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On the efficacy of the wavelet decomposition for high frequency vibration analyses



S. Zhang, L. Cheng*

Department of Mechanical Engineering, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong Special Administrative Region, PR China

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ABSTRACT

This paper reports the extraordinary ability of the wavelet decomposition for vibration analyses under the framework of Rayleigh–Ritz method. Using a beam as an example, Daubechies wavelet scale functions are used as admissible functions for decomposing the flexural displacement of the structure, along with the artificial springs at the boundary, to predict vibration of an Euler–Bernoulli beam in an extremely large frequency range. It is shown that the use of wavelet basis allows reaching very high frequencies, typically covering more than 1000 modes using conventional computational facility within the available numerical dynamics of the computers with no particular care needed for round-off errors. As a side benefit, the use of spring boundary also allows handling any elastic boundary conditions through a dynamic contribution in the Hamiltonian of the beam. The wavelet decomposed approach combines the flexibility of the global methods and the accuracy of local methods by inheriting the versatility of the Rayleigh–Ritz approach and the superior fitting ability of the wavelets. Numerical results on both free and forced vibrations are given, in excellent agreement with predictions of classical methods.

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1. Introduction

Vibration analyses of structures in the mid-to-high frequency range pose formidable technical changes to the existing simulation methods [1]. Particular problems arising are mainly twofold. First, wavelengths get shorter as the frequency increases, so that more structural details need to be considered in the model, leading to exorbitant computational cost. Second, due to the high sensitivity of the vibrational behavior to structural details and model parameters at higher frequencies, any inaccuracy in the model or round-off errors in numerical simulation will lead to erroneous results. In order to satisfy the demand for the vast amount of parametric and optimization studies in the preliminary design stage of structures, versatile and efficient simulation methods which can potentially cover a wide frequency spectrum is of paramount importance.

Among various so-called semi-analytical methods based on spatial discretization [2], Rayleigh–Ritz method with assumed admissible functions is probably the most popular and most studied one. It consists in approximating the unknown structural displacement through a linear combination of assumed global basis functions. The basis function should, a priori, satisfies the geometric boundary conditions, which can be tactically avoided through the use of artificial spring along

* Corresponding author. Tel.: +852 2766 6769; fax: +852 2365 4703.
 E-mail address: li.cheng@polyu.edu.hk (L. Cheng).

structural boundaries [2]. Based on the Hamilton's principle, the elastic boundary conditions are apprehended in the elastic energy term in the Hamiltonian of the structures, the extremalization of which can be achieved using Lagrange's equation. There is huge amount of literature on the topic, and typical applications can be found in Cheng and Lapointe [3] and Berry et al. [4]. The properties of the basis functions significantly affect the computational efficiency, stability and accuracy of the solution. Popular candidates are polynomials [3,4] and their derivatives such as Jacobi polynomials [5], Legendre polynomials [6] or Chebyshev polynomials [7]. However, they all suffer from the ill-conditioned problem [8] at higher order. This is due to the mismatch of the high numerical dynamics of the polynomial coefficients and the limited numerical dynamics of the digital computers; which is unavoidable even for symbolic computing. It is reported that the highest polynomial order that could possibly be used is roughly 19, and 44 for symbolic case [8] for one dimensional structure. Using the rule of thumb that roughly one-third to half of the modes can be correctly predicted, only less than twenty modes can be, in principle, corrected calculated. Trigonometric functions [9] or combination of the trigonometric functions and lower-order polynomials [10,11] are also possible choice. However, trigonometric functions suffer from potential convergence problems on the boundary [11]. To the best of our knowledge, practically nothing in the open literature shows their efficiency in dealing with very high-order modes except for Beslin's work [8] in which natural modes of a plate were computed, roughly reaching what can be quantified as the mid-frequency range of the structure (880 natural frequencies within 1.5 percent of error in 2-D structure, roughly corresponding to 30 modes in 1-D scenario).

Wavelets have been widely used in signal processing in various applications including sound and vibration. However, their efficacy in vibration simulations, especially as a global basis under the Rayleigh–Ritz framework, has never been explored up to now. Inspired by the appealing features of Daubechies wavelet scale functions in numerical approximation and our recent success in acoustic simulation using Galerkin formulation [12], this paper examines the use of the Daubechies wavelet scale functions as basis functions for vibration prediction under the general framework of Rayleigh–Ritz method. Daubechies wavelet scale functions have been shown to exhibit strong ability to express any square integrable functions in finite interval, high vanishing moments and orthogonality [13]. Using a beam example, it is shown that the proposed wavelet decomposed Rayleigh–Ritz method (WDRM) allows reaching very high frequencies without being restrained by the limited numerical dynamics of the computer with no particular care for round-off errors. While inheriting all the merits of Rayleigh–Ritz method, the proposed method combines the flexibility of the global methods and the accuracy of the local methods [12]. For implementation, proper treatment should be made to evaluate the derivatives of Daubechies wavelet scale functions due to its strong oscillation and lack of closed-form expression.

It is shown in this paper that the Daubechies wavelet scale functions can be used as admissible functions in Rayleigh–Ritz method and exhibit remarkable features for the vibration analyses of beams with arbitrary elastic boundaries. The limitations of the method using conventional admissible functions can be overcome since the wavelet basis can significantly reduce the round-off errors to avoid the ill-conditioned problems generally encountered. As a result, using normal computation facility, over thousands of natural modes can be accurately predicted with an error capped at 1 percent, outperforming any other basis functions reported in the literature under the frame of Rayleigh–Ritz method. Calculation cases are chosen for demonstrating the accuracy of the proposed Wavelet-based approach. Results on natural modes (from low to high frequencies) were provided. Although individual modes are less important at extra high frequencies, they are useful parameters which allow comparisons with existing analytical solutions. We also calculate the forced vibration responses which do not require the prior calculations of the modes. Results on both free and forced vibrations allow demonstrating that the proposed method is capable of dealing with typical vibration problems in a very wide frequency range, especially at high frequencies, where most of existing methods fail.

The paper is organized as follows. Section 2 is devoted to the theoretical modeling. The formulation for the beam vibrations, detailed in Section 2.1, is based on the Hamilton's principle; the elastic boundary conditions appear through a dynamic contribution in the Hamiltonian of the beam. The extremalization of the Hamiltonian is achieved using Lagrange's equation. Numerical results are presented and discussed in Section 3. The eigen-frequency and mode shape are shown. Comparisons with predictions by classic methods are made both in free–free and pinned–pinned cases. Then the forced vibration response is presented and compared with analytical solution for a large frequency band involving thousands of modes under pinned–pinned boundary conditions.

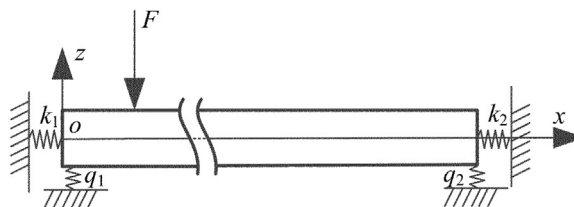


Fig. 1. Euler–Bernoulli beam model.

2. Method

2.1. Beam model

As shown in Fig. 1, the investigated structure is a uniform Euler–Bernoulli beam. The mounting conditions are simulated by a set of artificial translational and rotational springs (k_i in N/m for translation and c_i in N/rad). The use of springs allows the representation of a wide variety of boundary conditions by adjusting the stiffness values of the springs [3]. For example, simply supported boundaries can be obtained by using a sufficiently large k_i and a zero c_i . For the completeness of the paper, the classical Rayleigh–Ritz procedure is briefly recalled.

The flexural displacement $w(x, t)$ of the beam can be expanded as

$$w(x, t) = \sum_i a_i(t)\phi_i(x) \tag{1}$$

where $\phi_i(x)$ are the assumed admissible functions and $a_i(t)$ unknowns to be determined.

The Lagrange’s equations can then be applied to find the stationary state of the system with $a_i(t)$ as generalized coordinates:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{a}_i(t)} \right) - \frac{\partial L}{\partial a_i(t)} = 0 \quad , \quad (i = 0, 1, 2, \dots, n) \tag{2}$$

where L is the Lagrangian of the system expressed as

$$L = E_k - E_p + W \tag{3}$$

where the kinetic energy E_k , potential energy, E_p and the work, W , done by the external excitation force $f(t)$ write

$$E_k = \frac{1}{2} \rho B h \int_0^l \left(\frac{\partial w}{\partial t} \right)^2 dx \tag{4}$$

$$E_p = \frac{1}{2} \int_0^l EI \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx + \frac{1}{2} k_1 w(0, t)^2 + \frac{1}{2} q_1 \left(\frac{\partial w(0, t)}{\partial x} \right)^2 + \frac{1}{2} k_2 w(l, t)^2 + \frac{1}{2} q_2 \left(\frac{\partial w(l, t)}{\partial x} \right)^2 \tag{5}$$

$$W = f(t)w(x_F, t) \tag{6}$$

where l is the length of the beam; ρ is the density of the material; B and h are the width and height of the cross section area, respectively; x_F is the location of the excitation force.

Substituting Eqs. (4)–(6) into Eq. (3) yields the following linear equations in matrix form:

$$[\mathbf{M}]\{\ddot{a}(t)\} + [\mathbf{K}]\{a(t)\} = \{f(t)\} \tag{7}$$

where $[\mathbf{M}]$ and $[\mathbf{K}]$ are, respectively, the mass matrix and stiffness matrix. In a harmonic regime,

$$\{f(t)\} = \{F\}e^{j\omega t} \tag{8}$$

$$\{a(t)\} = \{A\}e^{j\omega t} \tag{9}$$

Eq. (7) can be rewritten as

$$[[\mathbf{K}] - \omega^2[\mathbf{M}]]\{A\} = \{F\} \tag{10}$$

the solution of which gives the natural frequencies and the corresponding mode shapes.

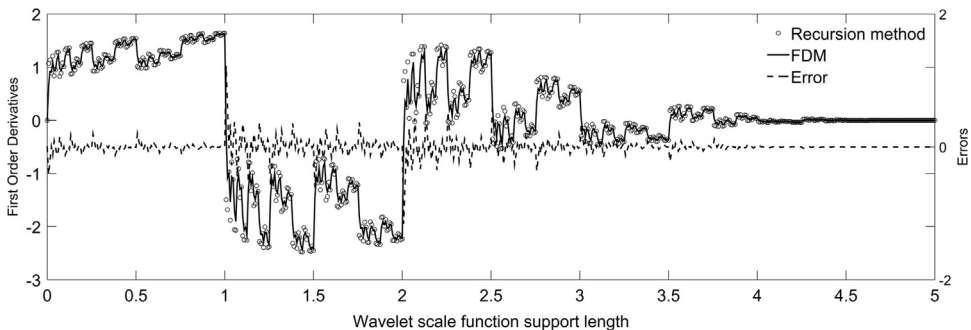


Fig. 2. First order derivatives of D6 scale function calculated by recursion method and FDM, with relative errors between them.

Table 1
Comparison of the eigen-frequencies in low frequency, middle and high frequency bands.

Low frequency band				
No.	F–F (A)	F–F (W)	P–P (A)	S–S (W)
5	0.1779E5	0.1779E5	0.1471E5	0.1471E5
6	0.2485E5	0.2485E5	0.2118E5	0.2118E5
7	0.3309E5	0.3309E5	0.2882E5	0.2882E5
8	0.4250E5	0.4250E5	0.3765E5	0.3765E5
9	0.5309E5	0.5309E5	0.4764E5	0.4764E5
10	0.6485E5	0.6485E5	0.5882E5	0.5882E5
11	0.7779E5	0.7779E5	0.7117E5	0.7117E5
12	0.9191E5	0.9191E5	0.8470E5	0.8470E5
13	1.0720E5	1.0720E5	0.9941E5	0.9941E5
14	1.2367E5	1.2367E5	1.1529E5	1.1529E5
15	1.4132E5	1.4132E5	1.3235E5	1.3235E5
16	1.6014E5	1.6014E5	1.5058E5	1.5058E5
17	1.8014E5	1.8014E5	1.6999E5	1.6999E5
18	2.0131E5	2.0131E5	1.9058E5	1.9058E5
19	2.2367E5	2.2367E5	2.1234E5	2.1234E5
20	2.4719E5	2.4719E5	2.3528E5	2.3528E5
21	2.7190E5	2.7190E5	2.5940E5	2.5940E5
22	2.9778E5	2.9778E5	2.8469E5	2.8469E5
23	3.2484E5	3.2484E5	3.1116E5	3.1116E5
24	3.5307E5	3.5307E5	3.3881E5	3.3881E5
25	3.8248E5	3.8248E5	3.6763E5	3.6763E5
26	4.1307E5	4.1307E5	3.9763E5	3.9763E5
27	4.4483E5	4.4483E5	4.2880E5	4.2880E5
28	4.7777E5	4.7777E5	4.6116E5	4.6116E5
29	5.1189E5	5.1189E5	4.9468E5	4.9468E5
30	5.4718E5	5.4718E5	5.2939E5	5.2939E5
31	5.8365E5	5.8365E5	5.6527E5	5.6527E5
32	6.2130E5	6.2130E5	6.0233E5	6.0233E5
33	6.6012E5	6.6012E5	6.4056E5	6.4056E5
34	7.0012E5	7.0012E5	6.7997E5	6.7997E5
Middle frequency band				
No.	F–F (A)	F–F (W)	P–P (A)	P–P (W)
505	1.5031E8	1.5030E8	1.5001E8	1.5000E8
506	1.5090E8	1.5090E8	1.5060E8	1.5060E8
507	1.5150E8	1.5150E8	1.5120E8	1.5119E8
508	1.5209E8	1.5209E8	1.5180E8	1.5179E8
509	1.5269E8	1.5269E8	1.5239E8	1.5239E8
510	1.5329E8	1.5329E8	1.5299E8	1.5299E8
511	1.5389E8	1.5389E8	1.5359E8	1.5359E8
512	1.5450E8	1.5450E8	1.5420E8	1.5419E8
513	1.5510E8	1.5510E8	1.5480E8	1.5479E8
514	1.5570E8	1.5570E8	1.5540E8	1.5540E8
515	1.5631E8	1.5631E8	1.5601E8	1.5600E8
516	1.5692E8	1.5692E8	1.5661E8	1.5661E8
517	1.5753E8	1.5753E8	1.5722E8	1.5722E8
518	1.5814E8	1.5813E8	1.5783E8	1.5783E8
519	1.5875E8	1.5875E8	1.5844E8	1.5844E8
520	1.5936E8	1.5936E8	1.5905E8	1.5905E8
521	1.5997E8	1.5997E8	1.5966E8	1.5966E8
522	1.6058E8	1.6058E8	1.6028E8	1.6027E8
523	1.6120E8	1.6120E8	1.6089E8	1.6089E8
524	1.6182E8	1.6182E8	1.6151E8	1.6150E8
525	1.6243E8	1.6243E8	1.6213E8	1.6212E8
526	1.6305E8	1.6305E8	1.6274E8	1.6274E8
527	1.6367E8	1.6367E8	1.6336E8	1.6336E8
528	1.6429E8	1.6429E8	1.6398E8	1.6398E8
529	1.6492E8	1.6492E8	1.6460E8	1.6460E8
530	1.6554E8	1.6554E8	1.6523E8	1.6522E8
531	1.6616E8	1.6616E8	1.6585E8	1.6585E8
532	1.6679E8	1.6679E8	1.6648E8	1.6647E8
533	1.6742E8	1.6742E8	1.6710E8	1.6710E8
534	1.6805E8	1.6804E8	1.6773E8	1.6773E8

Table 1 (continued)

High frequency band				
No.	F–F (A)	F–F (W)	P–P (A)	P–P (W)
1005	5.9470E8	5.9469E8	5.9411E8	5.9409E8
1006	5.9588E8	5.9588E8	5.9529E8	5.9527E8
1007	5.9707E8	5.9706E8	5.9647E8	5.9646E8
1008	5.9825E8	5.9825E8	5.9766E8	5.9764E8
1009	5.9944E8	5.9944E8	5.9884E8	5.9883E8
1010	6.0063E8	6.0062E8	6.0003E8	6.0002E8
1011	6.0182E8	6.0181E8	6.0122E8	6.0121E8
1012	6.0301E8	6.0300E8	6.0241E8	6.0240E8
1013	6.0420E8	6.0420E8	6.0360E8	6.0359E8
1014	6.0539E8	6.0539E8	6.0479E8	6.0478E8
1015	6.0658E8	6.0658E8	6.0599E8	6.0597E8
1016	6.0778E8	6.0778E8	6.0718E8	6.0717E8
1017	6.0898E8	6.0897E8	6.0838E8	6.0836E8
1018	6.1017E8	6.1017E8	6.0958E8	6.0956E8
1019	6.1137E8	6.1137E8	6.1077E8	6.1076E8
1020	6.1257E8	6.1257E8	6.1197E8	6.1196E8
1021	6.1377E8	6.1377E8	6.1317E8	6.1316E8
1022	6.1498E8	6.1497E8	6.1437E8	6.1436E8
1023	6.1618E8	6.1618E8	6.1558E8	6.1556E8
1024	6.1738E8	6.1738E8	6.1678E8	6.1677E8
1025	6.1859E8	6.1859E8	6.1799E8	6.1797E8
1026	6.1980E8	6.1979E8	6.1919E8	6.1918E8
1027	6.2101E8	6.2100E8	6.2040E8	6.2038E8
1028	6.2221E8	6.2221E8	6.2161E8	6.2159E8
1029	6.2343E8	6.2342E8	6.2282E8	6.2280E8
1030	6.2464E8	6.2463E8	6.2403E8	6.2401E8
1031	6.2585E8	6.2585E8	6.2524E8	6.2523E8
1032	6.2706E8	6.2706E8	6.2646E8	6.2644E8
1033	6.2828E8	6.2828E8	6.2767E8	6.2766E8
1034	6.2950E8	6.2949E8	6.2889E8	6.2887E8

Expressions for $[M]$ and $[K]$ are

$$M_{ij} = \rho B h \int \phi_i(x) \phi_j(x) dx \quad (11)$$

$$K_{ij} = \frac{EBh^3}{12} \int \phi''_i(x) \phi''_j(x) dx + k_1 \phi_i(0) \phi_j(0) + q_1 \phi_i(0) \phi_j(0) + k_2 \phi_i(l) \phi_j(l) + q_2 \phi_i(l) \phi_j(l) \quad (12)$$

2.2. Wavelet decomposition

The mathematical theory on wavelet analysis is readily available. Details can be found in reference [13]. Here, we only focus on the idea of the decomposition into orthogonal functions of different scales and locations. The wavelet basis we choose is an ensemble of wavelet series with its members being represented as linear combinations of finite sets of scaled, translated, copies of the scaling function.

Daubechies wavelet is a family of compactly supported orthogonal wavelets including both highly localized and highly smooth members. Each wavelet member is governed by a set of L (an even integer) coefficients $\{h_j; j=0, 1, \dots, L-1\}$ through two-scale relations

$$\varphi(x) = \sum_{j=0}^{L-1} h_j \varphi(2x-j) \quad (13)$$

and

$$\psi(x) = \sum_{j=2-L}^1 (-1)^j h_{1-j} \varphi(2x-j) \quad (14)$$

where $\varphi(x)$ and $\psi(x)$ are the scaling function and mother wavelet, respectively. The fundamental support of the scaling function $\varphi(x)$ is the interval $[0, L-1]$ while that of the corresponding wavelet $\psi(x)$ is in the interval $[1-L/2, L/2]$. The coefficients h_j in the two-scale relation (13) are the wavelet filter coefficients. Values of these coefficients can be found in the

literature. The scaling function $\varphi(x)$ has the following properties:

$$\int_{-\infty}^{\infty} x^k \psi(x) dx = 0, k = 0, 1, \dots, L/2 - 1 \tag{15}$$

Note that the property expressed above is equivalent to that the elements of the set $\{1, x, \dots, x^{L/2-1}\}$ are a linear combination of $\varphi(x-k)$, which is a integer translate of $\varphi(x)$. So the scaling function $\varphi(x)$ can be used to approximate a function within an interval [14].

The assumed admissible functions is therefore defined as

$$\phi_i(x) = 2^{m/2} \varphi(2^m x - i) \tag{16}$$

where m is the resolution of wavelet scale functions.

Eq. (12) requires the derivatives of wavelet bases to be evaluated. Since the derivatives of compactly supported wavelets are highly oscillatory, it is unfeasible and numerically unstable to compute the derivatives numerically. In order to overcome this problem, dedicated algorithms called recursion method have been derived for the exact evaluation of derivatives [14,15].

The n th derivative of the scaling function, denoted by $\varphi^{(n)}(x)$ writes

$$\varphi^{(n)}(x) = \frac{d^n \varphi(x)}{dx^n} = \frac{d}{dx} \varphi^{(n-1)}(x), \varphi^{(0)}(x) = \varphi(x) \tag{17}$$

Substituting the two-scale relationship equation (13) into the above equation yields

$$\varphi^{(n)}(x) = 2^n \sum_{k=0}^{L-1} h_k \varphi^{(n)}(2x - k) dx \tag{18}$$

Eq. (18) is the two-scale relationship for $\varphi^{(n)}(x)$, which can be used to compute $\varphi^{(n)}(x)$ at all dyadic points $x = k/2^j, j = 1, 2, \dots, \infty$, provided that the values of $\varphi^{(n)}(k), k = 1, 2, \dots, L-2$ are given. The above can be referred to as recursive procedure.

To get $\varphi^{(n)}(x)$ at integer points, we substitute $x = 1, 2, \dots, L-2$ into Eq. (18), resulting in the following homogeneous linear system:

$$(\mathbf{H} - 2^{-n} \mathbf{I}) \Phi = \mathbf{0} \tag{19}$$

where

$$\Phi = [\varphi^{(n)}(1) \ \varphi^{(n)}(2) \ \dots \ \varphi^{(n)}(L-2)]^T \tag{20}$$

\mathbf{H} is a $(L-2) \times (L-2)$ matrix

$$\mathbf{H} = [h_{2s-t}]_{1 \leq s, t \leq L-2} \tag{21}$$

with s being the row index and t the column index. Eq. (19) indicates that the unknown vector Φ is the eigenvector of the matrix \mathbf{H} corresponding to the eigenvalues 2^{-n} . The values $\varphi^{(n)}(x)$ at integer points, $x = 1, 2, \dots, L-2$, can be determined after getting the eigenvector of \mathbf{H} which corresponds to the eigenvalues 2^{-n} , and then be normalized as

$$\sum_{k=1}^{L-2} (-k)^n \varphi^{(n)}(k) = n! \tag{22}$$

Upon getting $\varphi^{(n)}(x)$, Eq. (18) can be written as

$$\varphi^{(n)}\left(\frac{k}{2^j}\right) = 2^n \sum_{l=0}^{L-1} h_l \varphi^{(n)}\left(\frac{k}{2^{j-1}} - l\right) \tag{23}$$

The use of above equation and $\varphi^{(n)}(x) = 0$ outside the support length ($x \leq 0$ and $x \geq L-1$) allows one to determine the values of $\varphi^{(n)}(x)$ at all dyadic points $x = k/2^j, k = 1, 2, \dots, 2^j(L-1) - 1$ and $j = 1, 2, \dots$

A comparison is made with the traditional numerical scheme based on finite difference method (FDM), expressed as

$$\varphi'(x_0) \approx \frac{\varphi(x_0 + h) - \varphi(x_0)}{h} \tag{24}$$

As shown in Fig. 2, the discrepancy of the 1st order derivatives of D6 wavelet scale function between recursive method and FDM are significant. Compared with the exact value calculated by recursive method at dyadic points, FDM has large errors. Therefore, the FDM cannot be directly used.

Integration needs to be carried out to obtained the mass and stiffness matrices expressed by Eqs. (11) and (12). To that end, the previously obtained two-scale relationship in terms of derivative calculation can be used in Eq. (13). By following the same procedure, recursive relationship could also be established for integration calculation [15,16]. Or alternatively, numerical scheme could be used to directly calculate the integration terms [17].

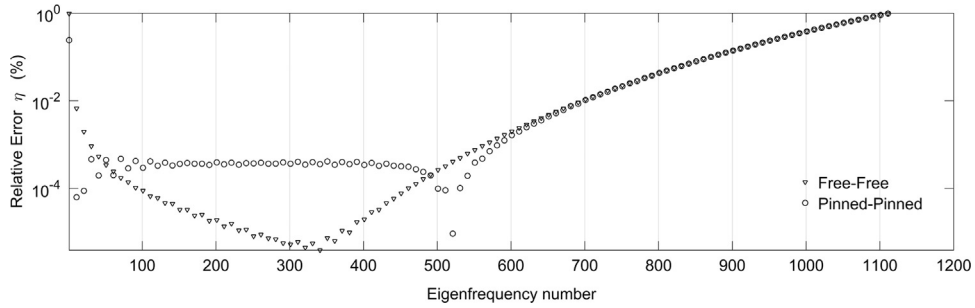


Fig. 3. Relative error compared with exact solution, using wavelet resolution $m=8$, support length $L=16$, considering both the free-free and pinned-pinned boundary conditions.

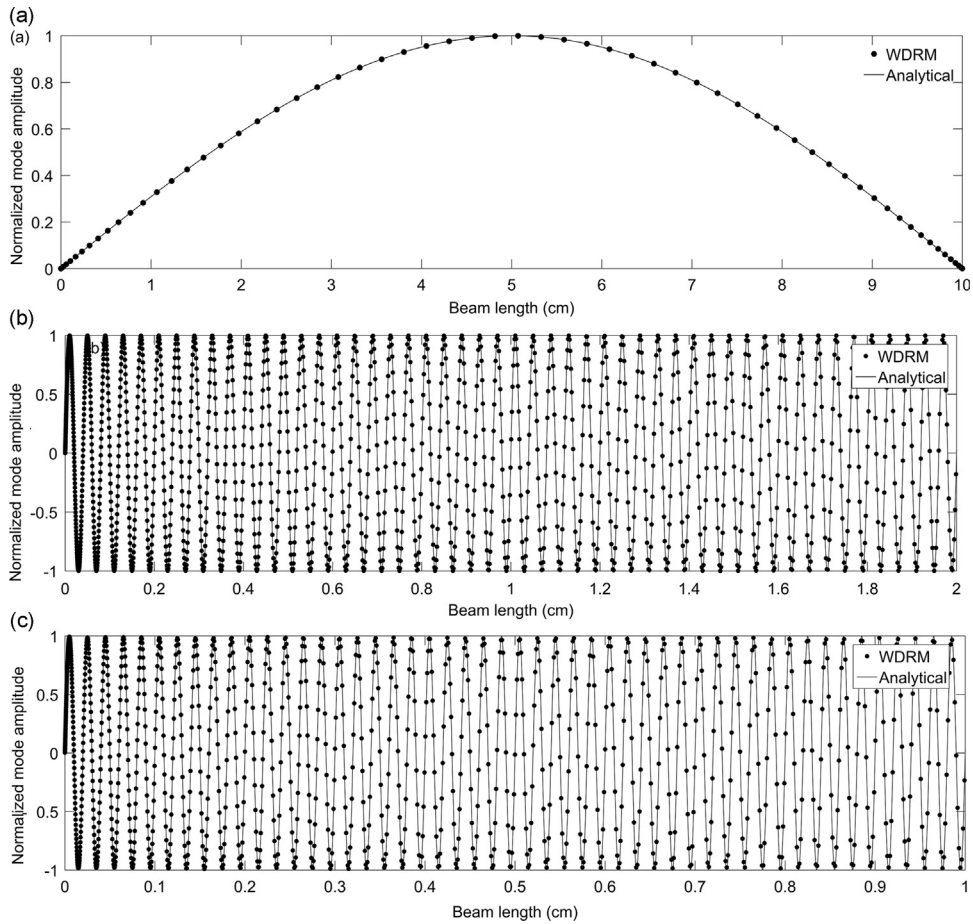


Fig. 4. Mode shape comparison between WDRM and analytical method: (a) 1st mode within beam length 0–10 cm, (b) 500th mode within beam length 0–2 cm and (c) 1000th mode with beam length 0–1 cm.

3. Results and discussions

The application of the WDRM in the analysis of high-frequency vibrations is demonstrated in this section using a uniform Euler–Bernoulli beam. This is a good benchmark problem which allows exact solutions. Results on both free and forced vibration problems are presented in the following two sections. It should be mentioned that the validity of the Euler–Bernoulli assumption at high-frequency range is well known and is beyond the main scope of the present paper, since the main purpose is for illustrating the efficiency and the accuracy of the method through comparison with a simple benchmark problem with exact solutions.

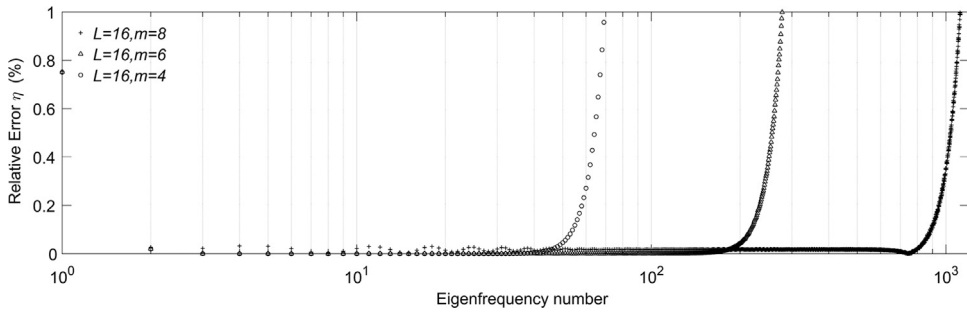


Fig. 5. Relative error within 1 percent threshold for different Daubechies wavelet scale function resolution m while the support length L is fixed at 16.

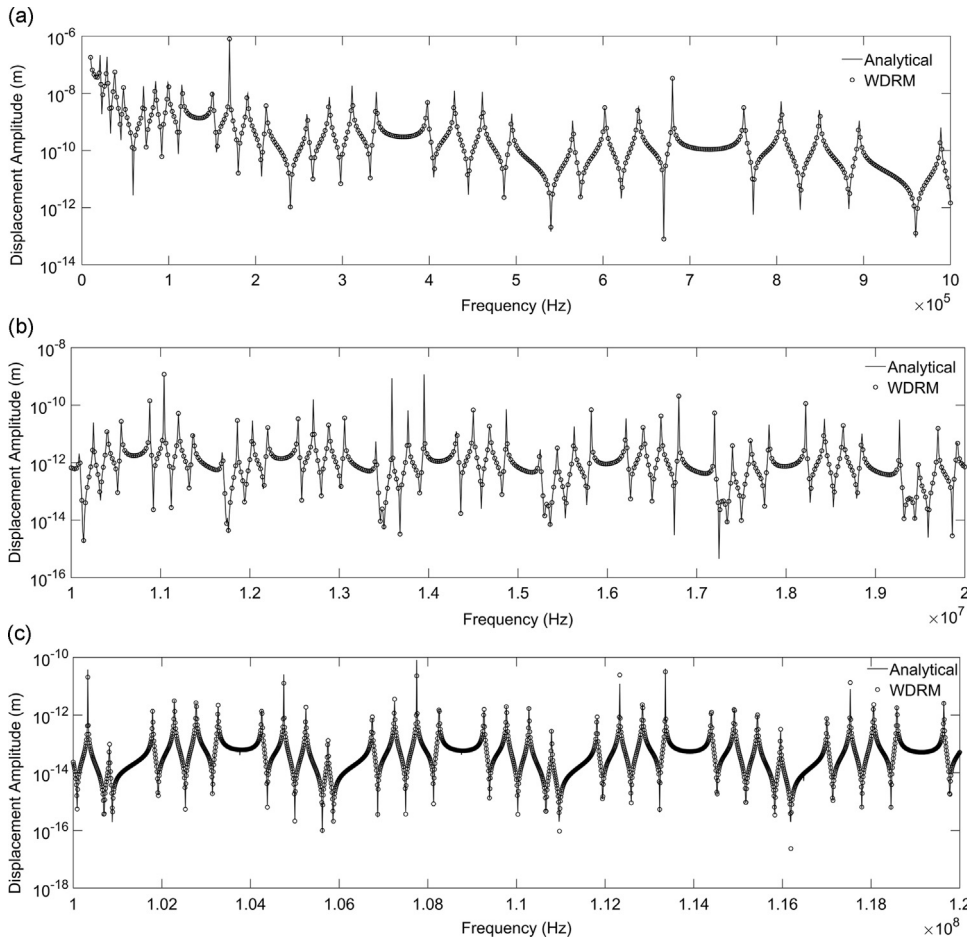


Fig. 6. Displacement amplitude at specific locations in different frequency bands: (a) $x=9$ cm with the frequency band from 100 Hz to 10 kHz, (b) $x=9$ cm with the frequency band from 10 MHz to 20 MHz and (c) $x=9$ cm with the frequency band from 0.1 GHz to 0.12 GHz.

3.1. Eigen-frequency study

The eigen-frequency problem described by Eq. (10) is first considered in this section. Both free–free and pinned–pinned boundary conditions are considered. The eigen-frequencies are respectively given below [18]

$$f_{n_1} = \frac{(n+1/2)^2 \pi}{2l^2} \sqrt{\frac{EI}{\rho A}}, \quad \text{for } n > 5, \text{ free – free} \tag{25}$$

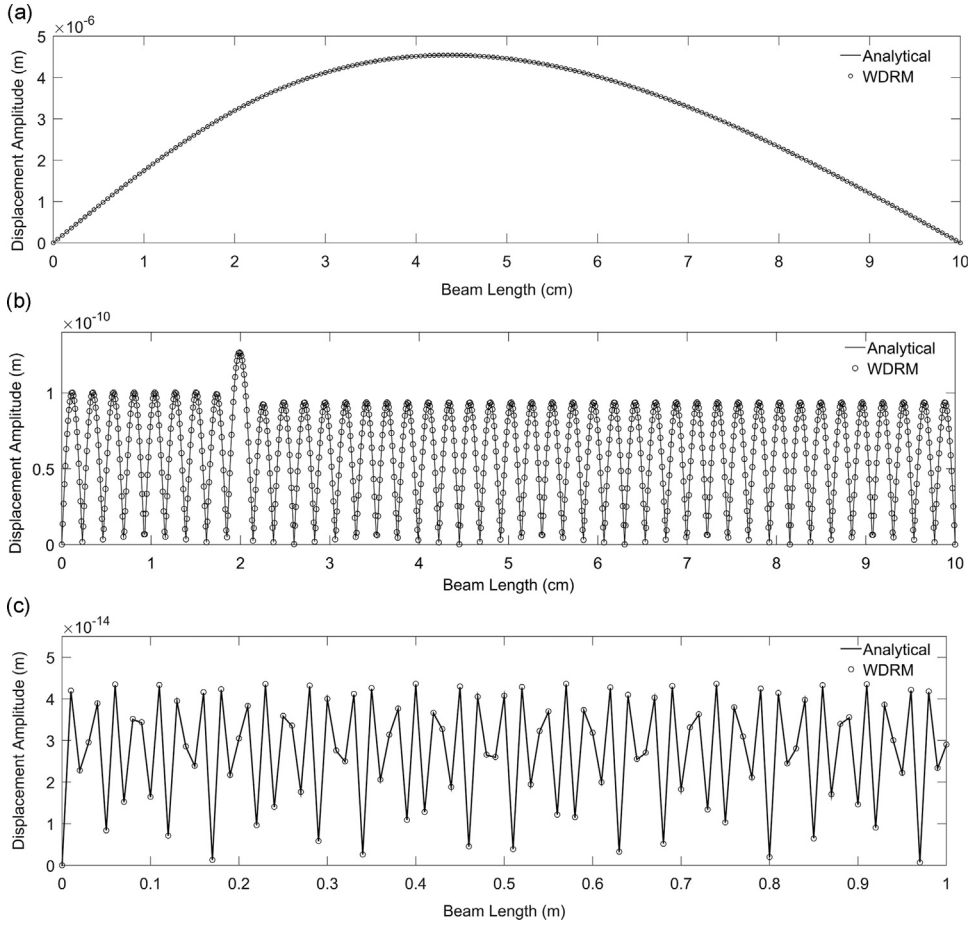


Fig. 7. Displacement amplitude along the beam at specific frequencies: (a) 100 Hz from 0 cm to 10 cm, (b) 1.1 MHz from 0 cm to 10 cm and (c) 0.1 GHz from 0 cm to 1 cm (only part of the beam).

$$f_{n_2} = \frac{n^2 \pi}{2l^2} \sqrt{\frac{EI}{\rho A}} \quad \text{pinned – pinned} \quad (26)$$

where A is the cross section area of the beam. In the calculations, parameters of the beam are chosen as $\rho = 7800\text{kg/m}^3$, $l = 0.1\text{ m}$, $B = 0.01\text{ m}$, $h = 0.025\text{ m}$, $E = 2.1\text{E}11\text{Pa}$. Various stiffness terms are set as $k_1 = k_2 = 0$, $q_1 = q_2 = 0$ for free–free boundary conditions; and $k_1 = k_2 = 5\text{E}16\text{ N/m}$, $q_1 = q_2 = 0$ for pinned–pinned boundaries. Compared with the nominal stiffness of the beam ($k_0 = EI/l^3$), these values correspond to a non-dimensional parameter $k_i/k_0 = 1.83\text{E}13$ for both k_1 and k_2 . The convergence has been carefully checked to show that further increases in k_i would not lead to noticeable changes in the calculated results.

The accuracy of the WDRM is tested through error analyses and convergence study respectively. This leads to the final selection of the parameters used in the calculation: $L = 16$ and $m = 8$, resulting in 2574 wavelet scale function items. The calculated natural frequencies are tabulated in Table 1, in comparison with analytical ones. Note all first 1100 modes were successfully obtained. For the sake of brevity, only three representative groups, corresponding to low, middle and high-frequency bands, are tabulated in Table 1. The first column in each sub-table is the mode number, including free–free (F–F) and pinned–pinned (P–P) boundary conditions of the beam. “(A)” and “(W)” are abbreviation for analytical method and WDRM, respectively. It is noted that the WDRM achieves remarkable accuracy as compared with the analytical results in both boundary cases in all three frequency bands.

In order to quantify the accuracy of the proposed WDRM, the relative error for each eigen-frequency is defined as

$$\eta = \frac{|f_{\text{exact}} - f_w|}{f_{\text{exact}}} \times 100\% \quad (27)$$

where f_{exact} is the exact frequency value and f_w the numerically calculated eigen-frequency by WDRM. The relative error for all 1100 calculated natural frequencies is shown in Fig. 3. Generally speaking, the relative error for both boundary conditions is very small for basically the first half of modes. Errors start to increase for higher-order modes. Nevertheless, the agreement between the WDRM results and analytical ones is capped under 1 percent for all modes, which is excellent considering

the fact that it is a 1-D configuration. For a 2-D case, with all cross terms among the two directions, the number of modes that could be correctly predicted should be phenomenal.

Typical mode shapes are also checked against the analytical ones. Shown in Fig. 4, three modes (1st, 500th and 1000th) are illustrated, showing perfect agreement with analytical results. In order to have a clear view of the mode shape, only part of the beam is used in the illustrations for mode 500 (0–2 cm) and mode 1000 (0–1 cm) due to their very small wavelengths.

To examine the convergence of the proposed WDRM, the relative error is examined with varying resolution parameter m , as shown in Fig. 5. As can be seen, m needs to be increased for the same support length L for more modes to be correctly calculated. More specifically, when $m=4$, about 70 eigen-frequencies can be obtained under 1 percent error. The number increases to about 280 for $m=6$; and 1100 for $m=8$. Therefore, the effect of the resolution parameter is obvious. When fixing m , however, results show little influence of the support length, L , (not shown here).

3.2. Forced vibration response

Forced vibration response of the beam is calculated to further verify the accuracy of the proposed method. A unit excitation force is located at $x_F=2$ cm as shown in Fig. 1. The response of the pinned–pinned beam, at an arbitrarily chosen location $x=9$ cm, is given in Fig. 6, covering a wide frequency band between 100 Hz and 100 kHz. The analytical solution, used as reference for comparison, writes [18]

$$w(x) = \frac{2}{\rho A l} \sum_{n=1}^{\infty} \frac{1}{\omega_n^2 - \omega^2} \sin\left(\frac{n\pi x_F}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \quad (28)$$

To generate accurate analytical results, 10,000 modes were used in the calculation using Eq. (28). In all three frequency bands, low in Fig. 6(a), middle in Fig. 6(b) and high in Fig. 6(c), the displacement responses calculated by the WDRM are almost identical to the analytical solutions, demonstrating again the remarkable accuracy of the method, even for very high frequencies.

Furthermore, Fig. 7 shows the displacement amplitude along the beam at three representative frequencies, taken from Fig. 6. As can be seen, for all three frequencies, WDRM and analytical method provide almost the same results. Note again that, due to the extremely tightness of the wave at high frequency, only one-tenth of the beam length is shown in Fig. 7(c).

It is relevant to comment on a few important issues related to Daubechies wavelet scale functions and their use as decomposition basis under the current Rayleigh–Ritz framework. First, Daubechies wavelet scale functions are known to possess unique features such as compact support, flexible scaling and translation, and the ability to express any square integrable function on the real axis. This leads to their extraordinary fitting ability in approximating both highly oscillating and smooth functions, as demonstrated earlier in acoustic predictions [12], and again confirmed here in the present vibration study. This guarantees much better convergence as opposed to the trigonometric functions. Meanwhile, it could also minimize the round-off errors which could lead to ill-conditioned matrices like in the case of polynomials. Second, the calculation efficiency of the propose WDRM is worth noting. Owing to the compact support property of the Daubechies wavelet scale functions, the resulting mass and stiffness matrices are band limited, with all non-zero terms concentrated along the diagonal band. The bandwidth is $L-1$, where L is the support length of the Daubechies wavelet scale functions. Thanks to this bandlimited property, the solving process is straightforward and very efficient. As an indicative example of the present 1-D problem, using $L=16$, $m=6$ leads to 654×654 matrices, the calculation of the natural modes roughly takes about 1 min on a conventional 64-bit Dell workstation with a 3 GHz CPU to get the first 280th mode within 1 percent error. Limited by the maximum ram of the workstation (64 GB), the true limit of the method was not reached in our investigation. Lastly, it is relevant to mention that, the established approach exhibits its flexibility in dealing with various structural boundaries. In that sense, the Daubechies wavelet scale functions can be generally applied since the parameters involved are independent of the physical boundary conditions of the structure.

As a general remark, it is relevant to note that an Euler–Bernoulli beam is used as a representative dynamic system to illustrate the issue. There should be no particular technical obstacles to apply the proposed Wavelet-decomposed approach to more complex systems, which would also benefit from the efficacy of the approach, similar to what has been demonstrated in this paper.

4. Conclusions

This paper explores the use of wavelet decomposition under the general framework of Rayleigh–Ritz method for high-frequency vibration analyses. Using a benchmark problem of Euler–Bernoulli beam, the efficacy of the proposed wavelet-decomposed Rayleigh–Ritz method is demonstrated. Wavelet basis provides localized compact support, which leads to sparse representations of the system matrices. Along with its exceptional fitting ability and the treatment of the structural boundary using artificial springs, the proposed approach exhibits tremendous computational advantages and allows reaching extremely high frequency range. The proposed approach combines the flexibility of the global methods and the accuracy of local methods by inheriting the versatility of the Rayleigh–Ritz approach and the unique feature of the wavelets. More specifically, more than 1000 modes are correctly predicted in the present configuration with an error of less than

1 percent using conventional computational facility. Forced vibration responses also show excellent agreement with analytical solutions within the range covered by the same group of modes. To our knowledge, the frequency coverage and the level of accuracy achieved outperform what can be found in the open literature using Rayleigh–Ritz methods. It can be anticipated that, upon extension to 2-D cases, the number of modes that could be accurately predicted should be at least a few order of magnitude higher, which potentially might accommodate the need for structural analyses and design in a very broad frequency range.

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