A Modified Fourier Series Solution for a Thermo-Acoustic Tube with Arbitrary Impedance Boundaries

Xue Xing*, Qi Xu†, Jingtao Du*,§,∥ Li Cheng*,¶ and Zhigang Liu*

*College of Power and Energy Engineering
Harbin Engineering University
Harbin 150001, P. R. China
†School of Mechanics and Engineering
Southwest Jiaotong University
Chengdu 610031, P. R. China
‡Department of Mechanical Engineering
The Hong Kong Polytechnic University
HongKong 999077, P. R. China
§ dujingtao@hrbeu.edu.cn
¶ li.cheng@polyu.edu.hk

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Rijke tube is a benchmark model widely used in thermo-acoustic community. As an alternative to existing modeling methods, this work proposes a modified Fourier series solution for modal characteristic analyses of a one dimensional (1D) thermo-acoustic system. The proposed modeling framework allows the consideration of arbitrary impedance boundaries owing to the special feature of the Fourier expansion series enriched by boundary smoothing polynomial terms. Thermo-acoustic Helmholtz governing equation coupled with a first-order heat release model is discretized through Galerkin procedure. Thermo-acoustic modal parameters are obtained by solving a standard quartic matrix characteristic equation, different from conventionally used root searching based on a transcendental equation. Numerical examples are presented to validate the proposed model through comparisons with results reported in the literature. Influences of boundary impedance are analyzed. Results reveal a quantitative relationship between the thermo-acoustic instability and heat source position with respect to the acoustic mode shapes. Results also show the existence of a sensitive zone, in which the thermo-acoustic modal behavior of the impedance-ended (IE) tube shows drastic changes with the boundary impedance. Meanwhile, a stable zone can be achieved upon a proper setting of the boundary impedance through suitable combination of the real and imaginary parts of the impedance.

Keywords: Thermo-acoustic mode; Rijke tube; modified Fourier series; arbitrary impedance boundary.

∥Corresponding authors.
1. Introduction

Thermo-acoustic oscillation (TAO), a self-excited phenomenon generated by the feedback interaction between the unsteady heat release and an acoustic field, is a common phenomenon in various combustion systems, such as rocket motors, gas turbine engines and industrial burners, etc. [Culick and Yang, 1992; Hubbard and Dowling, 2001; Laera and Camporeale, 2017]. TAO not only affects the combustion process, but also leads to detrimental effects such as vibrational fatigue damage of combustion chambers [Lieuwen, 2003; Lieuwen and Yang, 2005]. The complexity of the problem associated with the TAO is well recognized in the literature. Among many factors, the acoustic property of combustors plays an important role in the TAO process [Marx and Mac, 2003; Walz and Krebs, 1993; Emmert and Bomberg, 2017]. For its mitigation, it is necessary to understand the inherent characteristics of thermo-acoustic coupling systems, and to determine the dominating factors which affect their thermo-acoustic behavior.

During the past decades, various modeling approaches have been developed for the coupled thermo-acoustic analyses. Existing methods can be roughly classified into two categories. The first one aims at solving the three dimensional (3D) fluid dynamics of the whole system numerically in time domain, such as LES [Selle and Lartigue, 2004; Palies and Durox, 2011]. The other one consists in separating the whole system into the heat source part and simple-geometry acoustic modules, which could be treated in simpler manners [Nicoud and Benoit, 2007]. For instance, the heat release process can be modeled by using the heat transfer function that describes the coupled mechanism between the heat release and local acoustics [Lieuwen, 2003; Merk, 1957]; as to the acoustic modules, acoustic waves propagating inside are often treated linearly and solved using the network presentation or a Helmholtz solver [Courtine and Selle, 2015; Chen and Ayton, 2013; Olgac and Zalluhoglu, 2014; Mukherjee and Shiria, 2017]. Finite element analysis (FEA) is also utilized for more complex acoustic configurations, in which linear acoustics is assumed and a Helmholtz solver can be used for each element [Han and Li, 2013; Li and Xia, 2017]. Based on these approaches, TAO can be analyzed mainly in two stages: the triggering of the thermo-acoustic instability (TAI) and the limit cycle oscillation formed by the saturation of system response due to the TAI and system nonlinearity. For the oscillation modes, it is commonly accepted that the TAO frequency is usually close to the pure acoustic frequency (without heat source) [Lieuwen, 2003; Lieuwen and Yang, 2005; Courtine and Selle, 2015; Chen and Ayton, 2013]. Recent results also indicate that for combustors, the combustion may generate flame-induced modes that are apparently different from the pure acoustic modes [Emmert and Bomberg, 2015, 2017; Mukherjee and Shiria, 2017]. Meanwhile, other TAO related issues, such as parameter identification [Lieuwen and Yang, 2005] and control techniques [Raun and Beckstead, 1993] have also been studied.
A Solution for a Thermo-Acoustic Tube

Rijke tube, invented in the middle of the 19th century [Rijke, 1859], has been widely used as a benchmark to investigate TAO problems mostly because of its simple 1D geometry. In the late 19th century, Lord Rayleigh gave the first stability criterion of Rijke tube based on energy conservation [Rayleigh, 1878], and various mathematical models have been proposed since then [Lieuwen, 2003; Merk, 1957; Courtime and Selle, 2013; Chen and Ayton, 2019; Raun and Beckstead, 1993]. The heat release process is commonly described by two linear heat release models, i.e., the first-order model [Merk, 1957] and \( n-\tau \) model [Lieuwen, 2003; Chen and Ayton, 2019]. These models formulate the relationship between the heat release rate fluctuation and the air particle velocity, with the consideration of the time delay effect and the weakening effect due to the boundary layer surrounding the heat source [Lighthill, 1954]. Various nonlinear models have also been proposed, considering factors like amplitude saturation [Chen and Ayton, 2019; Orchini et al., 2015], single or multiple delay effects [Crocco, 1965; Sé Ducruix et al., 2003; Heckl, 2013]. Along with the heat release process through which acoustic wave gains energy, the boundary of the tube also affects the energy transport process due to the wave reflection and absorption. An ideal Rijke tube has two open ends, which can be described as the pressure release condition [Hoeijmakers et al., 2014]. Sound absorption coefficients are often used to describe the acoustical energy absorption at the boundary, which implies that the amplitude of the reflected sound waves is weakened due to the energy dissipation [Olgac and Zalluhoglu, 2014]. Meanwhile, sound absorption coefficients also enable parametric studies of the boundary condition varying from open to closed (rigid) ends [Mukherjee and Shrira, 2017]. It has been shown that the boundary condition greatly affects the entire TAO process, not only the onset of the TAI, but also the oscillation behavior such as modal frequency and mode shape [Andreini et al., 2008; Noiray et al., 2009].

In practice, acoustic boundary conditions can be altered and eventually manipulated through a proper design of the boundary or using acoustic control devices. In characterizing the acoustic property of a wall surface, the acoustic impedance proves to be a better parameter to use than the sound absorption coefficient, since it describes not only the sound energy dissipation, but also the phase changes between the incident and reflected waves [Bolt and Brown, 1940; Zhang et al., 2018; Mi et al., 2018a; Mi et al., 2018b]. Boundary conditions imposed by control devices (such as cooling tube rows or cavities connected to the tube) can also be described by the acoustic impedance which combines the acoustic pressure to the acoustic velocity perpendicular to the surface [Seo and Kim, 2003; Field and Frick, 1998; Baumann et al., 2013; Bellucci et al., 2004]. Therefore, through adjusting the real and imaginary parts of the wall impedance, all classical boundary conditions (open/closed), their combinations as well as the intermediate cases with passive control devices can be simulated. Cases of TAO with classical boundary conditions have been extensively investigated. Meanwhile, the consideration of impedance boundaries becomes necessary for some applications,
X. Xing et al.

for examples, a thermo-acoustic engine (TAE), which shares similar configuration with a 1D Rijke tube but with a stack of temperature gradient inside and a vibrating membrane or other data acquisition system at one of the tube ends, which has been investigated to generate electricity by installing electroacoustic transformers on the membrane. Along the same line, both Chen et al. and Matveev et al. studied TAI systems with impedance boundary implemented through the use of the TAE membrane [Gascón-Pérez, 2018; Chen et al., 2018; Matveev and Jung, 2011]. Meanwhile, a 3D time delayed TAO system with impedance boundaries was also considered, alongside the proposal of an iterative method to solve the derived transcendental characteristic equation based on FEM techniques [Nicoud and Benoit, 2007; Selle et al., 2004]. These studies addressed various aspects of a TAO system. While shedding lights on some important TAO phenomena, the need for a better understanding of the effects of the impedance boundaries on TAI behavior has also been recognized. Modal parameters can play an important role in the design of a thermo-acoustic system as well as the implementation of effective TAI mitigation means. From such a viewpoint, a systematic modal analysis with the consideration of general impedance boundaries and other TAO parameters relating to the thermo-acoustic interaction is vital to reveal the physical mechanism of the TAI triggering and the subsequent TAO process. Although the traditional modeling approaches, such as the wave propagation method, can treat the complex boundary condition in 1D configuration using the concept of reflection coefficient, their extension to 3D cases is still challenging. On the other hand, numerical methods like FEM exhibit versatility in dealing with complex-geometry systems, at the expense of high computational cost, but also show their limitations when dealing with large systems or high frequency problems. For these reasons, there is a need to seek alternative solutions.

Motivated by this, a semi-analytical model of a 1D Rijke-like tube with general impedance boundary condition, referred to as impedance-ended tube (IE tube), is presented in this paper. The proposed approach is intended as an alternative to the existing numerical modeling approaches, which is efficient and conducive to modal analyses. Thermo-acoustic Helmholtz equation with impedance boundaries is employed for the system dynamic description. To ensure the smoothness of the field function in the entire solving region with arbitrary boundary conditions, a modified Fourier series enriched with auxiliary polynomial terms is constructed for the decomposition of the sound pressure description. A quartic system matrix characteristic equation is derived through the Galerkin discretization procedure [Zhang and Cheng, 2015], which, after the truncation of the series, allows obtaining the coupled thermo-acoustic modal parameters, instead of commonly used search for the infinite number of roots of a transcendental equation. Numerical examples are then given to verify the reliability of the proposed model through comparisons with the results from the literature. Based on the established model, thermo-acoustic modal
characteristics of the IE tubes with various boundary conditions are investigated and discussed. Effects of the thermo-acoustic interactions on the modal characteristics are then scrutinized and physically explained. Finally, concluding remarks are drawn.

2. Theoretical Formulation

2.1. Model description

Figure 1 illustrates a 1D IE tube with a heat release source inside and air flow convection along the $x$-positive direction. The tube has a uniform cross-sectional area $S$ and a length $L$. The heat source is located at $x_q$. The mean temperatures $T$ and the density $\rho$ in the pre- and after-heat regions are constant. Impedance boundary conditions are imposed to both ends, with $Z_0$ and $Z_L$ being the acoustic impedances at $x = 0$ and $x = L$, respectively. By adjusting the real and/or the imaginary parts of the impedance value, various boundary conditions can be readily simulated accordingly. By setting the acoustic impedance to zero, the IE tube degenerates to the conventional Rijke tube.

The 1D sound pressure inside the tube, $p(x)$, can be written as

$$p(x, t) = p(x)e^{j\omega t},$$

where $j$ is the imaginary unit and $\omega$ is a complex number, whose real part, denoted by $\text{Re}(\omega)$, represents the oscillation frequency, and the minus of the imaginary part of $\omega$, $-\text{Im}(\omega)$, is the growth rate of the oscillation amplitude. When the growth rate is positive, the pressure amplitude grows exponentially with time, producing the so-called TAI.

For the arbitrary impedance boundary considered in this work, the imaginary and real parts of its complex impedance indicate the change of phase and the dissipation of the sound energy, respectively. The matching relationship between the sound pressure and the boundary impedance can be written as

$$\frac{\partial p}{\partial x} = -j\frac{\rho c}{Z_i}kp \quad (i = 0, 1),$$

Fig. 1. Schematic of a 1D thermo-acoustic system with arbitrary impedance boundaries.
X. Xing et al.

where \( c \) denotes the sound speed in the acoustic medium (air) and \( k = \omega / c \) is the wave number.

### 2.2. Fourier series construction of the pressure field and its constraint equations

For a 1D rigid-walled acoustic tube, Fourier series is commonly employed as

\[
p(x) = \sum_{m=0}^{\infty} a_m \cos \lambda_m x,
\]

where \( a_m \) is the Fourier series coefficients, and \( \lambda_m = m\pi / L \). However, for the current IE tube, due to its spatial differential property at the both ends, such Fourier series representation cannot be applied, which can be explained as follows:

\[
\left. \frac{\partial p(x)}{\partial x} \right|_{x=0} = \sum_{m=0}^{\infty} -\lambda_m a_m \sin \lambda_m (0) = 0,
\]

\[
\left. \frac{\partial p(x)}{\partial x} \right|_{x=L} = \sum_{m=0}^{\infty} -\lambda_m a_m \sin \lambda_m (L) = 0.
\]

The above two equations show that the traditional Fourier series expansion in Eq. (2.3) fails to cope with the non-conventional impedance boundaries, since the first-order spatial derivative of the sound pressure (the particle velocity) is not always zero at the boundary. To overcome this differential discontinuity at the impedance walls [Du et al., 2011], a modified Fourier series expansion is employed here, namely

\[
p(x) = \sum_{m=0}^{\infty} a_m \cos \lambda_m x + b_1 \xi_1(x) + b_2 \xi_2(x),
\]

in which \( \xi_1(x) \) and \( \xi_2(x) \) are the supplementary functions with \( b_1 \) and \( b_2 \) being the corresponding weight coefficients. The purpose of introducing \( \xi_1(x) \) and \( \xi_2(x) \) is to avoid the derivative discontinuity at both impedance boundaries as mentioned above. Mathematically, the basic requirement for \( \xi_1(x) \) and \( \xi_2(x) \) is the boundary-smoothness property, thus these two functions may not be unique. However, an appropriate choice can simplify the subsequent formulation, such as

\[
\xi_1(x) = x \left( \frac{x}{L} - 1 \right)^2,
\]

\[
\xi_2(x) = \frac{x^2}{L} \left( \frac{x}{L} - 1 \right),
\]

where these supplementary functions are chosen based on the criteria of removing the first differential discontinuities encountered in the general impedance boundary
A Solution for a Thermo-Acoustic Tube

Fig. 2. Graphs of the supplementary functions and their first-order derivatives.

conditions given in Eq. (2.2). It can be verified that

\[
\begin{bmatrix}
\xi_1(0) & \xi_2(0) \\
\xi_1(L) & \xi_2(L)
\end{bmatrix} =
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]  
(2.9)

and

\[
\begin{bmatrix}
\xi_1'(0) & \xi_2'(0) \\
\xi_1'(L) & \xi_2'(L)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]  
(2.10)

To better demonstrate the properties of the supplementary functions, their corresponding functional graphs are presented in Fig. 2. It can be seen that the differential discontinuity associated with Fourier series Eqs. (2.3)–(2.5) can be overcome by these two supplementary boundary-smoothing polynomials. Then, the modified Fourier series expansion of acoustic cavity Eq. (2.6) will be sufficiently smooth in the whole solving domain, including both arbitrary impedance boundaries, which ensures the convergence and accuracy of the subsequent solution.

Substituting Eq. (2.6) into Eq. (2.2) yields

\[
b_1 = -j \frac{\rho \alpha}{Z_0} \sum_{m=0}^{\infty} a_m,
\]  
(2.11)

\[
b_2 = -j \frac{\rho \alpha}{Z_L} \sum_{m=0}^{\infty} (-1)^m a_m.
\]  
(2.12)

The above two equations, referred to as constraint equations, actually define the constraint relationship between the coefficients of the Fourier series and those of the supplementary polynomials. In numerical implementations, all the Fourier series will be truncated to an integer \( M \), thus allowing Eqs. (2.11) and (2.12) to be written in...
a matrix form as

$$\omega \mathbf{H} \mathbf{A} = \mathbf{B}$$  \hspace{1cm} (2.13)

in which,

$$\mathbf{H} = -j\rho \begin{bmatrix} \frac{1}{Z_0} & \frac{1}{Z_0} & \cdots & \frac{1}{Z_0} \\ (-1)^0 \frac{1}{Z_L} & (-1)^1 \frac{1}{Z_L} & \cdots & (-1)^M \frac{1}{Z_L} \end{bmatrix}$$  \hspace{1cm} (2.14)

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_M \end{bmatrix}^T$$  \hspace{1cm} (2.15)

and

$$\mathbf{B} = \begin{bmatrix} b_1 & b_2 \end{bmatrix}^T.$$  \hspace{1cm} (2.16)

### 2.3. Solution of thermo-acoustic Helmholtz governing equation

The sound pressure inside the IE tube is governed by the thermo-acoustic Helmholtz equation, namely

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = \frac{\gamma - 1}{c^2} \frac{\partial q}{\partial t}$$  \hspace{1cm} (2.17)

in which $\gamma$ is the ratio of specific heats, and $q$ represents the heat release rate per unit area. The term on the right-hand side describes how the heat addition generates pressure disturbances in the IE tube. For simplicity, the first-order model [Merk, 1957] is chosen to describe the heat release, which takes the following form:

$$q(x, t) = \frac{Q(t)}{S} \delta(x - x_q),$$  \hspace{1cm} (2.18)

$$\tau \frac{dQ(t)}{dt} = -Q(t) + bu(x, t),$$  \hspace{1cm} (2.19)

where $Q$ is the heat release rate; $b$ a local interaction index; $\tau$ a thermal inertia factor and $u$ the vibrational velocity of particle. Momentum conservation principle gives

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}.$$  \hspace{1cm} (2.20)

Combining Eqs. (2.6) and (2.20) leads to

$$u(x, t) = -\frac{1}{j\omega \rho} \sum_{m=0}^{\infty} -\lambda_m a_m \sin \lambda_m x + b_1 \xi_1'(x) + b_2 \xi_2'(x) \right] e^{j\omega t}.$$  \hspace{1cm} (2.21)
Substituting Eqs. (2.6), (2.20), and (2.21) into Eq. (2.17) yields

\[
\begin{align*}
-\frac{\omega^2}{c^2} \left[ \sum_{m=0}^{\infty} a_m \cos \lambda_m x + b_1 \xi_1(x) + b_2 \xi_2(x) \right] \\
- \left[ \sum_{m=0}^{\infty} -\lambda_m^2 a_m \cos \lambda_m x + b_1 \xi'_1(x) + b_2 \xi'_2(x) \right] \\
= - \frac{j\omega(\gamma - 1)}{c^2} \frac{b}{S(1+j\omega\tau)} \frac{1}{j\omega \rho} \\
\times \left[ \sum_{m=0}^{\infty} -\lambda_m a_m \sin \lambda_m x + b_1 \xi''_1(x) + b_2 \xi''_2(x) \right] \delta(x - x_q) .
\end{align*}
\] (2.22)

By following Galerkin discretization procedure and using \( \cos(\lambda_n x) \) as the weighting function, one obtains the following thermo-acoustic coupling equations in matrix form (details in Appendix A)

\[
T(O_1A + DB) + \frac{\omega^2}{c^2} T(EA + CB) - \frac{b(\gamma - 1)}{S\rho c^2(1+j\omega\tau)} (R_1O_2A + R_2GB) = 0,
\] (2.23)

where \( E \) is a unit matrix and

\[
T^{n,m} = \delta(n, m) \begin{cases} L, & m = 0 \\ \frac{L}{2}, & m \neq 0, \end{cases}
\] (2.24)

\[
O_{1}^{n,m} = -\lambda_m^2 \delta(n, m),
\] (2.25a)

\[
O_{2}^{n,m} = -\lambda_m \delta(n, m),
\] (2.25b)

\[
G^n = \begin{bmatrix} g_{1n} & g_{2n} \end{bmatrix},
\] (2.25c)

\[
D^n = \begin{bmatrix} d_{1n} & d_{2n} \end{bmatrix},
\] (2.26a)

\[
C^n = \begin{bmatrix} c_{1n} & c_{2n} \end{bmatrix}
\] (2.26b)

and

\[
R_{1}^{n,m} = \sin \lambda_m x_q \cos \lambda_n x_q, \quad R_{2}^{n,m} = \cos \lambda_m x_q \cos \lambda_n x_q.
\] (2.27)

Considering the constraint relationship Eq. (2.13) and substituting it into Eq. (2.23), one can finally derive a quartic characteristic matrix for the thermo-acoustic system as

\[
(K + \omega Z + \omega^2 M + \omega^3 X + \omega^4 Y)A = 0,
\] (2.28)
X. Xing et al.

where \( K = TO_1 + rR_1O_2, \) \( Z = j\tau TO_1 + TDH + rR_2GH, \) \( M = j\tau TDH + TE/c^2, \) \( X = (j\tau TE + TCH)/c^2, \) \( Y = j\tau TCH/c^2, \) and \( r = -b(\gamma - 1)/S\rho c^2. \)

Equation (2.28) is cast into the state space form, and can be solved using any standard eigenvalue solver. This procedure gives the whole set of system modal parameters, including the coupled thermal-acoustic modal frequencies and corresponding growth rates. Substituting the corresponding Eigen-vectors into Eq. (2.6), one can get the corresponding mode shapes.

3. Results and Discussions

Numerical examples are given to demonstrate the correctness and effectiveness of the proposed solution, through comparisons with the results from literature. Based on the established model, the thermo-acoustic modal characteristics of the IE tube with various boundary conditions are then analyzed.

3.1. Model validation

As the first example, a Rijke tube from Olgac [2014] is used as a reference for comparison purpose. The model parameters used are tabulated in Table 1 in Olgac and Zalluhoglu [2014], the boundary condition of the tube is described by reflection coefficients, which can be further converted into its corresponding acoustic impedance as follows:

\[
Z = \rho c \left( \frac{1 + R}{1 - R} \right)
\]

in which \( R \) is the boundary reflection coefficient, and \( Z \) is its corresponding acoustic impedance.

Figure 3 presents the natural frequencies and the corresponding growth rates of the first five coupled thermo-acoustic modes for various heat release locations, namely \( x_q = 0.203 \text{ m}, 0.406 \text{ m}, 0.762 \text{ m}, \) and \( 0.864 \text{ m}, \) respectively. In each subplot, the \( x \)-axis coordinate is the natural frequency, and the \( y \)-axis coordinate represents the modal growth rate for which positive values indicate unstable modes.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_0 )</td>
<td>-0.93</td>
<td>—</td>
</tr>
<tr>
<td>( R_L )</td>
<td>-0.93</td>
<td>—</td>
</tr>
<tr>
<td>( c )</td>
<td>340</td>
<td>( \text{m s}^{-1} )</td>
</tr>
<tr>
<td>( \rho )</td>
<td>1.2</td>
<td>( \text{kg m}^{-3} )</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>1.4</td>
<td>—</td>
</tr>
<tr>
<td>( S )</td>
<td>0.00075</td>
<td>( \text{m}^3 )</td>
</tr>
<tr>
<td>( b )</td>
<td>200</td>
<td>—</td>
</tr>
<tr>
<td>( \tau )</td>
<td>0.002</td>
<td>s</td>
</tr>
</tbody>
</table>
Fig. 3. The lowest five frequencies and corresponding growth rates of the 1D thermo-acoustic system with the heat release located at: (a) \( x_q = 0.203 \) m, (b) \( x_q = 0.406 \) m, (c) \( x_q = 0.762 \) m, and (d) \( x_q = 0.864 \) m.

mentioned in Sec. 2, both the natural frequencies and the corresponding growth rates are obtained from the Eigen-solutions of Eq. (2.28).

It can be seen that the current prediction on the thermo-acoustic modal behavior agrees well with Olgac and Zalluhoglu [2014]. The agreement in terms of modal frequency is particularly good. Despite the minor differences observed, the two sets of growth rates carry the same signs, indicating a consistency in the prediction of the stability. The heat release locations exert obvious influence on thermo-acoustic modal behavior, especially on the stability of each mode, in agreement with the common understanding on Rijke tube.

As a series expansion method, the truncation order used in the expansion series is an important parameter to guarantee the prediction accuracy. To demonstrate the convergence property of the proposed model, the first five thermo-acoustic frequencies and the corresponding growth rates are calculated using different truncation orders, with the corresponding results presented in Table 2. It can be found that a few expansion terms can lead to a relatively stable and accurate prediction of the
### 3.2. Two simple boundary conditions

#### 3.2.1. Rigid and pressure-release ends

In the current model, various boundary conditions can be realized by tactically setting the real and imaginary parts of the acoustic impedance. Here, the configuration with a rigid left-end and a pressure release right-end is firstly studied as an example. The perfectly rigid condition $Z_0$ is defined by $Z_0 = j\infty$, and the pressure release boundary condition $Z_L$ is defined by $Z_L = 0$. In the numerical simulations, $Z_0 = j10^8$ is used.

Figure 4 presents the first (lowest) six frequencies and their corresponding growth rates for various heat release parameters $(b, \tau, x_q)$ of such a left-rigid

![Fig. 4. The lowest six frequencies and corresponding growth rates of the 1D thermo-acoustic system with a rigid end on the left, a pressure release end on the right, and the heat release located at $x_q = 0.203\text{ m}, 0.406\text{ m}, 0.762\text{ m}, \text{ and } 0.864\text{ m},$ respectively.](image-url)
and right-pressure-release IE tube, with other parameters being kept the same as tabulated in Table I. The case with $b = 0$ and $\tau = 0$ is actually an acoustic tube without heat source, whose modes are denoted by the blue squares in the figure, in agreement with the analytical modal solution $f = c(2m + 1)/4L$. The presence of the heat source and its position variation do not significantly affect the natural frequencies of the coupled thermo-acoustic system. The influence on the modal growth rate, however, is much more obvious, especially for the lower order modes.

To further explain the observed phenomena, let us examine the relationship between the heating position in relation to the acoustic mode shapes and the system instability. The so-called acoustic mode shapes are calculated in the absence of heat source. Figure 5 shows this relationship for the first six modes with the heating position continuously changing within $[0, L]$, where the color bar represents the value of modal growth rate, and the dash lines indicate the heating positions which trigger the TAI of IE tube.

It can be seen from the sub-figures that, when the heat location $x_q$ varies from a peak/valley to a node of an acoustic mode shape, along the positive direction of $x$, the thermo-acoustic mode of the IE tube is always stable; while the thermo-acoustic mode becomes unstable when $x_q$ varies from a node to a peak/valley of an

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**Fig. 5.** (Color online) The first six thermo-acoustic modal growth rates of a tube (with a left-rigid end and a right-pressure-release end) with the heating position changing within $[0, L]$: (a) the first order, (b) the second order, (c) the third order, (d) the fourth order, (e) the fifth order, (f) the sixth order. The level of the growth rates is indicated by the color bars. The pure acoustic mode shapes are given as reference, on which the dotted portion corresponds to these heating positions which trigger the TAI.
acoustic mode shape. In addition, the instability first increases and is then weakened as $x_q$ approaches a peak/valley. To explain this physically, one needs to note that the positive direction of $x$ is the convection direction of the mean airflow in the IE tube. In the node-to-peak/valley section of an acoustic mode, the air particles move along $x$, the corresponding sound pressure amplitude should be strengthened if no heating is involved. Meanwhile, since the heat source is placed also in the node-to-peak/valley section, the heating process then further enhances the sound pressure amplitude, which eventually leads to the instability of sound pressure. This quantitative observation of the IE tube enriches the explanations on a Rijke tube given by Lord Rayleigh [Rijke, 1859], who stated that TAI happens when the fresh air comes into the field at the moment of greatest condensation.

3.2.2. Pressure-release and pressure-release ends

The second example investigates an IE tube with both pressure release ends, defined by $Z_0 = Z_L = 0$. Similarly, simulation results are given in Figs. 6 and 7. A comparison between Figs. 6 and 4 shows that the boundary condition greatly affects the thermo-acoustic modes: On one hand, there is an obvious shift of the oscillation frequencies; on the other hand, the stabilities of thermo-acoustic modes are also changed, although the heat release remains the same. Compared with Fig. 5, it can be seen from Fig. 7 that the acoustic mode shapes are quite different from those given in Fig. 5, as a result of the changes in the boundary condition. Thus, even with the same heating position, a totally different TAI behavior (modal growth rates) is generated because of the boundary changes. Nevertheless, the observations and the explanations given before, in terms of relationship among the heating position, the acoustic mode shape and the instability behavior, still remain true.

Alongside the above observations, the simulation model established in Sec. 2 also enables the calculation of the coupled thermo-acoustic mode shapes. Figure 8...

Fig. 6. The lowest six frequencies and the corresponding growth rates of the 1D thermo-acoustic system with pressure release boundary at both ends, and the heat releases are located at $x_q = 0.203$ m, 0.406 m, 0.762 m, and 0.864 m, respectively.
Fig. 7. (Color online) The first six thermo-acoustic modal growth rates of a tube (with both pressure release ends) with the heating position changing within \([0, L]\): (a) the first order, (b) the second order, (c) the third order, (d) the fourth order, (e) the fifth order, (f) the sixth order. The level of the growth rates is indicated by the color bars. The pure acoustic mode shapes are given as reference, on which the dotted portion corresponds to these heating positions which trigger the TAI.

Fig. 8. The first and second mode shapes of pure acoustical and thermo-acoustic IE tubes with the same pressure-release boundary condition, in which the unsteady heat release is located at 0.203 m, 0.406 m, 0.762 m, and 0.864 m, respectively: (a) the 1st mode; (b) the 2nd mode.

presents the first and the second coupled mode shapes of the IE tube for different heating positions. It can be seen that the thermo-acoustic mode shapes basically share similarities with their acoustic counterparts (without heating). However, for certain cases \((x_q = 0.203 \text{ m} \text{ and } x_q = 0.406 \text{ m}) for the first mode, \(x_q = 0.762 \text{ m} \text{ for the} \)
second mode), an abrupt change in thermo-acoustic mode shapes can be observed at the heating position, and the overall mode shapes also undergo observable changes; while for some other cases, the corresponding heating positions may not significantly affect the pressure oscillation distribution. This implies that the extent to which the heat source affects the mode shapes may need case-by-case analyses.

3.3. Arbitrary impedance boundaries

The above two examples show that different classical boundary conditions can yield different stabilities. More specifically, the first two thermo-acoustic modes in Fig. 4 are stable with a rigid left-end and a pressure release right-end for \( x_q = 0.203 \text{ m} \), while in Fig. 6 they become unstable with both pressure release ends. To illustrate the transition process, the acoustic impedance on the left end of the IE tube is gradually changed from pressure release (\( Z_0 = j 10^{-5} \)) to rigid (\( Z_0 = j 10^6 \)), while the right end being kept as pressure release with a fixed heating position at \( x_q = 0.203 \text{ m} \).

Figure 9 presents the corresponding thermo-acoustic modal frequencies and the corresponding growth rates. During the variation of boundary impedance, there exists a sensitive zone in which thermoacoustic modal behavior undergoes drastic changes, i.e., when \( \bar{Z}_0 = |Z_0|/\rho c = 1 \). Apart from this sensitive zone, the modal characteristics remain nearly unchanged. Therefore, the observed sensitive zone deserves more attention for the design/control of the system thermo-acoustic stability though adjusting the boundary impedance. Similar to Figs. 4 and 7 by choosing \( Z_0 = j 10^2 \), Fig. 10 presents the modal growth rates of the first two thermo-acoustic modes, with respect to the heating positions and the acoustic mode shapes. It can be observed that the same relationship among the three factors (heating position, mode shapes, and TAI) still holds, even for such general boundary conditions.

Fig. 9. Variation of the first two thermo-acoustic modal parameters with continuous change of the acoustic impedance on left end from \( Z_0 = j 10^{-5} \) to \( Z_0 = j 10^6 \), in which \( \bar{Z}_0 = |Z_0|/\rho c \): (a) modal frequency, (b) modal growth rate.
A Solution for a Thermo-Acoustic Tube

Fig. 10. The first two thermo-acoustic modal growth rates of a tube (with a left impedance-end $Z_0 = j10^2$ and a right open end) with the heating position changing within $[0, L]$: (a) the first order, (b) the second order. The level of the growth rates is indicated by the color bars. The pure acoustic mode shapes are given as reference, on which the dotted portion corresponds to these heating positions which trigger the TAI.

Fig. 11. Contour plots of the first two thermo-acoustic modal growth rates with the combined variation of the heating position within $[0, L]$ and the left end acoustic impedance from $j10^{-5}$ to $j10^8$: (a) the first order, (b) the second order.

In addition, to show the influence of the left end impedance variation on the thermo-acoustic stability more globally, Fig. 11 gives the stability charts of the IE tube for the first and second modes. The left-end impedance and the heating position vary in $(Z_0, x_q) = [j10^{-5}, j10^8] \times [0, L]$, and the color bar represents the value of the modal growth rate. It can be clearly seen that an unstable mode can be transformed to a stable one, and vice versa, by adjusting the heating position and/or the boundary impedance values, which points at the possibility of controlling or triggering TAI.
X. Xing et al.

As the final example, a dissipative boundary condition is considered, in which the real part of the acoustic impedance takes nonzero values, implying a loss of energy at the boundary. The acoustic impedance on the left end of the IE tube is gradually changed with Re($Z_0$) varied from $10^{-5}$ to $10^{8}$ and Im($Z_0$) from j$10^{-5}$ to j$10^{8}$, while the right end being kept as pressure release with a fixed heating position at $x_q = 0.203$ m. Plotted in Fig. 12 is the influence of the left-end impedance on the first two thermo-acoustic modal frequencies and their growth rates. In this figure, the $x$-axis and $y$-axis represent the power index of the real and imaginary parts of left-end impedance, respectively, while the $z$-axis denotes the thermo-acoustic modal frequency, with its color denoting the growth rate. It can be seen that there also

![Contour plots of the first two thermo-acoustic modal frequencies and growth rates with continuous variation of the acoustic impedance on left end, where re($Z_0$) is varied from $10^{-5}$ to $10^{8}$ and Im($Z_0$) from j$10^{-5}$ to j$10^{8}$: (a) the first order, (b) the second order.](image1)

![Contour plot of the stable zone for the first two thermo-acoustic modes with continuous change of the real and imaginary acoustic impedance on left end, in which Re($Z_0$) is varied from $10^{-5}$ to $10^{8}$ and Im($Z_0$) from j$10^{-5}$ to j$10^{8}$.](image2)
exists a sensitive zone in which boundary acoustic impedance shows vital influence on the modal frequencies and stabilities of the system, and stable modes can be achieved by choosing appropriate matching combination of the real and imaginary parts of the acoustic impedance.

To show a global influence trend of the left-end impedance on system stability, the common stable zone for the first two thermo-acoustic modes with continuous change of the real and imaginary parts of acoustic impedance on left end is given in Fig. 13, which can serve as a useful guidance for the control of thermo-acoustic system instability through setting the boundary impedances. In practice, various TAO dampers, such as acoustic liners, Helmholtz resonators, can all be regarded as effective tools to bring about boundary impedance changes. To enlarge the stable range for various thermo-acoustic modes, appropriate impedance matching between the combustor and damping devices would be an effective option.

4. Conclusions

In this paper, a modified Fourier series solution for the thermo-acoustic modal characteristics analyses of 1D IE tube is established. The proposed model tackles the arbitrary impedance boundaries by introducing two boundary smooth auxiliary polynomials to the traditional Fourier series, which overcomes the differential discontinuity issue encountered at the impedance boundaries. The coupled thermo-acoustic system is then characterized and cast into a quartic matrix characteristic equation, which gives the modal parameters with high computational efficiency, rather than solving a transcendental equation or resorting to FEM method.

The effectiveness of the proposed model is demonstrated and validated against the results from the literature in terms of modal parameters like frequencies, growth rates and mode shapes. The proposed method allows establishing a quantitative relationship between the modal stability of the IE tube with the heat release position and the pure acoustic mode shape (without heat source). It is shown that the TAI is triggered when the heat source is placed at those locations where the sound pressure is strengthened for lossless boundaries. For the non-dissipative case, an abrupt change of thermo-acoustic stability appears around the characteristic impedance of the acoustic medium; and for its dissipative counterpart, a common stable zone for the first two thermo-acoustic modes can be achieved through an appropriate matching combination of real and imaginary parts of the boundary impedance. Thus, to obtain an enlarged stable range of the thermo-acoustic modes in a combustion system, a due consideration of the whole set of system parameters, such as the heating position and boundary impedance variation range as well as their interaction, is of paramount importance for the TAI control.

Although this study is confined to 1D configuration, the proposed modeling framework can be extended to more complex 3D cases with various acoustic devices being included into the model. In that sense, the presently proposed modeling approach provides an attractive alternative to the existing analysis tools for
X. Xing et al.

analyzing the complex interplay among various system parameters. As a continuation of this work, relevant experimental studies will be conducted in the future.

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Appendix A
To obtain the thermo-acoustic characteristic matrix form via the Galerkin discretization procedure with \(\cos(\lambda_n x)\) as the weighting function, the auxiliary polynomials are expanded into Fourier series, and Eq. (22) is translated into Eq. (A.1) as

\[
-\frac{\omega^2}{c^2} \left[ \sum_{m=0}^{\infty} (a_m + b_1 c_{1m} + b_2 c_{2m}) \cos \lambda_m x \right] - \left[ \sum_{m=0}^{\infty} (-\lambda_m^2 a_m + b_1 d_{1m} + b_2 d_{2m}) \cos \lambda_m x \right] = -\frac{j\omega(\gamma - 1)b}{c^2 S} \frac{1}{1 + j\omega\rho} \sum_{m=0}^{\infty} \left[ -\lambda_m a_m \sin \lambda_m x \right] + (b_1 g_{1m} + b_2 g_{2m}) \cos \lambda_m x] \delta(x - x_q),
\]

where

\[
\xi_1(x) = x \left( \frac{x}{L} - 1 \right)^2 = \sum_{m=0}^{\infty} c_{1m} \cos \lambda_m x
\]

\[
c_{1m} = \begin{cases} \frac{L}{12} & m = 0 \\ -\frac{2L}{m^2\pi^2} & m \neq 0 \end{cases}
\]

\[
\xi_2(x) = \frac{x^2}{L} \left( \frac{x}{L} - 1 \right) = \sum_{m=0}^{\infty} c_{2m} \cos \lambda_m x
\]

\[
c_{2m} = \begin{cases} -\frac{L}{12} & m = 0 \\ \frac{2L}{m^2\pi^2(-1)^m} & m \neq 0 \end{cases}
\]
A Solution for a Thermo-Acoustic Tube

\[ \xi_1'(x) = \frac{3x^2}{L^2} - \frac{4x}{L} + 1 = \sum_{m=0}^{\infty} g_{1m} \cos \lambda_m x \quad (A.6) \]

\[ g_{1m} = \begin{cases} 0 & m = 0 \\ \frac{4[m\pi(-1)^m + 2m\pi]}{m^3\pi^3} & m \neq 0 \end{cases} \quad (A.7) \]

\[ \xi_2'(x) = \frac{3x^2}{L^2} - \frac{2x}{L} = \sum_{m=0}^{\infty} g_{2m} \cos \lambda_m x \quad (A.8) \]

\[ g_{2m} = \begin{cases} 0 & m = 0 \\ \frac{2[4m\pi(-1)^m + 2m\pi]}{m^3\pi^3} & m \neq 0 \end{cases} \quad (A.9) \]

\[ \xi_1''(x) = \frac{6x}{L^2} - \frac{4}{L} = \sum_{m=0}^{\infty} d_{1m} \cos \lambda_m x \quad (A.10) \]

\[ d_{1m} = \begin{cases} -\frac{1}{L} & m = 0 \\ \frac{12[-1 + (-1)^m]}{Lm^2\pi^2} & m \neq 0 \end{cases} \quad (A.11) \]

\[ \xi_2''(x) = \frac{6x}{L^2} - \frac{2}{L} = \sum_{m=0}^{\infty} d_{2m} \cos \lambda_m x \quad (A.12) \]

and

\[ d_{2m} = \begin{cases} \frac{1}{L} & m = 0 \\ \frac{12[-1 + (-1)^m]}{Lm^2\pi^2} & m \neq 0 \end{cases} \quad (A.13) \]

References


X. Xing et al.


X. Xing et al.

