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Vibrational analysis of point-coupled structures

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Abstract

Vibration Analysis of systems with beam components is performed. A previously developed method called Internal Loading Decomposition is applied to this particular configuration. A beam is used as the master structure to which are added one or more substructures. The method is assessed for several variations of this configuration using finite element analysis. Numerical examples demonstrate how the method can be advantageously applied to analyze periodical structures. Finally, a sensitivity study is performed to illustrate the effect of compliance uncertainties on the vibrational response.

Keywords: Vibration; Beam; Compliance; Coupled structure; Sensitivity; Damping; Mobility

1. Introduction

Vibration analysis of coupled structures is of great importance in structural dynamics. Significant progress has been made over the last decades to simulate the vibrational response of complex systems composed of many substructures. Apart from standard numerical methods like Finite Element Analysis and some purely experimental approaches, classical methods such as modal synthesis method [1], mobility power flow [2,3] and statistical energy analysis [4] are well known methods for investigating low, medium and high frequency ranges, respectively. Other alternatives have also been proposed in the literature. Some examples are: substructure synthesis analysis [5], receptance method [6], Lagrange multiplier methods [7,8],

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mobility power flow approach [9,10], frequency window method [11] and modal sampling method [12].

Generally speaking, most of these methods can be classified in three categories: analytical or semi-analytical, numerical and experimental methods. Every one of them has its own advantages and restrictions. Analytical methods are accurate and make it easier to investigate the effect of different parameters of simple systems. Numerical methods, however, still require good system models and become computationally demanding when a large number of substructures are coupled together. Experimental methods are applicable to any system but the fact that they cannot be used as a provisional tool make them less attractive. As far as structural modeling is concerned, the fact that most of these methods try to model all structural components limit their applications especially when a great number of substructures are coupled together. In addition to that, in a real-life system, there may be some effects which can hardly be modeled with reasonable cost and of which only experimental tests are capable of taking into account. New strategies should be developed to tackle these problems.

In the past few years, efforts were made to develop more physical approaches capable of analyzing structures with a certain degree of complexity. The essence of these techniques is to use a more physical base before purely numerical treatments are engaged. In fact, although a system may be very complex, elements like beams, plates and shells are usually the main components in a vast amount of structural systems. Under some circumstances, these structures can be reasonably well analyzed by analytical or so-called semi-analytical approaches based on variational principle using artificial spring systems [13–15]. However, these approaches fail when a great number of substructures are involved.

Recently, a new substructuring technique called "Coupling Load Decomposition" [16] is proposed to perform vibration analysis of line-coupled structures. In dividing the whole structure into a master structure and several auxiliary structures, a variational formulation is used to model the master structure, enabling one to introduce the effects of all auxiliary structures by using their compliance characteristics at several observation points along the junction. This new formulation permits the direct use of the compliance of the junction, which may be obtained analytically, numerically or experimentally. Simulation results on the plane structures show good agreement in low and medium frequency ranges. The general procedure is shown in Fig. 1. Up to now, compliance data used in the previous simulations are supposed to contain no error. In practice however, compliance of substructures should be obtained, more often than not, experimentally. Errors are then inevitable. Analyses should then be performed to investigate the effects of compliance component uncertainties. This analysis should help to determine the tolerance on compliance measurements in the application of the technique.

In this article, this modeling philosophy is applied to one dimensional point coupled structures. In many vibrating systems, the contact zone between coupled elements is small enough compared to the wavelength of the structure so that it can be considered as a single point. The objectives are two fold: (1) to assess the suitability of the technique for point-coupled cases and (2) to investigate the effect of

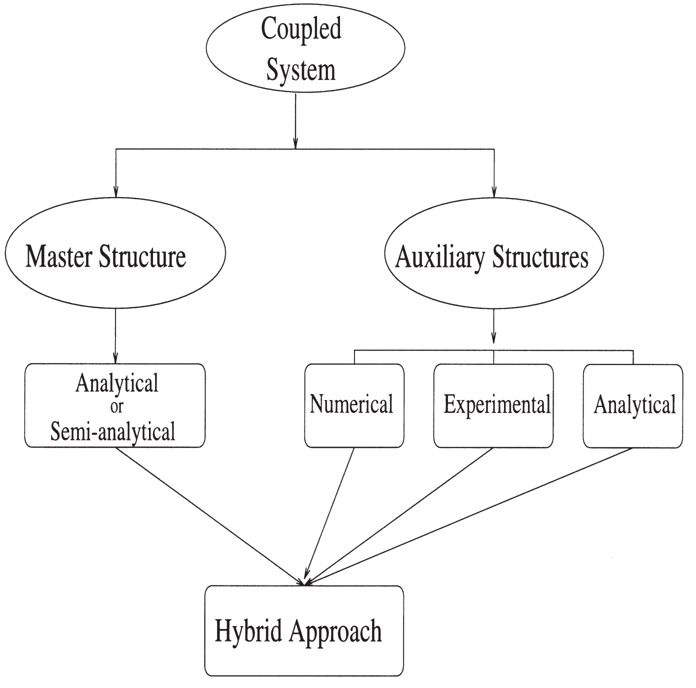

Fig. 1. Flow chart of the method.

biased compliance measurement on the accuracy of the prediction. To this end, a general formulation is proposed for a general beam as the main structure, coupled to arbitrary substructures. The method is assessed for various configurations by comparing numerical results to finite element simulations. Finally, a sensitivity analysis is performed.

2. Theoretical formulation

The basic idea behind the method is to divide the whole structure in two parts: a master structure and auxiliary substructures, as illustrated in Fig. 1. The effects of substructures are introduced as forces and moments acting at the contact points. These forces and moments are related to a deformation vector at the contact point by a frequency dependent complex matrix. This matrix is the compliance matrix which describes the dynamic characteristics of the substructure at the contact point. For point coupling, the compliance β^i_{km} is defined as

$$\beta^i_{km}=\frac{D^i_m}{F^i_k} \tag{1}$$

where F^i_k is the component of the load vector applied at point "i" in direction "k"

and D_m^i is the component of the induced deformation vector at point "*i*" in direction "*m*". Using a variational approach, the effects of substructures may be introduced by their related energy term E_{cp}^i which can be written as

$$E_{cp}^i=\frac{1}{2}\sum_k^N F_k^i D_k^i \tag{2}$$

where N is the number of degrees of freedom at the contact point "*i*". Using Eq. (1) yields

$$E_{cp}^i=\frac{1}{2}\sum_k^N\sum_m^N \eta_{km}^i D_m^i D_k^i \tag{3}$$

Where η is the inverse of compliance matrix. This procedure can be repeated for all contact points and the total energy related to the substructures can be written as

$$E_{cp}=\frac{1}{2}\sum_i^{NP}\sum_k^N\sum_m^N \eta_{km}^i D_m^i D_k^i \tag{4}$$

where NP is the number of contact points.

In this article, the method is assessed by investigating the dynamics of a system composed of a beam to which a number of substructures are attached as illustrated in Fig. 2. Two flexible supports at points c_1 and c_2 are used to define the supporting

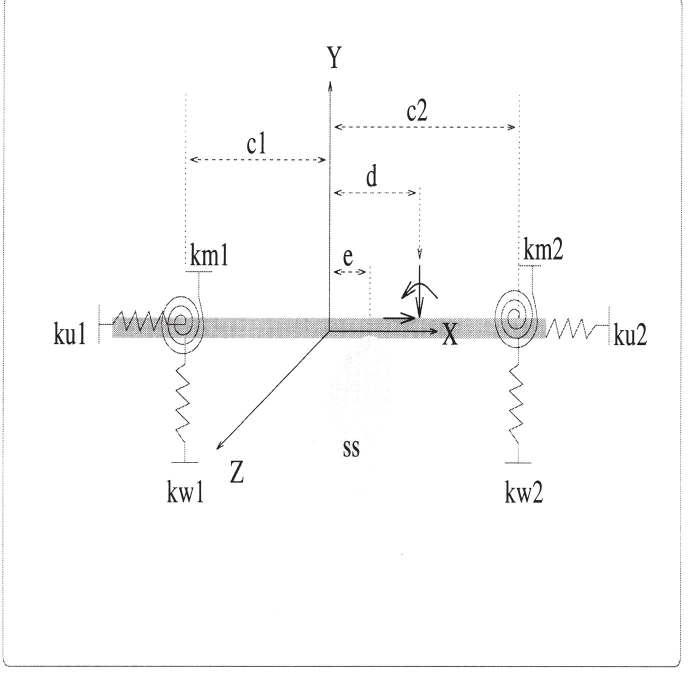

Fig. 2. A uniform beam with general boundary conditions, coupled with an arbitrary substructure.

conditions of the beam. Using a coordinate system located at the middle point of the beam, the displacement field of the beam can be represented by a sinusoidal decomposition as:

$$w=\sum_{n=1}^{\infty}[a_{1n}(t)\sin(n\pi x/L)+a_{2n}(t)\cos(n\pi x/L)] \tag{5}$$

$$u=\sum_{n=1}^{\infty}[b_{1n}(t)\sin(n\pi x/L)+b_{2n}(t)\cos(n\pi x/L)] \tag{6}$$

where w and u are respectively the transversal (Y axis) and longitudinal (X axis) displacements and L, the length of the beam.

Using thin beam theory with small deformations, decoupled displacement fields can be used for flexural and longitudinal vibrations. The kinetic energy E_c of the master structure and its potential energy E_p are

$$E_c=1/2\rho S\int_{-L/2}^{L/2}(\dot{u}^2+\dot{w}^2)\,dx \tag{7}$$

$$E_p=1/2ES\int_{-L/2}^{L/2}(\partial u/\partial x)^2dx+1/2EI\int_{-L/2}^{L/2}[\partial^2 w/\partial x^2]^2dx \tag{8}$$

where ρ, E, S and I are respectively the mass density, the Young's modulus, the cross sectional area and the moment of inertia of the beam. Stretching energy E_s, of the springs at positions $x=c_1$, c_2 is

$$\begin{aligned}E_s\ &=[1/2k_{w1}w^2+1/2k_{u1}u^2+1/2k_{m1}[\partial w/\partial x]^2]_{x=c_1}\\ &+[1/2k_{w2}w^2+1/2k_{u2}u^2+1/2k_{m2}[\partial w/\partial x]^2]_{x=c_2}\end{aligned} \tag{9}$$

where k_{w1} and k_{w2} are the translational stiffnesses of supporting springs at points c_1 and c_2 respectively, and k_{m1} and k_{m2}, the corresponding rotational stiffnesses.

The work E_f done by external forces and moments and the energy E_{cp} related to the substructures at the contact points can be written as

$$E_f=\left[F_w w+F_u u+M\left(\frac{\partial w}{\partial x}\right)\right]_{x=d} \tag{10}$$

$$E_{cp}=\frac{1}{2}\left[F_w w+F_u u+M\left(\frac{\partial w}{\partial x}\right)\right]_{x=e} \tag{11}$$

where F_u and F_w, are longitudinal and transversal forces respectively, and M is the flexural moment.

The Lagrangian of the system is defined as

$$L=E_c-(E_p-E_s)+E_f+E_{cp} \tag{12}$$

After minimizing the Lagrangian of the system with respect to coefficients in Eqs. (5) and (6), the forced vibration of the system may be obtained by solving the following system of equations

$$[\boldsymbol{M}]\{\ddot{\boldsymbol{A}}\}+[\boldsymbol{K}]\{\boldsymbol{A}\}=\{\boldsymbol{F}\} \tag{13}$$

where $[M]$ and $[K]$ are mass and stiffness matrices respectively. $\{F\}$ is the force vector and $\{A\}$ is the unknown vector of coefficients used in the displacement decomposition. Note that the dynamic effects of a substructure is considered through its frequency dependent, compliance components which have been introduced into the stiffness matrix. Hence the stiffness matrix K is a frequency dependent complex matrix.

The theoretical formulation developed above is quite general. Each matrix should be calculated depending on the coupling nature and substructure type. Details for each particular configuration are given hereafter.

3. Applications

In all examples treated hereafter, a steel beam is used as the main structure with a modulus of elasticity E=2.0E11 N/m^2 and a mass density ρ=7800 kg/m^3 . The beam is 60 cm long and its cross section is A=15×3 mm^2. A unit transversal harmonic force excitation is applied to the beam at x=9 cm and the transversal response of the structure is calculated at the same point. In all cases, a damping factor of 1% is introduced into the model by using a complex Young's Modulus. The main structure was modelled semi-analytically using a series expansion with sufficient terms to ensure a convergent solution, In the results presented hereafter, the number of terms varies according to the configuration and the frequency range.

3.1. Dynamic absorber as a substructure

As a first example, a dynamic absorber spring-mass-damper system is attached to the simply supported beam at point x=9 cm. The mass (m_d=21.16 g) and the stiffness (k_d=10 000.0 N/m) of the dynamic absorber are chosen in such a way that its compliance is in the same order of magnitude as that of the beam at the contact point to ensure an effective coupling with the beam. The damper has a damping factor of ξ=1%. This, simple substructure can be analytically solved to obtain the compliance term at the contact point, as follows:

$$\beta=\frac{\left(\left(\frac{\omega_n}{\omega}\right)^2-1\right)^2+4\xi^2\left(\frac{\omega_n}{\omega}\right)^2}{k_d\left(\left(\frac{\omega_n}{\omega}\right)^2+4\xi^2-1\right)-2k_d\frac{\xi_w}{\omega_n}i} \tag{14}$$

where ω_n is the natural frequency of the damper and i the imaginary number.

The displacement of the beam is shown in Fig. 3, as a function of frequency. It can be seen that all resonant frequencies originally higher than the fundamental frequency of the spring-mass system increased and those originally lower than this frequency decreased. This result agrees with the conclusions of Dowell [7] on general properties of combined dynamical systems. In addition, the absorbing effect of the dynamic absorber on the beam at its natural frequency is quite noticeable.

3.2. Single beam as a substructure

The method was also validated using a substructure involving more complex modal behavior. Fig. 4 shows a system composed of two perpendicular beams, clamped at one end and coupled together at the other end. The horizontal beam has the same characteristics as the one described previously and it is considered as the main structure. The vertical beam, defined as the auxiliary structure, has similar cross sectional area and material properties, and it has a length L_s=2/3L. The effects of the vertical beam may be introduced by its three compliance components. Its longitudinal motion affects the flexural vibration of the master beam while its flexural vibration affects the longitudinal and flexural behavior. In this case, the relationship between

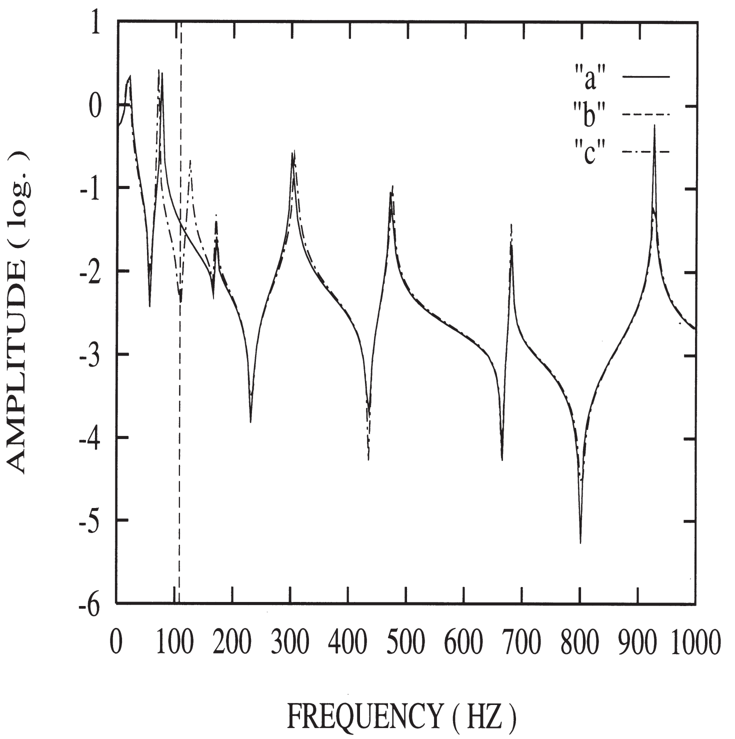

Fig. 3. Response of a simply supported beam coupled with a dynamic absorber ——— beam without spring-mass system – – – natural frequency of spring-mass system – · – response of coupled system.

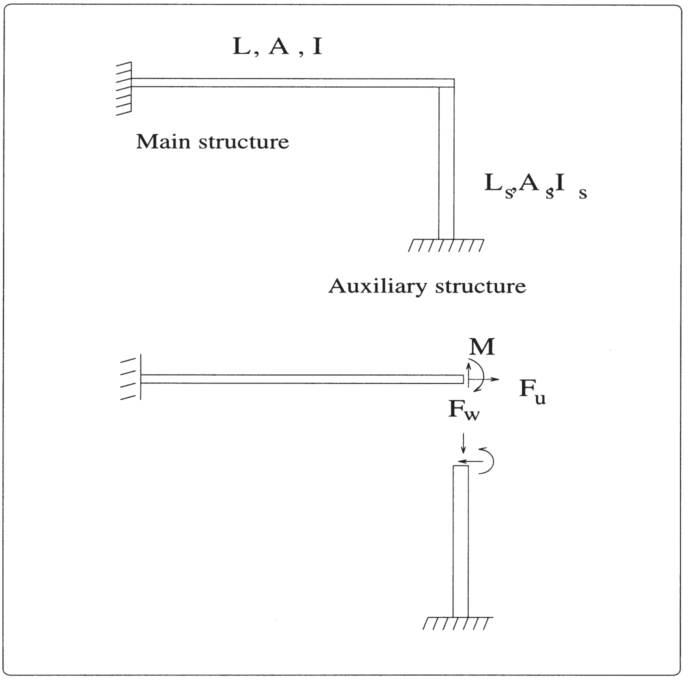

Fig. 4. Coupling between two perpendicular beams.

the generalized forces (two forces and one moment in this case) and the generalized displacements (two translations and one rotation) can be expressed by a compliance matrix:

$$\begin{Bmatrix} w^p \\ u^p \\ \theta^p \end{Bmatrix} = \begin{bmatrix} \beta^p_{11} & \beta^p_{12} & \beta^p_{13} \\ \beta^p_{21} & \beta^p_{22} & \beta^p_{23} \\ \beta^p_{31} & \beta^p_{32} & \beta^p_{33} \end{bmatrix} \begin{Bmatrix} f^p_w \\ f^p_u \\ m^p \end{Bmatrix} = [\beta^p] \begin{Bmatrix} f^p_w \\ f^p_u \\ m^p \end{Bmatrix} \tag{15}$$

In the above expression, the terms β^p_{ij} of the compliance matrix are defined by Eq. (1) and f^p_w, f^p_u and m^p are, respectively, the transversal and longitudinal forces and moment about the axis normal to the plane formed by the beams. The flexural and longitudinal displacements at the contact point are noted w^p and u^p while the rotation normal to the plane is noted θ^p.

Assuming decoupling between longitudinal and bending vibrations, the compliance terms β^p_{12}, β^p_{21}, β^p_{23} and β^p_{32} are equal to zero. In addition, the compliance matrix is symmetric due to the reciprocity principle of linear components [17]. Hence, only three components of the compliance matrix must be determined.

For the supporting beam, no analytical solution is available. Therefore, different compliance components are calculated semi-analytically using the same method previously explained for the horizontal beam. Using, respectively, the appropriate excitation, the two components related to the flexural motion of the beam are calculated using the series decomposition (5); whilst the one related to the longitudinal motion is calculated using Eq. (6).

Fig. 5 shows the response of this coupled system. The results are compared to the finite element method using linear beam elements (33 elements per unit length of the beam). One can observe that both methods give very similar results. Validation of the present method with a vertical beam is rather representative since the vertical beam exhibits relatively complex dynamic behavior with six natural modes included in the frequency band of interest.

3.3. Systems with repeated substructures

Periodical structures or systems with repeated substructures form an important family of vibrating systems. Such a system was investigated to illustrate the efficiency of the present method for handling this kind of situation. The previous

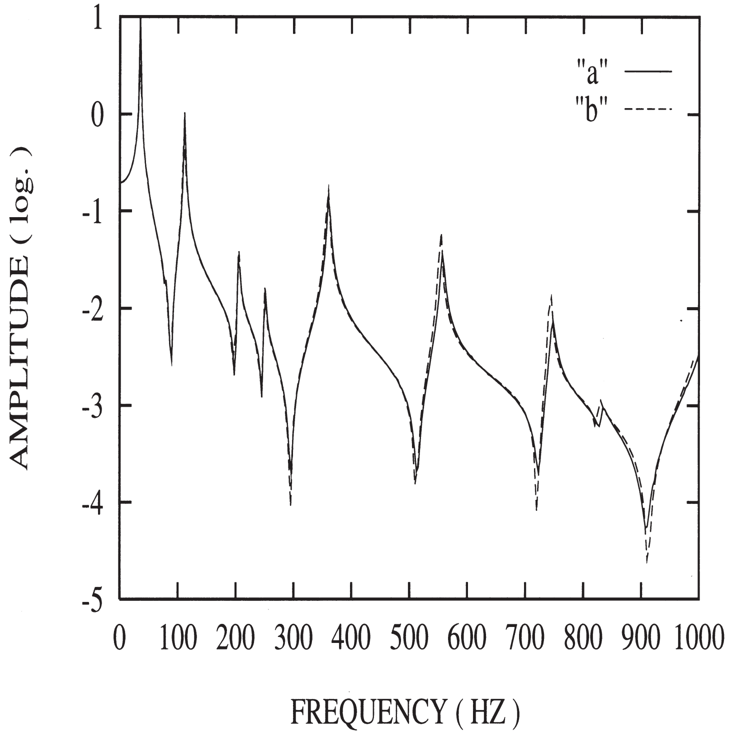

Fig. 5. Vibrational response of an L-shape beam. ——— the present approach – – – Finite Element method.

example of two perpendicular beams was extended by adding a number of supporting beams as shown in Fig. 6. All supporting beams have the same dimensions and mechanical properties of the beams previously described, are clamped at one end and are connected to a horizontal cantilevered beam at the other end.

Fig. 7 compares the vibrational responses of three different systems. The first is a single horizontal cantilevered beam without any connected substructure; the second and the third responses represent the cantilever beam response supported respectively by one and four vertical beams. In all cases, the supporting beams are equally distributed along the horizontal beam starting from the right end. It can be seen from Fig. 7 that the presence of the supporting beams increases the stiffness of the system so that all natural frequencies of the horizontal beam are shifted towards higher frequencies. This frequency increase becomes more obvious when the number of supporting beams is increased. In addition, the presence of supporting beams introduces additional peaks. These peaks should correspond to the motion controlled by the supporting beams near their own resonances [15]. It can also be observed that the four supporting beams significantly attenuate the vibrational level at low frequencies due to the stiffening effect. However, this effect becomes less obvious at higher frequencies when the wave length is comparable or smaller than the span between the two supporting beams. The case of four supporting beams has also been compared to Finite Element simulations, and similar results were obtained.

It is generally admitted that the modal density in a coupled system increases with the number of substructures. For any method that uses modal superposition, the computing time and memory requirements is significantly increased. For example, in the case of two perpendicular beams, there exists 21 modes of vibration (including the vibrational modes normal to the plane of the frame) up to 1000 Hz. The number

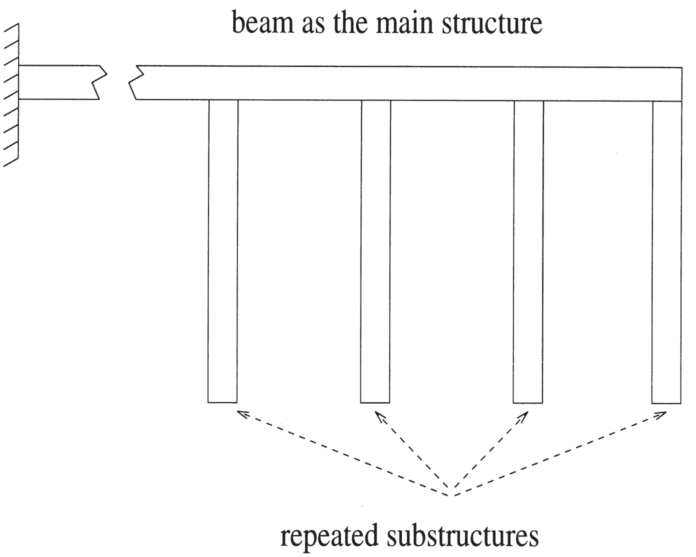

Fig. 6. An example of a coupled system with repeated substructures.

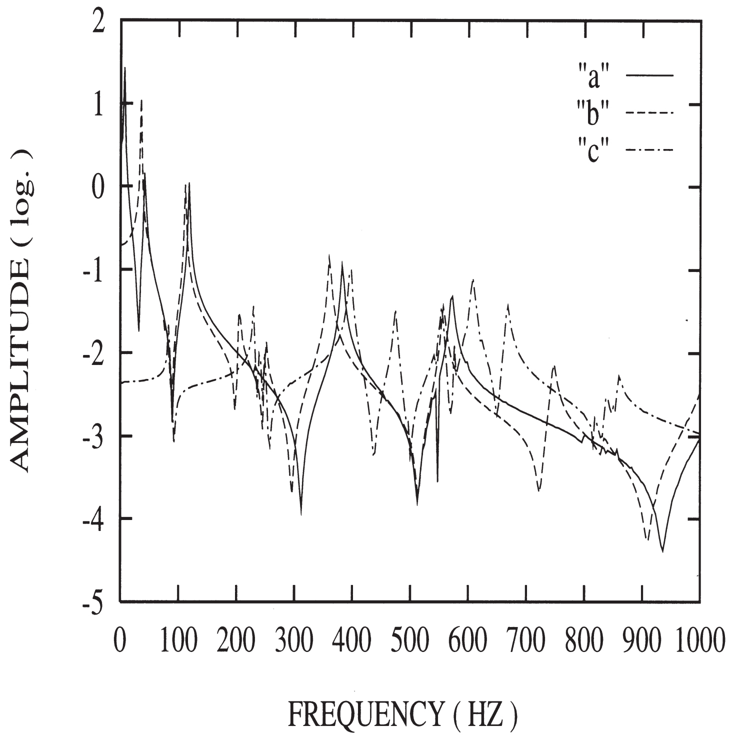

Fig. 7. Beam with periodical substructures: ——— Beam without any substructures. – – – two perpendicular beam. – · – beam to which are attached 4 vertical beams.

of modes reaches 28 and 44 for the case of two and four vertical beams respectively. This increase in modal density, in most methods, is more time consuming, because of additional data preparation procedures, and it imposes further calculations. Using the present method. there is no need to recalculate the compliance components when the number of identical substructures increases. Hence the necessary time and memory does not increase significantly.

Fig. 8 illustrates the processing time required for dynamic analysis when the number of vertical beams in Fig. 6 is increased. The analysis was made using a HP735 unix station and the processing time of a horizontal cantilevered beam without any substructure was used as a time reference. The analysis was performed up to 1000 Hz. When four substructures are included, there is less than 30% increase in processing time with respect to a simple cantilevered beam; with 10 substructures, the increase is only 80%. These results show the excellent efficiency of the method to solve systems with periodic substructures.

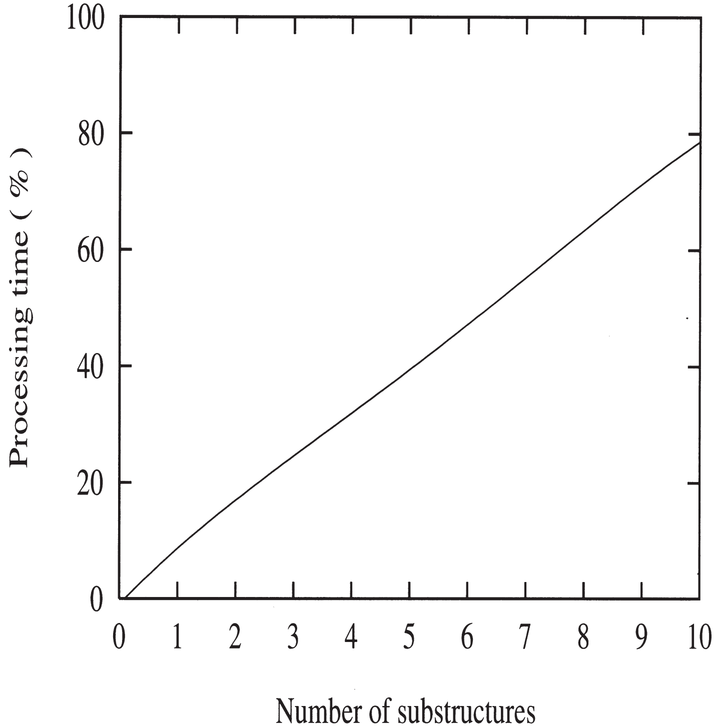

Fig. 8. Percentage of time increasing vs the number of substructures.

4. Sensitivity analysis

In all previous examples, simple substructures were used and their compliance matrices were analytically calculated. In real applications, however, complex substructures have no analytical solution so that either numerical or experimental approaches should be used to derive their compliance characteristics. In such cases, errors in the compliance estimations are unavoidable. In this section, a sensitivity analysis is performed in order to investigate the effects of different error types on the present method. Three types of errors may exist. The first one is an amplitude error. It may result from an inaccurate estimation of the compliance. The second one is a phase difference error which may be more closely related to the damping estimation of the system. The third one is a frequency shift in the measured data. This is the kind of error often encountered in experimental tests, since devices (such as accelerometers or exciters) added to the structure tend to interfere with the system so that resonant frequencies may shift [18]. In the following examples, these errors will be artificially introduced to study their effects on the vibrational response of the whole structure. The L-shaped beam configuration is again used.

Fig. 9 shows the effect of amplitude error on the response of the L-shaped beam. Amplitude errors are artificially introduced to compliance components by multiplying

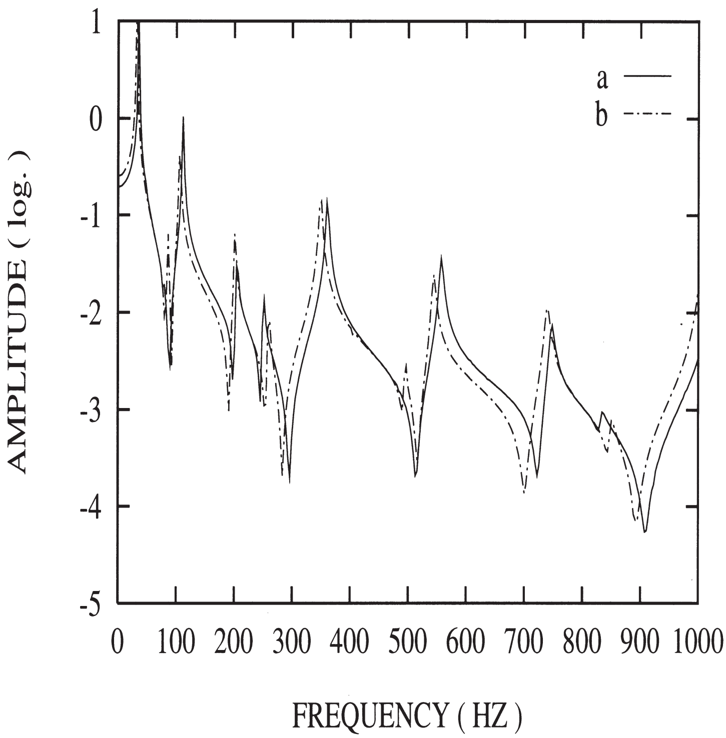

Fig. 9. Effect of compliance amplitude error on overall response of two coupled perpendicular beams. ——— Response without any error – · – Amplitude coefficient of 4.0.

their amplitude values by a factor four in linear scale, corresponding to a deviation of about 6 dB in logarithmic scale. It can be seen that the increase in compliance amplitude systematically changes the response of the structure at all frequencies. In fact, when increasing the compliance amplitude, the substructure becomes more flexible and hence shifts all resonant frequencies towards lower frequencies.

The effects of phase errors on the same coupled system are shown in Fig. 10. Results for a phase difference of 20 degrees in the compliance are compared with those of the coupled system, without any phase error. Results show that a phase difference error has noticeable effects on the system response only in the vicinity of the structure resonances. This is understandable since the phase angle of the compliance is more likely to be related to the damping characteristics of the substructures. This observation is quite beneficial for the application of the method, since it permits a less harsh tolerance for the phase measurement of compliances. In fact, it is well known that the phase angle is quite difficult to be accurately measured experimentally.

Fig. 11 shows the influence of a frequency shift in compliance components over the structural response of the L-shaped beam. A shift of 25 Hz towards low frequencies is introduced in the compliance curve. Two results with and without fre-

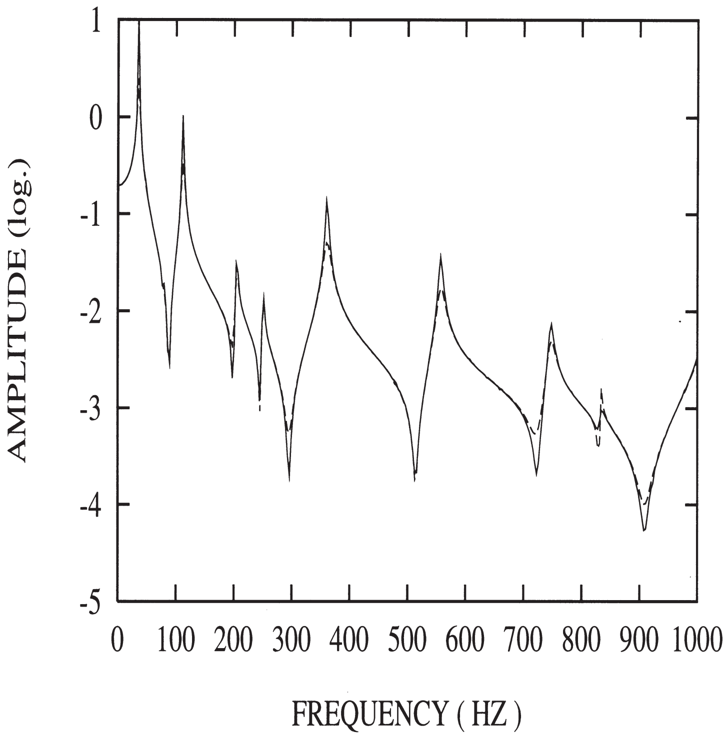

Fig. 10. Effect of compliance phase angle error on overall response of two coupled perpendicular beams. ——— Response without any error – – – 20 degrees of error in phase difference of compliance components.

quency shift are first compared in Fig. 11. It can be observed that the response curve is also systematically shifted toward low frequencies. In addition to that, the perturbation is not uniform and rather frequency dependent. In order to better understand this, the response of the horizontal beam with clamped-clamped boundary conditions is plotted in the same figure. As can one observe, the response is more sensitive to the frequency shift in frequency ranges where the substructure significantly changes the behavior of the main structure. For other frequencies, the effect of the frequency shift errors seems to be reasonably limited for the present configuration.

Similar sensitivity analysis has equally been performed for other types of substructures. The observations seem to be consistent with the above analyzes.

5. Conclusion

Vibrations of point-coupled structures are investigated in the present paper. The method is particularly suitable to analyze the effects of structural modifications: adding one or more elements to the system does not require to recalculate or measure

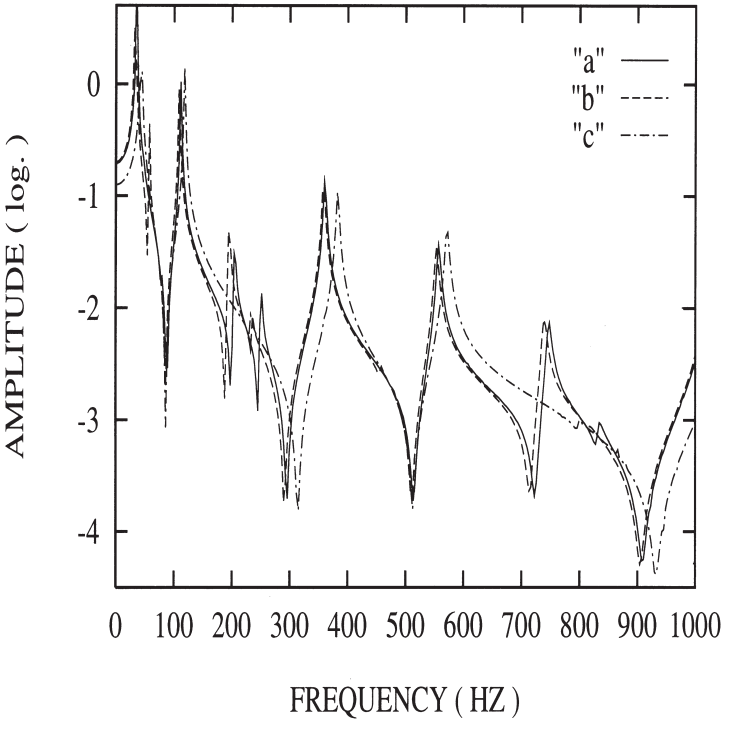

Fig. 11. Effect of frequency shift in compliance data ——— Response without any error – – – with a frequency shift of 25 Hz – · – Response of a clamped-clamped beam.

previous elements of the system. It has also been shown that the method is quite powerful to treat periodical structures without sensitively increasing computing time and storage space. Since the compliance of substructures can be obtained in different ways, the proposed formulation can be classified as a hybrid method. A sensitivity analysis was also conducted to define the tolerances for the compliance measurement. It was shown that among three types of errors that may be encountered in practice, amplitude errors are the most important errors to be controlled, since they will affect the system response at all frequencies. Errors related to the frequency shift have more considerable effects in frequency ranges where the dynamic behavior of substructures are strongly involved. Finally, the structural response is not quite sensitive to phase difference errors except at the resonant frequencies of the system.

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