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A unified theoretical modeling framework for dynamic behaviors of grillage structures using the dynamic stiffness method

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ABSTRACT

A unified theoretical modeling framework based on the dynamic stiffness method (DSM) is proposed for analyzing dynamic behaviors of grillage structures. The unification of the proposed framework is reflected in two aspects: arbitrary grillage configurations and boundary conditions can be treated using the same multi-plate dynamic stiffness formulation, coordinate transformation, and matrix assembly procedure, while free vibration, forced vibration, wave propagation, and displacement field reconstruction are derived and implemented within the same theoretical framework. The grillage structures are modeled as coupled multi-plate systems, where plate-plate coupling and the associated in-plane/out-of-plane coupling are achieved through displacement continuity and interface force/moment balance along joint edges, together with coordinate transformations. A zero-crossing method is developed to extract the eigenfrequencies from the frequency-dependent dynamic stiffness matrix. The proposed framework is applied to five grillage configurations, including two stiffened grillage structures and three sandwich grillage structures, to investigate natural frequencies, transmission characteristics, and band structures. Furthermore, a displacement field reconstruction method is proposed to systematically recover the mode shapes, the total displacement fields of the eigenmodes, and the forced vibration responses. Comparisons with finite element results and published experimental data reported in the literature verify the accuracy and applicability of the proposed framework. Convergence and computational efficiency studies are also conducted to provide practical guidance for selecting the DSM truncation order. The results show that the proposed framework achieves the prescribed accuracy with significantly reduced computational cost compared with the finite element method.

1. Introduction

Grillage structures are extensively used in marine, aerospace, and civil engineering due to their high stiffness-to-weight ratio and excellent load-bearing performance [1–7]. Despite their diverse configurations, these structures share a common feature: they are composed of multiple plate components that are mechanically connected, leading to strong interactions between adjacent plates [8–10]. As a result, their dynamic responses cannot be interpreted by considering each plate in isolation, but must instead be understood in terms of the coupled dynamic

behavior of the plate assemblies. Grillage structures are inevitably subjected to vibration and noise excitations during service. Under such excitations, the strong coupling between plate components may induce resonance, modal interaction, vibration transmission, and wave propagation through the connected structural paths. Therefore, evaluating the dynamic behavior of grillage structures from both vibration and wave propagation perspectives is important for understanding how dynamic energy is transmitted, redistributed, attenuated, or suppressed in coupled plate systems [11–16].

To investigate the dynamic behavior of grillage structures, a variety of numerical and theoretical approaches have been developed. Among

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Nomenclature

a	Half-length of the rectangular plate in the x -direction	M_i	Bending moment on boundary i
a_1, a_2	Coefficients determined by Poisson's ratio in the in-plane governing equations	N_x, N_y, N_{xy}	In-plane force resultants
b	Half-width of the rectangular plate in the y -direction	$\mathbf{0}$	Zero matrix or zero vector
$\mathbf{C}_{ab}$	Unknown coefficient vector of the ab symmetry component	$\mathbf{q}, \mathbf{Q}$	Generalized displacement and force vectors
c_p	Longitudinal wave velocity	$\mathbf{q}_b, \mathbf{q}_{\text{Model I}}$	Displacement vector associated with the i -th edge and full displacement vector of the Model I unit cell
D	Bending stiffness of the plate	$\mathbf{q}_E, \mathbf{q}_I$	Displacement vectors of external and internal edges
$\mathbf{D}_i$	Displacement coefficient matrix of the i -th plate element	$\mathbf{q}_G, \mathbf{q}_S, \mathbf{q}_{E-S}$	Global edge displacement vector and reduced displacement vectors after Bloch transformation or condensation
$\mathbf{\bar{D}}_{ab}$	Projection displacement matrices associated with the four symmetry components	$\mathbf{q}_i^l$	Local edge displacement vector of the i -th plate element
E	Young's modulus	$\mathbf{T}$	Coordinate transformation matrix
F, S, C	Free, simply supported and clamped boundary conditions, respectively	$\mathbf{T}_{ix}, \mathbf{T}_{iy}, \mathbf{T}_i$	Transformation matrices associated with rotations about the global axes and with the i -th plate element
F_0	Magnitude of the applied concentrated force	$\mathbf{T}_S, \mathbf{T}_{E-S}$	Bloch transformation matrices before and after internal-edge condensation
$\mathbf{\tilde{F}}_{ab}$	Projection force matrices associated with the four symmetry components	Trans	Vibration transmission level
$\mathbf{F}_{\text{total}}$	External force vector applied to the assembled structure	$\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}$	Matrices obtained from singular value decomposition
h	Plate thickness	u, v, w	Displacement components in the x -, y - and z -directions
$\mathbf{H}_{ab}$	Projection function of the ab symmetry component	$\mathbf{u}_i(x, y)$	Reconstructed displacement field of the i -th plate element
$\mathbf{I}$	Identity matrix	V_i	Transverse shear force on boundary i
$\mathbf{K}$	Dynamic stiffness matrix in the local coordinate system	$\mathbf{v}_k$	k -th column of $\mathbf{V}$
$\mathbf{K}^g$	Dynamic stiffness matrix in the global coordinate system	$w_{\text{in}}, w_{\text{out}}$	Displacements at the input and output points
$\mathbf{K}(\omega)$	Frequency-dependent dynamic stiffness matrix	x, y, z	Cartesian coordinate components
$\mathbf{K}_{\text{Model I}}, \mathbf{K}_{\text{Model I-per}}$	Assembled dynamic stiffness matrix of Model I and finite periodic Model I system	$x^l, y^l, z^l; x^g, y^g, z^g$	Local and global coordinate systems
$\tilde{\mathbf{K}}_{ij}^{\text{in}}, \tilde{\mathbf{K}}_{ij}^{\text{tra}}$	In-plane and transverse dynamic stiffness submatrices associated with boundaries i and j	y_F	y -coordinate of the excitation point
$\mathbf{K}_S, \mathbf{K}_E, \mathbf{K}_{E-S}$	Bloch-reduced, external-edge condensed and condensed Bloch-reduced dynamic stiffness matrices	$\Delta\omega$	Initial frequency step
$\mathbf{K}_{EE}, \mathbf{K}_{EI}, \mathbf{K}_{IE}, \mathbf{K}_{II}$	Submatrices associated with external and internal edge partitions	$\varepsilon_1, \varepsilon_2$	Phase constants in the two periodic directions
k_u, k_v, k_w, k_φ	Translational and torsional constraint spring stiffnesses	$\varepsilon_{\omega}, \varepsilon_\lambda$	Prescribed frequency and eigenvalue tolerances
k_u, k_v, k_w, k_φ	Non-dimensional translational and torsional spring stiffnesses	θ	Rotation angle between the local and global coordinate systems
k_x, k_y	Components of the Bloch wave vector in the two periodic directions	$\lambda_i(\omega), \lambda_s(\omega)$	i -th eigenvalue and selected signed eigenvalue with the smallest modulus
L	Length of the rectangular plate	ν	Poisson's ratio
M	Truncation order of the Fourier series	ρ	Mass density
		σ_i	i -th singular value in singular value decomposition
		$\Phi(x, y; \omega)$	Matrix of analytical shape functions
		ω	Angular frequency
		$\omega_n, \omega_b, \omega_L, \omega_R, \omega^*$	Natural angular frequency, sampling frequency, bisection bounds and refined candidate frequency

them, the finite element method (FEM) has been widely adopted owing to its strong modeling flexibility and its capability to handle complex geometries and boundary conditions [11, 17–25]. FEM-based approaches are particularly effective for low and mid frequency vibration analyses and have become a standard tool in engineering practice. However, when addressing high-frequency dynamics, wave propagation phenomena, or repeated parametric analyses, FEM often suffers from prohibitive computational cost and dispersion errors, as increasingly fine meshes are required to maintain accuracy. These limitations motivate the development of efficient theoretical approaches that can provide accurate dynamic predictions for grillage structures with significantly reduced computational effort. For example, Vergote et al. [26] developed a wave based method (WBM) rooted in the indirect Trefftz principle to investigate the forced vibration response of grillage structures, employing wave function expansions and a weighted residual formulation to enforce interface continuity. Li et al. [27] investigated the flexural wave bandgap characteristics of infinite periodic stiffened plates under free boundary conditions based on beam-plate coupling theory and the plane wave expansion method (PWEM). Wu et al. [28] applied the spectral element method (SEM) to the vibration analysis of

sandwich panels with corrugated cores, and derived spectral equations containing in-plane and out-of-plane components to reveal the bandgap behavior of one-directional periodic structures with two opposite simply supported edges. Although these methods provide useful insights into vibration and wave propagation characteristics, their formulations are often tailored to specific structural configurations, boundary conditions, or either vibration response or wave propagation analysis, making it difficult to describe both vibration and wave propagation problems of grillage structures with arbitrary configurations and boundary constraints within a unified framework.

The dynamic stiffness method (DSM) has a frequency dependent matrix formulation and an assembly procedure similar to that used in FEM. These features enable various structural configurations, boundary constraints, and Bloch periodic conditions to be incorporated within the same matrix framework, thereby providing a suitable basis for the unified analysis of vibration and wave propagation behavior in grillage structures. Early studies on the DSM primarily focused on single plate elements. Nefovska-Danilovic et al. [29] and Banerjee et al. [30] derived the exact dynamic stiffness matrices for in-plane and out-of-plane plate elements, respectively, and investigated their free vibration

Table 1

Comparison of the present framework with existing DSM, SEM, WBM, and PWEM formulations.

Method	Theoretical basis	Applicable boundary conditions	Analysis capabilities
DSM[29–40]	Dynamic stiffness	Classical & elastic	Free & forced vibration
SEM[28]	Lévy-type	Two opposite edges SS	Forced vibration; wave propagation
WBM[26]	Trefftz	Classical	Forced vibration
PWEM[27]	Bloch/plane-wave	Bloch periodic	Wave propagation
Present framework	Dynamic stiffness	Classical & elastic	Free & forced vibration; wave propagation; displacement

characteristics. Liu et al. [31,32] integrated the spectral method with the classical DSM to propose a novel spectral dynamic stiffness method (SDSM), incorporating an improved Wittrick-Williams algorithm to efficiently analyze the free vibration of plates. Rai and Gupta [33,34] further developed DSM formulations for the free vibration analysis of functionally graded nanoplates and porous functionally graded nanoplates on elastic foundations. Subsequently, the DSM was extended to periodic plate structures. Jin et al. [35] and Zhou et al. [36] employed the SDSM and the SEM, respectively, to study the forced vibration responses of periodic Kirchhoff and Mindlin plates. For grillage structures, which can be regarded as a class of coupled plate structures, the DSM has also been gradually developed. Liu and Banerjee [37,38] proposed an exact spectral dynamic stiffness method for the free flexural vibration analysis of orthotropic composite plate assemblies. Zhang et al. [39] applied the DSM to three-dimensionally coupled plate structures, in which both flexural and in-plane vibrations were considered, and analyzed their free and forced vibration responses. Zhang et al. [40] further established a dynamic model based on the DSM for periodically coupled plate structures and investigated their harmonic responses and bandgap characteristics. Table 1 summarizes the main features of

existing DSM, SEM, WBM, PWEM, and the present framework, including their theoretical basis, applicable boundary conditions, and analysis capabilities.

Although the aforementioned studies have significantly advanced the application of the DSM to plate elements, periodic plate structures, and grillage structures, most existing DSM studies still mainly focus on free vibration and forced vibration problems. Vibration analysis can accurately evaluate the dynamic response of finite grillage structures under external excitations; however, it is limited by structural boundaries and therefore cannot fully reveal the dispersion relations and bandgap mechanisms inherent to infinitely periodic systems. Therefore, incorporating wave propagation analysis into the same DSM framework is important for comprehensively evaluating the dynamic behavior of grillage structures from both vibration and wave perspectives. Moreover, existing studies usually present the results mainly in numerical form, while direct visualization of the corresponding structural response fields has received relatively limited attention. This limits the intuitive interpretation of the physical mechanisms underlying the dynamic behavior of grillage structures. Accordingly, for grillage structures with strong plate to plate coupling and in plane/out of plane coupling, it is necessary to establish a unified DSM framework that incorporates free vibration, forced vibration, wave propagation characteristics, and displacement field reconstruction into the same theoretical formulation for visualization and physical interpretation.

In this paper, a unified theoretical modeling framework based on the DSM is proposed for analyzing the vibration and wave behaviors of grillage structures. The unification of the proposed framework is reflected in two aspects. First, arbitrary grillage configurations and boundary conditions can be treated using the same multi-plate dynamic stiffness formulation, coordinate transformation, and matrix assembly procedure. Second, free vibration, forced vibration, wave propagation, and displacement field reconstruction are derived and implemented within the same theoretical framework. In this formulation, grillage structures are modeled as coupled multi-plate systems, where inter-plate coupling and the associated in-plane/out-of-plane coupling are achieved through displacement continuity, interface force/moment equilibrium

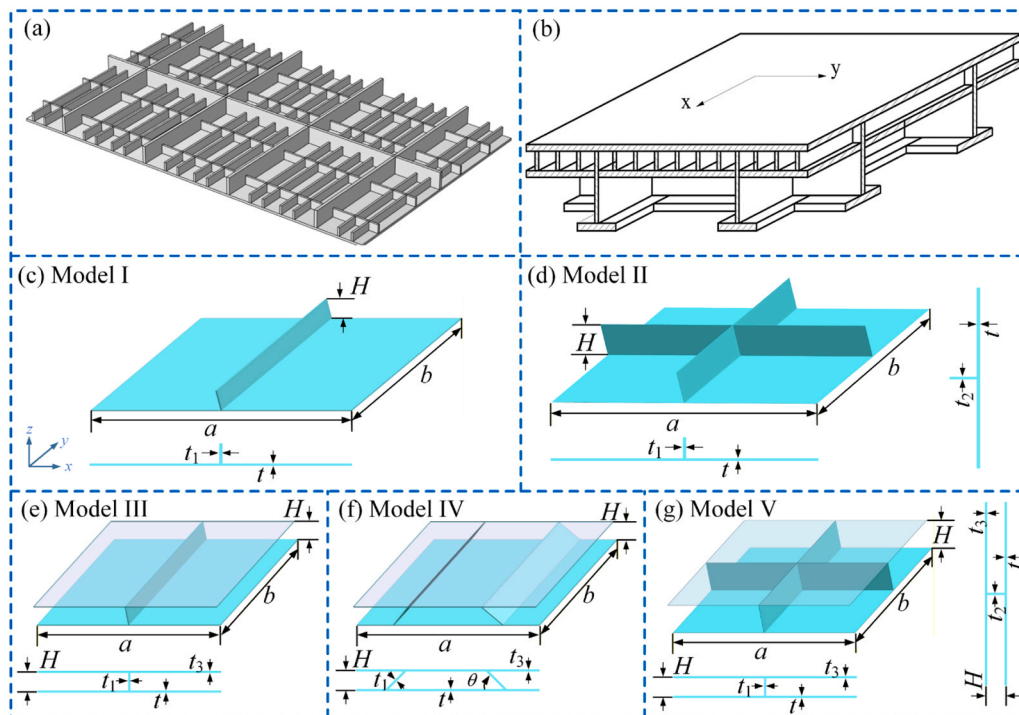


Fig. 1. Typical engineering grillage structures and the five representative models used in this study: (a) A typical stiffened grillage structure [27]; (b) A typical sandwich grillage structure [41]; (c-g) Schematic diagrams and dimensions of Models I-V.

Table 2

Geometric parameters of the five grillage structure models.

a/mm	b/mm	H/mm	t/mm	t_1/mm	t_2/mm
200	200	20	1	1	1
t_3/mm	θ	E/Pa	$\rho/(\text{kg}/\text{m}^3)$	ν	
1	$\pi/4$	2.1×10^{11}	7850	0.3	

along the connected edges, and coordinate transformations. A zero-crossing method is developed to extract eigenfrequencies from the frequency-dependent dynamic stiffness matrix. Using the proposed framework, the dynamic behaviors of five representative grillage configurations are investigated, including free vibration characteristics, forced vibration responses, transmission characteristics, and band structures. Furthermore, a displacement field reconstruction method is proposed to obtain mode shapes, the total displacement fields of eigenmode and forced vibration responses in a consistent manner. Comparisons with finite element results and published experimental data reported in the literature verify the accuracy and applicability of the proposed framework. In addition, convergence and computational efficiency analyses are conducted to provide practical guidelines for selecting the DSM truncation order and FEM mesh size.

2. Physical model and theoretical formulation

2.1. Physical model

This paper develops a unified theoretical modeling framework for arbitrary grillage structures. Fig. 1(a) and (b) show typical stiffened and sandwich grillage structures [27,41]. Based on these typical configurations, five representative models are selected for the analysis reported in this paper, as illustrated in Fig. 1(c-g): a unidirectionally stiffened plate (Model I), an orthogonally stiffened plate (Model II), an I-core sandwich plate (Model III), a corrugated-core sandwich plate (Model IV), and an orthogonal-core sandwich plate (Model V). The geometric schematics of these representative configurations are depicted in Fig. 1(c-g), while their corresponding structural and material parameters are detailed in Table 2. By abstracting these diverse geometries into coupled plate assemblies, the general modeling procedure can be systematically applied.

2.2. Theoretical formulation

This section presents the unified theoretical framework for modeling the dynamic behavior of grillage structures. In contrast to existing separated workflows that usually focus on isolated vibration problems, specific structural configurations, or individual response characteristics, the proposed framework achieves unification in two aspects. First, arbitrary grillage configurations and boundary conditions can be treated

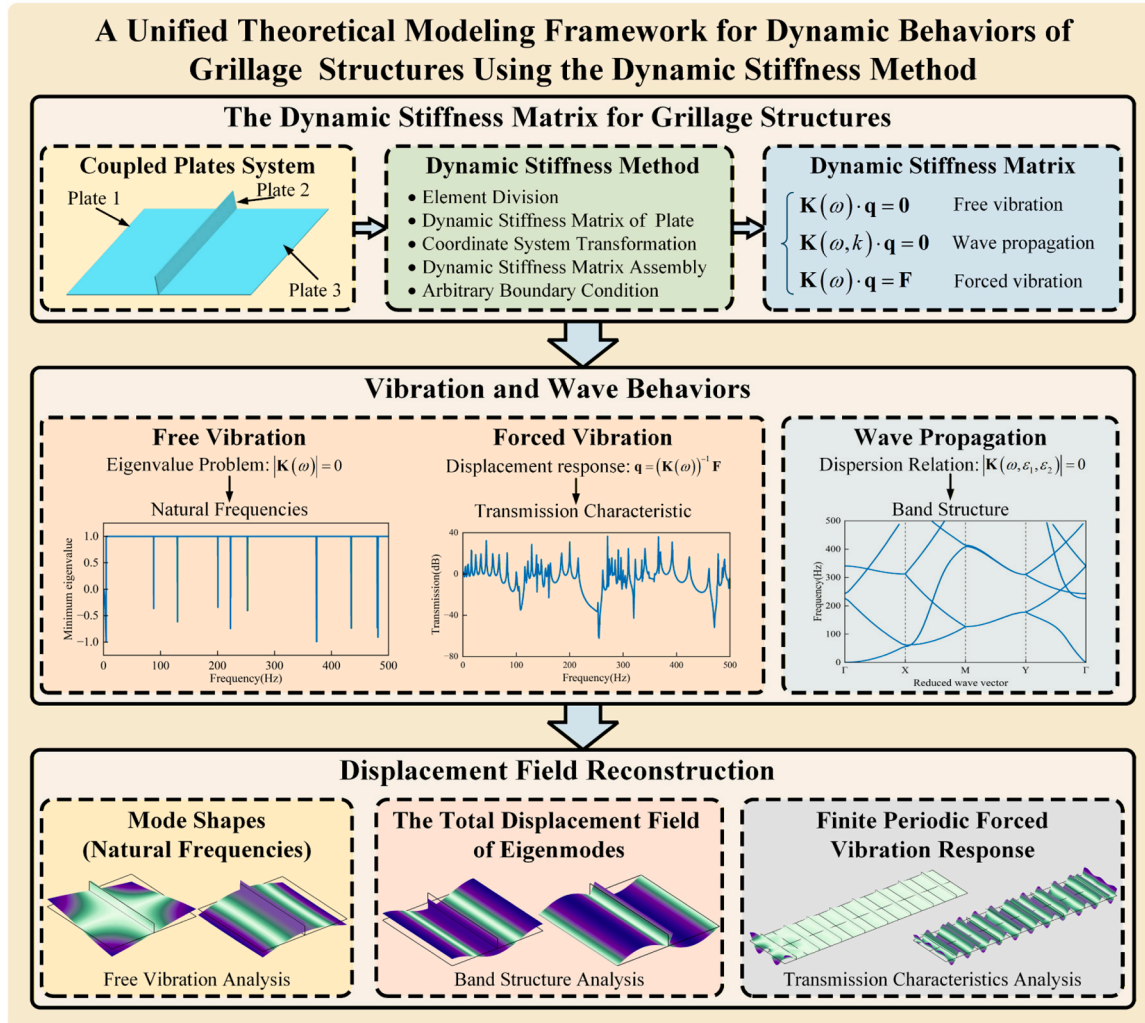


Fig. 2. Schematic of the unified theoretical modeling framework of dynamic behaviors for grillage structures. The overall procedure includes dynamic stiffness matrix construction, free vibration analysis, forced vibration analysis, wave propagation analysis based on Bloch periodicity, and displacement field reconstruction.

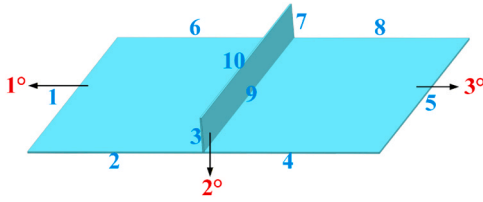


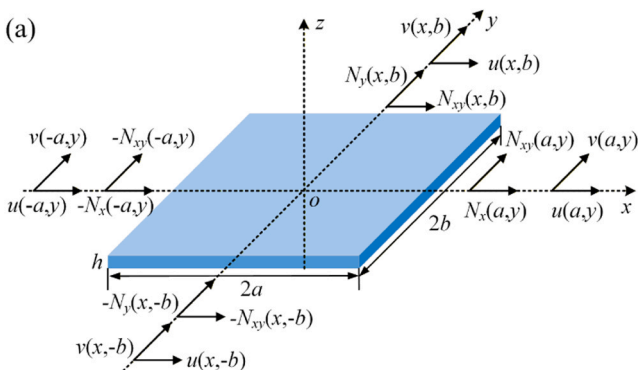
Fig. 3. Division of elements for the Model I unit cell.

using the same multi-plate dynamic stiffness formulation, coordinate transformation, and matrix assembly procedure. Second, free vibration, forced vibration, wave propagation, and displacement field reconstruction are derived and implemented within the same theoretical framework. An overview of the overall formulation procedure is illustrated in Fig. 2, highlighting the main steps from constructing the dynamic stiffness matrix to characterizing the vibration and wave behaviors, and finally to reconstructing the displacement fields.

Specifically, the dynamic stiffness matrix of grillage structures is derived in Section 2.2.1, which serves as the fundamental building block of the proposed framework. The present formulation is established within the framework of Kirchhoff thin plate theory and is therefore intended for thin plate-type grillage structures, for which transverse shear deformation is neglected. Based on this formulation, Section 2.2.2 establishes the theoretical formulations for vibration behavior, including the eigenvalue problems for free vibration and the governing equations for forced responses. Section 2.2.3 develops the wave behavior formulation by incorporating Bloch periodicity to construct the solution model for the band structures within the same framework. Finally, Section 2.2.4 presents a displacement field reconstruction method, enabling the recovery of modal shapes, Bloch eigen-displacement fields of infinite periodic structures, and forced vibration responses of finite periodic structures.

2.2.1. The dynamic stiffness matrix for grillage structures

In this subsection, the dynamic stiffness matrix of grillage structures is formulated. First, the dynamic stiffness matrix of a rectangular plate is presented. Then, coordinate transformations are performed, and the transformed matrices are assembled into the global dynamic stiffness matrix, with the relevant boundary conditions taken into account. The detailed derivation has been reported in our previous study [42], and only the key steps are summarized here. For more details, please refer to the relevant literature.



(1) Element division

Before deriving the dynamic stiffness matrix of rectangular plates, the grillage unit cell is first divided into plate elements. Taking the unidirectionally stiffened plate (Model I) as an example, the unit cell is partitioned into three plate elements with ten edges, as shown in Fig. 3. The labels 'X' and 'X°' shown in the diagram denote the edge number and plate number, respectively.

(2) Dynamic stiffness matrix for plate element

Grillage structures used in engineering applications generally belong to lightweight thin-walled structures, in which the plates are much thinner than their in-plane dimensions. Therefore, the rectangular plates considered in this work are modeled using Kirchhoff thin plate theory, and the corresponding dynamic stiffness matrices are formulated accordingly. Fig. 4 illustrates the geometric model of the elastic rectangular plate and the associated boundary resultants. A Cartesian coordinate system Oxyz is constructed, taking the plate center as the origin. The plate dimensions are 2a (length), 2b (width), and thickness h. The translational displacement components in the x, y, and z directions are defined as u, v, and w, respectively.

The governing equations corresponding to in-plane and transverse vibration behaviors can be expressed as follows:

$$\begin{aligned} \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{\rho h}{D} \frac{\partial^2 w}{\partial t^2} &= 0 \\ \frac{\partial^2 u}{\partial x^2} + a_1 \frac{\partial^2 u}{\partial y^2} + a_2 \frac{\partial^2 v}{\partial x \partial y} - \frac{\rho h}{D} \frac{\partial^2 u}{\partial t^2} &= 0 \\ \frac{\partial^2 v}{\partial y^2} + a_1 \frac{\partial^2 v}{\partial x^2} + a_2 \frac{\partial^2 u}{\partial x \partial y} - \frac{\rho h}{D} \frac{\partial^2 v}{\partial t^2} &= 0 \end{aligned} \quad (1)$$

where D corresponds to the bending stiffness of the plate, defined as: $D = \frac{Eh^3}{12(1-\nu^2)}$, $a_1 = \frac{1-\nu}{2}$ and $a_2 = \frac{1+\nu}{2}$.

Applying the Fourier transform to Eq. (1) yields [29,30]:

$$\begin{aligned} \frac{\partial^4 \hat{w}}{\partial x^4} + 2 \frac{\partial^4 \hat{w}}{\partial x^2 \partial y^2} + \frac{\partial^4 \hat{w}}{\partial y^4} - \frac{\rho h \omega^2}{D} \hat{w} &= 0 \\ \frac{\partial^2 \hat{u}}{\partial x^2} + a_1 \frac{\partial^2 \hat{u}}{\partial y^2} + a_2 \frac{\partial^2 \hat{v}}{\partial x \partial y} + \frac{\omega^2}{c_p^2} \hat{u} &= 0 \\ \frac{\partial^2 \hat{v}}{\partial y^2} + a_1 \frac{\partial^2 \hat{v}}{\partial x^2} + a_2 \frac{\partial^2 \hat{u}}{\partial x \partial y} + \frac{\omega^2}{c_p^2} \hat{v} &= 0 \end{aligned} \quad (2)$$

where the longitudinal wave velocity is defined as $c_p = \sqrt{\frac{E}{\rho(1-\nu^2)}}$.

By applying Gorman's superposition method [43], the displacements

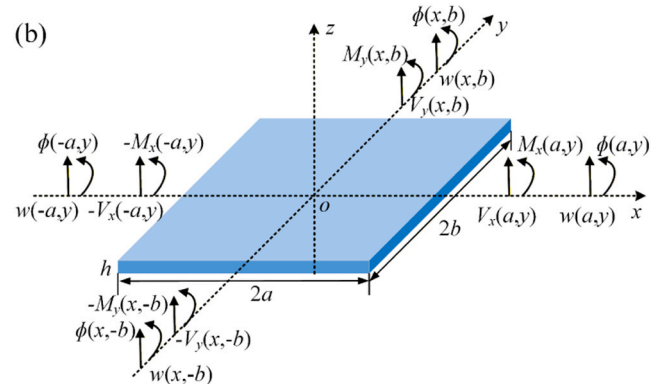


Fig. 4. Schematic of a Kirchhoff rectangular plate illustrating displacement components, rotations, forces, and bending moments. (a) In-plane; (b) Transverse.

can be decomposed into four distinct components according to their symmetry properties:

$$\begin{aligned}\hat{u}(x, y) &= \hat{u}_{SS}(x, y) + \hat{u}_{SA}(x, y) + \hat{u}_{AA}(x, y) + \hat{u}_{AS}(x, y) \\ \hat{v}(x, y) &= \hat{v}_{SS}(x, y) + \hat{v}_{SA}(x, y) + \hat{v}_{AA}(x, y) + \hat{v}_{AS}(x, y) \\ \hat{w}(x, y) &= \hat{w}_{SS}(x, y) + \hat{w}_{SA}(x, y) + \hat{w}_{AA}(x, y) + \hat{w}_{AS}(x, y)\end{aligned}\quad (3)$$

Accordingly, the four symmetry components of the transverse displacement can be explicitly expressed as follows:

$$\begin{aligned}\hat{w}_{SS}(x, y) &= \sum_{m=0}^M W_{SSm}(y) \cos \frac{m\pi x}{a} + \sum_{m=0}^M W_{SSm}(x) \cos \frac{m\pi y}{b} \\ \hat{w}_{AA}(x, y) &= \sum_{m=1}^M W_{AAm}(y) \sin \frac{(2m-1)\pi x}{2a} + \sum_{m=1}^M W_{AAm}(x) \sin \frac{(2m-1)\pi y}{2b} \\ \hat{w}_{SA}(x, y) &= \sum_{m=0}^M W_{SAm}(y) \cos \frac{m\pi x}{a} + \sum_{m=1}^M W_{SAm}(x) \sin \frac{(2m-1)\pi y}{2b} \\ \hat{w}_{AS}(x, y) &= \sum_{m=1}^M W_{ASm}(y) \sin \frac{(2m-1)\pi x}{2a} + \sum_{m=0}^M W_{ASm}(x) \cos \frac{m\pi y}{b}\end{aligned}\quad (4)$$

where M is the truncated series.

The expressions for the force and displacement at the boundary are as follows:

$$\begin{aligned}\mathbf{q}_{SS}^T &= \left[\hat{w}_{SS}(a, y) - \frac{\partial \hat{w}_{SS}}{\partial x}(a, y) \hat{w}_{SS}(x, b) \frac{\partial \hat{w}_{SS}}{\partial y}(x, b) \right] \\ \mathbf{Q}_{SS}^T &= [\hat{T}_{xSS}(a, y) - \hat{M}_{xSS}(a, y) \hat{T}_{ySS}(x, b) \hat{M}_{ySS}(x, b)] \\ \mathbf{q}_{SA}^T &= \left[\hat{w}_{SA}(a, y) - \frac{\partial \hat{w}_{SA}}{\partial x}(a, y) \hat{w}_{SA}(x, b) \frac{\partial \hat{w}_{SA}}{\partial y}(x, b) \right] \\ \mathbf{Q}_{SA}^T &= [\hat{T}_{xSA}(a, y) - \hat{M}_{xSA}(a, y) \hat{T}_{ySA}(x, b) \hat{M}_{ySA}(x, b)] \\ \mathbf{q}_{AA}^T &= \left[\hat{w}_{AA}(a, y) - \frac{\partial \hat{w}_{AA}}{\partial x}(a, y) \hat{w}_{AA}(x, b) \frac{\partial \hat{w}_{AA}}{\partial y}(x, b) \right] \\ \mathbf{Q}_{AA}^T &= [\hat{T}_{xAA}(a, y) - \hat{M}_{xAA}(a, y) \hat{T}_{yAA}(x, b) \hat{M}_{yAA}(x, b)] \\ \mathbf{q}_{AS}^T &= \left[\hat{w}_{AS}(a, y) - \frac{\partial \hat{w}_{AS}}{\partial x}(a, y) \hat{w}_{AS}(x, b) \frac{\partial \hat{w}_{AS}}{\partial y}(x, b) \right] \\ \mathbf{Q}_{AS}^T &= [\hat{T}_{xAS}(a, y) - \hat{M}_{xAS}(a, y) \hat{T}_{yAS}(x, b) \hat{M}_{yAS}(x, b)]\end{aligned}\quad (5)$$

The projection functions associated with the displacement components take the form:

$$\begin{aligned}h_{nSS}(x) &= \cos \frac{n\pi x}{a}, h_{nSS}(y) = \cos \frac{n\pi y}{b}, n = 0, 1, \dots, M \\ h_{nAA}(x) &= \sin \frac{(2n-1)\pi x}{2a}, h_{nAA}(y) = \sin \frac{(2n-1)\pi y}{2b}, n = 0, 1, \dots, M \\ h_{nSA}(x) &= \cos \frac{n\pi x}{a}, h_{nSA}(y) = \sin \frac{(2n-1)\pi y}{2b}, n = 0, 1, \dots, M \\ h_{nAS}(x) &= \sin \frac{(2n-1)\pi x}{2a}, h_{nAS}(y) = \cos \frac{n\pi y}{b}, n = 0, 1, \dots, M\end{aligned}\quad (6)$$

The projection vector can be expressed as:

$$\begin{aligned}\tilde{\mathbf{q}}_{SS} &= \frac{2}{L} \int_s \mathbf{H}_{SS} \mathbf{q}_{SS} ds = \frac{2}{L} \int_s \mathbf{H}_{SS} \Phi_b \mathbf{C}_{SS} ds = \tilde{\mathbf{D}}_{SS} \mathbf{C}_{SS} \\ \tilde{\mathbf{Q}}_{SS} &= \frac{2}{L} \int_s \mathbf{H}_{SS} \mathbf{Q}_{SS} ds = \frac{2}{L} \int_s \mathbf{H}_{SS} \mathbf{G}_b \mathbf{C}_{SS} ds = \tilde{\mathbf{F}}_{SS} \mathbf{C}_{SS} \\ \tilde{\mathbf{q}}_{AA} &= \frac{2}{L} \int_s \mathbf{H}_{AA} \mathbf{q}_{AA} ds = \frac{2}{L} \int_s \mathbf{H}_{AA} \Phi_b \mathbf{C}_{AA} ds = \tilde{\mathbf{D}}_{AA} \mathbf{C}_{AA} \\ \tilde{\mathbf{Q}}_{AA} &= \frac{2}{L} \int_s \mathbf{H}_{AA} \mathbf{Q}_{AA} ds = \frac{2}{L} \int_s \mathbf{H}_{AA} \mathbf{G}_b \mathbf{C}_{AA} ds = \tilde{\mathbf{F}}_{AA} \mathbf{C}_{AA} \\ \tilde{\mathbf{q}}_{SA} &= \frac{2}{L} \int_s \mathbf{H}_{SA} \mathbf{q}_{SA} ds = \frac{2}{L} \int_s \mathbf{H}_{SA} \Phi_b \mathbf{C}_{SA} ds = \tilde{\mathbf{D}}_{SA} \mathbf{C}_{SA} \\ \tilde{\mathbf{Q}}_{SA} &= \frac{2}{L} \int_s \mathbf{H}_{SA} \mathbf{Q}_{SA} ds = \frac{2}{L} \int_s \mathbf{H}_{SA} \mathbf{G}_b \mathbf{C}_{SA} ds = \tilde{\mathbf{F}}_{SA} \mathbf{C}_{SA} \\ \tilde{\mathbf{q}}_{AS} &= \frac{2}{L} \int_s \mathbf{H}_{AS} \mathbf{q}_{AS} ds = \frac{2}{L} \int_s \mathbf{H}_{AS} \Phi_b \mathbf{C}_{AS} ds = \tilde{\mathbf{D}}_{AS} \mathbf{C}_{AS} \\ \tilde{\mathbf{Q}}_{AS} &= \frac{2}{L} \int_s \mathbf{H}_{AS} \mathbf{Q}_{AS} ds = \frac{2}{L} \int_s \mathbf{H}_{AS} \mathbf{G}_b \mathbf{C}_{AS} ds = \tilde{\mathbf{F}}_{AS} \mathbf{C}_{AS}\end{aligned}\quad (7)$$

where $\mathbf{C}_{ab}$ ($ab=SS, AA, SA, AS$) represents the vector containing the unknown coefficients, $\mathbf{H}_{ab}$ is the projection function, and $\tilde{\mathbf{D}}_{ab}$ and $\tilde{\mathbf{F}}_{ab}$ are projection displacement and force matrices, which are composed of four components of displacement.

Upon elimination of $\mathbf{C}$ in Eq. (7), the result is expressed as:

$$\begin{aligned}\tilde{\mathbf{Q}}_{SS} &= \tilde{\mathbf{F}}_{SS} \mathbf{C} = \tilde{\mathbf{F}}_{SS} \tilde{\mathbf{D}}_{SS}^{-1} \tilde{\mathbf{D}}_{SS} \mathbf{C} = \tilde{\mathbf{F}}_{SS} \tilde{\mathbf{D}}_{SS}^{-1} \tilde{\mathbf{q}}_{SS} = \tilde{\mathbf{K}}_{SS}(\omega) \tilde{\mathbf{q}}_{SS} \\ \tilde{\mathbf{Q}}_{AA} &= \tilde{\mathbf{F}}_{AA} \mathbf{C} = \tilde{\mathbf{F}}_{AA} \tilde{\mathbf{D}}_{AA}^{-1} \tilde{\mathbf{D}}_{AA} \mathbf{C} = \tilde{\mathbf{F}}_{AA} \tilde{\mathbf{D}}_{AA}^{-1} \tilde{\mathbf{q}}_{AA} = \tilde{\mathbf{K}}_{AA}(\omega) \tilde{\mathbf{q}}_{AA} \\ \tilde{\mathbf{Q}}_{SA} &= \tilde{\mathbf{F}}_{SA} \mathbf{C} = \tilde{\mathbf{F}}_{SA} \tilde{\mathbf{D}}_{SA}^{-1} \tilde{\mathbf{D}}_{SA} \mathbf{C} = \tilde{\mathbf{F}}_{SA} \tilde{\mathbf{D}}_{SA}^{-1} \tilde{\mathbf{q}}_{SA} = \tilde{\mathbf{K}}_{SA}(\omega) \tilde{\mathbf{q}}_{SA} \\ \tilde{\mathbf{Q}}_{AS} &= \tilde{\mathbf{F}}_{AS} \mathbf{C} = \tilde{\mathbf{F}}_{AS} \tilde{\mathbf{D}}_{AS}^{-1} \tilde{\mathbf{D}}_{AS} \mathbf{C} = \tilde{\mathbf{F}}_{AS} \tilde{\mathbf{D}}_{AS}^{-1} \tilde{\mathbf{q}}_{AS} = \tilde{\mathbf{K}}_{AS}(\omega) \tilde{\mathbf{q}}_{AS}\end{aligned}\quad (8)$$

By assembling the four component matrices, the complete governing equation for the transverse motion of the plate under fully free boundary conditions, together with the corresponding dynamic stiffness matrix, can be obtained. Following the same procedure, the governing equation for the in-plane motion under fully free boundary conditions and its associated dynamic stiffness matrix can also be derived.

$$\begin{aligned}\tilde{\mathbf{Q}}_{(16M+8) \times 1}^{tra} &= \tilde{\mathbf{K}}_o^{tra} \tilde{\mathbf{q}}^{tra} \\ &= \begin{bmatrix} \tilde{\mathbf{K}}_{SS}(\omega) & & & \\ & \tilde{\mathbf{K}}_{AS}(\omega) & & \\ & & \tilde{\mathbf{K}}_{SA}(\omega) & \\ & & & \tilde{\mathbf{K}}_{AA}(\omega) \end{bmatrix}_{(16M+8) \times (16M+8)} \tilde{\mathbf{q}}_{(16M+8) \times 1}^{tra}\end{aligned}\quad (9)$$

It should be noted that $\tilde{\mathbf{K}}_o^{tra}$ is formed in the symmetry component basis and is not yet the final dynamic stiffness matrix expressed in the physical boundary degree of freedom order. The complete dynamic stiffness matrix is obtained through the transformation $\tilde{\mathbf{K}}^{tra} = \frac{1}{2}(\mathbf{T}^{tra})^T \tilde{\mathbf{K}}_o^{tra} \mathbf{T}^{tra}$, where $\mathbf{T}^{tra}$ is the transformation matrix used to combine the four symmetry components into the physical boundary displacement and force vectors.

The complete dynamic stiffness matrix takes the following form:

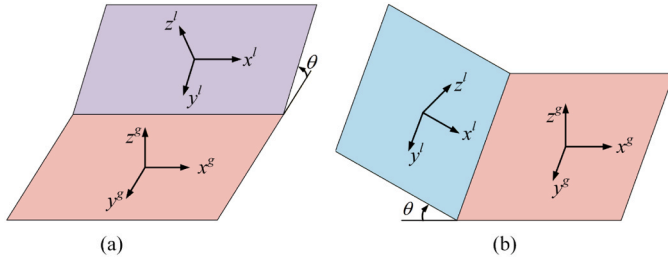


Fig. 5. Local-to-global coordinate system transformation: (a) Transformation associated with rotation by an angle θ about the global x -axis; (b) Transformation associated with rotation by an angle θ about the global y -axis. The local coordinate system (x^l, y^l, z^l) and the global coordinate system (x^g, y^g, z^g) are shown for the coordinate transformation required in the dynamic stiffness formulation.

$$\begin{aligned} \tilde{\mathbf{Q}} &= \tilde{\mathbf{K}} \tilde{\mathbf{q}} \\ \begin{bmatrix} 1 \tilde{\mathbf{Q}}^{in} \\ 1 \tilde{\mathbf{Q}}^{tra} \\ 2 \tilde{\mathbf{Q}}^{in} \\ 2 \tilde{\mathbf{Q}}^{tra} \\ 3 \tilde{\mathbf{Q}}^{in} \\ 3 \tilde{\mathbf{Q}}^{tra} \\ 4 \tilde{\mathbf{Q}}^{in} \\ 4 \tilde{\mathbf{Q}}^{tra} \end{bmatrix}_{(32M+12) \times 1} &= \begin{bmatrix} \tilde{\mathbf{K}}_{11}^{in} & \tilde{\mathbf{K}}_{12}^{in} & \tilde{\mathbf{K}}_{13}^{in} & \tilde{\mathbf{K}}_{14}^{in} & \tilde{\mathbf{K}}_{11}^{tra} & \tilde{\mathbf{K}}_{12}^{tra} & \tilde{\mathbf{K}}_{13}^{tra} & \tilde{\mathbf{K}}_{14}^{tra} \\ \tilde{\mathbf{K}}_{21}^{in} & \tilde{\mathbf{K}}_{22}^{in} & \tilde{\mathbf{K}}_{23}^{in} & \tilde{\mathbf{K}}_{24}^{in} & \tilde{\mathbf{K}}_{21}^{tra} & \tilde{\mathbf{K}}_{22}^{tra} & \tilde{\mathbf{K}}_{23}^{tra} & \tilde{\mathbf{K}}_{24}^{tra} \\ \tilde{\mathbf{K}}_{31}^{in} & \tilde{\mathbf{K}}_{32}^{in} & \tilde{\mathbf{K}}_{33}^{in} & \tilde{\mathbf{K}}_{34}^{in} & \tilde{\mathbf{K}}_{31}^{tra} & \tilde{\mathbf{K}}_{32}^{tra} & \tilde{\mathbf{K}}_{33}^{tra} & \tilde{\mathbf{K}}_{34}^{tra} \\ \tilde{\mathbf{K}}_{41}^{in} & \tilde{\mathbf{K}}_{42}^{in} & \tilde{\mathbf{K}}_{43}^{in} & \tilde{\mathbf{K}}_{44}^{in} & \tilde{\mathbf{K}}_{41}^{tra} & \tilde{\mathbf{K}}_{42}^{tra} & \tilde{\mathbf{K}}_{43}^{tra} & \tilde{\mathbf{K}}_{44}^{tra} \end{bmatrix}_{(32M+12) \times (32M+12)} \begin{bmatrix} 1 \tilde{\mathbf{q}}^{in} \\ 1 \tilde{\mathbf{q}}^{tra} \\ 2 \tilde{\mathbf{q}}^{in} \\ 2 \tilde{\mathbf{q}}^{tra} \\ 3 \tilde{\mathbf{q}}^{in} \\ 3 \tilde{\mathbf{q}}^{tra} \\ 4 \tilde{\mathbf{q}}^{in} \\ 4 \tilde{\mathbf{q}}^{tra} \end{bmatrix}_{(32M+12) \times 1} \end{aligned} \quad (10)$$

where $\tilde{\mathbf{Q}}$, $\tilde{\mathbf{q}}$ denote the force vector, displacement vector.

(3) Coordinate system transformation

Since each plate component is formulated in its own local coordinate system, the corresponding dynamic stiffness matrix must be transformed into a unified global coordinate system before matrix assembly. In this work, the grillage structure is modeled as an assembly of rectangular plate components connected along straight edges. For the non-coplanar grillage configurations considered here, the local coordinate system of each rectangular plate component is related to the global coordinate system through the rotation determined by its geometric orientation.

As illustrated in Fig. 5(a), when the local coordinate system is obtained by rotating the global coordinate system by an angle θ about the global x -axis, the coordinate relation can be expressed as:

$$\begin{bmatrix} x^l \\ y^l \\ z^l \\ \phi^l \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^g \\ y^g \\ z^g \\ \phi^g \end{bmatrix} \quad (11)$$

where, x^l, y^l, z^l , and ϕ^l denote the components in the local coordinate system, while x^g, y^g, z^g , and ϕ^g denote the corresponding components in the global coordinate system.

As illustrated in Fig. 5(b), when the local coordinate system is obtained by rotating the global coordinate system by an angle θ about the global y -axis, the corresponding transformation is:

$$\begin{bmatrix} x^l \\ y^l \\ z^l \\ \phi^l \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^g \\ y^g \\ z^g \\ \phi^g \end{bmatrix} \quad (12)$$

For the i -th edge of a rectangular plate component, the local and global edge generalized displacement and force variables are defined as:

$$\tilde{\mathbf{q}} = \begin{bmatrix} i \tilde{\mathbf{u}} \\ i \tilde{\mathbf{v}} \\ i \tilde{\mathbf{w}} \\ i \tilde{\Phi} \end{bmatrix}, \quad i \tilde{\mathbf{q}}^g = \begin{bmatrix} i \tilde{\mathbf{u}}^g \\ i \tilde{\mathbf{v}}^g \\ i \tilde{\mathbf{w}}^g \\ i \tilde{\Phi}^g \end{bmatrix}, \quad i \tilde{\mathbf{Q}} = \begin{bmatrix} i \tilde{\mathbf{N}}_{x/y} \\ i \tilde{\mathbf{N}}_{xy} \\ i \tilde{\mathbf{V}}_{x/y} \\ i \tilde{\mathbf{M}}_{xy} \end{bmatrix}, \quad i \tilde{\mathbf{Q}}^g = \begin{bmatrix} i \tilde{\mathbf{N}}_{x/y}^g \\ i \tilde{\mathbf{N}}_{xy}^g \\ i \tilde{\mathbf{V}}_{x/y}^g \\ i \tilde{\mathbf{M}}_{xy}^g \end{bmatrix} \quad (13)$$

According to the coordinate relations in Eqs. (11) and (12), the edge generalized variables are projected between the local and global coordinate systems as:

$$i \tilde{\mathbf{q}} = \mathbf{T}_{ix/y} i \tilde{\mathbf{q}}^g, \quad i \tilde{\mathbf{Q}} = \mathbf{T}_{ix/y} i \tilde{\mathbf{Q}}^g \quad (14)$$

By stacking the generalized variables of the four edges, the plate generalized displacement and force vectors are obtained. Accordingly, the local-to-global transformation of a complete plate component can be written as:

$$\tilde{\mathbf{q}} = \mathbf{T} \tilde{\mathbf{q}}^g, \quad \tilde{\mathbf{Q}} = \mathbf{T} \tilde{\mathbf{Q}}^g \quad (15)$$

where $\mathbf{T}$ is the transformation matrix corresponding to the orientation of the plate component.

The transformation matrix of the complete plate component is written in block-diagonal form because the same coordinate transformation is applied to the generalized variables associated with each plate edge:

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{1x/y} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{2x/y} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{T}_{3x/y} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T}_{4x/y} \end{bmatrix}, \quad \mathbf{T}_{ix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\cos \theta} & \mathbf{I}_{\sin \theta} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{\sin \theta} & \mathbf{I}_{\cos \theta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}_{(8M+3) \times (8M+3)}, \quad \mathbf{T}_{iy} = \begin{bmatrix} \mathbf{I}_{\cos \theta} & \mathbf{0} & \mathbf{I}_{\sin \theta} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_{\sin \theta} & \mathbf{0} & \mathbf{I}_{\cos \theta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}_{(8M+3) \times (8M+3)} \quad (16)$$

where $\mathbf{T}_{ix}$ and $\mathbf{T}_{iy}$ denote the transformation matrices associated with rotations about the global x - and y -axes, respectively.

Substituting the displacement transformation into the local dynamic stiffness relation and transforming the generalized force vector into the global coordinate system gives:

$$\tilde{\mathbf{Q}}^g = \mathbf{T}^T \tilde{\mathbf{K}} \mathbf{T} \tilde{\mathbf{q}}^g, \tilde{\mathbf{K}}^g = \mathbf{T}^T \tilde{\mathbf{K}} \mathbf{T} \quad (17)$$

(4) Dynamic stiffness matrix assembly

As can be seen in Fig. 6, the dynamic stiffness matrix for Model I is obtained by assembling the dynamic stiffness matrices of its three plate elements according to their connectivity relations. In a manner analogous to the assembly procedure in the finite element method, the contributions of the individual plate elements are combined by enforcing displacement compatibility and force equilibrium at the shared boundaries. From a physical point of view, this assembly process also represents the coupling mechanism between adjacent plate elements: displacement compatibility ensures continuous deformation along the shared boundaries, while force equilibrium allows bending moments, shear forces, and in-plane resultants to be transferred across the interfaces. Together with the local-to-global coordinate transformation, this coupling mechanism introduces in-plane/out-of-plane interaction at the structural level and is embedded in the assembled global dynamic stiffness matrix.

For Model I, the complete dynamic stiffness matrix is given by:

$$\tilde{\mathbf{Q}}_{\text{Model I}} = \tilde{\mathbf{K}}_{\text{Model I}} \tilde{\mathbf{q}}_{\text{Model I}} \quad (18)$$

(5) Application of arbitrary boundary conditions

Three classical boundary conditions are considered: free (F), simply supported (S) and clamped (C), which are given by:

Transverse vibration: F: $V_i = 0$, $M_i = 0$; S: $w_i = 0$, $M_i = 0$; C: $w_i = 0$, $\phi_i = 0$;

In-plane vibration: F: $N_{x/yi} = N_{xyi} = 0$; S: $v_i = 0$, $N_{xyi} = 0$ (or $u_i = 0$, $N_{xyi} = 0$); C: $u_i = 0$, $v_i = 0$.

Under a given boundary condition, the stiffness matrix of the plate element is formed by removing the rows and columns corresponding to vanishing displacement components.

In addition, constraint springs can be used to impose arbitrary boundary conditions. Along the boundary i , the force-displacement equilibrium takes the form:

$$\begin{aligned} V_i - k_{wi} w_i &= 0 \\ M_i - k_{\phi i} \phi_i &= 0 \end{aligned} \quad (19)$$

$$\begin{aligned} N_{xi} - k_{xi,u} u_i &= 0 \quad \text{or} \quad N_{yi} - k_{yi,v} v_i = 0 \\ N_{xyi} - k_{xyi,v} v_i &= 0 \quad \text{or} \quad N_{xyi} - k_{xyi,u} u_i = 0 \end{aligned} \quad (20)$$

where k_u , k_v , and k_w denote the translational spring stiffnesses, and k_ϕ is

the torsional constraint stiffness.

The restraint intensity depends on the value of the spring stiffness. To describe arbitrary boundary conditions in a more systematic and reproducible manner, the spring stiffnesses are further expressed in non-dimensional form. For the in-plane constraints, the non-dimensional spring stiffnesses can be defined as:

$$\hat{k}_u = \frac{k_u L(1 - \nu^2)}{E}, \quad \hat{k}_v = \frac{k_v L(1 - \nu^2)}{E} \quad (21)$$

For the transverse and torsional constraints, the non-dimensional spring stiffnesses can be defined as:

$$\hat{k}_w = k_w L^3 / D, \quad \hat{k}_\phi = k_\phi L / D \quad (22)$$

Under these definitions, a zero spring stiffness corresponds to the free boundary condition, whereas the clamped boundary condition is obtained by increasing the spring stiffness until the results converge; intermediate values model elastically restrained boundary conditions. Therefore, arbitrary boundary conditions can be continuously represented by the non-dimensional spring stiffnesses within a unified parameterized framework.

The present formulation uses Kirchhoff-based dynamic stiffness matrices for rectangular plates. Its applicability is therefore governed by the thin-plate assumption and by the flexural wavelength within the frequency range of interest. For relatively thick plates or high-frequency responses where transverse shear deformation becomes non-negligible, the plate dynamic stiffness matrices need to be re-derived according to Mindlin/Reissner or higher-order shear deformation plate theories, whereas the general modeling procedure, including coordinate transformation, matrix assembly, and boundary condition implementation, remains applicable.

Furthermore, material damping may be introduced in future studies through complex-valued material properties, while damping associated with boundary restraints or supports may be represented by complex, frequency-dependent boundary impedances, such as spring-dashpot restraints, within the frequency-domain DSM framework.

2.2.2. Vibration behavior formulation

(1) Free vibration formulation

Although the Wittrick-Williams (WW) algorithm [44] is widely regarded as a standard tool for preventing missing eigenvalues in dynamic stiffness formulations, its direct application to the grillage structures considered in this study is not straightforward. For example, in grillage structures, coordinate transformations can lead to strong coupling between in-plane and transverse deformations. In addition, the stiffness scales associated with

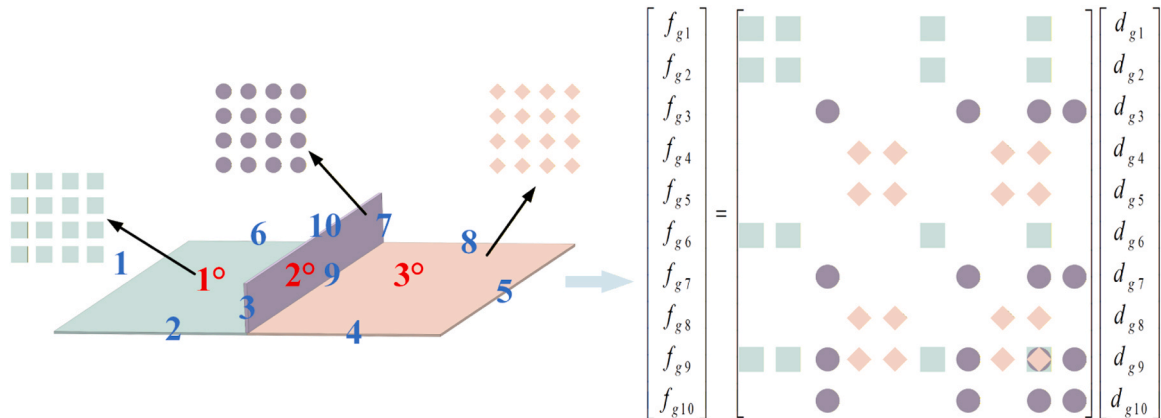


Fig. 6. Schematic of the assembly procedure for the dynamic stiffness matrices of the grillage structure. The figure illustrates how the dynamic stiffness matrices of individual components are transformed and assembled into the global dynamic stiffness matrix according to the connectivity relations of the structure.

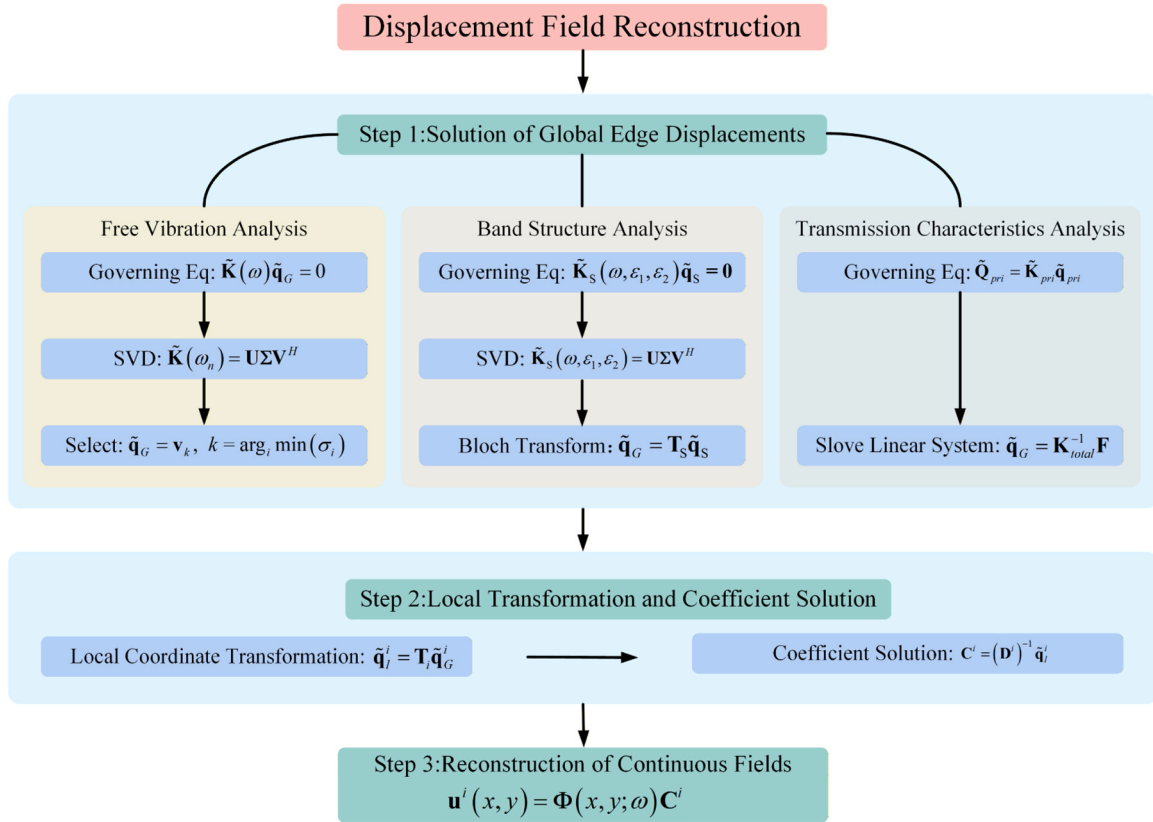


Fig. 7. Procedure for displacement field reconstruction in the proposed DSM framework, including global edge displacement solution, local transformation and coefficient solution, and continuous field reconstruction.

in-plane deformation and transverse deformation can be significantly different, which may increase the numerical sensitivity of the sign-counting procedure when the global dynamic stiffness matrix is factorized. Under such conditions, the LDL^T decomposition required by the WW algorithm may require additional numerical treatment to ensure robust sign counting in the present coupled multi-plate formulation. Moreover, the WW algorithm strictly depends on the availability of element-specific clamped frequency counts (J_0). While such counting rules have been rigorously derived for classical elements such as beams [44,45] and plates [31,37], extending the framework to other structural components, such as cylindrical shells [46], would necessitate a separate derivation of the corresponding J_0 . This requirement complicates the extension of the present grillage framework to more general structural assemblies.

Therefore, in the present study, a direct eigenvalue identification strategy is adopted. For free vibration analysis, the governing equation of the grillage structures can be written as:

$$\tilde{\mathbf{K}}(\omega)\tilde{\mathbf{q}} = 0 \quad (23)$$

A non-trivial solution exists only when the matrix $\tilde{\mathbf{K}}(\omega)$ becomes singular, which leads to the classical characteristic condition.

$$\det[\tilde{\mathbf{K}}(\omega)] = 0 \quad (24)$$

Eq. (24) is therefore the characteristic equation of the reduced dynamic stiffness system, whose roots give the natural frequencies of the grillage structure. For the solution of Eq. (24), a scalar indicator is introduced by selecting the signed eigenvalue of $\tilde{\mathbf{K}}(\omega)$ with the smallest modulus:

$$\lambda_s(\omega) = \lambda_k(\omega), \quad k = \operatorname{argmin}_i |\lambda_i(\omega)| \quad (25)$$

where $\lambda_i(\omega)$ is the i -th eigenvalue of $\tilde{\mathbf{K}}(\omega)$. At a root of the characteristic equation, $\tilde{\mathbf{K}}(\omega)$ becomes singular, and at least one eigenvalue vanishes. Hence, the vanishing of $\lambda_s(\omega)$ is used as the numerical indicator for identifying the roots of Eq. (24).

The frequency interval of interest is first discretized as:

$$\omega_i = \omega_{\min} + i\Delta\omega, \quad i = 0, 1, \dots, N \quad (26)$$

where $\Delta\omega$ is the initial frequency step. The value of $\lambda_s(\omega_i)$ is evaluated at each sampling point. A candidate interval is detected when a sign change occurs between two adjacent sampling frequencies,

$$\lambda_s(\omega_i)\lambda_s(\omega_{i+1}) < 0 \quad (27)$$

This sign change is used as a bracketing condition for possible roots of the characteristic equation. Once a candidate interval $[\omega_i, \omega_{i+1}]$ is detected, a bisection procedure is applied to refine the root location. At each iteration, the midpoint frequency is evaluated, and the subinterval that still contains a sign change is retained for the next iteration. The refinement is terminated when

$$|\omega_R - \omega_L| < \varepsilon_\omega \quad (28)$$

where ω_L and ω_R are the current lower and upper bounds, and ε_ω is the prescribed frequency tolerance. The refined frequency is denoted by ω^* .

The candidate frequency ω^* is accepted as a root only when the corresponding indicator satisfies:

$$|\lambda_s(\omega^*)| < \varepsilon_\lambda \quad (29)$$

where ε_λ is the eigenvalue tolerance.

This condition verifies that the selected eigenvalue actually converges to zero at the candidate frequency. The sign change in Eq. (27) is used only to identify a candidate interval. The candidate is accepted as a physical eigenfrequency only when the refined frequency also satisfies the residual condition in Eq. (29).

This criterion is used to distinguish physical eigenfrequencies

with a refined frequency step. For repeated roots, the sign change only indicates the presence of a root, so that the number of coincident roots at the same frequency cannot be determined.

For clarity, the overall implementation procedure for DSM assembly and eigenfrequency extraction is summarized in Algorithm 1, including the construction and assembly of the dynamic stiffness matrices and the subsequent zero-crossing-based root search. Algorithm 1 Implementation procedure for DSM assembly and eigenfrequency extraction.

Input: Geometric and material parameters; plate connectivity; boundary conditions; truncation order M ; frequency range $[\omega_{\min}, \omega_{\max}]$; initial frequency step $\Delta\omega$; tolerances ε_ω and ε_λ .

Output: Identified eigenfrequencies.

- 1: Initialize an empty eigenfrequency list.
 - 2: Discretize $[\omega_{\min}, \omega_{\max}]$ using $\Delta\omega$.
 - 3: **for** each sampling frequency ω_i **do**
 - 4: construct the component dynamic stiffness matrices, transform them into the global coordinate system, and assemble the reduced dynamic stiffness matrix.
 - 5: compute the selected signed eigenvalue $\lambda_s(\omega_i)$ according to Eq. (23).
 - 6: **end for**
 - 7: **for** each adjacent interval $[\omega_i, \omega_{i+1}]$ **do**
 - 8: **if** $\lambda_s(\omega_i)\lambda_s(\omega_{i+1}) < 0$ **then**
 - 9: set $\omega_L \leftarrow \omega_i$ and $\omega_R \leftarrow \omega_{i+1}$.
 - 10: **while** $|\omega_R - \omega_L| > \varepsilon_\omega$ **do**
 - 11: evaluate λ_s at the midpoint frequency and retain the subinterval containing the sign change.
 - 12: **end while**
 - 13: set $\omega^* \leftarrow (\omega_L + \omega_R)/2$.
 - 14: **if** $|\lambda_s(\omega^*)| < \varepsilon_\lambda$ **then**
 - 15: record ω^* as an eigenfrequency.
 - 16: **end if**
 - 17: **end if**
 - 18: **end for**
 - 19: **return** the identified eigenfrequencies.
-

from spurious sign changes and pole-related singularities. For a physical eigenfrequency, the reduced dynamic stiffness matrix becomes singular, and the selected eigenvalue continuously approaches zero after bisection refinement. In contrast, a pole-related singularity may produce an abrupt sign reversal of $\lambda_s(\omega)$, but this is associated with the eigenvalue diverging rather than passing through zero, so that the refined candidate frequency does not satisfy Eq. (29) and is therefore rejected. These poles may arise from frequency-dependent element matrices or dynamically condensed substructures, and they do not correspond to zeros of the assembled characteristic equation. To reduce the possibility of missing closely spaced roots in high-modal-density regions, the zero-crossing search can be repeated

(2) Forced vibration formulation

Next, the transmission characteristics of the periodic grillage structures subjected to harmonic forcing are considered. Similar to the FEM, multiple unit cells can be assembled to form a finite periodic system.

$$\tilde{\mathbf{Q}}_{\text{Model } 1\text{-per}} = \tilde{\mathbf{K}}_{\text{Model } 1\text{-per}} \tilde{\mathbf{q}}_{\text{Model } 1\text{-per}} \quad (30)$$

By applying the prescribed boundary conditions and external excitations to the resulting global dynamic stiffness equation and solving it, the displacement responses of the grillage structures can be obtained. Extracting the displacements at the input and output nodes of the grillage structures, the transmission characteristics are then expressed as:

Table 3

Natural frequencies of the Model I.

Boundary condition	Method	Natural frequency (Hz)					
		1	2	3	5	10	20
FFFF	Present	88.0	129.7	202.7	252.5	664.0	1310.9
	COMSOL	87.8	129.7	202.5	252.1	662.9	1309.4
	Relative error	0.20%	0.03%	0.10%	0.16%	0.16%	0.12%
FSFS	Present	103.5	140.6	294.9	466.5	818.5	1579.1
	COMSOL	103.4	140.3	294.6	466.0	816.1	1576.9
	Relative error	0.15%	0.23%	0.09%	0.11%	0.30%	0.14%
FCFC	Present	166.6	193.3	421.0	495.3	893.6	1831.3
	COMSOL	166.5	193.1	420.9	494.7	892.1	1827.1
	Relative error	0.09%	0.10%	0.03%	0.13%	0.17%	0.23%

Table 4

Natural frequencies of the Model II.

Boundary condition	Method	Natural frequency (Hz)					
		1	2	3	5	10	20
FFFF	Present	93.4	208.6	208.6	438.7	987.5	1809.5
	COMSOL	93.2	208.5	208.5	438.1	985.3	1804.4
	Relative error	0.24%	0.04%	0.04%	0.15%	0.22%	0.28%
FSFS	Present	94.3	300.6	320.2	557.4	933.7	1689.2
	COMSOL	94.2	300.3	319.9	555.2	930.6	1687.1
	Relative error	0.13%	0.11%	0.10%	0.39%	0.33%	0.12%
FCFC	Present	148.7	423.4	437.6	732.8	1117.7	1837.7
	COMSOL	148.6	423.0	437.2	732.2	1113.2	1834.2
	Relative error	0.07%	0.10%	0.08%	0.09%	0.41%	0.19%

$$\text{Trans} = 20 \log_{10} \left| \frac{w_{\text{out}}}{w_{\text{in}}} \right| \quad (31)$$

where w_{in} and w_{out} correspond to the displacements of the input and output points of the grillage structures.

Specifically, for the boundary i subjected to transverse line excitation, its expression follows from the Fourier series definition as:

$$\begin{aligned} {}^1T_{S_0} &= \frac{1}{2b} \int_{-b}^b \hat{f}(y, \omega) dy \\ {}^1T_{S_m} &= \frac{1}{b} \int_{-b}^b \hat{f}(y, \omega) \cos \frac{m\pi y}{b} dy \\ {}^1T_{A_m} &= \frac{1}{b} \int_{-b}^b \hat{f}(y, \omega) \sin \frac{(2m-1)\pi y}{2b} dy \end{aligned} \quad (32)$$

The vertical concentrated force in the z -direction is written as:

$$\begin{aligned} {}^1T_{S_0} &= \frac{1}{2b} F_0 \\ {}^1T_{S_m} &= \frac{1}{b} F_0 \cos \frac{m\pi y_F}{b} \\ {}^1T_{A_m} &= \frac{1}{b} F_0 \sin \frac{(2m-1)\pi y_F}{2b} \end{aligned} \quad (33)$$

where F_0 is the magnitude of the force, and y_F denotes the y -coordinate of the excitation point.

The displacement is computed as:

$$w(a, y) = {}^1w_{S_0} + \sum_{m=1}^M {}^1w_{S_m} \cos \frac{m\pi y}{b} + \sum_{m=1}^M {}^1w_{A_m} \sin \frac{(2m-1)\pi y}{2b} \quad (34)$$

2.2.3. Wave behavior formulation

This subsection addresses the wave propagation formulation of grillage structures, in which both band structure and transmission analyses are considered. The band structure is obtained from the Bloch eigenvalue problem of the grillage unit cell, whereas the transmission characteristics are evaluated through the forced vibration response of finite periodic assemblies. In the following, the band structure

formulation for an infinitely periodic grillage structure is first presented.

Taking Model I as an example, the unit cell is discretized into ten edges, as illustrated in Fig. 3. The displacement vector of the unit cell is accordingly defined by collecting the displacement components along all edges, which is given by:

$$\tilde{\mathbf{q}}_{\text{Model I}} = [\tilde{\mathbf{q}}_1^T \quad \tilde{\mathbf{q}}_2^T \quad \cdots \quad \tilde{\mathbf{q}}_{10}^T]^T \quad (35)$$

where $\tilde{\mathbf{q}}_i^T$ denotes the displacement vector associated with the i -th edge of the unit cell.

According to Bloch's theorem, the displacement vectors on corresponding edges of two adjacent unit cells are related through a phase shift determined by the Bloch wave vector. For a periodic grillage structure, the displacement continuity can be written as:

$$\begin{aligned} \tilde{\mathbf{q}}_5 &= e^{-i\varepsilon_1} \tilde{\mathbf{q}}_1, \quad \tilde{\mathbf{q}}_6 = e^{-i\varepsilon_2} \tilde{\mathbf{q}}_2 \\ \tilde{\mathbf{q}}_7 &= e^{-i\varepsilon_2} \tilde{\mathbf{q}}_3, \quad \tilde{\mathbf{q}}_8 = e^{-i\varepsilon_2} \tilde{\mathbf{q}}_4 \end{aligned} \quad (36)$$

where ε_1 and ε_2 are the phase constants associated with the two periodic directions. They represent the phase shift between corresponding boundary displacement vectors of adjacent unit cells, as determined by the Bloch wave vector. Specifically, $\varepsilon_1 = 2ak_x$ and $\varepsilon_2 = 2bk_y$, where k_x and k_y are the components of the Bloch wave vector in the two periodic directions.

Hence, the displacement vectors $\tilde{\mathbf{q}}_{\text{Model I}}$ take the following simplified form:

$$\tilde{\mathbf{q}}_{\text{Model I}} = \mathbf{T}_S \tilde{\mathbf{q}}_{\text{Model I-S}} \quad (37)$$

where $\tilde{\mathbf{q}}_{\text{Model I-S}} = [\tilde{\mathbf{q}}_1^T \quad \tilde{\mathbf{q}}_2^T \quad \tilde{\mathbf{q}}_3^T \quad \tilde{\mathbf{q}}_4^T \quad \tilde{\mathbf{q}}_9^T \quad \tilde{\mathbf{q}}_{10}^T]^T$; $\mathbf{T}_S$ is the transformation matrix introduced to enforce the Bloch periodic boundary conditions in matrix form, by expressing the full boundary displacement vector of the unit cell in terms of a reduced set of independent boundary degrees of freedom. Based on the above phase relations, $\mathbf{T}_S$ can be written as:

Table 5
Natural frequencies of the Model III.

Boundary condition	Method	Natural frequency (Hz)					
		1	2	3	5	10	20
FFFF	Present	64.8	83.7	87.0	133.5	253.2	543.8
	COMSOL	64.8	83.5	87.0	133.1	252.9	543.6
	Relative error	0.05%	0.20%	0.01%	0.27%	0.11%	0.04%
FSFS	Present	123.5	133.0	141.5	302.4	570.3	823.5
	COMSOL	123.3	132.8	141.3	302.2	569.5	822.3
	Relative error	0.15%	0.18%	0.16%	0.07%	0.14%	0.15%
FCFC	Present	179.7	186.9	193.3	426.5	594.5	897.2
	COMSOL	179.6	186.5	193.2	426.1	593.6	895.7
	Relative error	0.07%	0.22%	0.05%	0.10%	0.15%	0.17%

Table 6
Natural frequencies of the Model IV.

Boundary condition	Method	Natural frequency (Hz)					
		1	2	3	5	10	20
FFFF	Present	209.2	213.5	242.4	299.8	402.2	738.9
	COMSOL	209.2	213.4	242.5	299.7	401.7	738.9
	Relative error	0.00%	0.03%	−0.02%	0.03%	0.12%	0.00%
FSFS	Present	251.8	263.0	345.7	436.1	689.6	1147.5
	COMSOL	251.5	262.6	345.3	435.3	687.7	1145.6
	Relative error	0.14%	0.16%	0.11%	0.18%	0.28%	0.16%
FCFC	Present	289.0	298.9	374.7	530.7	764.1	1276.0
	COMSOL	288.7	298.5	374.5	530.6	763.1	1273.9
	Relative error	0.11%	0.12%	0.06%	0.03%	0.12%	0.17%

Table 7
Natural frequencies of the Model V.

Boundary condition	Method	Natural frequency (Hz)					
		1	2	3	5	10	20
FFFF	Present	135.2	154.2	154.2	172.3	498.9	626.9
	COMSOL	135.2	154.1	154.1	172.2	498.7	626.4
	Relative error	0.02%	0.05%	0.05%	0.06%	0.03%	0.08%
FSFS	Present	358.2	369.9	394.9	409.4	819.7	1119.2
	COMSOL	358.2	369.9	394.6	408.5	819.0	1118.5
	Relative error	0.01%	0.01%	0.08%	0.23%	0.08%	0.07%
FCFC	Present	494.3	503.0	540.3	556.4	914.0	1374.0
	COMSOL	494.1	502.9	539.9	556.2	912.5	1372.8
	Relative error	0.05%	0.01%	0.07%	0.03%	0.17%	0.09%

$$\mathbf{T}_S = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}e^{-ie_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}e^{-ie_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}e^{-ie_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}e^{-ie_2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \quad (38)$$

By substituting Eq. (37) into Eq. (23) and pre-multiplying both sides by the transpose of $\mathbf{T}_S$, the following expression is obtained:

$$\mathbf{T}_S^T(\varepsilon_1, \varepsilon_2) \tilde{\mathbf{K}} \mathbf{T}_S(\varepsilon_1, \varepsilon_2) \tilde{\mathbf{q}}_S = \tilde{\mathbf{K}}_S(\omega, \varepsilon_1, \varepsilon_2) \tilde{\mathbf{q}}_S = \mathbf{0} \quad (39)$$

It should be noted that the size of the matrix increases rapidly with the truncation order of the series expansion adopted in the dynamic stiffness formulation. As a result, to improve computational efficiency, the boundary edges of the unit cell are classified into internal edges and external edges according to whether they participate in the periodicity of the unit cell.

$$\tilde{\mathbf{K}}_{\text{Model I}} \tilde{\mathbf{q}}_{\text{Model I}} = \begin{bmatrix} \tilde{\mathbf{K}}_{EE} & \tilde{\mathbf{K}}_{EI} \\ \tilde{\mathbf{K}}_{IE} & \tilde{\mathbf{K}}_{II} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{q}}_E \\ \tilde{\mathbf{q}}_I \end{bmatrix} = \mathbf{0} \quad (40)$$

The dynamic stiffness matrix of the external edge can be expressed as:

$$\tilde{\mathbf{K}}_{\text{Model I-E}} = \tilde{\mathbf{K}}_{EE} - \tilde{\mathbf{K}}_{EI} \tilde{\mathbf{K}}_{II}^{-1} \tilde{\mathbf{K}}_{IE} \quad (41)$$

Similarly, based on Bloch's theorem, the displacement vector of the external edges takes the following reduced form:

$$\tilde{\mathbf{q}}_E = \mathbf{T}_{E-S}(\varepsilon_1, \varepsilon_2) \tilde{\mathbf{q}}_{E-S} \quad (42)$$

$$\text{where for Model I, } \tilde{\mathbf{q}}_{E-S} = \begin{bmatrix} \tilde{\mathbf{q}}_1^T & \tilde{\mathbf{q}}_2^T & \tilde{\mathbf{q}}_3^T & \tilde{\mathbf{q}}_4^T \end{bmatrix}^T.$$

Similarly, after condensation of the internal edges, the displacement vector of the external edges can be expressed in terms of the independent external boundary degrees of freedom through the transformation matrix $\mathbf{T}_{E-S}$. For Model I, the explicit form of $\mathbf{T}_{E-S}$ is given as follows:

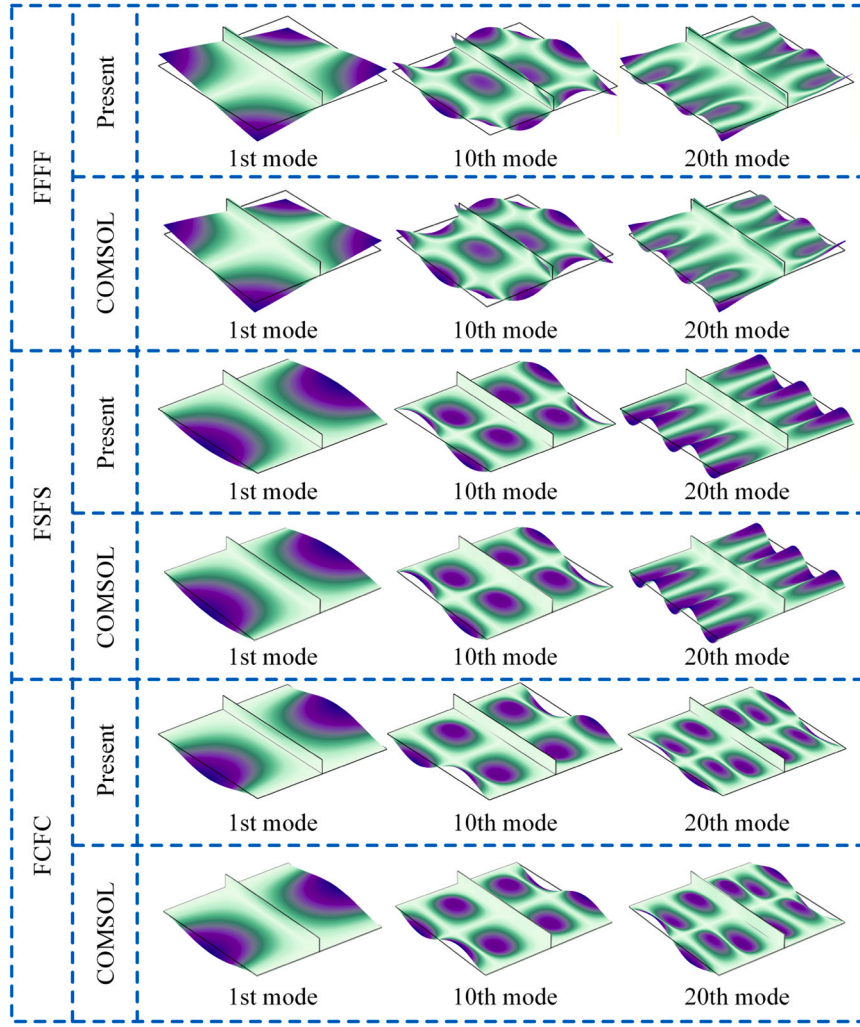


Fig. 8. Comparison of mode shapes between the present method and COMSOL for Model I under FFFF, FSFS, and FCFC boundary conditions. The 1st, 10th, and 20th modes are selected as representative cases.

$$\mathbf{T}_{E-S} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{I}e^{-i\varepsilon_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}e^{-i\varepsilon_2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}e^{-i\varepsilon_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}e^{-i\varepsilon_2} \end{bmatrix} \quad (43)$$

Similarly, the following expression is obtained:

$$\mathbf{T}_{E-S}^T(\varepsilon_1, \varepsilon_2) \tilde{\mathbf{K}}_E \mathbf{T}_{E-S}(\varepsilon_1, \varepsilon_2) \tilde{\mathbf{q}}_{E-S} = \tilde{\mathbf{K}}_{E-S}(\omega, \varepsilon_1, \varepsilon_2) \tilde{\mathbf{q}}_{E-S} = \mathbf{0} \quad (44)$$

For a given Bloch wave vector (k_x, k_y) , or equivalently for a given set of phase constants $(\varepsilon_1, \varepsilon_2)$, the dynamic stiffness equation above is formally equivalent to a free vibration problem, and its eigenvalues can be determined using the approach described in Section 2.2.2. By varying the Bloch wave vector (k_x, k_y) along the prescribed path in the irreducible Brillouin zone and evaluating the corresponding phase constants, the complete band structure can be obtained.

In addition, flexural waves dominate the low-frequency response of the grillage structures and largely determine their dynamic behavior, making it essential to compute the flexural wave band structure separately. To isolate the flexural wave bands, the in-plane displacements of the face sheets are constrained to zero, while the in-plane degrees of freedom of the stiffeners or core plates are retained. This ensures that the transverse motion is accurately captured without interference from the

in-plane deformation. The corresponding dynamic stiffness matrix is then solved following the same procedure.

2.2.4. Displacement field reconstruction

In the FEM, intra-element displacements are typically interpolated using static polynomials. In contrast, the DSM constructs the field using frequency-dependent transcendental functions derived from exact analytical solutions. Due to this non-polynomial nature, standard interpolation techniques are inapplicable. Therefore, the visualization of mode shapes and wave propagation fields requires a dedicated post-processing procedure to reconstruct the continuous displacement field from the discrete nodal degrees of freedom (DOFs). In the present formulation, this reconstruction is performed after the natural frequencies, transmission characteristics, or band structures have been obtained, and its accuracy is mainly governed by the accuracy of the corresponding frequency solution, the solved boundary DOFs, and the truncation order adopted in the DSM series expansion.

This procedure consists of three sequential steps: the solution of global edge displacements, local transformation and coefficient solution, and the reconstruction of continuous fields, as illustrated in Fig. 7.

(1) Solution of Global Edge Displacements

The first step is to obtain the global edge displacement vector, denoted as $\tilde{\mathbf{q}}_G$, which varies depending on the analysis type.

For free vibration analysis: The governing equation for the

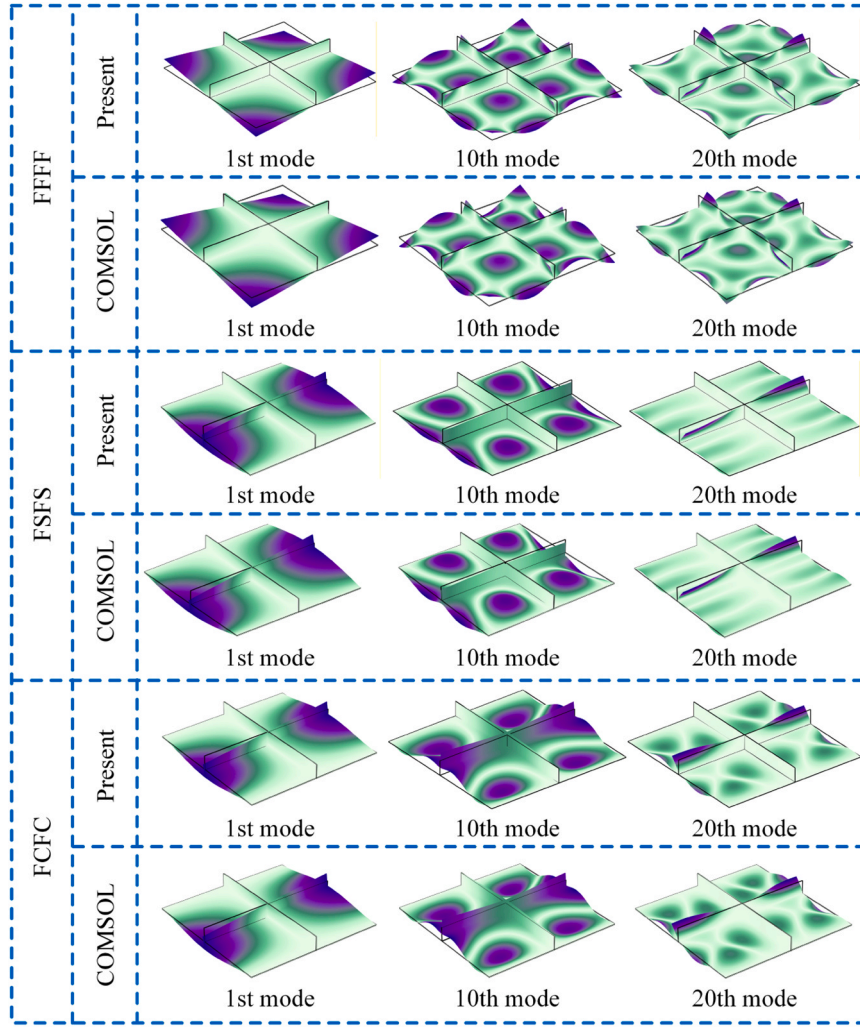


Fig. 9. Comparison of mode shapes between the present method and COMSOL for Model II under FFFF, FSFS, and FCFC boundary conditions. The 1st, 10th, and 20th modes are selected as representative cases.

finite structure is $\tilde{\mathbf{K}}(\omega)\tilde{\mathbf{q}} = \mathbf{0}$. At a natural frequency ω_n , the dynamic stiffness matrix becomes singular. To ensure numerical stability, Singular Value Decomposition (SVD) is employed to solve the non-trivial solution:

$$\tilde{\mathbf{K}}(\omega_n) = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H \quad (45)$$

The global displacement vector is identified as the right singular vector corresponding to the minimum singular value:

$$\tilde{\mathbf{q}}_G = \mathbf{v}_k, \quad k = \arg_i \min(\sigma_i) \quad (46)$$

where σ_i represents the singular values on the diagonal of $\mathbf{\Sigma}$, and $\mathbf{v}_k$ denotes the k -th column of $\mathbf{V}$.

For band structure analysis: The governing equation involves the Bloch-reduced stiffness matrix $\tilde{\mathbf{K}}_S(\omega, \varepsilon_1, \varepsilon_2)$ or $\tilde{\mathbf{K}}_{E-S}(\omega, \varepsilon_1, \varepsilon_2)$. Similar to the free vibration, SVD is applied to the Bloch-reduced stiffness matrix to obtain the generalized displacement vector $\tilde{\mathbf{q}}_S$ or $\tilde{\mathbf{q}}_{E-S}$ in the reduced coordinates. The global displacement $\tilde{\mathbf{q}}_G$ of the unit cell is then recovered using the Bloch transformation matrix $\mathbf{T}_S$ or $\mathbf{T}_{E-S}$:

$$\tilde{\mathbf{q}}_G = \mathbf{T}_S \tilde{\mathbf{q}}_S, \quad \tilde{\mathbf{q}}_G = \mathbf{T}_{E-S} \tilde{\mathbf{q}}_{E-S} \quad (47)$$

For transmission characteristics analysis: The grillage

structures are subjected to an external force vector. The global displacement response is directly computed by solving the linear system:

$$\tilde{\mathbf{q}}_G(\omega) = \mathbf{K}_{total}^{-1}(\omega)\mathbf{F}(\omega) \quad (48)$$

(2) Local Transformation and Coefficient Solution

Once $\tilde{\mathbf{q}}_G$ is obtained, the displacement subset associated with the i -th plate element is extracted and transformed from the global coordinate system (x^g, y^g, z^g) to the local element coordinate system (x^l, y^l, z^l) . Let $\mathbf{T}_i$ be the coordinate transformation matrix accounting for the spatial rotation of the i -th plate element. The local edge displacement vector $\tilde{\mathbf{q}}_l^i$ is given by:

$$\tilde{\mathbf{q}}_l^i = \mathbf{T}_i \tilde{\mathbf{q}}_G^i \quad (49)$$

In the DSM formulation, the relationship between the edge displacements and the vector of coefficients $\mathbf{C}^i$ is defined by the plate element dynamic stiffness matrix $\mathbf{D}^i$. Therefore, the coefficients can be solved by inverting this relationship:

$$\mathbf{C}^i = (\mathbf{D}^i)^{-1} \tilde{\mathbf{q}}_l^i \quad (50)$$

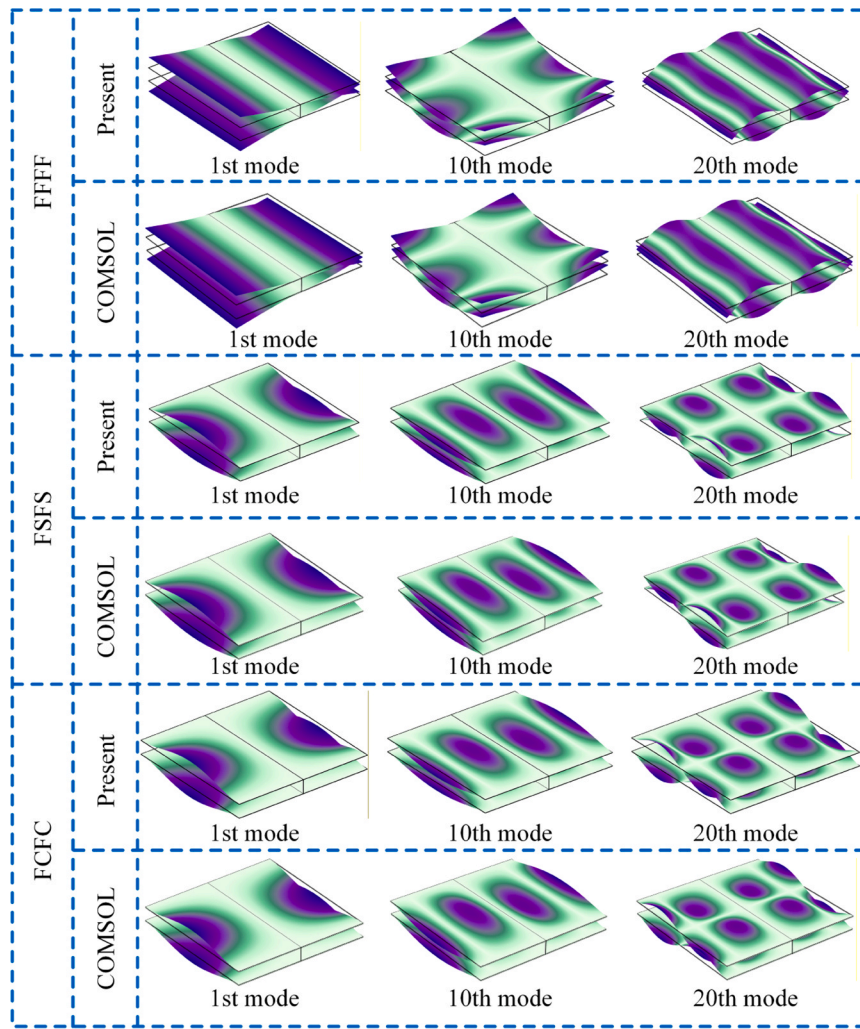


Fig. 10. Comparison of mode shapes between the present method and COMSOL for Model III under FFFF, FSFS, and FCFC boundary conditions. The 1st, 10th, and 20th modes are selected as representative cases.

(3) Reconstruction of Continuous Fields

Finally, the continuous displacement field $\mathbf{u}^i(x, y) = [u, v, w]^T$ at any point (x, y) within the i -th element is reconstructed by superposing the analytical shape functions. Let $\Phi(x, y; \omega)$ represent the matrix of shape functions (composed of trigonometric and hyperbolic terms obtained from the governing differential equations). The displacement field is given by:

$$\mathbf{u}^i(x, y) = \Phi(x, y; \omega) \mathbf{C}^i \quad (51)$$

By repeating this process for all plate elements and assembling them according to their global topology, the complete mode shape or wave propagation field of the coupled structure is visualized.

3. Results and discussion

In this section, the dynamic behaviors of the grillage structures are systematically investigated based on the theoretical framework developed in Section 2. The proposed unified theoretical modeling framework is verified by comparisons with finite element simulations for the considered dynamic behaviors and with published experimental data for wave propagation behavior. The free vibration, wave propagation, and vibration transmission characteristics of the grillage structures are studied in detail. Particular attention is paid to the effects of boundary conditions on the bandgap. Unless otherwise stated, the results

discussed in this section are obtained within the framework of Kirchhoff thin plate theory.

3.1. Free vibration analysis

The natural frequencies of the five grillage structure models under various boundary conditions are first evaluated using the proposed unified framework, as summarized in Tables 3-7. For validation, the present results are compared with those obtained from COMSOL Multiphysics. In the finite element model, all plate components are modeled using shell elements with a quadrilateral mesh. The same material properties and geometric parameters are adopted in both approaches. The natural frequencies of the first, second, third, fifth, tenth, and twentieth natural frequencies are evaluated for the FFFF, FSFS, and FCFC boundary conditions, where F, S, and C denote free, simply supported, and clamped edges, respectively. For example, FFFF indicates that all four plate edges are free. The comparison shows that the maximum relative error between the present results and the COMSOL results does not exceed 0.4%, demonstrating the accuracy of the proposed zero-crossing method and unified theoretical modeling framework. It is also observed that, as the boundary constraint strength increases, the natural frequencies gradually increase, which can be attributed to the additional stiffness introduced by the constraints, making the structure more resistant to deformation.

To verify the accuracy of the displacement field reconstruction

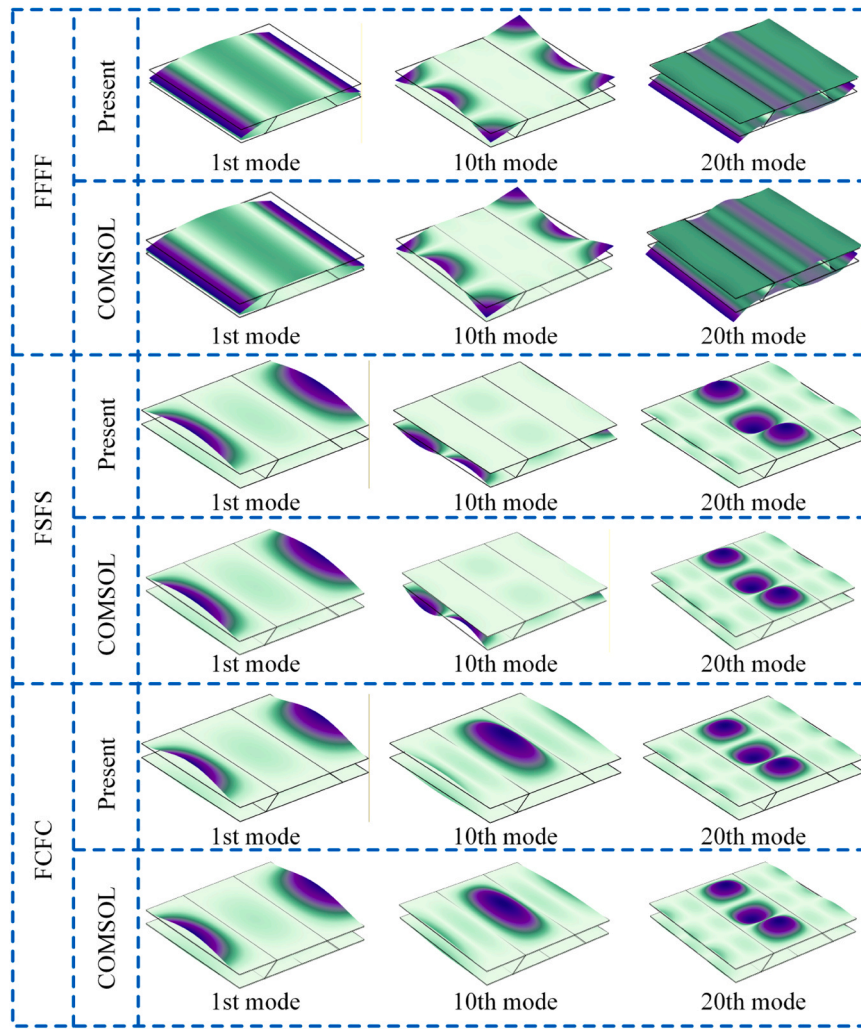


Fig. 11. Comparison of mode shapes between the present method and COMSOL for Model IV under FFFF, FSFS, and FCFC boundary conditions. The 1st, 10th, and 20th modes are selected as representative cases.

method proposed in this study, the mode shapes of the five grillage structure models are further examined and compared with those obtained from COMSOL Multiphysics. For each model, the 1st, 10th, and 20th mode shapes are selected as representative cases, as shown in Figs. 8–12. The reconstructed mode shapes agree well with the COMSOL results, confirming the accuracy of the proposed displacement field reconstruction method. It should be noted that the reconstruction is carried out as an independent post-processing step for selected representative mode shapes. In the present implementation, reconstructing the displacement field of a single selected mode shape takes approximately 1 s, indicating that the additional computational overhead is limited.

In addition, the free vibration characteristics of the grillage structures are influenced by the boundary conditions. For the first mode, both the natural frequency and mode shape under the FFFF boundary condition differ markedly from those under the FSFS and FCFC conditions, whereas the FSFS and FCFC cases exhibit nearly identical first-mode characteristics. For higher-order modes, the similarity between FSFS and FCFC modes depends on the structural type. For the stiffened plates, the 10th modes are nearly identical, while the 20th modes begin to differ. For the sandwich plates, the modes remain almost identical up to the 20th mode, except for the corrugated-core sandwich plate where the 10th mode exhibits slight differences. These observations indicate that the sensitivity of free vibration modes to boundary constraints varies with both the mode order and the structural configuration, while the

mode shapes of the FFFF boundary condition always show significant differences due to the lack of boundary stiffness.

3.2. Wave propagation characteristics

This subsection discusses the wave propagation characteristics of grillage structures. Using the proposed unified theoretical modeling framework based on the DSM and the zero-crossing method, the band structures and transmission characteristics are computed for different grillage structures. In addition, the corresponding eigenmodes of the unit cell and the forced vibration response of a finite periodic structure are reconstructed to further reveal the wave propagation behavior. It should be noted that the results presented here are obtained within the framework of Kirchhoff thin plate theory, and possible quantitative shifts may occur in the higher-frequency range due to the neglect of transverse shear effects.

3.2.1. Band structures and transmission characteristics

The band structures and transmission characteristics of Models I–V are calculated using the proposed unified framework, as shown in Figs. 13–17. Specifically, for Models II and V, the wave vector scanning path is set along the boundaries of the triangular irreducible Brillouin zone (Γ -X-M- Γ). For Models I, III, and IV, the path is extended to the boundary of the rectangular irreducible Brillouin zone (Γ -X-M-Y- Γ). In addition to the band structures including all wave modes, the band

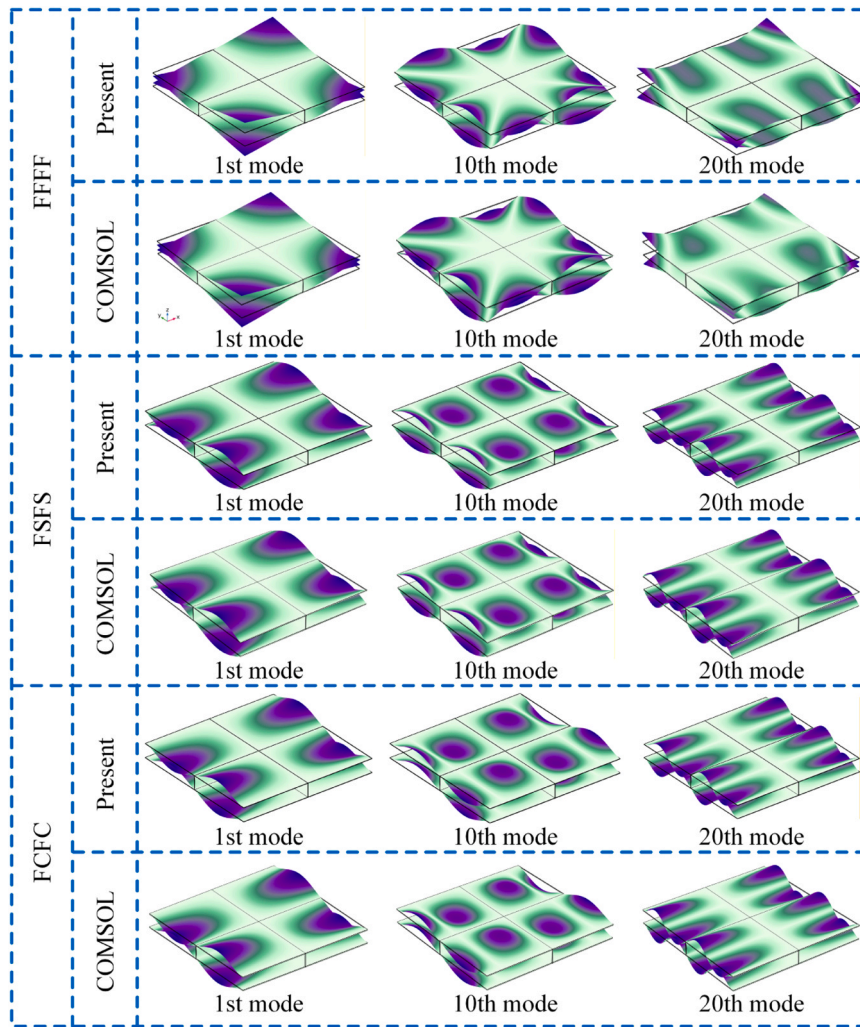


Fig. 12. Comparison of mode shapes between the present method and COMSOL for Model V under FFFF, FSFS, and FCFC boundary conditions. The 1st, 10th, and 20th modes are selected as representative cases.

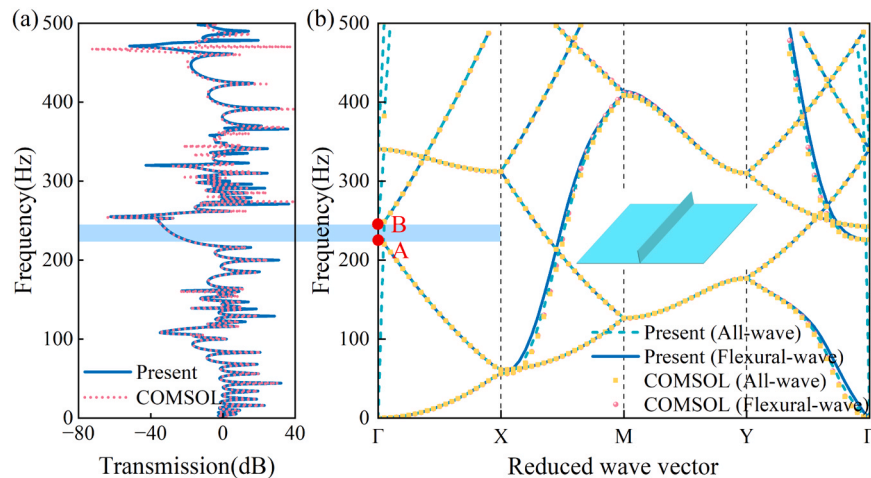


Fig. 13. Comparison of (a) transmission characteristics and (b) band structures of Model I between the present method and COMSOL for the all-wave and flexural-wave cases.

structures corresponding to pure flexural waves are also extracted for comparison, in order to distinguish the flexural-dominated branches from the coupled in-plane and out-of-plane wave modes and to reveal the role of flexural motion in the formation of bandgap characteristics.

To this end, the in-plane displacement components on the unit-cell boundaries are constrained, so that only the out-of-plane flexural-related motion is retained. The transmission characteristics are obtained using a finite periodic array of 8×2 unit cells. The bandgap frequency

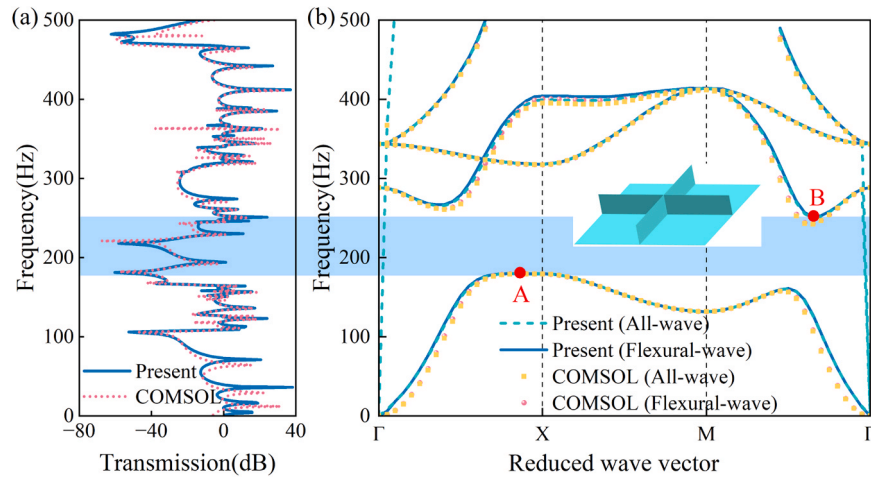


Fig. 14. Comparison of (a) transmission characteristics and (b) band structures of Model II between the present method and COMSOL for the all-wave and flexural-wave cases.

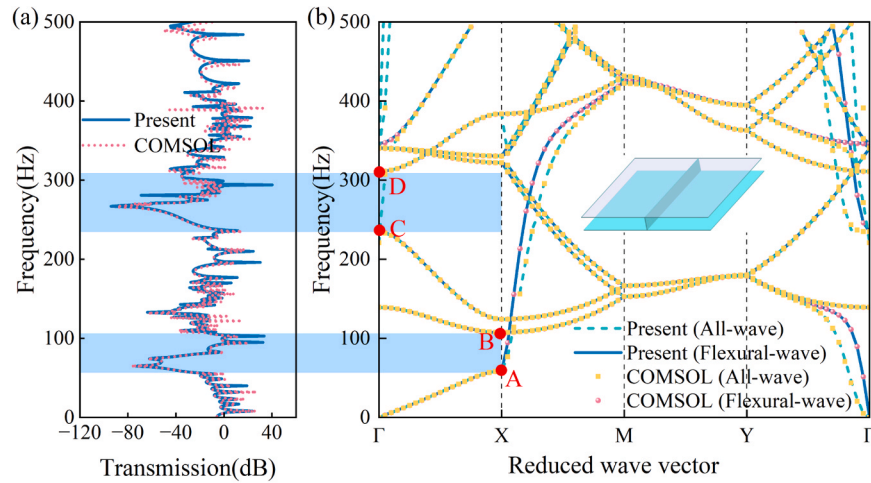


Fig. 15. Comparison of (a) transmission characteristics and (b) band structures of Model III between the present method and COMSOL for the all-wave and flexural-wave cases.

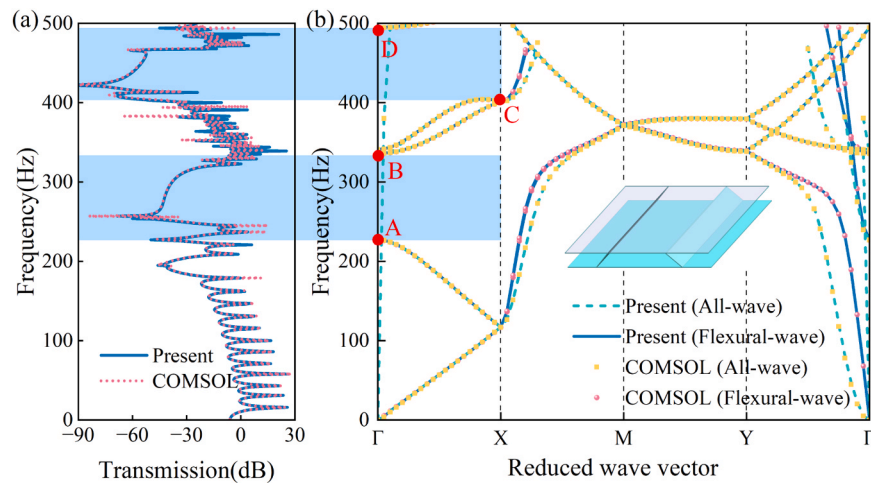


Fig. 16. Comparison of (a) transmission characteristics and (b) band structures of Model IV between the present method and COMSOL for the all-wave and flexural-wave cases.

ranges (blue shaded area) in the five models match well with the vibration attenuation zones, verifying the correctness of the proposed framework. To further verify the results of the band structure and the

transmission characteristics, numerical simulations are performed using COMSOL with material and structural parameters identical to those in the proposed framework. All plates are modeled using shell elements

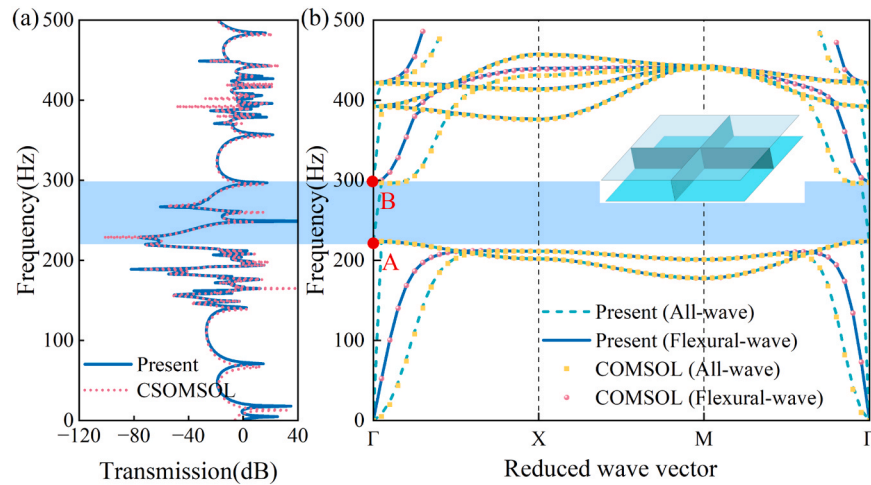


Fig. 17. Comparison of (a) transmission characteristics and (b) band structures of Model V between the present method and COMSOL for the all-wave and flexural-wave cases.

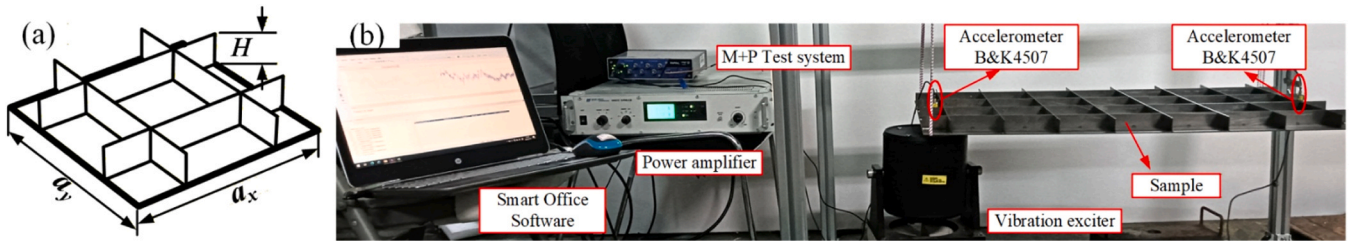


Fig. 18. (a) Unit cell model, (b) Experimental device [27].

with a quadrilateral mesh. Floquet periodic boundary conditions are applied in the x and y directions for band structure calculations, whereas a finite periodic array of 8×2 unit cells is constructed to evaluate the vibration transmission characteristics. The present results show excellent agreement with the COMSOL simulations for both the band structures and the transmission characteristics, further validating the accuracy of the proposed unified framework.

In addition, comparing the extracted flexural-wave band structures with the corresponding all-wave results shows that the longitudinal and shear wave branches are effectively eliminated, while the remaining flexural-wave dispersion curves exhibit excellent agreement with their counterparts in the full-wave results. This comparison indicates that the additional branches in the all-wave band structures are mainly associated with in-plane longitudinal and shear-dominated motions. Since the flexural-wave branches remain almost unchanged after these in-plane-dominated branches are removed, their influence on the flexural-wave bandgap boundaries is limited for the frequency range and grillage configurations considered here. This observation further confirms the validity and accuracy of the proposed flexural-wave band structure calculation approach for grillage structures.

The bandgap characteristics of the grillage structures in the 0–500 Hz frequency range are further analyzed. Model I exhibits a unidirectional flexural-wave bandgap (225.9–242.6 Hz) along the x -direction, while Model II shows a complete bandgap ranging from 179.8 to 251 Hz. Model III presents two unidirectional flexural-wave bandgaps (58.59–107.53 Hz and 234.81–310.98 Hz) along the x -direction, and Model IV also features two unidirectional flexural-wave bandgaps (226.99–336.05 Hz and 400.93–494.63 Hz) along the x -direction. Model V exhibits a complete bandgap from 224.03 to 297.1 Hz. The comparison indicates that grillage structures with orthogonal symmetry (Models II and V) are capable of forming complete bandgaps, whereas the grillage structures with predominantly one-directional periodicity

mainly exhibit unidirectional flexural-wave bandgaps.

The proposed framework is further validated against published experimental data to assess its accuracy in predicting wave propagation in grillage structures. Specifically, the feasibility and accuracy of the unified framework are verified using the structural model and the corresponding measurement data reported in Ref. [27]. The configuration of a representative unit cell and the corresponding experimental setup are shown in Fig. 18.

The band structures computed by the proposed framework are compared with the corresponding numerical and experimental transmission results from Ref. [27], as shown in Fig. 19. The predicted bandgaps are generally consistent with the vibration attenuation region observed in the measured transmission characteristics, confirming the reliability of the present framework. Some differences nevertheless appear in the high-frequency band structure. To clarify the source of these differences, the eigenmodes corresponding to points A–D in Fig. 19 (b) are extracted, as shown in Fig. 20(a)–(d). It is found that the additional bands within the bandgap primarily arise from the motion of the stiffeners in the grillage structure, while the host plate remains essentially stationary. These high-frequency differences originate from the different modeling assumptions adopted: the present study employs a plate–plate coupling model that explicitly captures the dynamic behavior of the stiffeners, whereas Ref. [27] adopts a plate–beam coupling model in which this effect is represented in a simplified manner. By characterizing the dynamic behavior of both the host plate and the stiffeners, the present plate–plate coupling approach provides a more realistic prediction of the band structures of grillage structures. The present results also agree closely with numerical simulations based on the same plate–plate coupling model, further confirming the accuracy of the proposed framework in predicting wave propagation behavior.

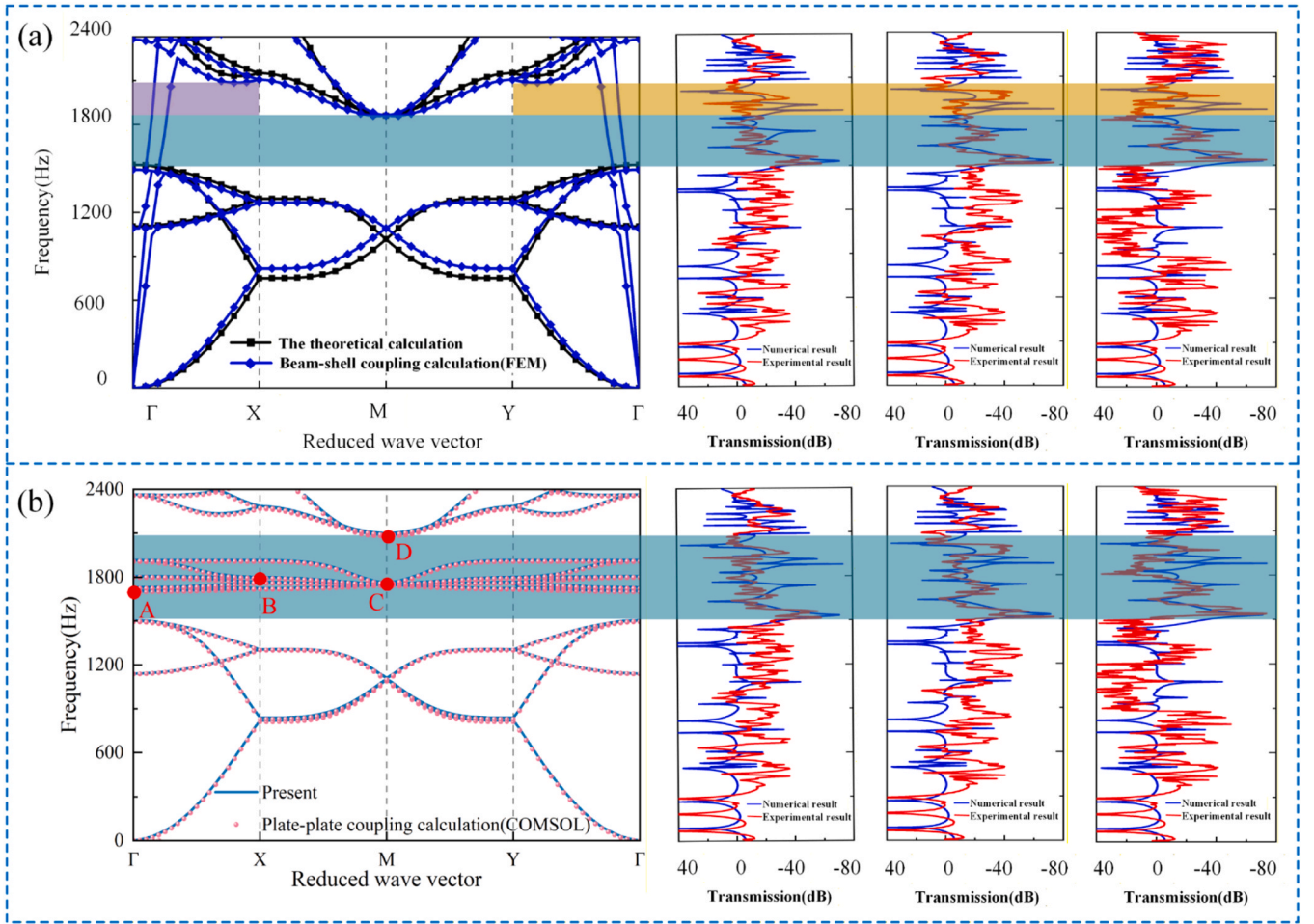


Fig. 19. (a) Band structures and transmission characteristics from the Ref. [27]; (b) Band structures from the present work compared with the transmission characteristics from the Ref. [27].

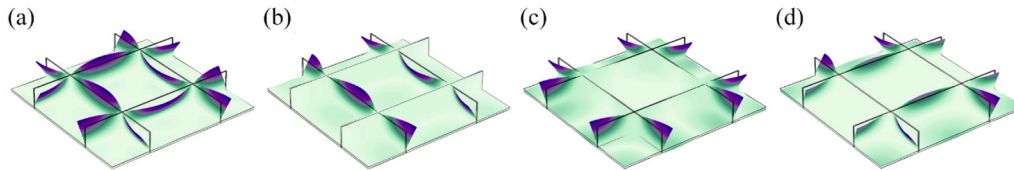


Fig. 20. Eigenmodes of unit cell marked as A, B, C, D in Fig. 19(b).

3.2.2. Flexural wave modes and propagation

To more intuitively elucidate the formation mechanisms of flexural wave bandgaps in grillage structures and to validate the accuracy of the displacement field reconstruction method proposed in this study, the total displacement fields of the unit cells in the specific wave vectors marked in the band structure diagrams are extracted using the present method and COMSOL simulations, respectively. Fig. 21 presents the total displacement fields of eigenmodes corresponding to the lower and upper edges of the flexural wave bandgaps for Models I-V. The present results show excellent agreement with those from COMSOL results, confirming the accuracy of the proposed displacement field reconstruction approach. Moreover, owing to the plate-plate coupling formulation adopted in this study, the vibration modes of the face plates, core plates, and stiffeners can be clearly displayed in the reconstructed displacement fields, providing a more intuitive description of the underlying wave motion.

In addition, it can be observed that for the grillage structure models

with unidirectional symmetry (Models I, III, and IV), flexural-wave bandgaps occur only along the Γ -X direction. This directional bandgap behavior is consistent with the unidirectional periodic geometry of these models. As shown in Fig. 21, the bandgap-edge modes of Models I, III, and IV are dominated by flexural deformation of the face plates between periodically arranged stiffeners or core plates, and the displacement variation is mainly governed by the structural periodicity along the Γ -X direction. The eigenmodes at the lower and upper edges of these bandgaps exhibit different standing-wave-like flexural deformation patterns of the face plates with respect to the periodically arranged stiffeners or core plates. At the lower bandgap edge, the vibration is dominated by the global flexural motion of the grillage structures, whereas at the upper edge the face plates still undergo overall flexural deformation while the stiffeners or core plates remain nearly stationary. Therefore, the opening of the unidirectional bandgaps can be interpreted from the displacement fields as a transition between two flexural-wave states constrained by the same one-directional periodic geometry.

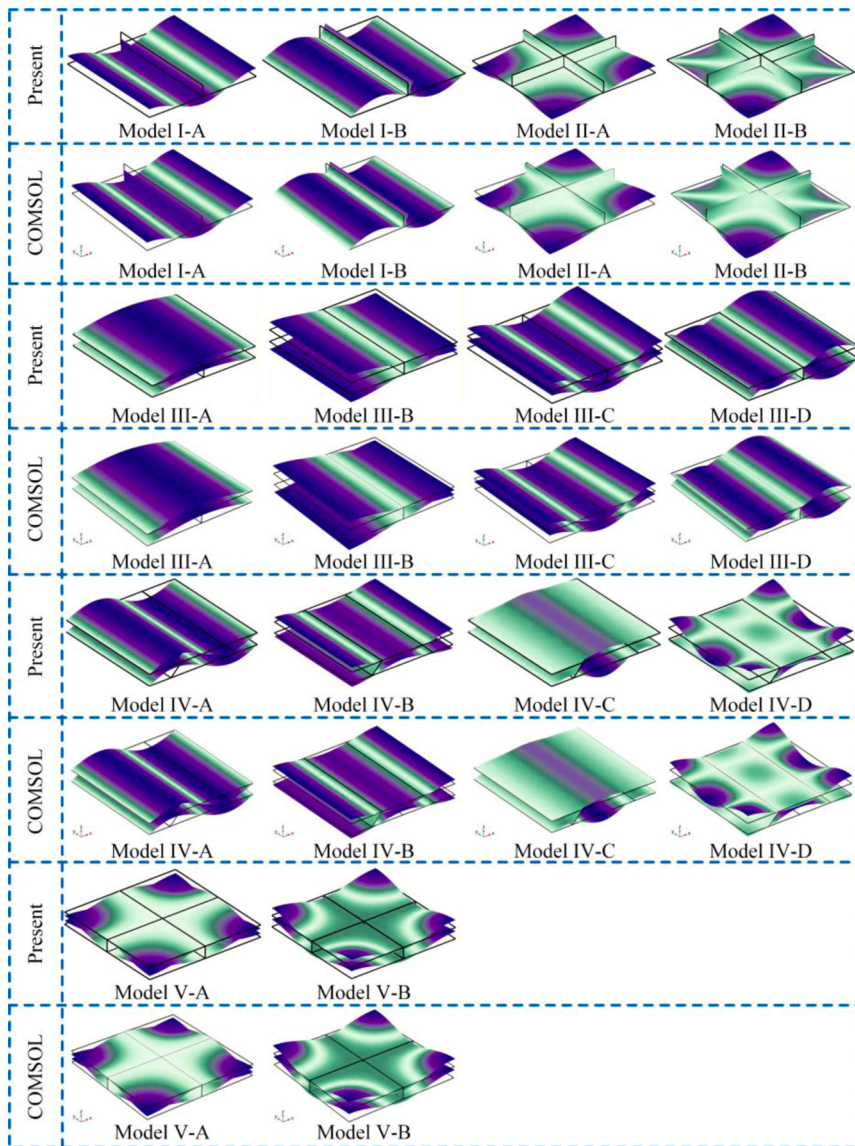


Fig. 21. Comparison of the total displacement fields of the eigenmodes at the lower and upper edges of the flexural wave bandgaps for Models I-V, obtained by the present method and COMSOL simulations.

For the grillage structure models with orthogonal symmetry (Models II and V), the scattering effect induced by the orthogonally arranged stiffeners or core plates leads to pronounced deformation near the four corners of the unit cell, while these elements themselves remain almost stationary. In these two models, the orthogonal stiffeners or core plates divide the unit cell in both principal directions, and the reconstructed displacement fields show deformation patterns controlled by this two-directional periodic constraint. This explains why Models II and V can form complete bandgaps, rather than only directional bandgaps along Γ -X. Similar to the unidirectional cases, the lower- and upper-edge eigenmodes exhibit different flexural standing-wave states with respect to the periodic scatterers. These total displacement fields suggest that the flexural-wave bandgaps originate predominantly from Bragg-type scattering caused by the periodic modulation of the coupled plate components, as reflected by the transition between different flexural standing-wave states.

In addition, to provide a more intuitive illustration of the bandgap suppression effect on flexural wave propagation, the displacement fields inside and outside the flexural wave bandgaps for the finite periodic structures of Models I-V are calculated using both the present method

and COMSOL simulations, as shown in Fig. 22. It can be found that the present results are highly consistent with the COMSOL results. Specifically, at frequencies outside the bandgap, the flexural vibrations propagate along the finite periodic structures with minimal attenuation, while at frequencies inside the bandgap, their propagation is strongly suppressed, indicating a significant attenuation effect induced by the bandgaps. These observations not only demonstrate the effectiveness of the flexural wave bandgaps in controlling wave propagation but also confirm the accuracy of the proposed displacement field reconstruction method.

3.2.3. Effect of boundary conditions on bandgaps

To investigate the effect of boundary conditions on bandgap characteristics, the center frequencies f_c of the first bandgap of Models I-V under five representative boundary conditions, namely FFFF, FTFT1, FSFS, FTFT2, and FCFC, are calculated using the proposed unified theoretical modeling framework, as shown in Fig. 23. The two transition cases, FTFT1 and FTFT2, are introduced between the classical boundary conditions by prescribing two intermediate levels of non-dimensional boundary restraint between the free and clamped limits. These cases

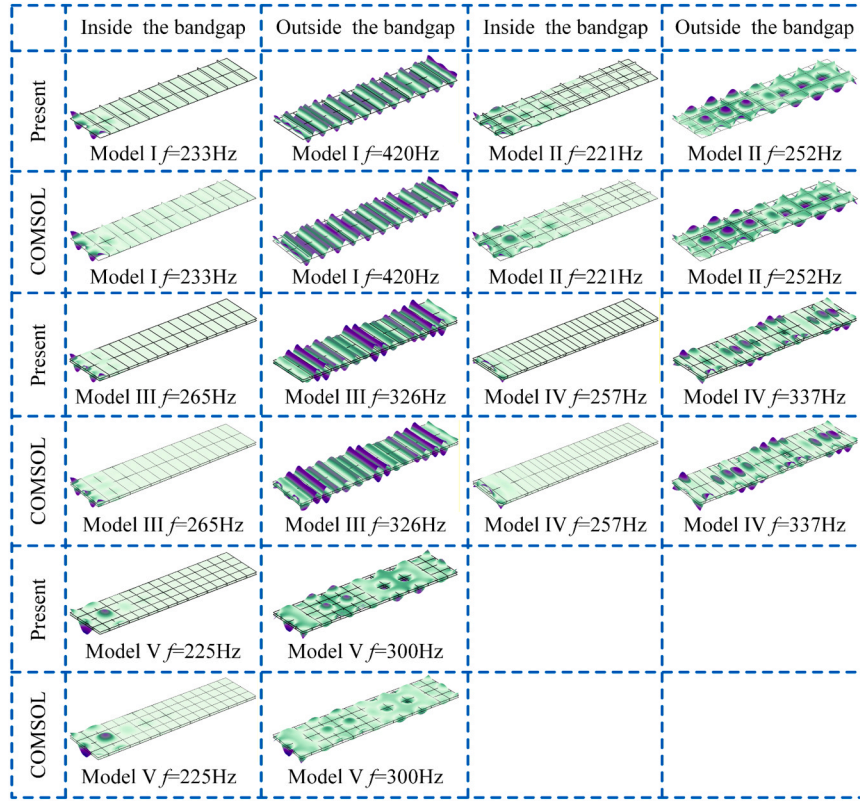


Fig. 22. Comparison of the flexural wave propagation displacement fields of finite periodic structures inside and outside the bandgaps for Models I–V, obtained by the present method and COMSOL simulations.

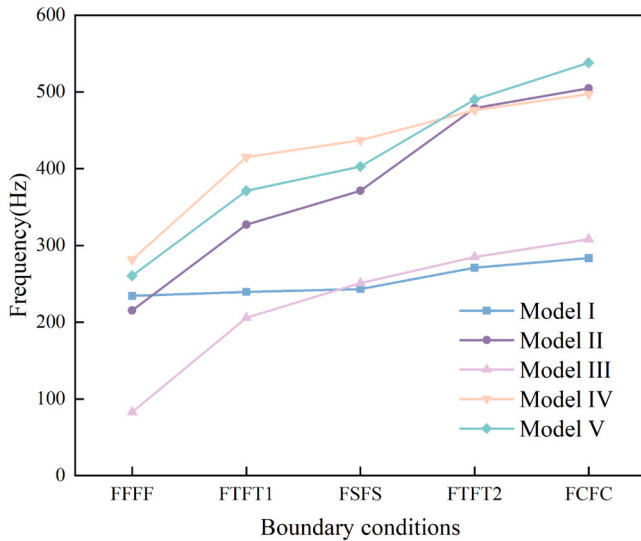


Fig. 23. Effect of boundary conditions on the bandgap characteristics of Models I–V, showing the variation of bandgap ranges under different boundary constraints.

are used to illustrate the applicability of the proposed framework to general elastic boundary conditions and to capture the gradual variation of bandgap characteristics with increasing boundary constraint. The results reveal that the bandgap center frequencies generally exhibit a continuous upward trend as the boundary constraint strength increases from FFFF to FCFC. This behavior arises because stronger boundary constraints restrict rotational and translational degrees of freedom, thereby enhancing the global bending stiffness of the grillage structures

and increasing the equivalent phase velocity c_{eff} of flexural waves. According to the Bragg scattering condition, the bandgap center frequency scales as $f_c \propto c_{eff}/a$. Consequently, the increase in equivalent phase velocity c_{eff} directly leads to an overall upshift of the bandgap center frequency f_c .

To more intuitively illustrate the impact of boundary conditions on bandgap characteristics, the total displacement fields of the eigenmodes at the lower and upper edges of the first bandgap for Models I–V are extracted, as shown in Fig. 24. The results indicate that boundary conditions significantly modify the global mode shapes of the grillage structures by constraining displacements at the plate edges. Under the FFFF boundary condition, the eigenmodes at the lower and upper edges of the bandgaps are dominated by low order global flexural modes. As the boundary condition changes to FSFS or FCFC, the displacements near the constrained edges are markedly reduced, and the primary deformation region shifts toward the interior of the grillage structures. This modal transition suppresses low order global modes and promotes higher order deformation modes, which effectively increases structural stiffness and leads to an upward shift of the bandgap center frequency.

4. Convergence and computational efficiency study

This section provides practical guidance for selecting the DSM truncation order and FEM mesh size and compares the two methods in terms of degrees of freedom (DOFs) and convergence rate.

For the DSM, the displacement field is represented using a truncated Fourier series expansion. As the calculation accuracy is intrinsically dependent on the number of retained Fourier terms, the series must be truncated to a finite order M for practical computation. To ensure that the highest retained harmonic is capable of resolving the spatial variations associated with the maximum wavenumber k_{max} (evaluated at the upper frequency limit), the maximum boundary wavenumber $M\pi/L$ must satisfy the condition $M\pi/L \geq k_{max}$. This yields a direct criterion for

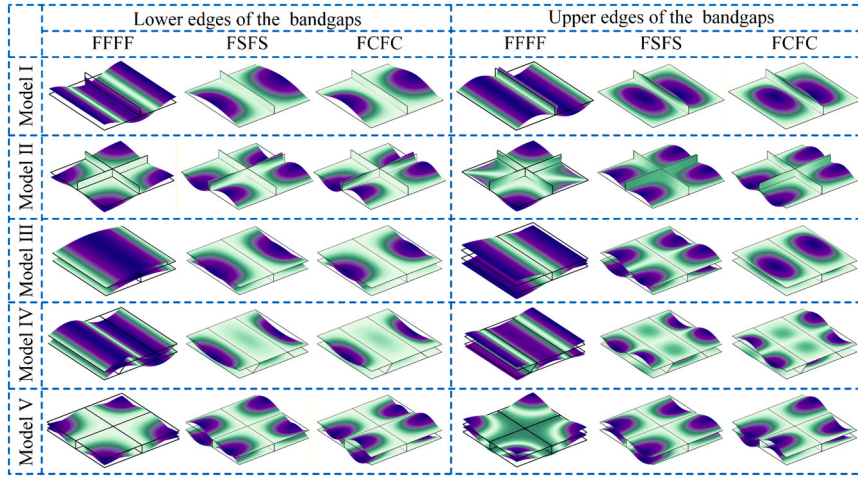


Fig. 24. Comparison of the eigenmodes at the lower and upper edges of the flexural wave bandgaps for Models I-V under FFFF, FSFS, and FCFC boundary conditions.

Table 8
Discretization cases for the convergence analysis (Model I, 0–500 Hz).

Case	Truncation Order (M)	Scaling Factor (r)	h_p (mm)
1	1	4	93.50
2	2	3	70.12
3	3	2	46.75
4	4	1	23.37
5	5	0.8	18.70
6	6	0.6	14.02
7	8	0.4	9.35
8	12	0.1	2.34

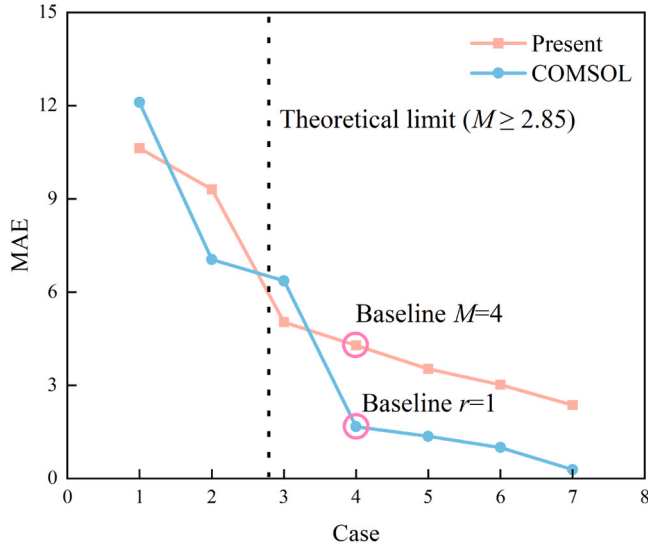


Fig. 25. Convergence of the MAE for the present method and COMSOL with respect to the DSM truncation order and FEM mesh size, respectively.

selecting the truncation order:

$$M \geq \frac{2L}{\lambda_{\min}} \quad (52)$$

where L denotes the length of the plate.

To quantify $\lambda_{\min}$, one must consider the dispersion relation of flexural waves. For a thin plate with thickness h , density ρ , and bending stiffness D , the wavenumber k at an angular frequency ω is governed by the characteristic equation $\omega^2 = (D/\rho h)k^4$. Consequently, the wavelength at

the upper frequency limit $f_{\max}$ is expressed as:

$$\lambda(\omega) = \frac{2\pi}{k(\omega_{\max})} = 2\pi \left(\frac{D}{\rho h (2\pi f_{\max})^2} \right)^{1/4} \quad (53)$$

In comparison, the discretization accuracy of the FEM is strictly dictated by the mesh density relative to this same minimum wavelength. To accurately capture global deformations using quadratic shell elements, the maximum global element size h_p is constrained by the criterion $h_p \leq \lambda_{\min}/6$.

To validate these criteria and evaluate computational efficiency, a convergence benchmark is conducted on Model I within the 0–500 Hz frequency range. Based on the material and geometric parameters, the minimum wavelength at the upper frequency limit is calculated as $\lambda_{\min} \approx 140.2$ mm. Applying the DSM criterion to Model I, the lower limit of the truncation order is determined to be $M \geq 2.85$, indicating that a low truncation order of $M_0 = 4$ is sufficient as a conservative baseline. Conversely, applying the FEM criterion requires a global mesh size of approximately $h_{p0} \approx 23.4$ mm.

Based on the above criteria, a series of DSM truncation orders and FEM mesh refinements are constructed for the convergence study. The corresponding discretization parameters are summarized in Table 8, where eight representative cases are considered. For the DSM, the truncation order M is gradually increased from an under-resolved level ($M=1$) to a sufficiently large value ($M=12$). For the FEM, a single refinement factor r is introduced to scale both the global mesh size h_p and the local stiffener mesh size h_l in a consistent manner, with $r=1$ corresponding to the baseline mesh predicted by the wavelength based criterion.

The convergence behavior of the two methods is evaluated in terms of the accuracy of the transmission characteristics. Fig. 25 compares the mean absolute error (MAE) of the transmission characteristics relative to the finest discretization for both DSM and FEM across the eight computational cases defined in Table 8. For the DSM (pink line), a rapid reduction in MAE is observed as the case index increases (corresponding to an increase in M). The error drops significantly once the theoretical lower bound $M \geq 2.85$ (indicated by the vertical dashed line) is satisfied. Notably, the chosen baseline at Case 4 ($M=4$) yields a stable transmission error. Beyond this point, further increasing the truncation order leads to diminishing improvements, indicating that the DSM solution has effectively reached its converged spectral regime. This behavior confirms the validity of the proposed truncation criterion and demonstrates that the dominant spatial features of the response are accurately captured once the half-wave requirement is met. Similarly, the FEM convergence curve (blue line) exhibits a consistent monotonically decreasing trend as the mesh density increases. A significant reduction

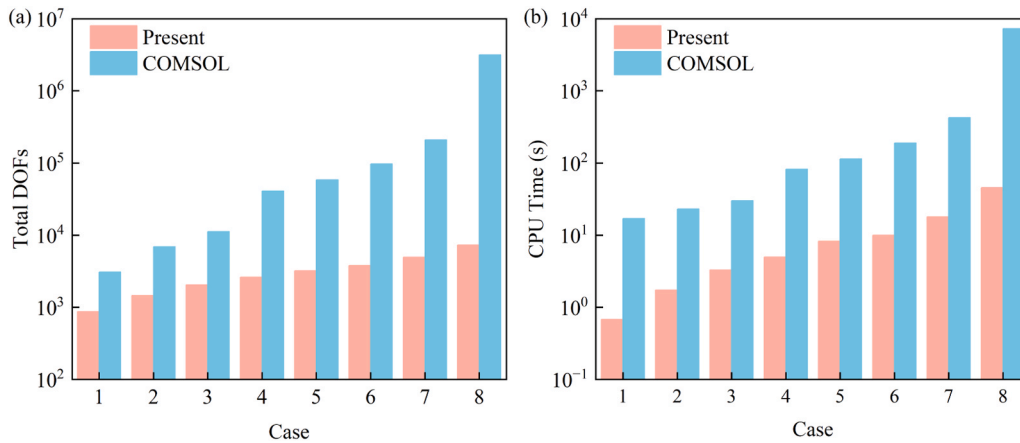


Fig. 26. Comparison of the total number of DOFs and computational cost between the present method and COMSOL: (a) total number of DOFs across the tested cases; (b) CPU time across the tested cases.

in error is observed as the model transitions from the coarse mesh (Case 1–3) to the refined mesh regime. At the baseline Case 4 ($r = 1$), the FEM solution reaches a low error level comparable to that of the DSM. This consistency between the two distinct methods reinforces the validity of the proposed theoretical convergence criteria, confirming that the physical wavelength resolution defined at Case 4 is indeed sufficient to capture structural dynamic behavior accurately.

To further quantify the computational implications of the convergence behavior, the total degrees of freedom (DOFs) and CPU time associated with each case are reported in Fig. 26 (a) and (b), respectively. Since the tested cases correspond to progressively increased numerical resolution, including higher DSM truncation orders and refined FEM meshes, Fig. 26 also provides a comparison of numerical scalability under progressive refinement. For all discretization levels, the DSM consistently requires orders of magnitude fewer DOFs than the FEM. At the baseline setting, the DSM model involves on the order of 10^3 DOFs, whereas the corresponding FEM model approaches 10^5 DOFs. This disparity becomes increasingly pronounced as the discretization is refined, reflecting the fundamentally different scaling characteristics of the two methods.

A similar trend is observed in the computational cost. As shown in Fig. 26(b), the CPU time required by the FEM increases rapidly with mesh refinement, reaching several thousand seconds for the finest case. In contrast, the DSM computation time remains comparatively low across all truncation orders, even for the largest value of M considered. This slower growth of computational cost with increasing numerical resolution indicates the better scalability of the DSM in the present refinement study. All computations are performed on a workstation equipped with an Intel Core i7–13700K CPU, an NVIDIA GeForce RTX 4070 Ti GPU, and 64 GB of RAM.

Overall, the convergence analysis provides direct evidence of the advantages of the proposed unified DSM-based framework over the conventional FEM in studying the dynamic behaviors of grillage structures. Specifically, the DSM exhibits a steeper reduction in the solution error under progressive refinement, indicating a faster convergence rate, and reaches the prescribed accuracy at a substantially lower computational cost, as reflected by the reduced CPU time required across the tested cases. Moreover, the much slower increase in DOFs and CPU time under progressive numerical refinement demonstrates the favorable numerical scalability of the proposed framework. These features make the proposed framework particularly suitable for repeated parametric studies and design-oriented evaluations where both accuracy and computational efficiency are critical.

5. Conclusions

In this study, a unified theoretical modeling framework based on the DSM is proposed for analyzing both vibration and wave behaviors in grillage structures. Using the proposed framework, the dynamic behaviors of five grillage configurations, including two stiffened grillages and three sandwich grillages, are systematically investigated in terms of natural frequencies, transmission characteristics, band structures, and the corresponding displacement fields. The main conclusions are summarized as follows:

- (1) The proposed framework treats different grillage configurations and boundary conditions using the same multi-plate dynamic stiffness formulation, coordinate transformation, and matrix assembly procedure, while integrating free vibration, forced vibration, wave propagation, and displacement field reconstruction within a consistent theoretical procedure. Comprehensive comparisons with numerical simulations, together with comparisons against published experimental data for wave propagation behavior, verify the accuracy and applicability of the proposed framework.
- (2) The flexural-wave bandgaps are mainly associated with Bragg-type scattering induced by the periodic arrangement of stiffeners or core plates. This mechanism is reflected by the transition between different flexural standing-wave states at the bandgap edges. One-directional periodicity mainly modulates and scatters flexural waves along the corresponding periodic direction, leading to unidirectional bandgaps, whereas orthogonal periodicity provides two-directional modulation and promotes the formation of complete bandgaps.
- (3) The boundary-condition analysis indicates that increasing boundary constraint strength progressively restricts both rotational and translational degrees of freedom, thereby enhancing the global bending stiffness of the grillage structures. This stiffness increase leads to higher effective phase velocities of flexural waves and results in an overall upward shift of the bandgap center frequencies.
- (4) The convergence and computational-efficiency analyses provide practical guidance for selecting the DSM truncation order and the FEM mesh size. The proposed DSM-based framework achieves prescribed accuracy with substantially reduced computational cost relative to the FEM, making it suitable for repeated parametric studies and performance-oriented design of grillage structures.

The present study focuses on periodic grillage structures, and future

studies may further investigate the effects of unit-cell asymmetry and irregular periodicity on bandgap evolution. In addition, the present framework is developed for isotropic plate components, and future studies may extend it to grillage structures composed of laminated composite, functionally graded, or metamaterial-inspired plate components to further broaden its applicability.

CRediT authorship contribution statement

Dingkang Chen: Writing – original draft, Validation, Software, Methodology, Investigation, Formal analysis. **Xunyu Li:** Writing – review & editing, Formal analysis. **Xiaobin Li:** Writing – review & editing, Supervision, Formal analysis. **Kian Meng Lim:** Writing – review & editing, Supervision, Formal analysis, Conceptualization. **Li Cheng:** Writing – review & editing, Supervision, Formal analysis, Conceptualization. **Yinggang Li:** Writing – review & editing, Supervision, Project administration, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data will be made available on request.

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