potential wells. When the absolute value of $\bar{k}_{h3}$ is relatively small (less than 3.93), the system transitions from a tristable state to a monostable state. Figure 23c depicts the variation of bifurcation points with $\bar{k}_{h3}$. It is worth noting that an additional curve for bifurcation point 5 (B5) is included in the figure. This bifurcation point corresponds to the stable LCO emerging from a Hopf bifurcation in the lateral potential wells. If the LCO is stable (supercritical Hopf bifurcation, occurring when $\bar{k}_{h3}$ ranges from -3.93 to -4.15), B5 coincides with B1. Conversely, if the LCO is unstable (subcritical Hopf bifurcation, occurring when the absolute value of $\bar{k}_{h3}$ exceeds 4.15), B5 represents the saddle-node bifurcation where the stable LCO appears on the periodic solution branch.

The position of B2 remains unaffected by $\bar{k}_{h3}$ because the Hopf bifurcation within the central potential well is solely influenced by $\bar{k}_{h1}$. As the absolute value of $\bar{k}_{h3}$ decreases, the distance between B3 and B4 diminishes, indicating a reduction in the purely chaotic range, where only chaotic attractors exist. However, this trend is not particularly pronounced: as $\bar{k}_{h3}$ varies from -4.6 to -4 , the U_r corresponding to B3 changes only slightly from 5.5 to 5.51 , whereas the U_r for B4 shifts from 6.2 to 6.0 . Nevertheless, when $\bar{k}_{h3}$ becomes too small, the U_r for B5 increases, implying a higher required wind speed for the system to transition from a fixed point in the lateral potential well to an LCO. This behavior is unfavorable for energy harvesting at low wind speeds.

We now discuss the influence of the nonlinear stiffness $\bar{k}_{a3}$ in the pitching direction on the tristable FEH, as illustrated in Fig. 24. The pitching motion of the tristable FEH exhibits only monostable characteristics, as shown in Figs. 24a and 24b. A larger $\bar{k}_{a3}$ results in greater stiffness for the pitching motion, which is indicative of a hardening behavior. From Fig. 24c, it can be observed that $\bar{k}_{a3}$ has little effect on U_r corresponding to the two Hopf bifurcation points. However, it significantly influences B3 and B4. As $\bar{k}_{a3}$ decreases, U_r for both B3 and B4 decreases. Notably, when $\bar{k}_{a3} < 0.9$, the U_r for B4 becomes smaller than that for B3, suggesting that large-amplitude LCOs will emerge before the appearance of chaotic attractors. Consequently, there exists no pure chaotic range theoretically (although coexistence of chaotic attractors and large-amplitude LCOs may occur), which is a desirable parameter regime for energy harvesting. To examine the system's output response when $\bar{k}_{a3} = 0$, the numerical solution of the system using the Runge–Kutta method is shown in Fig. 25.

To illustrate the response of the tristable FEH under different initial conditions, two distinct initial values of $y(0)$ are chosen: $y(0) = 0.01$ and $y(0) = 1$. The initial values of all other state variables are set to zero. When $y(0) = 0.01$, the RMS value of the dimensionless voltage $\bar{v}$ exhibits a response trend similar to that shown in Fig. 25a, featuring intrawell oscillations, chaotic

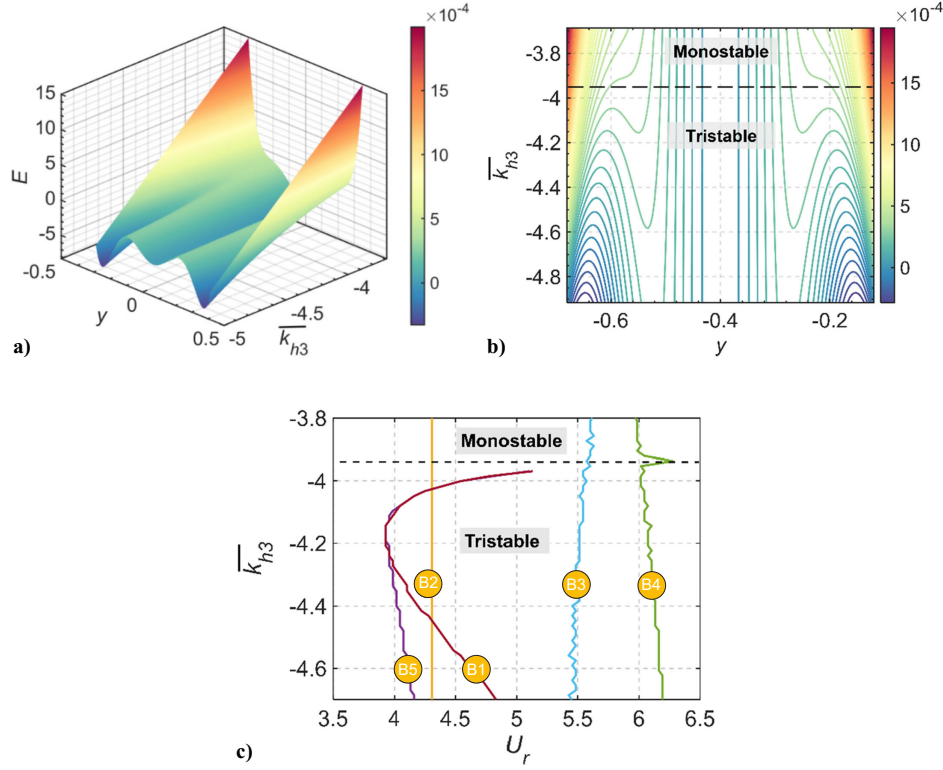


Fig. 23 Effect of $\overline{k}_{h3}$ on tristable FEH: a) surface of potential energy vs y and $\overline{k}_{h3}$ ($\alpha = 0$), b) contour of potential energy, and c) U_r at bifurcation points vs $\overline{k}_{h3}$.

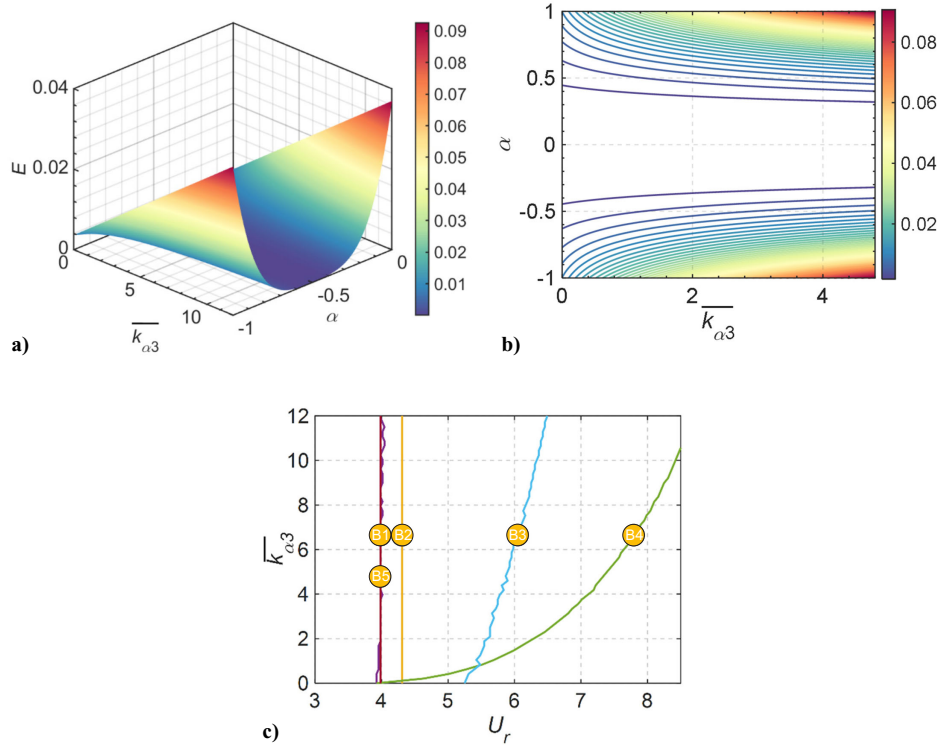


Fig. 24 Effect of $\overline{k}_{a3}$ on tristable FEH: a) surface of potential energy vs α and $\overline{k}_{a3}$ ($y = 0$), b) contour of potential energy, and c) U_r at bifurcation points vs $\overline{k}_{a3}$.

responses, and large-amplitude interwell LCOs. In contrast, when $y(0) = 1$, the system directly transitions to a high-output response state characterized by large-amplitude LCOs from $U_r = 4$, bypassing intrawell periodic oscillations and chaotic responses. These results validate the conclusions drawn from Fig. 24: a smaller $\overline{k}_{a3}$ can significantly advance the onset of large-amplitude LCOs, even

causing their disappearance before the Hopf bifurcation points. In other words, a smaller $\overline{k}_{a3}$ not only ensures that the wind-speed range for large-amplitude LCOs overlaps with or entirely supercedes the ranges for chaotic responses and small-amplitude intrawell oscillations, but it also enhances the output performance at both low and medium wind speeds. Although different initial conditions may

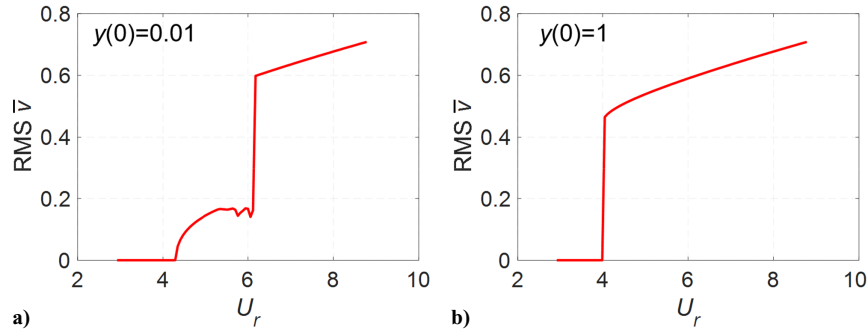


Fig. 25 RMS values of the dimensionless voltage $\bar{v}$ for the tristable FEH when $\bar{k}_{a3} = 0$: a) initial value of y set to 0.01, and b) initial value of y set to one.

still lead to these low-output states, the aforementioned findings provide a design guideline: reducing $\bar{k}_{a3}$ can be an effective strategy for conceiving high-performance FEHs.

V. Discussion and Design Guidelines

Based on the parametric analysis, several practical design guidelines emerge for tristable FEH development.

1) For the nonlinear stiffness design strategy, one should prioritize reducing the cubic pitching stiffness coefficient to $\bar{k}_{a3}$, which effectively eliminates pure chaotic regions and enables direct transition to high-output interwell LCOs. Optimal performance can be achieved when $\bar{k}_{a3}$ approaches zero, allowing large-amplitude responses to emerge at relatively low wind speeds around $U_r \approx 4$.

2) The cubic plunging stiffness should be properly tuned, maintaining $|\bar{k}_{h3}| > 3.93$ to preserve the essential tristable characteristics, keeping $|\bar{k}_{h3}| \approx 4.2$ to provide an effective balance between the potential well depth and transition thresholds, and using $|\bar{k}_{h3}| > 4.15$ to achieve subcritical Hopf bifurcations for lower cut-in wind speeds.

3) For initial condition sensitivity management, practical implementation requires specific design strategies to ensure the system accesses high-energy states rather than remaining trapped in low-output configurations. This can be achieved through mechanical reset mechanisms that provide large initial displacements, impact-based triggering systems, or controlled perturbation devices that momentarily drive the system away from central equilibrium toward the lateral potential wells. Such initialization strategies enable the system to bypass undesirable chaotic or small-amplitude intrawell responses and directly access the high-performance interwell oscillation regime that maximizes energy harvesting efficiency.

This study employs the two-dimensional incompressible unsteady Theodorsen aerodynamic theory [35,45]. This model does not account for three-dimensional flow effects such as finite aspect ratio and wingtip vortices, and it assumes the flow remains attached, neglecting the influence of dynamic stall. Rather than conducting numerical simulations incorporating the aforementioned aerodynamic corrections, this work attempts to elucidate the underlying physical mechanism perspective and the potential impact of complex aerodynamic effects.

More specifically, three-dimensional aerodynamic effects primarily manifest as quantitative corrections to the lift curve slope and aerodynamic damping coefficients. For a finite-aspect-ratio airfoil section, wingtip vortices lead to a significant reduction in the lift curve slope as compared to the two-dimensional theoretical values, with the typical reduction increasing as the aspect ratio decreases. Concurrently, three-dimensionality-induced drag and wingtip vortex dissipation contribute to aerodynamic damping [46]. These corrections directly alter the amplitude of the linear aerodynamic forces and the overall system damping level, which in the model are collectively reflected as numerical shifts in the bifurcation critical wind speeds and LCO amplitudes [47]. Dynamic stall [48,49] primarily affects the large-amplitude interwell oscillation stage. Its vortex shedding time lag and nonlinear damping characteristics may alter the stability boundaries of LCOs. However, this effect is not activated under experimental conditions where the

angle of attack does not deeply enter the stall regime, and its influence on the bifurcation topology similarly manifests as a shift in parameter thresholds rather than a reconfiguration of the sequence.

The core dynamic mechanism revealed in this paper (namely, the symmetry-breaking and period-doubling route to chaos induced by the central potential well) is dominated by the nonlinearity in stiffness and the inherent phase lag in fluid–structure interaction. It is not fundamentally linked to the specific dimensionality of the aerodynamic model or the stall state. To verify the robustness of the conclusions reached in this work, a structural parameter sensitivity analysis is conducted. By varying the mass ratio and the damping coefficient within a range of 0.8–1.2 times their baseline values, the results show that the topological features such as period-doubling bifurcation sequence and symmetry-breaking bifurcation behavior remain persistent and stable, with slight shifting of only specific numerical values like the critical wind speed.

The two-dimensional Theodorsen model is a classical framework for airfoil flutter analysis, and its physical rationality and capability to capture the salient dynamic phenomena are widely recognized. A rigorous correction for three-dimensional effects involves a series of complex issues, including finite-aspect-ratio lifting surface theory, coupling with spanwise bending modes, and three-dimensional wake modeling, etc. Accurate simulation of dynamic stall relies on semiempirical state-space models or CFD-based reduced-order methods, both requiring substantial and independent research endeavors. Therefore, the precise implementation and correction of aerodynamic forces are not pursued in this paper. Instead, while acknowledging this limitation, it focuses on revealing the mechanistic pathway of chaotic evolution in the tristable FEH. This core finding retains a fundamental reference value for refined models that would incorporate complex aerodynamic effects.

VI. Conclusions

This paper investigates a tristable flutter-based energy harvester to elucidate the chaotic response characteristics of strongly nonlinear piezoelectric flutter systems. The harmonic balance method is employed to derive approximate analytical solutions, whereas Lyapunov's first method is applied to determine equilibrium points' stability. Through bifurcation theory, Lyapunov exponents, and numerical simulations, the evolutionary pathways and the underlying physical mechanism from periodic oscillations to chaos are revealed. The results demonstrate that, as wind speed increases, both lateral wells ($U_r = 3.96$) and the central well ($U_r = 4.28$) generate local limit cycle oscillations (LCOs) via Hopf bifurcations. When $U_r > 5.4$, symmetry-breaking bifurcations trigger period-doubling routes, ultimately forming chaotic attractors with a one-positive one-zero multiple-negative Lyapunov exponent spectrum in the $U_r = 5.8$ – 6.5 range. The nonrepeatability of phase-space trajectories and the broadband spectral features validate chaotic behavior. Parametric analysis shows that reducing cubic stiffness in the pitching direction $\bar{k}_{a3}$ significantly lowers the critical wind speed for large-amplitude interwell LCOs ($U_r < 4.0$). By weakening $\bar{k}_{a3}$ (< 0.9), chaotic responses can be suppressed with an initial dimensionless displacement of $y = 1$, enabling a direct transition

from Hopf bifurcations to high-output LCO states, thereby mitigating the adverse effects of chaos on energy harvesting efficiency.

This study offers guidance for optimizing nonlinear aeroelastic energy harvesters while maintaining stability–performance balance. The identified bifurcation mechanisms and chaos pathways advance flutter suppression theory for assessing instability thresholds in strongly nonlinear systems.

Acknowledgments

This work was supported by the National Key Research and Development Program of China (grant number 2022YFB2603200), the National Natural Science Foundation of China (grant number 12472023), and the Innovation Foundation for Doctor Dissertation of Northwestern Polytechnical University (grant number CX2024002). The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this study.

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