Minimization of the mean square velocity response of dynamic structures using an active-passive dynamic vibration absorber

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An optimal design of a hybrid vibration absorber (HVA) with a displacement and a velocity feedback for minimizing the velocity response of the structure based on the H_2 optimization criterion is proposed. The objective of the optimal design is to reduce the total vibration energy of the vibrating structure under wideband excitation, i.e., the total area under the velocity response spectrum is minimized in this criterion. One of the inherent limitations of the traditional passive vibration absorber is that its vibration suppression is low if the mass ratio between the absorber mass and the mass of the primary structure is low. The active element of the proposed HVA helps further reduce the vibration of the controlled structure, and it can provide very good vibration absorption performance even at a low mass ratio. Both the passive and active elements are optimized together for the minimization of the mean square velocity of the primary system as well as the active force required in the HVA. The proposed HVA was tested on single degree-of-freedom (SDOF) and continuous vibrating structures and compared to the traditional passive vibration absorber.
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I. INTRODUCTION

The traditional passive dynamic vibration absorber (PVA) is an auxiliary mass-spring- damper system that, when correctly tuned and attached to a vibrating system subject to harmonic excitation, gives rise to a reduced steady-state motion at the point to which it is attached. An early research conducted at the beginning of the 20th century considered an undamped PVA tuned to the frequency of the disturbing force.[1] Such an absorber is of a narrow-band type as it is not effective in reducing structural vibration when there is any change in the disturbing frequency.

Finding the optimum parameters of a PVA with a viscous damper in single degree-of- freedom (SDOF) system drew the attention of many scholars. One common optimization method is the H_∞ optimization that aims at minimizing the resonant vibration amplitude of the dynamic structure. The standard H_∞ optimum design method of PVA applied to SDOF system is well documented by Den Hartog.[2] H_2 optimization is another optimization method that aims at minimizing the mean square motion of the dynamic structure. The H_2 optimization is more desirable than the H_∞ optimization when the vibrating system is subjected to random excitation rather than sinusoidal excitation. The H_2 optimum design method of PVA applied to SDOF system is proposed by Warburton.[3,4] However, the major disadvantage of PVA is that its vibration suppression performance depends very much on the ratio between the mass of the absorber and that of the vibrating structure.[5] This mass ratio is small in cases where the primary mass is big such as buildings and ships. Therefore PVA is not commonly applied to civil and large mechanical structures.

To improve the performance of the PVA, some researchers rearranged the elements of the absorber[6–9] and some researchers applied multiple vibration absorbers and Helmholtz resonators to reduce the vibration and sound transmission of a wide frequency band.[10–12] On the other hand, some researchers added an active force actuator to a PVA to form a hybrid vibration absorber (HVA) as illustrated in Fig. 1. Various methods were proposed for the control of the active force of the HVA including neural network,[13] delayed resonator,[14] modal feedback control[15–19] and closed-loop poles by modal feedback.[20,21] Most of these control methods of HVA are very complicated, and only the active elements but not the passive elements are optimized.

In this article, an optimized design of a HVA using displacement and velocity feedback for the minimization of the kinetic energy of the vibrating structure based on the H_2 optimization criterion is proposed. The objective of the H_2 optimization is to reduce the mean square velocity of the vibrating system, i.e., the total area under the velocity response spectrum is minimized. Both the active and passive elements of the HVA are optimized together analytically. It is proved analytically that the proposed HVA can provide good vibration absorption performance even at a low mass ratio and the use of the velocity feedback can reduce the amount of active force required in the HVA. The theoretical analysis improves our understanding and the design of HVA. The proposed HVA is tested numerically on both a SDOF system and a continuous beam structure and compared to the optimized PVA.[22,23] The test results shows that the proposed HVA has a much better vibration suppression performance than the traditional PVA.

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II. HVA APPLIED TO A SINGLE DEGREE-OF-FREEDOM SYSTEM

A. Mathematical model

A HVA coupled with an undamped primary system is shown in Fig. 1, where x, M, and K denote, respectively, displacement, mass, damping, and spring coefficients of the primary system; and x_a, m, c, and k are those of the absorber.

The equations of motion of the primary mass M and the absorber mass m may be written as

$$\begin{cases} M\ddot{x} = -Kx - k(x - x_a) - c(\dot{x} - \dot{x}_a) - f_a + F, \\ m\ddot{x}_a = -k(x_a - x) - c(\dot{x}_a - \dot{x}) + f_a \end{cases} \quad (1)$$

where F is a disturbance and $f_a = a_0 x + a_0 \dot{x}$ is the active force applied by the actuator as illustrated in Fig. 1. Performing Laplace transformation of Eq. (1), the transfer function of the velocity response of the primary mass M may be written as

$$G(p) = \frac{\dot{X}}{\omega_n\left(F/K\right)} = \frac{\left(\gamma^2 + 2\zeta\gamma p + p^2\right)p}{\left(1+p^2\right)\left(\gamma^2+p^2\right) + \left(\mu\gamma^2 + 2\alpha_0\right)p^2 + 2\zeta\gamma p\left[1 + (1+\mu+2\alpha_1\mu)p^2\right]}, \quad (2)$$

where $p = s/\omega_n$, $\mu = m/M$, $\omega_n = \sqrt{K/M}$, $\omega_a = \sqrt{k/m}$, $\gamma = \omega_a/\omega_n$, $\zeta = c/2\sqrt{mk}$, $\alpha_0 = a_0/2K$, and $\alpha_1 = a_1/2c$.

The frequency response function of mass M can be obtained by replacing p in Eq. (2) by $\mathrm{j}\lambda$ where $\lambda = \omega/\omega_n$ and $\mathrm{j}^2 = -1$. The frequency response function of mass M may be written as

$$G(\mathrm{j}\lambda) = \frac{-2\zeta\gamma\lambda^2 + \mathrm{j}\lambda\left(\gamma^2 - \lambda^2\right)}{\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right) - \left(\mu\gamma^2 + 2\alpha_0\right)\lambda^2 + 2\mathrm{j}\zeta\gamma\lambda\left(1 - (1+\mu+2\alpha_1\mu)\lambda^2\right)}. \quad (3)$$

The amplitude of the velocity response function in Eq. (3) may be written as

$$|G(\lambda)| = \sqrt{\frac{\left(2\zeta\gamma\lambda^2\right)^2 + \lambda^2\left(\gamma^2-\lambda^2\right)^2}{\left(\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right) - \left(\mu\gamma^2+2\alpha_0\right)\lambda^2\right)^2 + \left(2\zeta\gamma\lambda\right)^2\left(1-(1+\mu+2\alpha_1\mu)\lambda^2\right)^2}}. \quad (4)$$

The mean square velocity of the primary mass M may be written as[24]

$$E\left[\dot{x}^2\right] = \int_{-\infty}^{\infty} |G|^2 S_y(\omega)\, d\omega \quad (5)$$

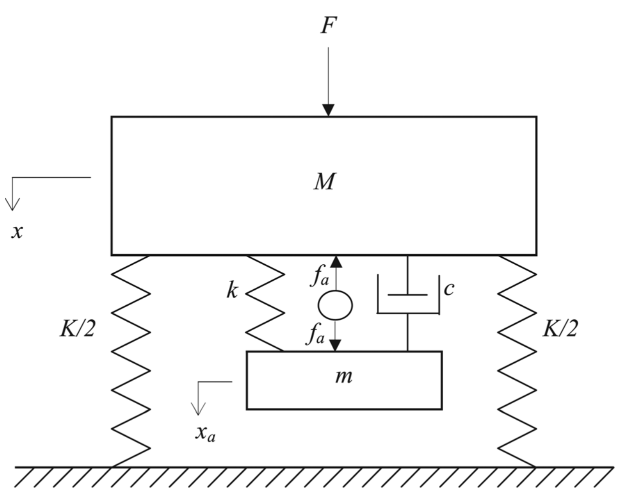

FIG. 1. Schematic diagram of the proposed hybrid vibration absorber (m-k-c-f_a system) attached to the primary (M-K) system.

where G is the velocity response function of the primary mass and $S_y(\omega)$ is the input mean square spectral density function.

If the input spectrum is assumed to be ideally white, i.e., $S_y(\omega) = S_0$, a constant for all frequencies, Eq. (5) can then be rewritten as

$$E\left[\dot{x}^2\right] = S_0 \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega. \quad (6)$$

The non-dimensional mean square velocity of mass M may be defined as[24]

$$E\left[\dot{x}^2\right] = \frac{\omega_n S_0}{2\pi} \int_{-\infty}^{\infty} |G(\lambda)|^2 d\lambda. \quad (7)$$

A useful formula of Gradshteyn and Ryzhik[25] written as Eq. (8) in the following text is used for solving Eq. (7).

If

$$G(\omega) = \frac{-\mathrm{j}\omega^3 B_3 - \omega^2 B_2 + \mathrm{j}\omega B_1 + B_0}{\omega^4 A_4 - \mathrm{j}\omega^3 A_3 - \omega^2 A_2 + \mathrm{j}\omega A_1 + A_0} \quad (8a)$$

then

$$\int_{-\infty}^{\infty}|G(\omega)|^2 d\omega = \frac{\pi\left[\dfrac{B_0^2(A_2A_3-A_1A_4)}{A_0}+A_3\left(B_1^2-2B_0B_2\right)+A_1\left(B_2^2-2B_1B_3\right)+\dfrac{B_3^2(A_1A_2-A_0A_3)}{A_4}\right]}{A_1(A_2A_3-A_1A_4)-A_0A_3^2}. \tag{8b}$$

Comparing Eqs. (3) and (8a), we may write

$$\begin{aligned}&A_0=\gamma^2,\quad A_1=2\zeta\gamma,\quad A_2=1+2\alpha_0+\gamma^2+\mu\gamma^2,\quad A_3=2\zeta\gamma(1+(1+2\alpha_1)\mu),\quad A_4=1,\\&B_0=0,\quad B_1=\gamma^2,\quad B_2=2\zeta\gamma\quad\text{and}\quad B_3=1.\end{aligned} \tag{9}$$

Using Eqs. (8) and (9), Eq. (7) may be rewritten as

$$E\left[\dot{x}^2\right]=\frac{\pi\omega_n S_0}{2\zeta\gamma}\left(\frac{1+2\alpha_0-2\gamma^2\left(1-2\zeta^2+\mu\alpha_1\right)+\gamma^4(1+\mu(1+2\alpha_1))}{\mu+2\alpha_0(1+\mu)+2\mu\alpha_1+4\mu\alpha_0\alpha_1-2\mu\gamma^2\alpha_1(1+\mu+2\mu\alpha_1)}\right). \tag{10}$$

The dimensionless mean square velocity of the primary mass M is calculated using Eq. (10) with $\mu=0.2$, $\alpha_0=0.1$, and $\alpha_1=0.3$, and the results are plotted in Fig. 2 for illustration. The dimensionless mean square velocity of the primary mass M has a minimum as marked in Fig. 2. The optimum frequency and damping of the HVA for this minimum may be found by solving $\partial/\partial\gamma\, E[\dot{x}^2]=\partial/\partial\zeta E[\dot{x}^2]=0$. Using Eq. (10), we may consider

$$\frac{\partial}{\partial\zeta}E\left[\dot{x}^2\right]=\frac{\omega_n S_0}{2\zeta^2\gamma}\left(\frac{1+2\alpha_0-2\gamma^2\left(1+2\zeta^2+\mu\alpha_1\right)+\gamma^4(1+\mu(1+2\alpha_1))}{\mu+2\alpha_1(1+\mu)+2\mu\alpha_1+4\mu\alpha_0\alpha_1-2\mu\gamma^2\alpha_1(1+\mu+2\mu\alpha_1)}\right)=0, \tag{11}$$

$$\frac{\partial}{\partial\gamma}E\left[\dot{x}^2\right]=\frac{\omega_n S_0}{2\zeta^2\gamma}\left(\frac{C_0\zeta^2-C_1}{\left(\mu+2\alpha_0(1+\mu)+2\mu\alpha_1+4\mu\alpha_0\alpha_1-2\mu\gamma^2 a_1(1+\mu+2\mu\alpha_1)\right)^2}\right)=0, \tag{12}$$

where

$$\begin{cases}C_0=8\mu\gamma^4\alpha_1(1+\mu+2\mu\alpha_1)+4\gamma^2((1+2\alpha_0)(1+2\alpha_1)\mu+2\alpha_0),\\ C_1=2\mu\alpha_1(1+\mu+2\mu\alpha_1)^2\gamma^6\\ \qquad+(1+\mu+2\mu\alpha_1)\left(4\mu^2\alpha_1^2-(12\alpha_0\alpha_1+3+2\alpha_1+6\alpha_0)\mu-6\alpha_0\right)\gamma^4\\ \qquad-(4\alpha_1(1+2\alpha_0)(1+2\alpha_1)\mu^2+(-2-4\alpha_0+2\alpha_1-4\alpha_1\alpha_2)\mu-4\alpha_0)\gamma^2\\ \qquad+(1+2\alpha_0)((1+2\alpha_0)(1+2\alpha_1)\mu+2\alpha_0).\end{cases}$$

Solving Eqs. (11) and (12), the optimum tuning ratio may be written, respectively, as

$$\gamma_{\mathrm{opt}}=\sqrt{\frac{D_0\mu^2+D_1\mu+D_2}{(1+\mu+2\mu\alpha_1)(D_0\mu^2+D_1\mu+D_2+a_1(1+2\alpha_0+4\alpha_0\alpha_1)\mu^2+2\alpha_0 a_1\mu)}}, \tag{13}$$

$$\zeta_{\mathrm{opt}}=\frac{(\mu+2\alpha_0+2\mu\alpha_0+4\mu\alpha_0\alpha_1)}{2\sqrt{(D_0\mu^2+D_1\mu+D_2)}}\sqrt{1+\frac{\alpha_1^2\mu^2}{(D_0\mu^2+D_1\mu+D_2+\alpha_1(1+2\alpha_0+4\alpha_0\alpha_1)\mu^2+2\alpha_0\alpha_1\mu)}}, \tag{14}$$

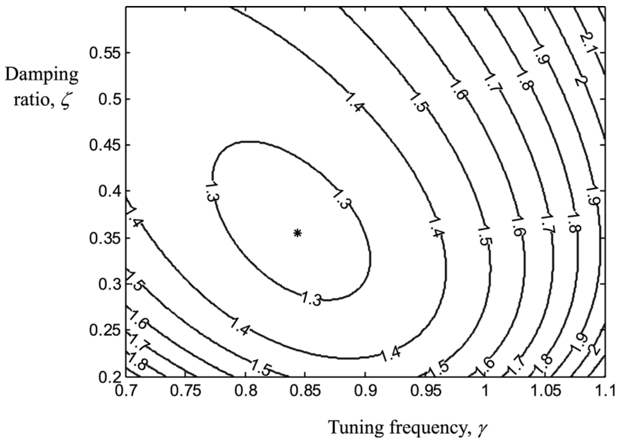

FIG. 2. Contours of the mean square velocity of mass M, $E[|G|^2]/(\pi\omega_n S_0)$, of Fig. 1 with $\mu = 0.2$, $\alpha_0 = 0.1$, and $\alpha_1 = 0.3$. (*) – Optimum tuning frequency [Eq. (13)] and damping [Eq. (14)].

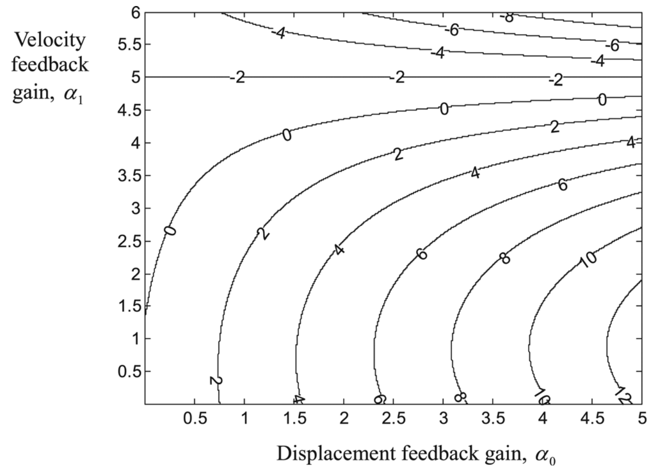

FIG. 3. Contours of $D_0\mu^2 + D_1\mu + D_2$ with $\mu = 0.2$.

where

$$\begin{cases} D_0 = -\alpha_1(1+2\alpha_0)(1+2\alpha_1), \\ D_1 = 1 + 2\alpha_0 + 2\alpha_0\alpha_1, \\ D_2 = 2\alpha_0. \end{cases} \quad (15)$$

Both the optimum frequency ratio γ_{opt} and damping ratio ζ_{opt} exist if

$$D_0\mu^2 + D_1\mu + D_2 > 0. \quad (16)$$

Using Eq. (15) and the inequality (16), α_1 may be stated as

$$\alpha_1 < \frac{-(2\alpha_0\mu + \mu - 2\alpha_0) + \sqrt{(2\alpha_0\mu + \mu - 2\alpha_0)^2 + 8(1+2\alpha_0)(\mu + 2\mu\alpha_0 + 2\alpha_0)}}{4\mu(1+2\alpha_0)} = \alpha_{1\_max}. \quad (17)$$

$D_0\mu^2 + D_1\mu + D_2$ is calculated with different values of α_0 and α_1 with $\mu = 0.2$ and plotted in Fig. 3 for illustration. As shown in Fig. 3, α_1 has a range of values such that $D_0\mu^2 + D_1\mu + D_2 > 0$. The values of α_1 corresponding to the contour $D_0\mu^2 + D_1\mu + D_2 = 0$ in Fig. 3 are α_{1_max} with $\mu = 0.2$.

Substituting Eqs. (13) and (14) into Eq. (10), the mean square velocity of the primary mass M using the optimum frequency and damping may be written as

$$\begin{aligned} E[\dot{x}^2]_{HVA\_opt} &= \frac{\sqrt{D_0\mu^2 + D_1\mu + D_2 + \alpha_1(1 + 2\alpha_0 + \alpha_1 + 4\alpha_0\alpha_1)\mu^2 + 2\alpha_0\alpha_1\mu}}{(2\alpha_0\alpha_1\mu + D_1\mu + D_2)\sqrt{2\alpha_1\mu + \mu + 1}} \\ &= \frac{\sqrt{(1 + 2\alpha_0 + 4\alpha_0\alpha_1)\mu + 2\alpha_0 - \alpha_1^2\mu^2}}{[(1 + 2\alpha_0 + 4\alpha_0\alpha_1)\mu + 2\alpha_0]\sqrt{1 + \mu + 2\alpha_1\mu}}. \end{aligned} \quad (18)$$

It can be shown using Eq. (18) that the mean square velocity of the primary mass M, $E[\dot{x}^2]_{HVA\_opt}$, decreases when μ or α_1 increases. Differentiating $E[\dot{x}^2]_{HVA\_opt}$ with respect to α_0 using Eq. (18), we may write

$$\frac{\partial}{\partial\alpha_0}E[\dot{x}^2]_{HVA\_opt} = -\frac{1}{E[\dot{x}^2]_{HVA\_opt}}\left(\frac{D_0\mu^2 + D_1\mu + D_2 + \alpha_1(1 + 2\alpha_0 + 4\alpha_0\alpha_0)\mu^2 + 2\alpha_0\alpha\mu}{((1 + 2\alpha_0 + 4\alpha_0\alpha_1)\mu + 2\alpha_0)^3}\right) < 0. \quad (19)$$

Moreover, Eq. (19) shows that $E[\dot{x}^2]_{HVA\_opt}$ decreases as α_0 increases. Therefore there is no optimum value for these parameters such that the mean square velocity of the primary mass M becomes a minimum.

If only displacement feedback signal is used, we have $\alpha_1 = 0$ and $f_a = a_0 x$. The optimum frequency, damping and the mean square velocity of mass M may be obtained by substituting $\alpha_1 = 0$ into Eqs. (13), (14), and (18), respectively, and we may write

$$\gamma_{\rm opt} = \sqrt{\frac{1}{1+\mu}}, \tag{20a}$$

$$\zeta_{\rm opt} = \sqrt{\frac{(1+2\alpha_0)\mu + 2\alpha_0}{4}}, \quad \text{and} \tag{20b}$$

$$E[\dot{x}^2]_{\rm HVA\_opt} = \frac{1}{\sqrt{[(1+2\alpha_0)\mu + 2\alpha_0](1+\mu)}}. \tag{20c}$$

If only velocity feedback signal is used, we have $\alpha_0 = 0$ and $f_a = a_1 \dot{x}$. The optimum frequency, damping and mean square velocity of mass M may be obtained by substituting $\alpha_0 = 0$ into Eqs. (13), (14), and (18), respectively, and we may write

$$\gamma_{\rm opt} = \sqrt{\frac{1 - \mu\alpha_1(1+2\alpha_1)}{(1+\mu(1+2\alpha_1))(1-2\mu\alpha_1^2)}}, \tag{21a}$$

$$\zeta_{\rm opt} = \sqrt{\frac{\mu(1-\mu\alpha_1^2)}{4(1-\mu\alpha_1(1+2a_1))(1-2\mu\alpha_1^2)}}, \quad \text{and} \tag{21b}$$

$$E[\dot{x}^2]_{\rm HVA\_opt} = \sqrt{\frac{1-\mu\alpha_1^2}{\mu(1+\mu+2\mu\alpha_1)}}. \tag{21c}$$

If no feedback signal and no active force are applied, $\alpha_0 = \alpha_1 = 0$, $f_a = 0$. The absorber becomes the traditional passive dynamic vibration absorber. The mean square velocity of the primary mass M is calculated according to Eq. (18) with $\alpha_0 = \alpha_1 = 0$ and is written as

$$E[\dot{x}^2]_{\rm PVA\_opt} = \sqrt{\frac{1}{\mu(1+\mu)}}. \tag{22}$$

As shown in Eq. (22), $E[\dot{x}^2]_{\rm PVA\_opt}$ tends to infinity as μ tends to zero. On the other hand, $E[\dot{x}^2]_{\rm HVA\_opt}$ tends to $1/\sqrt{2\alpha_0}$ according to Eq. (18) as μ tends to zero. Therefore the traditional passive vibration absorber is not effective in suppressing the mean square velocity of the primary system if the mass ratio μ is small while the proposed hybrid absorber can be effective even though the mass ratio is very small if α_0 is not too small. The velocity response amplitude of the primary mass M, $|G(\lambda)|$, at $\mu = 0.01$ with the proposed optimum frequency and damping ratios are calculated according to Eqs. (4), (13), and (14) with $\alpha_0 = \alpha_1 = 0$, and $\alpha_0 = 0.3$ and $\alpha_1 = 0$, respectively, and the results are plotted in Fig. 4 for illustration. It shows that the proposed optimized HVA is very effective in suppressing the velocity response of the vibrating mass M but the passive PVA is not effective when the mass ratio μ is small.

Using Eqs. (18) and (22), the dimensionless mean square velocity of the primary mass M may be defined as

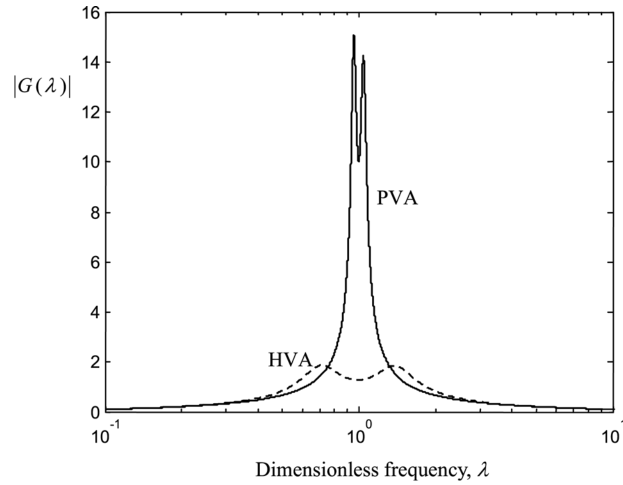

FIG. 4. The velocity response of the primary mass M of the SDOF in Fig. 1 with $\mu = 0.01$. ——, $\alpha_0 = \alpha_1 = 0$; - - - - - -, $\alpha_0 = 0.3$ and $\alpha_1 = 0$.

$$\frac{E[\dot{x}^2]_{\rm HVA\_opt}}{E[\dot{x}^2]_{\rm PVA\_opt}} = \frac{\sqrt{\left[(1+2\alpha_0+4\alpha_0\alpha_1)\mu + 2\alpha_0 - \alpha_1^2\mu^2\right](1+\mu)\mu}}{[(1+2\alpha_0+4\alpha_0\alpha_1)\mu + 2\alpha_0]\sqrt{1+\mu+2\alpha_1\mu}}. \tag{23}$$

Equation (23) shows the relation between the mean square velocity $E[\dot{x}^2]_{\rm HVA\_opt}/E[\dot{x}^2]_{\rm PVA\_opt}$ and the parameters of the HVA including the mass ratio μ and the feedback gains α_0 and α_1. The mean square velocity of the primary mass M is calculated according to Eq. (23) with different values of α_0 and α_1 and the contours of the percentage reduction of $E[\dot{x}^2]$ are plotted in Fig. 5 for illustration. As shown in Fig. 5, $E[\dot{x}^2]_{\rm HVA\_opt}/E[\dot{x}^2]_{\rm PVA\_opt}$ decreases when α_0 or α_1 increases. There is no limit in selecting the value of α_0, but α_1 has a maximum value according to Eq. (17), and it is plotted as the dotted line in Fig. 5.

Besides the mean square velocity of the mass M, another important factor to be considered would be the active force required in the HVA to achieve the desired mean square velocity of the mass M. Recall from Eq. (1) that $f_a = a_0 x + a_1 \dot{x} = 2K\alpha_0 x + 2x\alpha_1 \dot{x}$, the transfer function of the active force in the absorber may be written as

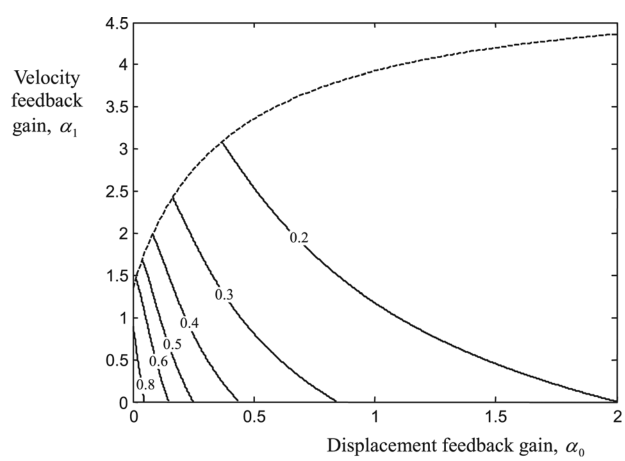

FIG. 5. Contours of $E[\dot{x}^2]_{\rm HVA\_opt}/E[\dot{x}^2]_{\rm PVA\_opt}$ according to Eq. (23). $\mu = 0.2$. - - -, $\alpha_1 = \alpha_{1\_\max}$ [Eq. (17)].

$$\frac{F_a}{F}=\left(-\frac{2j\alpha_0}{\lambda}+4\alpha_1\mu\zeta\gamma\right)\frac{\dot{X}}{\omega_n F/_{\mathrm{K}}}. \quad (24)$$

Using Eqs. (2) and (24), we may write

$$\frac{F_a}{F}=\frac{-4j\mu\zeta\gamma\lambda^3\alpha_1-2\left(\alpha_0+4\mu\zeta^2\gamma^2\alpha_1\right)\lambda^2+4j\zeta\gamma\lambda(\mu\gamma^2\alpha_1+\alpha_0)+2\gamma^2\alpha_0}{\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right)-(\mu\gamma^2+2\alpha_0)\lambda^2+2j\zeta\gamma\lambda\left(1-(1+\mu+2\alpha_1\mu)\lambda^2\right)}. \quad (25)$$

The mean square actuation force may be written as

$$E\left[f_a^2\right]=\frac{\omega_n S_0}{2\pi}\int_{-\infty}^{\infty}\left|\frac{F_a}{F}\right|^2 d\lambda. \quad (26)$$

Comparing Eqs. (8a) and (25), we may write

$$\begin{aligned}&A_0=\gamma^2,\quad A_1=2\zeta\gamma,\quad A_2=1+2\alpha_0+\gamma^2+\mu\gamma^2,\quad A_3=2\zeta\gamma(1+(1+2\alpha_1)\mu),\quad A_4=1,\quad B_1=2\alpha_0\gamma^2,\\&B_1=4\zeta\gamma\left(\alpha_0+\alpha_1\mu\gamma^2\right),\quad B_2=2\left(\alpha_0+4\alpha_1\mu\zeta^2\gamma^2\right),\quad B_3=4\alpha_1\mu\zeta\gamma.\end{aligned} \quad (27)$$

Using Eqs. (8b) and (26), the mean square actuation force may be written as

$$E\left[f_a^2\right]=\frac{\omega_n S_0}{2\zeta\gamma}\left(\frac{G_0\gamma^6+G_1\gamma^4+G_2\gamma^2+G_3}{\mu+2\alpha_0(1+\mu)+2\mu\alpha_1+4\mu\alpha_0\alpha_1-2\mu\alpha_1\gamma^2(1+\mu+2\mu\alpha_1)}\right), \quad (28a)$$

and the mean square actuation force of the HVA under optimal tuning condition may be written using Eqs. (13), (14), and (28a) as

$$E\left[f_a^2\right]_{\mathrm{HVA\_opt}}=\frac{\omega_n S_0}{2\zeta_{\mathrm{opt}}\gamma_{\mathrm{opt}}}\left(\frac{G_0\gamma_{\mathrm{opt}}^6+G_1\gamma_{\mathrm{opt}}^4+G_2\gamma_{\mathrm{opt}}^2+G_3}{\mu+2\alpha_0(1+\mu)+2\mu\alpha_1+4\mu\alpha_0\alpha_1-2\mu\alpha_1\gamma_{\mathrm{opt}}^2(1+\mu+2\mu\alpha_1)}\right) \quad (28b)$$

where

$$\begin{cases}G_0=4\mu^2\zeta^2\alpha_1^2(1+\mu+2\mu\alpha_1),\\G_1=16\mu^2\zeta^4\alpha_1^2-8\mu^2\zeta^2\alpha_1^2(1+\mu\alpha_1)+\alpha_0^2\left(1+2\mu\left(1+\alpha_1+\alpha_1^2\right)+\mu^2\right),\\G_2=4\zeta^2\left(\alpha_0^2+\mu\alpha_0^2(1+2\alpha_1)+\mu^2\alpha_1^2(1+2\alpha_0)\right)\\\qquad-\alpha_0^2(\mu-2\mu\alpha_0+2-2\alpha_0+2\mu\alpha_1-4\mu\alpha_0\alpha_1),\\G_3=\alpha_0^2.\end{cases} \quad (29)$$

As shown in Eq. (23), the mean square velocity $E[\dot{x}^2]_{\mathrm{HVA\_opt}}/E[\dot{x}^2]_{\mathrm{PVA\_opt}}$ of mass M is a function of μ, α_0, and α_1. The active force required in the HVA may be different with different values of the feedback gain α_0 and α_1. After selecting the mass ratio μ and the desired mean square velocity $E[\dot{x}^2]_{\mathrm{HVA\_opt}}/E[\dot{x}^2]_{\mathrm{PVA\_opt}}$, the possible sets of α_0 and α_1 that can satisfy Eq. (23) can be found numerically. These sets of α_0 and α_1 were used to calculate the active force required according to Eq. (28b) and the results are plotted in Fig. 6 with $\mu=0.2$ for illustration. Figure 6 shows the contours of $E[\dot{x}^2]_{\mathrm{HVA\_opt}}/E[\dot{x}^2]_{\mathrm{PVA\_opt}}$ at different values of mean square force $E[f_a^2]$ and feedback gains α_0 and α_1. It can be observed that for a desired mean square velocity $E[\dot{x}^2]_{\mathrm{HVA\_opt}}/E[\dot{x}^2]_{\mathrm{PVA\_opt}}$, there is a certain set of α_0 and α_1 such that the mean square active force $E\left[f_a^2\right]_{\mathrm{HVA\_opt}}$ is a minimum. It shows that a nonzero positive value of α_1 would help to reduce the active force required in the HVA if this value is selected properly. The proposed optimum feedback gain α_0 is the one that corresponds to the minimum points of the solid curves in Fig. 6. After the optimum feedback gain α_0 is determined, the corresponding optimum feedback gain α_1 can then be found from Fig. 5.

B. Stability analysis

According to Eq. (2), the characteristic equation of the control system may be written as

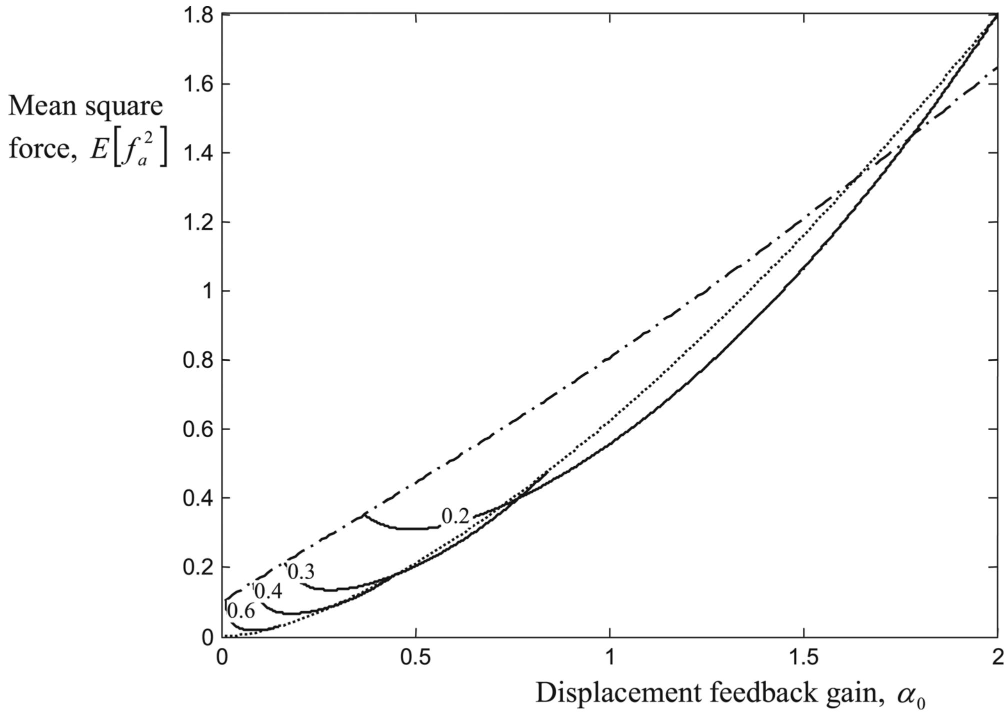

FIG. 6. Contours of $E[\dot{x}^2]_{\mathrm{HVA\_opt}}/E[\dot{x}^2]_{\mathrm{PVA\_opt}}$ at different values of mean sqaure force and feedback gains α_0 and α_1. — - — - -, $\alpha_1 = \alpha_{1\_\max}$ [Eq. 17]; ---------, $\alpha_1 = 0$, $\mu = 0.2$.

$$p^4 + 2\zeta\gamma(1+\mu+2\alpha_1\mu)p^3 + (1+2\alpha_0+\gamma^2+\mu\gamma^2)p^2 + 2\zeta\gamma p + \gamma^2 = 0 \tag{30}$$

where $\forall \zeta, \gamma, \mu, \alpha_0, \alpha_1 \in R^+$.

The system is stable if the real parts of all poles of Eq. (2) are negative. To apply the Routh's stability criterion,[26] the array of coefficient of Eq. (30) may be written as

$$\begin{array}{c|ccc} p^4 & 1 & 1+2\alpha_0+\gamma^2+\mu\gamma^2 & \gamma^2 \\ p^3 & 2\zeta\gamma(1+\mu+2\alpha_\mu) & 2\zeta\gamma & 0 \\ p^2 & \dfrac{(1+\mu+2\alpha_1\mu)(1+\mu)\gamma^2+2\alpha_0+\mu(1+2\alpha_0)(1+2\alpha_1)}{1+\mu+2\alpha_1\mu} & \gamma^2 & \\ p & \dfrac{2\zeta\gamma(2\alpha_0(1+\mu+2\mu\alpha_1)-\mu(2\alpha_1\gamma^2(1+\mu+2\mu\alpha_1)-1-2\alpha_1))}{(1+\mu+2\alpha_1\mu)(1+\mu)\gamma^2+2\alpha_0+\mu(1+2\alpha_0)(1+2\alpha_1)} & & \\ 1 & \gamma^2 & & \end{array} \tag{31}$$

All the coefficients of the array in Eq. (31) are positive if the following inequality is true.

$$2\alpha_0(1+\mu+2\mu\alpha_1) - \mu(2\alpha_1\gamma^2(1+\mu+2\mu\alpha_1) - 1 - 2\alpha_1) > 0. \tag{32}$$

If $\alpha_1 = 0$, the inequality in the preceding text will become $2\alpha_0(1+\mu)+\mu > 0$ and therefore the control system must be stable. However, it is shown in the previous section that the active force required in the HVA can be reduced if $\alpha_1 > 0$, and therefore both displacement and velocity feedbacks are used in the proposed HVA design.

Substituting the optimum frequency γ_{opt} of Eq. (13) into the inequality (32), we have

$$\frac{[(1+2\alpha_0+4\alpha_0\alpha_1)\mu+2\alpha_0]^2}{D_0\mu^2+D_1\mu+D_2+\alpha_1(1+2\alpha_0+\alpha_1+4\alpha_0\alpha_1)\mu^2+2\alpha_0\alpha_1\mu} > 0. \tag{33}$$

According to the Routh's stability criterion, the dynamic system is stable if the inequality (33) is true. As shown in Eqs. (13) and (14), optimum frequency γ_{opt} and damping ζ_{opt} of the HVA exist if $D_0\mu^2+D_1\mu+D_2>0$. Therefore, the dynamic system is stable if $D_0\mu^2+D_1\mu+D_2>0$ such that all the poles of the characteristic equation as shown in Eq. (30) have negative real parts. The root locus and zero locus of Eq. (30) are calculated with $\mu=0.2$, $\gamma=\gamma_{\mathrm{opt}}$, $\zeta=\zeta_{\mathrm{opt}}$, $\alpha_0=0.5$, and $0\le\alpha_1<\alpha_{1\_\max}$ and plotted in Figs. 7(a) and 7(b), respectively, for illustration. One of the root

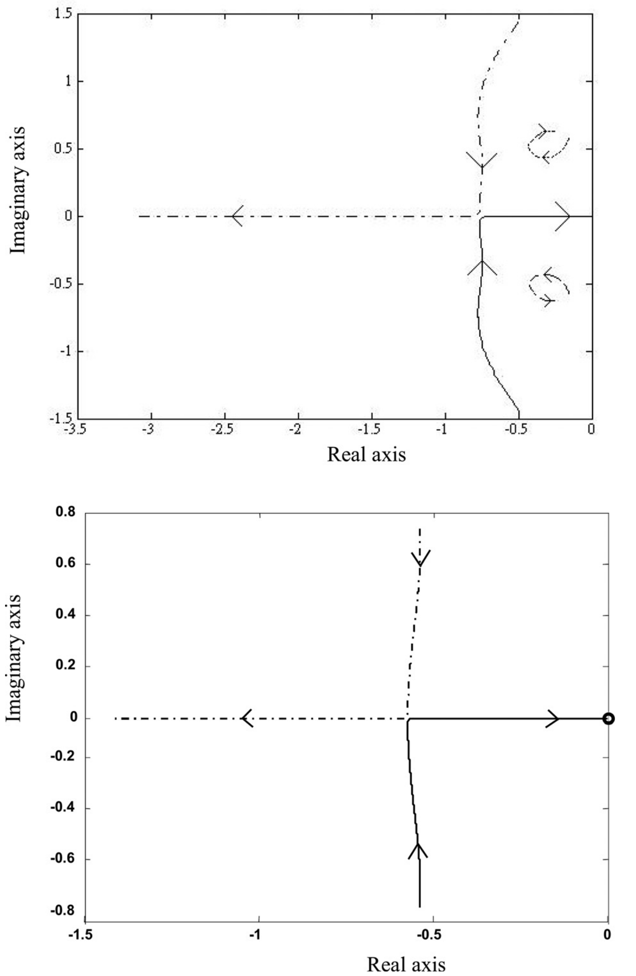

FIG. 7. (a) Root locus of the control system with $\mu = 0.2$, $\alpha_0 = 0.5$ and $0 \le \alpha_1 < \alpha_{1\_max}$. ——, root 1; - - - -, root 2; ·····, root 3; -·-·-, root 4 of Eq. (30). (b) Locus of the zeros of the control system with $\mu = 0.2$, $\alpha_0 = 0.5$ and $0 \le \alpha_1 < \alpha_{1\_max}$. ·····, zeros 1; ——, zero 2; **o** zero 3 of Eq. (2).

loci (solid curve) in Fig. 7(a) approaches the origin as α_1 approaches α_{1_max}. It shows that stability of the control system is reduced when α_1 approaches α_{1_max}.

C. Design guideline of the hybrid vibration absorber

A design guideline of the HVA is proposed as follows. An applicable mass ratio μ and the desired mean square velocity of the HVA are decided first. For example, if μ is selected to be 0.2 and the desired $E[\dot{x}^2]_{\mathrm{HVA\_opt}}$ is 0.4. The corresponding dimensionless mean square velocity $E[\dot{x}^2]_{\mathrm{HVA\_opt}}/E[\dot{x}^2]_{\mathrm{PVA\_opt}}$ can be determined to be 0.2 using Eq. (22). Many sets of α_0 and α_1 can then be determined by solving Eq. (23), and the result is plotted as the curve with label 0.2 in Fig. 5. These sets of α_0 and α_1 with the selected value of μ are used to calculate the active force using Eq. (28b), and the results are plotted as the thick solid curve in Fig. 6. One set of α_0 and α_1 leading to the minimum active force can then be found. The optimum frequency γ_{opt} and damping ζ_{opt} of the passive elements of the proposed HVA can then be determined according to Eqs. (13) and (14), respectively. In the present example, the optimum feedback gains can be found as $\alpha_0 = 0.488$ and $\alpha_1 = 2.589$. The active

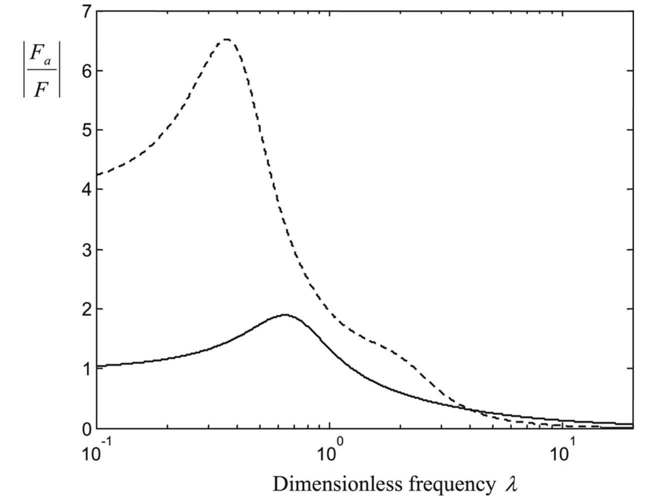

FIG. 8. Active force amplitude of the HVA, $|F_a/F|$. $\mu = 0.2$ and $E[\dot{x}^2]_{\mathrm{HVA\_opt}}/E[\dot{x}^2]_{\mathrm{PVA\_opt}} = 0.2$. ——, $\alpha_0 = 0.488$ and $\alpha_1 = 2.589$; - - - - - - $\alpha_0 = 0.488$ and $\alpha_1 = 0$.

force required in the HVA is calculated according to Eq. (25), and its amplitude is plotted as the solid curve in Fig. 8 and compared to the case of non-optimum feedback gains with $\alpha_0 = 2$ and $\alpha_1 = 0$ (dotted curve). It shows that both the maximum active force amplitude and the mean square force are reduced if α_1 is selected properly. As illustrated in Fig. 8, using both the feedback signal of the velocity and displacement of the primary mass can help reduce both the maximum and the mean square value of the active force required in the HVA. The required control force for the case with optimum feedback gains (solid curve) exceeds the excitation force only in a small frequency range $\lambda < 1.25$ and it approaches to zero at higher frequency. It is normal because we aim at reducing the control force at the entire frequency range instead of just one frequency. The optimum values of the feedback gains α_0 and α_1 are derived such that the mean square force $E[f_a^2] = (\omega_n S_0/2\pi)\int_{-\infty}^{\infty}|F_a/F|^2 d\lambda$ becomes a minimum. The dotted curve in Fig. 8 represents the case of non-optimum feedback gains and therefore the active force required in the HVA is higher than the case represented by the solid curve.

The effect of the active elements in the proposed hybrid absorber can be observed when the mass ratio μ is small. To illustrate this effect, the dimensionless mean square velocity $E[\dot{x}^2]/\omega_n S_0$ is assumed to be 0.4, and the corresponding optimum feedback gains α_0 and α_1 are derived according to the procedure as stated in Sec. II C. The corresponding dimensionless mean square forces $E[f_a^2]/w_n S_0$ are calculated according to Eq. (28b) and listed in Table I. It shows that to

TABLE I. Dimensionless mean square forces $E[f_a^2]/\omega_n S_0$ of the HVA with optimum feedback gains at five different mass ratios.

μ	$\frac{E[\dot{x}^2]}{\omega_n S_0}$	$\frac{E[\dot{x}^2]_{\mathrm{HVA\_opt}}}{E[\dot{x}^2]_{\mathrm{PVA\_opt}}}$	α_0 (Optimum value)	α_1 (Optimum value)	$\frac{E[f_a^2]}{\omega_n S_0} = \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty}\left|\frac{F_a}{F}\right|^2 d\lambda$
0.3	0.4	0.25	0.086	11.62	1.1
0.2	0.4	0.20	0.488	2.589	1.2
0.1	0.4	0.133	0.31	3.98	1.5
0.05	0.4	0.092	0.5497	11.46	1.63
0.02	0.4	0.057	0.5442	29.68	1.72

TABLE II. Dimensionless mean square forces $E[f_a^2]/\omega_n S_0$ of the HVA with displacement feedback gains only at five different mass ratios.

μ	$\frac{E[\dot{x}^2]}{\omega_n S_0}$	$\frac{E[\dot{x}^2]_{HVA\_opt}}{E[\dot{x}^2]_{PVA\_opt}}$	α_0 (Optimum value)	α_1 (Optimum value)	$\frac{E[f_a^2]}{\omega_n S_0} = \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty}\left\|\frac{F_a}{F}\right\|^2 d\lambda$
0.3	0.4	0.25	1.734	0	5.53
0.2	0.4	0.2	2	0	7.18
0.1	0.4	0.133	2.536	0	10.81
0.05	0.4	0.0917	2.808	0	12.94
0.02	0.4	0.0571	2.997	0	14.51

achieve a constant value of mean square velocity of the mass M, the mean square active force required in the proposed absorber increases when the mass of the absorber is reduced. This shows that the passive elements of the hybrid absorber help reduce the active force required for vibration control of the mass M.

An important finding in this research is the effect of the velocity feedback in the control scheme. To illustrate this effect, the cases as listed in Table I are considered again but this time with displacement feedback only. The displacement feedback gains to achieve the same mean square velocity $E[\dot{x}^2]/w_n S_0$ of mass M and the respective mean square active forces $E[f_a^2]/w_n S_0$ are calculated according to Eq. (28b), and they are listed in Table II. Comparing the mean square active forces $E[f_a^2]/w_n S_0$ listed in Tables I and II, it can be observed the active forces can be reduced significantly if the proposed optimum velocity feedback is used together with the displacement feedback in the control scheme.

III. HVA APPLIED TO SUPPRESSING VIBRATION OF FLEXIBLE BEAM STRUCTURE

A. Mathematical model

In this section, a general Euler–Bernoulli beam is considered as the primary system. The mean square velocity of the frequency response of a simply supported beam excited by an uniform distributed force as shown in Fig. 9 is to be suppressed with the proposed HVA attached at $x = x_0$. The equation of motion of the beam may be written as[27]

$$\rho A\frac{\partial^2 w}{\partial t^3} + EI\frac{\partial^4 w}{\partial x^4} = p(t)g(x) + F_h(t)\delta(x - x_0) \quad (34)$$

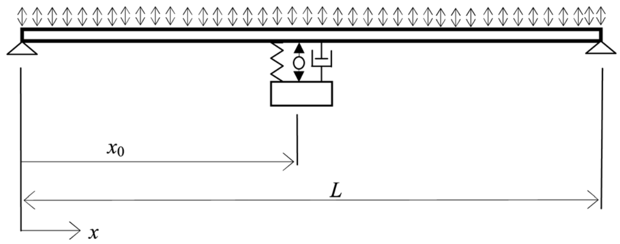

FIG. 9. Schematics of a simply supported beam with a hybrid vibration absorber excited by a uniform disturbed force.

where $p(t)g(x)$ is the externally applied forcing function and F_h is the force acting by the HVA onto the beam. The eigenfunction and the eigenvalues of the beam may be written as

$$\varphi_i(x) = \frac{2}{L}\sin(\beta_i x) \quad \text{where} \quad \beta_i = \frac{i\pi}{L}, \quad i \in N. \quad (35)$$

The spatial parts of the forcing functions and the Dirac delta function expanded in Fourier series written as

$$g(x) = \sum_{i=1}^{\infty} a_i\varphi_i(x), \quad \text{and} \quad (36a)$$

$$\delta(x - x_0) = \sum_{i=1}^{\infty} b_i\varphi_i(x) \quad (36b)$$

where the Fourier coefficients a_i and b_i are, respectively,

$$a_i = \frac{2}{n\pi L}, \quad i = 2n - 1, \quad n \in N \quad (37a)$$

else $a_i = 0$, and

$$b_i = \frac{2}{L}\sin\left(\frac{i\pi x_0}{L}\right), \quad (37b)$$

assuming the HVA is tuned to suppress the first vibration mode of the beam. The velocity response function of the beam can be derived using a similar approach of Ref. 23 and written as

$$\frac{\dot{W}(x,s)}{\omega_n P(s)} = \frac{j\lambda}{\rho A\omega_n^2}\left(\sum_{p=1}^{\infty}\left(\frac{a_p - b_p\left(\dfrac{\mu L\displaystyle\sum_{q=1}^{\infty}\frac{a_q\varphi_q(x_0)}{\gamma_q^2-\lambda^2}}{-\dfrac{\gamma^2-\lambda^2+2j\zeta\gamma\lambda}{\lambda^2\left(\gamma^2+\dfrac{2\alpha_0}{\varepsilon}+2j\zeta\gamma\lambda+4j\alpha_1\zeta\gamma\lambda\right)}+\mu L\displaystyle\sum_{r=1}^{\infty}\frac{b_r\varphi_r(x_0)}{\gamma_r^2-\lambda^2}}\right)}{\gamma_p^2-\lambda^2}\right)\varphi_p(x)\right) \quad (38)$$

where $\omega_n = \sqrt{EI\beta_1^4/\rho A}$, $\omega_r = \sqrt{EI\beta_r^4/\rho A}$, $\gamma_r = \omega_r/\omega_n$, $\mu = m/\rho AL$, $\varepsilon = \mu\varphi_1^2(x_0)$, and $\alpha_0 = a\varphi_1^2(x_0)/2EIL\beta_1^4$ and $\alpha_1 = a_1/2c$.

The mean square velocity of the beam averaged over the beam length can be written using Eq. (38) as

$$\frac{\int_0^L \left|\frac{\dot{W}(x,\lambda)}{\omega_n P(\lambda)}\right|^2 dx}{L} = \left(\frac{\lambda}{\rho A\omega_n^2}\right)^2 \sum_{p=1}^{\infty} \left| \frac{a_p - \dfrac{\mu L b_p \sum_{q=1}^{\infty} \dfrac{a_q \varphi_q(x_0)}{\gamma_q^2-\lambda^2}}{\dfrac{\gamma^2-\lambda^2+2j\zeta\gamma\lambda}{-\lambda^2\left(\gamma^2+\frac{2\alpha_0}{\varepsilon}+2j\zeta\gamma\lambda+4j\alpha_1\zeta\gamma\lambda\right)} + \mu L \sum_{r=1}^{\infty} \dfrac{b_r\varphi_r(x_0)}{\gamma_r^2-\lambda^2}}}{\gamma_p^2-\lambda^2} \right|^2. \tag{39}$$

For a structure with well-separated natural frequencies, the modal velocity response in the vicinity of the first natural frequency may be approximated by considering $p = q = r = 1$ and ignoring other vibration modes of the beam. Equation (38) may then be approximately written as

$$\frac{\dot{W}(x,s)}{\omega_n P(s)} = \frac{a_1\varphi_1(x)}{\rho A\omega_n^2} \frac{-2\zeta\gamma\lambda^2 + j\lambda\left(\gamma^2-\lambda^2\right)}{\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right) - \left(\varepsilon\gamma^2+2\alpha_0\right)\lambda^2 + 2j\zeta\gamma\lambda\left(1-\left(1+\varepsilon+2\alpha_1\varepsilon\right)\lambda^2\right)}. \tag{40}$$

Similarly, Eq. (39) may then be approximately written as

$$\frac{\int_0^L \left|\frac{\dot{W}(x,\lambda)}{\omega_n P(\lambda))}\right|^2 dx}{L} = \left(\frac{a_1}{\rho A\omega_n^2}\right)^2 \left|\frac{-2\zeta\gamma\lambda^2 + j\lambda\left(\gamma^2-\lambda^2\right)}{\left(1-\lambda^2\right)\left(\gamma^2-\lambda^2\right) - \left(\varepsilon\gamma^2+2\alpha_0\right)\lambda^2 + 2j\zeta\gamma\lambda\left(1-\left(1+\varepsilon+2\alpha_1\varepsilon\right)\lambda^2\right)}\right|^2. \tag{41}$$

Equation (40) has a similar form as Eq. (3) if ε in Eq. (40) is replaced by μ. Following the same procedure as described in Sec. II A, one can obtain the optimum tuning frequency and damping of the proposed HVA, respectively, to minimize the mean square velocity averaged over the beam length. These optimum frequency and damping of the proposed HVA may be written as

$$\gamma_{\mathrm{opt}} = \sqrt{\frac{D_0\mu^2 + D_1\mu + D_2}{(1+\mu+2\mu\alpha_1)(D_0\mu^2 + D_1\mu + D_2 + \alpha_1(1+2\alpha_0+4\alpha_0\alpha_1)\mu^2 + 2\alpha_0\alpha_1\mu)}}, \tag{42a}$$

$$\zeta_{\mathrm{opt}} = \frac{(\varepsilon + 2\alpha_0 + 2\varepsilon\alpha_0 + 4\varepsilon\alpha_0\alpha_1)}{2\sqrt{(D_0\varepsilon^2 + D_1\varepsilon + D_2)}} \sqrt{1 + \frac{\alpha_1^2\varepsilon^2}{(D_0\varepsilon^2 + D_1\varepsilon + D_2 + \alpha_1(1+2\alpha_0+4\alpha_0\alpha_1)\varepsilon^2 + 2\alpha_0\alpha_1\varepsilon)}}. \tag{42b}$$

B. Numerical example

Assuming the length of the beam is $L = 1$ m with cross section 0.025 m × 0.025 m. The proposed HVA is attached at $x = x_0 = 0.5$ m of the beam as shown in Fig. 9. The mass ratio μ is 0.05. $\varepsilon = \mu\varphi_1^2(x_0) = 0.2$. The material of the beam is aluminum with $\rho = 2710$ kgm^{-3} and $E = 6.9$ GPa. The desired dimensionless mean square velocity $E[\dot{x}^2]_{\mathrm{HVA\_opt}}/E[\dot{x}^2]_{\mathrm{PVA\_opt}}$ is assumed to be 0.2. As discussed in Sec. II A, the optimum feedback gain can be determined to be $\alpha_0 = 0.488$ and $\alpha_1 = 2.589$. The spatial mean square velocity of the beam, $(1/L)\int_0^L \left|\dot{W}(x,\lambda)/\omega_n P(\lambda)\right|^2 dx$, is calculated according to Eqs. (39), (42a), and (42b) and plotted in Fig. 10. The calculation is repeated for the case of using the passive absorber ($\alpha_0 = \alpha_1 = 0$), and the result is also plotted in Fig. 10 for comparison. As shown in Fig. 10, the suppression of beam vibration by the HVA is much better than by the PVA for both the fundamental and higher vibration modes of the beam. It

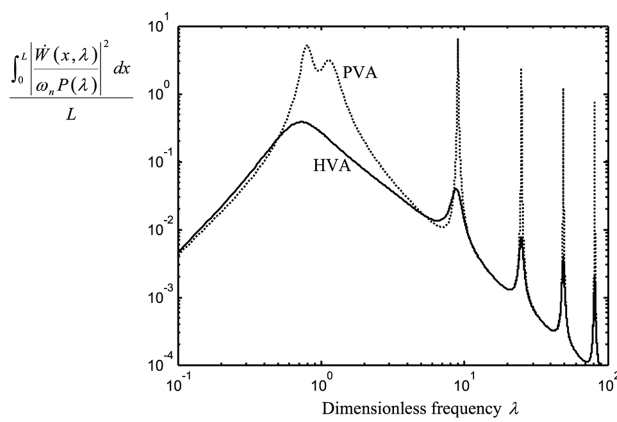

FIG. 10. Mean square velocity of the beam averaged over the beam length (Eq. 41) of the beam in Fig. 9. $x_0/L = 0.5$; $\mu = 0.05$ and $\varepsilon = 0.2$. ——, $\alpha_0 = 0.488$ and $\alpha_1 = 2.589$ (HVA with optimum feedback gains); ……, $\alpha_0 = \alpha_1 = 0$ (PVA).

shows that the proposed HVA may also be used for suppressing vibration of continuous vibrating systems.

IV. CONCLUSION

The hybrid vibration absorber (HVA) is an active-passive vibration control device that is attached to a vibrating body subjected to exciting force or motion. In this article, we propose an optimal design of a HVA with a displacement and a velocity feedback for minimizing the velocity response of the structure based on the H_2 optimization criterion. The objective of the optimal design is to reduce the total vibration energy of the vibrating structure under wideband excitation through minimization of the total area under the velocity response spectrum. One of the inherent limitations of the traditional passive vibration absorber is that its vibration suppression is low if the mass ratio (between the absorber mass and the mass of the primary structure) is low. The proposed HVA overcomes this limitation and provides good vibration suppression performance even at a low mass ratio. Optimal tuning frequency and damping of the passive elements of the HVA are derived for the minimization of the mean square velocity of the primary system. On the other hand, optimal feedback gains of the displacement and velocity of the primary system are derived to minimize the active force required in the HVA. The proposed HVA was tested on both a SDOF and a beam like structure and compared to the traditional passive vibration absorber.

The proposed vibration absorber has both passive and active element similar to the commercial inertial actuator. However, the inertial actuator has fixed secondary mass m, stiffness k and damping c using a standard proportional control based on the feedback displacement signal for generating the active force f_a. As shown in Sec. II, using velocity feedback signal together with displacement feedback signal to generate the active force would lead to smaller active forces for wideband vibration control. In this manuscript, we derive the optimum passive and active parameter values for the minimization of the mean square velocity of the primary system as well as the active force required in the HVA. The research results in the manuscript improve the understanding of the relations between the passive and active parameters of a HVA and provide a design guideline of the HVA and may motivate an improved design of the commercial inertial actuator.

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