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Optimization of a hybrid vibration absorber for vibration control of structures under random force excitation

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ABSTRACT

A recently reported design of a hybrid vibration absorber (HVA) which is optimized to suppress resonant vibration of a single degree-of-freedom (SDOF) system is re-optimized for suppressing wide frequency band vibration of the SDOF system under stationary random force excitation. The proposed HVA makes use of the feedback signals from the displacement and velocity of the absorber mass for minimizing the vibration response of the dynamic structure based on the H_2 optimization criterion. The objective of the optimal design is to minimize the mean square vibration amplitude of a dynamic structure under a wideband excitation, i.e., the total area under the vibration response spectrum is minimized in this criterion. One of the inherent limitations of the traditional passive vibration absorber is that its vibration suppression is low if the mass ratio between the absorber mass and the mass of the primary structure is low. The active element of the proposed HVA helps further reduce the vibration of the controlled structure and it can provide significant vibration absorption performance even at a low mass ratio. Both the passive and active elements are optimized together for the minimization of the mean square vibration amplitude of the primary system. The proposed HVA are tested on a SDOF system and continuous vibrating structures with comparisons to the traditional passive vibration absorber.

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1. Introduction

Passive dynamic vibration absorber (PVA) is an auxiliary mass-spring-damper system which, when correctly tuned and attached to a vibrating system subject to harmonic excitation, causes to cease the steady-state motion at the point to which it is attached. The first research conducted at the beginning of the twentieth century considered an undamped PVA tuned to the frequency of the disturbing force [1]. Such an absorber is a narrow-band device as it is unable to eliminate structural vibration after a change in the disturbing frequency.

Finding the optimum parameters of a viscous friction PVA in a SDOF system drew the attention of many scholars. One of the optimization methods is H_∞ optimization which aims to minimize the resonant vibration amplitude. The optimum design method of a PVA is called “Fixed-points theory”, which was well documented in the textbook by Den Hartog [2]. Another optimization method proposed by Warburton [3,4] for PVA is H_2 optimization which aims to minimize the mean square vibration amplitude of the dynamic structure over the entire frequency range. If the system is subjected to random excitation rather than sinusoidal excitation, the H_2 optimization is more desirable than the H_∞ optimization. However, the

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major disadvantage of PVA is that its vibration absorption performance depends on the mass ratio [5] and it is quite limited when the mass ratio is small.

In order to improve the performance of a PVA, some researchers added a force actuator to the PVA to form a hybrid vibration absorber (HVA) in order to provide an active force to counteract the vibration of the controlled structure. However, most of the designs of HVA found in literature are complicated with focus on the control of the active element only. Some common methods for controlling the active force in the HVA are neural network [6], delayed resonator [7], modal feedback control [8–12] and closed-loop poles by modal feedback [13,14].

In this article, a hybrid vibration absorber (HVA) which is proposed and optimized by Chatterjee [15] recently for suppressing resonant vibration of a single degree-of-freedom system is re-optimized for suppressing wide frequency band vibration under stationary random force excitation. H_2 optimization is applied to the proposed absorber such that the mean square vibration amplitude of the dynamic structure over the entire frequency range is minimized, i.e., the total area under the power spectrum response curve is minimized in this criterion. Damping in the primary vibrating structure was ignored by Chatterjee [15] before but its effect on the optimization process is reported in this article. It is found that optimum values of the system parameters do not exist if the damping in the primary system is relatively high.

2. HVA applied to a single degree-of-freedom system

2.1. Mathematical model

Fig. 1 shows three different configurations of vibration absorbers mounted on a SDOF vibrating system for suppressing the vibration of the primary mass M . Case A in Fig. 1 is the one proposed and optimized by Chatterjee [15] for suppressing resonant vibration of the primary mass M . Case B in Fig. 1 is similar to Case A but a viscous damper is added in the primary system so that the effect of damping in the primary system to the optimization of the absorber can be studied. Case B becomes Case A if damping coefficient in Case B, C , equals to zero. Since Case B is more general than Case A, the mathematical model of Case B is established and presented in the following. Case B as shown in Fig. 1 has a HVA coupled with a primary system where x , M , C and K denote, respectively, displacement, mass, damping and spring coefficients of the primary system and x_a , m and k represent those of the absorber. The equations of motion of the dynamic system may be written as

$$\begin{cases} M\ddot{x} = -Kx - C\dot{x} - k(x - x_a) - f_a + F \\ m\ddot{x}_a = -k(x_a - x) + f_a \end{cases} \quad (1)$$

where F is the exciting force and the active force $f_a = ax_a - b\dot{x}_a$ is proposed by Chatterjee [15]. Performing Laplace transformation of Eq. (1), the transfer function of the vibration response of the primary mass M may be written in term of dimensionless parameters as

$$\begin{aligned} H(p) &= \frac{X}{(F/K)} \\ &= \frac{\mu\gamma^2 - 2\alpha + 2\beta p + \mu p^2}{(1 + 2\zeta p + p^2)(\mu\gamma^2 - 2\alpha + 2\beta p + \mu p^2) + \mu^2\gamma^2 p^2} \end{aligned} \quad (2)$$

where

$$p = \frac{s}{\omega_n}, \quad \mu = \frac{m}{M}, \quad \omega_n = \sqrt{\frac{K}{M}}, \quad \omega_a = \sqrt{\frac{k}{m}}, \quad \gamma = \frac{\omega_a}{\omega_n}, \quad \zeta = \frac{C}{2\sqrt{mK}}, \quad \alpha = \frac{a}{2K}$$

and

$$\beta = \frac{b\omega_n}{2K}$$

The frequency response function of mass M can be obtained by replacing p in Eq. (2) by $j\lambda$ where $\lambda = \omega/\omega_n$ and $j^2 = -1$ and it may be written as

$$H(\lambda) = \frac{X}{F/K} = \frac{\mu\gamma^2 - 2\alpha - \mu\lambda^2 + 2j\beta\lambda}{(1 - \lambda^2 + 2j\zeta\lambda)(\mu\gamma^2 - 2\alpha - \mu\lambda^2 + 2j\beta\lambda) - \mu^2\gamma^2\lambda^2} \quad (3)$$

The mean square vibration amplitude of the primary mass M may be written as [16]

$$E[x^2] = \int_{-\infty}^{\infty} |H|^2 S_y(\omega) d\omega \quad (4)$$

where H is the frequency response function of the primary mass and $S_y(\omega)$ is the input mean square spectral density function.

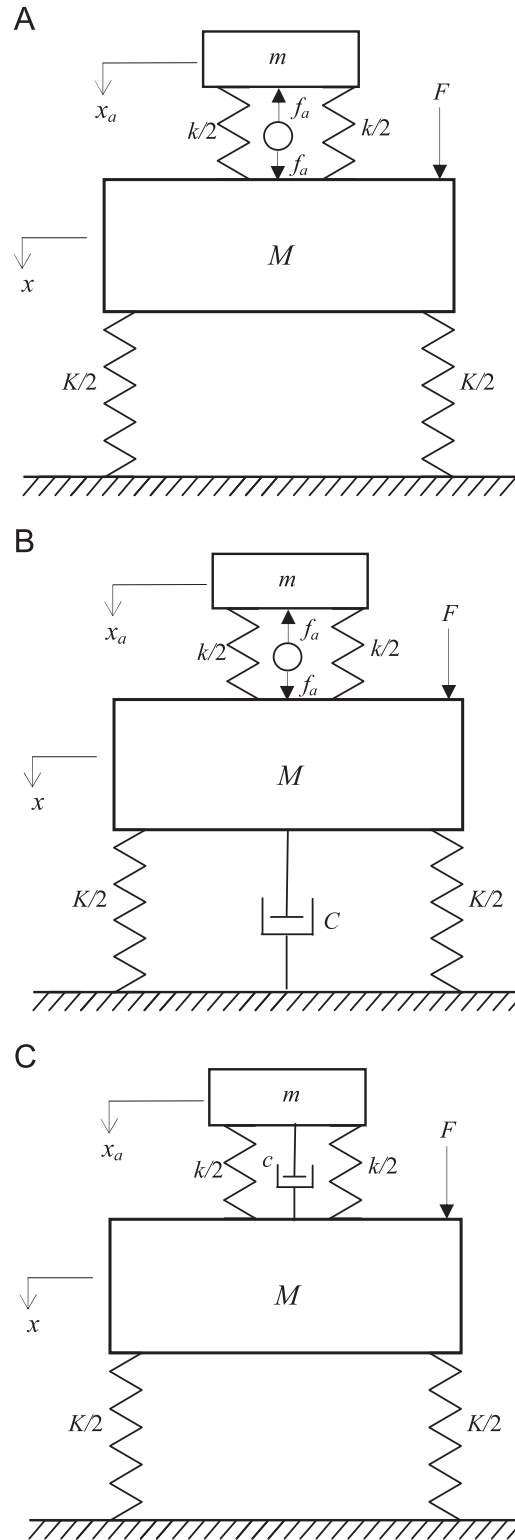


Fig. 1. Schematic diagrams of three different configurations of vibration absorber. Case A: Vibration control of a (M-K) system using the proposed hybrid vibration absorber (HVA), Case B: Vibration control of a (M-K-C) system using the proposed hybrid vibration absorber (HVA) and Case C: Vibration control of a (M-K) system using a passive vibration absorber (PVA).

If the input spectrum is assumed to be ideally white, i.e., $S_y(\omega) = S_0$, a constant for all frequencies, Eq. (4) can then be rewritten as

$$E[x^2] = S_0 \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega \tag{5}$$

The non-dimensional mean square vibration amplitude of mass M may be defined as [16]

$$E[x^2] = \frac{\omega_n S_0}{2\pi} \int_{-\infty}^{\infty} |H(\lambda)|^2 d\lambda \tag{6}$$

A useful formula of Gradshteyn and Ryzhik [17] written as Eq. (7) below is used for solving Eq. (6).

If

$$H(\omega) = \frac{-j\omega^3 B_3 - \omega^2 B_2 + j\omega B_1 + B_0}{\omega^4 A_4 - j\omega^3 A_3 - \omega^2 A_2 + j\omega A_1 + A_0} \tag{7a}$$

then

$$\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega = \pi \left[\frac{\frac{B_0^2(A_2 A_3 - A_1 A_4)}{A_0} + A_3(B_1^2 - 2B_0 B_2) + A_1(B_2^2 - 2B_1 B_3) + \frac{B_3^2(A_1 A_2 - A_0 A_3)}{A_4}}{A_1(A_2 A_3 - A_1 A_4) - A_0 A_3^2} \right] \tag{7b}$$

Comparing Eq. (3) with Eq. (7a), we may write

$$\begin{aligned} A_0 &= \mu\gamma^2 - 2\alpha, & A_1 &= 2\beta + 2\mu\zeta\gamma^2 - 4\alpha\zeta, & A_2 &= \mu + \mu\gamma^2 + \mu^2\gamma^2 - 2\alpha + 4\zeta\beta, \\ A_3 &= 2\mu\zeta + 2\beta, & A_4 &= \mu, & B_0 &= \mu\gamma^2 - 2\alpha, & B_1 &= 2\beta, & B_2 &= \mu, & B_3 &= 0. \end{aligned} \tag{8}$$

Using Eqs. (7) and (8), Eq. (6) may be rewritten as

$$E[x^2] = \frac{\omega_n S_0}{2} \left(\frac{C_0\gamma^4 + C_1\gamma^2 + C_2}{D_0\gamma^4 + D_1\gamma^2 + D_2} \right) \tag{9}$$

where

$$\begin{cases} C_0 = 2\mu^2(\mu^2\zeta + \mu\beta + \beta) \\ C_1 = -4\mu(\alpha\mu^2\zeta + (1 + \alpha - 2\zeta^2)\beta\mu + 2\beta(\alpha - \beta\zeta)) \\ C_2 = 2\beta(\mu^2 + (4\alpha - 8\alpha\zeta^2 + 4\beta\zeta)\mu + 4\alpha^2 + 4\beta^2 - 8\alpha\beta\zeta) \\ D_0 = 4\mu^2\zeta(\mu^2\zeta + \mu\beta + \beta) \\ D_1 = -4\mu((2\alpha\zeta - \beta)\zeta\mu^2 + (2\alpha\zeta + 2\zeta - 4\zeta^3 - \beta)\beta\mu + 4\beta\zeta(\alpha - \beta\zeta)) \\ D_2 = 4\beta\zeta(\mu^2 + (4\alpha - 8\alpha\zeta^2 + 4\beta\zeta)\mu + 4\alpha^2 + 4\beta^2 - 8\alpha\beta\zeta) \end{cases} \tag{10}$$

2.2. Stability analysis of the proposed HVA (Case A)

According to Eq. (2), the characteristic equation of the control system may be written as

$$(1 + p^2 + 2\zeta p)(\mu\gamma^2 - 2\alpha + \mu p^2 + 2\beta p) + \mu^2\gamma^2 p^2 = 0 \tag{11}$$

where

$$\forall \gamma, \mu, \alpha, \beta \in R^+$$

The dynamic system is stable if the real parts of all the roots of Eq. (11) are negative. Applying the Routh's stability criterion [18], the array of coefficient of Eq. (11) with $\zeta = 0$ may be written as

$$\begin{array}{c|ccc} p^4 & \mu & \mu + \mu\gamma^2 + \mu^2\gamma^2 - 2\alpha & \mu\gamma^2 - 2\alpha \\ p^3 & 2\beta & 2\beta & 0 \\ p^2 & \mu\gamma^2(1 + \mu) - 2\alpha & \mu\gamma^2 - 2\alpha & \\ p & \frac{2\beta\mu^2\gamma^2}{\mu\gamma^2(1 + \mu) - 2\alpha} & 0 & \\ 1 & \mu\gamma^2 - 2\alpha & & \end{array} \tag{12}$$

If $\mu\gamma^2 - 2\alpha > 0$ is assumed then all the terms in the array above will be positive and therefore the control system must be stable according to the Routh's stability criterion.

2.3. Optimization of HVA

The proposed HVA is optimized based on the H_2 optimization criterion. The objective of the optimal design is to reduce the mean square vibration amplitude of the dynamic structure over the entire frequency range, i.e., the total area under the vibration response spectrum is minimized. The optimum parameters of the HVA for undamped primary structure are derived analytically while those with a damped primary structure are derived numerically.

2.3.1. Optimum design of the HVA for an undamped SDOF vibrating system (Case A)

Substituting $\zeta=0$ into Eq. (9), the mean square vibration amplitude of the mass M may be written as

$$E[x^2] = \frac{\omega_n S_0}{4\beta\mu^2\gamma^2} \left(\mu^2(1+\mu)\gamma^4 - 2\mu(\mu+\alpha\mu+2\alpha)\gamma^2 + 4\alpha^2 + 4\beta^2 + 4\alpha\mu + \mu^2 \right) \quad (13)$$

There are four variables μ , γ and α and β in Eq. (13). The existence of an optimum set of these four variables requires that $(\partial/\partial\gamma)E[x^2] = (\partial/\partial\alpha)E[x^2] = (\partial/\partial\beta)E[x^2] = (\partial/\partial\mu)E[x^2] = 0$. Since the possible range of the mass ratio μ of the HVA is small in practice, it is chosen as a basic variable in the following optimization process of the HVA and the optimum values of other three variables are derived and expressed in terms of μ whenever possible.

If $(\partial/\partial\gamma)E[x^2] = (\partial/\partial\beta)E[x^2] = 0$ can be solved the HVA will have an optimum set of tuning frequency γ and feedback gain β in term of α and μ . We may therefore write

$$\frac{\partial E[x^2]}{\partial\gamma} = \frac{\omega_n S_0 \left(2\mu^2(1+\mu)\gamma^4 - 2(4\alpha^2 + 4\beta^2 + 4\alpha\mu + \mu^2) \right)}{4\beta\mu^2\gamma^3} = 0 \quad (14a)$$

$$\frac{\partial E[x^2]}{\partial\beta} = \frac{\omega_n S_0 \left(-\mu^2(1+\mu)\gamma^4 + 2\mu(\mu+\alpha\mu+2\alpha)\gamma^2 + 4\beta^2 - 4\alpha^2 - 4\alpha\mu - \mu^2 \right)}{4\beta^2\mu^2\gamma^2} = 0 \quad (14b)$$

Using Eq. (14a), we may write

$$\beta = \sqrt{\frac{\mu^2(1+\mu)\gamma^4 - 4\alpha^2 - 4\alpha\mu - \mu^2}{4}} \quad (15a)$$

Using Eq. (14b), we may write

$$\beta = \sqrt{\frac{\mu^2(1+\mu)\gamma^4 - 2\mu(\mu+\alpha\mu+2\alpha)\gamma^2 + 4\alpha^2 + 4\alpha\mu + \mu^2}{4}} \quad (15b)$$

The optimum tuning ratio γ_{opt} can be obtained by equating Eqs. (15a) and (15b) and solving for γ and it can be written in term of α and μ as

$$\gamma_{\text{opt}} = \sqrt{\frac{4\alpha^2 + 4\alpha\mu + \mu^2}{\mu(\mu+\alpha\mu+2\alpha)}} \quad (16)$$

Substituting γ_{opt} from Eq. (16) into Eqs. (15a) or (15b), the optimum velocity feedback may be written as

$$\beta_{\text{opt}} = \frac{\mu(2\alpha+\mu)\sqrt{\mu+2\alpha-\alpha^2}}{2(\mu+\alpha\mu+2\alpha)} \quad (17)$$

β_{opt} exists when $\mu+2\alpha-\alpha^2 > 0$. The dimensionless mean square vibration amplitude of the primary mass M may be written using Eq. (13) as

$$\frac{E[x^2]_{\text{HVA}}}{\omega_n S_0} = \frac{1}{4\beta\mu^2\gamma^2} \left(\mu^2(1+\mu)\gamma^4 - 2\mu(\mu+\alpha\mu+2\alpha)\gamma^2 + 4\alpha^2 + 4\beta^2 + 4\alpha\mu + \mu^2 \right) \quad (18)$$

Substituting Eqs. (16) and (17) into (18), the mean square vibration amplitude of the primary mass M with the optimized HVA may be written as

$$\frac{E[x^2]_{\text{HVA,opt}}}{\omega_n S_0} = \frac{\sqrt{\mu+2\alpha-\alpha^2}}{\mu+2\alpha} \quad (19)$$

Taking the partial derivative of Eq. (19) with respect to α , we may write

$$\frac{\partial}{\partial\alpha} \left(\frac{E[x^2]_{\text{HVA,opt}}}{\omega_n S_0} \right) = -\frac{\mu+2\alpha+\alpha}{2(\mu+2\alpha)^2\sqrt{\mu+2\alpha-\alpha^2}} < 0 \quad (20a)$$

Taking the partial derivative of Eq. (19) with respect to μ , we may write

$$\frac{\partial}{\partial \mu} \left(\frac{E[x^2]_{\text{HVA_opt}}}{\omega_n S_0} \right) = - \frac{\mu + 2\alpha - 2\alpha^2}{2(\mu + 2\alpha)^2 \sqrt{\mu + 2\alpha - \alpha^2}} < 0 \tag{20b}$$

The proof of $\mu + 2\alpha - 2\alpha^2 > 0$ is shown in Appendix A. Eqs. (20a) and (20b) show that no optimum displacement feedback gain α and mass ratio μ exist and the mean square vibration amplitude of the primary mass M , $E[x^2]_{\text{HVA_opt}}/\omega_n S_0$, can be reduced when the mass ratio μ or the displacement feedback gain α is increased.

To illustrate the variations of $E[x^2]_{\text{HVA_opt}}/\omega_n S_0$ with the tuning ratio γ and velocity feedback gain β , $E[x^2]_{\text{HVA_opt}}/\omega_n S_0$ is calculated according to Eq. (18) with $\mu = 0.2$ and $\alpha = 0.3$ and then plotted in Fig. 2. It can be observed in Fig. 2 that $E[x^2]_{\text{HVA_opt}}/\omega_n S_0$ becomes a minimum at $\beta_{\text{opt}} \approx 0.08$ and $\gamma_{\text{opt}} \approx 1.93$.

It can be derived as shown in Appendix A that $\alpha \leq 1 + \sqrt{1 + 2\mu}/2$ if γ_{opt} from Eq. (16) is substituted into $\mu\gamma^2 - 2\alpha > 0$. The root locus of Eq. (11) is plotted with α increasing from zero to $\alpha_{\text{max}} = (1 + \sqrt{1 + 2\mu})/2$ and $\mu = 0.2$ in Fig. 3 for illustration. As shown in Fig. 3, the control system has four complex roots when the feedback of the displacement α is small and it has two complex poles and two negative real poles when α approaches α_{max} . Substituting α_{max} into Eq. (19), the minimum

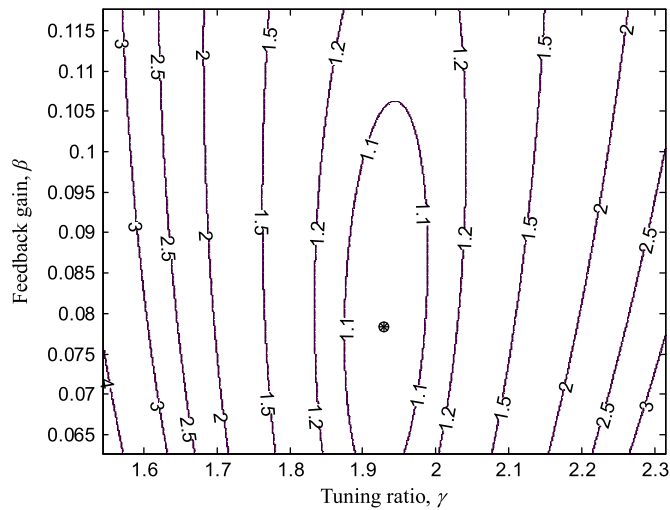


Fig. 2. Contour plot of $E[x^2]_{\text{HVA}}/\omega_n S_0$ of Case A with different tuning ratio γ and velocity feedback gain β . $\mu = 0.2$, $\alpha = 0.3$. * —Point of minimum of $E[x^2]_{\text{HVA}}/\omega_n S_0$.

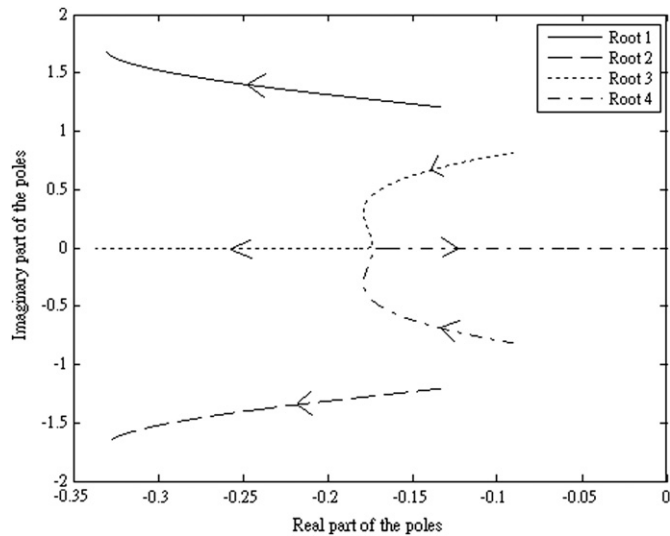


Fig. 3. Root locus of Case A with $\mu = 0.2$ and $\alpha \in (0, 1 + \sqrt{1 + 2\mu}/2)$.

mean square vibration amplitude of the primary mass M may be written in terms of μ as

$$\frac{E[x^2]_{\text{HVA_opt_min}}}{\omega_n S_0} = \lim_{\alpha \rightarrow \alpha_{\text{max}}} \left(\frac{E[x^2]_{\text{HVA_opt}}}{\omega_n S_0} \right) = \frac{1}{\sqrt{2(1+\mu+\sqrt{1+2\mu})}} < \frac{1}{2} \quad (21)$$

Eq. (21) shows that the larger the mass ratio μ , the smaller the minimum mean square vibration amplitude $E[x^2]_{\text{HVA_opt_min}}$, and $E[x^2]_{\text{HVA_opt_min}}/\omega_n S_0$ is smaller than 0.5 and it reduces as the mass ratio μ increases.

It is derived as shown in Appendix B that if the optimization process is started by solving $(\partial/\partial\gamma)E[x^2] = (\partial/\partial\alpha)E[x^2] = 0$ no real solution of μ , γ and β can be obtained. On the other hand, if the optimization process is started by solving $(\partial/\partial\alpha)E[x^2] = (\partial/\partial\beta)E[x^2] = 0$ the minimum mean square vibration amplitude $E[x^2]_{\text{HVA_opt_min}}/\omega_n S_0$ in that case becomes 0.5 as showed by Eq. (B11) of Appendix B. Comparing Eq. (21) with Eq. (B11), the proposed γ_{opt} and β_{opt} as shown in Eqs. (16) and (17), respectively, are the best choice of optimum parameters among the three cases of optimization considered in this Section.

2.3.2. Effect of primary damping on the optimum design of the HVA (Case B)

In Case B, the mean square vibration amplitude of mass M , $E[x^2]_{\text{HVA}}/\omega_n S_0$, has a minimum if $(\partial/\partial\gamma)E[x^2] = (\partial/\partial\beta)E[x^2] = 0$, the HVA has an optimum set of γ and β in terms of α . So we may write using Eq. (9)

$$\frac{\partial E[x^2]}{\partial \gamma} = \frac{16\mu^2\beta\gamma(\beta+\mu\zeta)(E_0\gamma^4+E_1)}{(D_0\gamma^4+D_1\gamma^2+D_2)^2} = 0 \quad (22a)$$

$$\frac{\partial E[x^2]}{\partial \beta} = \frac{8\mu^2\gamma^2(F_0\gamma^4+F_1\gamma^2+F_2)}{(D_0\gamma^4+D_1\gamma^2+D_2)^2} = 0 \quad (22b)$$

where

$$\begin{cases} E_0 = \mu^2(\beta + \mu\beta + \mu^2\zeta) \\ E_1 = -\beta(\mu^2 + (4\alpha + 4\beta\zeta - 8\alpha\zeta^2)\mu + 4\alpha^2 + 4\beta^2 - 8\alpha\beta\zeta) \\ F_0 = \mu^2(\beta^2 + \mu\beta^2 + 2\mu^2\beta\zeta + \mu^3\zeta^2) \\ F_1 = -2\mu(2\alpha\beta^2 + \mu\beta^2(1+\alpha) + 2\mu^2\alpha\beta\zeta + \alpha\mu^3\zeta^2) \\ F_2 = \beta^2(2\alpha + \mu - 2\mu\zeta - 2\beta)(2\alpha + \mu + 2\mu\zeta + 2\beta) \end{cases} \quad (23)$$

Using Eq. (22a), we may write

$$\gamma = \left(\frac{\beta(\mu^2 + (4\alpha + 4\beta\zeta - 8\alpha\zeta^2)\mu + 4\alpha^2 + 4\beta^2 - 8\alpha\beta\zeta)}{\mu^2(\beta + \mu\beta + \mu^2\zeta)} \right)^{\frac{1}{4}} \quad (24a)$$

Using Eq. (22b), we may write

$$\gamma = \sqrt{\frac{-F_1 - \sqrt{F_1^2 - 4F_0F_2}}{2F_0}} \quad (24b)$$

The optimum feedback gain of velocity β_{opt_d} can be obtained by equating Eqs. (24a) and (24b) and solving for β in terms of μ , α and ζ . The optimum tuning ratio γ_{opt_d} can then be derived by substituting β_{opt_d} into either Eqs. (24a) or (24b). To illustrate the effect of primary damping on the variation of the mean square vibration amplitude of mass M , $E[x^2]_{\text{HVA}}/\omega_n S_0$ are calculated according to Eqs. (9) and (10) with $\zeta=0.1$ for different values of velocity feedback gain β and tuning ratio γ . The results are plotted in Fig. 4 for the illustration of the optimum frequency γ_{opt_d} and feedback gain β_{opt_d} of the HVA. In comparison between Figs. 2 and 4, the minimum point of $E[x^2]_{\text{HVA}}/\omega_n S_0$ is further reduced because of the primary damping ζ in Case B and it also causes a saddle point in the contour plot in Fig. 3. It is found that β_{opt_d} does not exist if the primary damping ratio ζ is high such that the square roots terms in Eqs. ((22a) and (22b)) cannot be solved. Figs. 5 and 6 show the variations of β_{opt_d} and γ_{opt_d} , respectively with the primary damping ζ at $\mu=0.2$ and four different values of α for illustration. Both Figs. 5 and 6 show that the optimum tuning ratio γ_{opt_d} and the optimum feedback gain of velocity β_{opt_d} decrease as the primary damping ζ increases. The dimensionless mean square vibration amplitude of mass M , $E[x^2]_{\text{HVA_opt}}/\omega_n S_0$, which is then calculated using Eq. (9) with β_{opt_d} , γ_{opt_d} , $\mu=0.2$ and four different values of α is plotted in Fig. 7 for illustration. It shows that the mean square vibration amplitude of the primary mass M decreases as primary damping ζ increases.

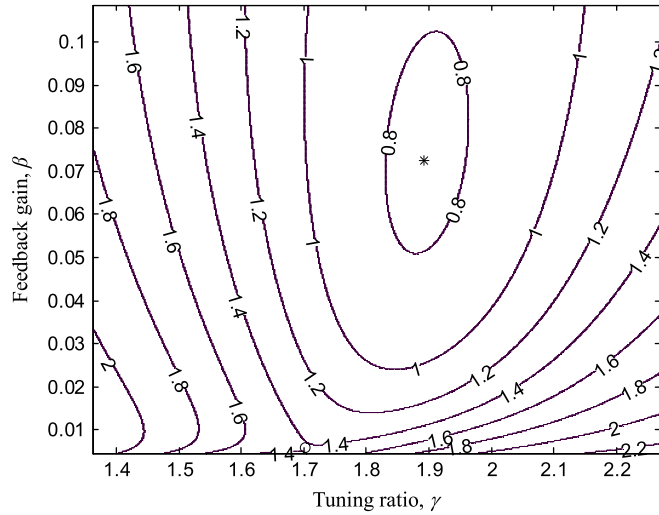


Fig. 4. Contour plot of $E[x^2]_{HVA}/\omega_n S_0$ of Case B with different tuning ratio γ and velocity feedback gain β . $\mu = 0.2$, $\alpha = 0.3$ and $\zeta = 0.1$. *—Point of minimum of $E[x^2]_{HVA}/\omega_n S_0$; o—Saddle point.

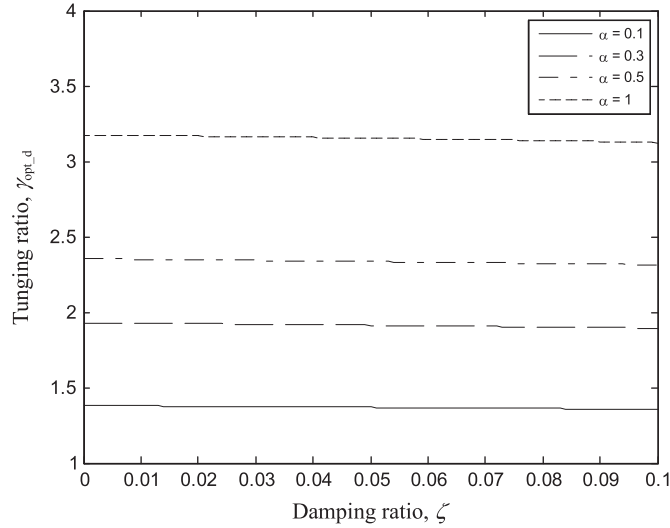


Fig. 5. Optimum tuning ratio $\gamma_{opt,d}$ of Case B versus damping ratio ζ . $\mu = 0.2$.

2.4. Comparison to a dynamic vibration absorber with passive damping (Case C)

Case A in Fig. 1 is compared to a passive dynamic vibration absorber (PVA), shown as Case C in Fig. 1 in this section. The frequency response function of the PVA in Case C may be written as [2]

$$\frac{X}{F/K} = \frac{\gamma^2 - \lambda^2 + 2j\zeta_a \gamma \lambda}{(1 - \lambda^2)(\gamma^2 - \lambda^2) - \mu \gamma^2 \lambda^2 + 2j\zeta_a \gamma \lambda (1 - \lambda^2 - \mu \lambda^2)} \tag{25}$$

where

$$\zeta_a = \frac{c}{2\sqrt{mk}}$$

The H_2 optimum tuning frequency and damping of the PVA may be written as [3]

$$\gamma = \sqrt{\frac{2 + \mu}{2(1 + \mu)^2}} \quad \text{and} \quad \zeta_a = \frac{1}{2} \sqrt{\frac{\mu(3\mu + 4)}{2(2 + \mu)(1 + \mu)}} \tag{26}$$

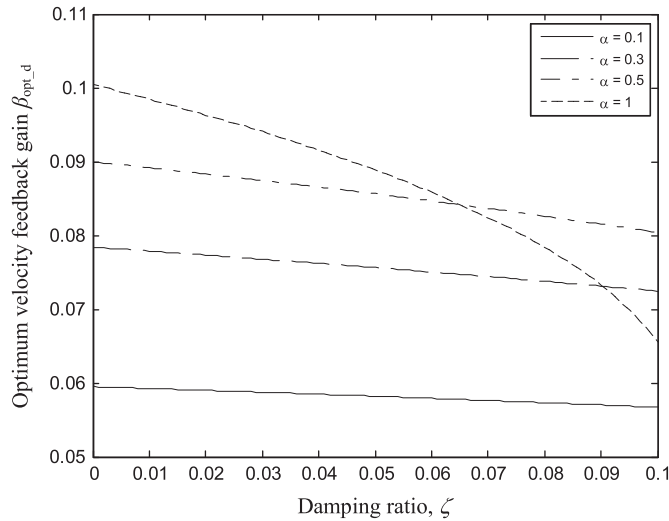


Fig. 6. Optimum velocity feedback gain $\beta_{opt,d}$ of Case B versus damping ratio ζ with $\mu = 0.2$.

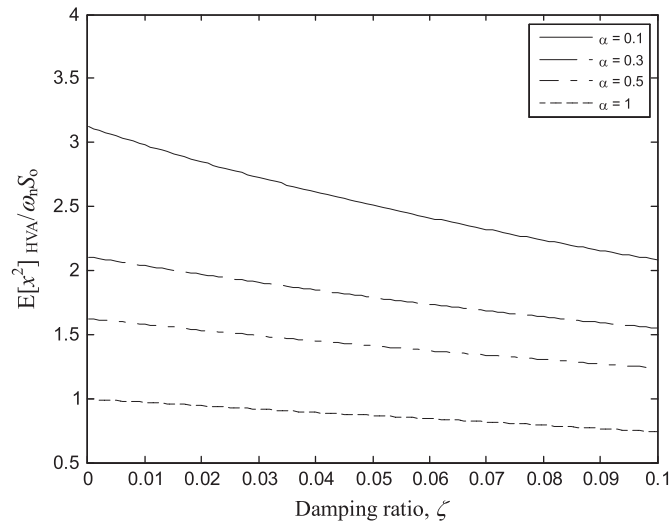


Fig. 7. The dimensionless mean square vibration amplitude of mass M of Case B, $E[x^2]_{HVA}/\omega_n S_0$, versus damping ratio ζ . $\mu = 0.2$.

The mean square vibration amplitude of the primary mass may be written as [3]

$$E[x^2]_{PVA_{opt}} = \frac{1}{2} \sqrt{\frac{4+3\mu}{\mu(1+\mu)}} \tag{27}$$

Eq. (27) shows that the minimum mean square vibration amplitude of Case C, $E[x^2]_{PVA_{opt}}$, tends to infinity as the mass ratio μ tends to zero while Eq. (21) shows that the minimum mean square vibration amplitude of Case A, $E[x^2]_{HVA_{opt_min}}/\omega_n S_0$ tends to 0.5 as the mass ratio μ tends to zero. These results show that the proposed HVA but not the PVA can produce significant vibration suppression of the SDOF vibrating system if the mass ratio is low.

The ratio between the mean square vibration amplitude of the primary mass of PVA and that of the undamped HVA can be derived using Eqs. (19) and (27) and written as

$$\frac{E[x^2]_{HVA_{opt}}}{E[x^2]_{PVA_{opt}}} = \frac{\sqrt{\frac{\mu+2\alpha-\alpha^2}{\mu+2\alpha}}}{\frac{1}{2} \sqrt{\frac{4+3\mu}{\mu(1+\mu)}}} = \frac{2}{(\mu+2\alpha)} \sqrt{\frac{\mu(1+\mu)(\mu+2\alpha-\alpha^2)}{4+3\mu}} \tag{28}$$

The mean square vibration amplitude motion ratio $E[x^2]_{HVA_{opt}}/E[x^2]_{PVA_{opt}}$ is calculated for different values of displacement feedback gain α and mass ratio μ and the results are plotted as contours in Fig. 8 for illustration. As shown in Fig. 8, the ratio

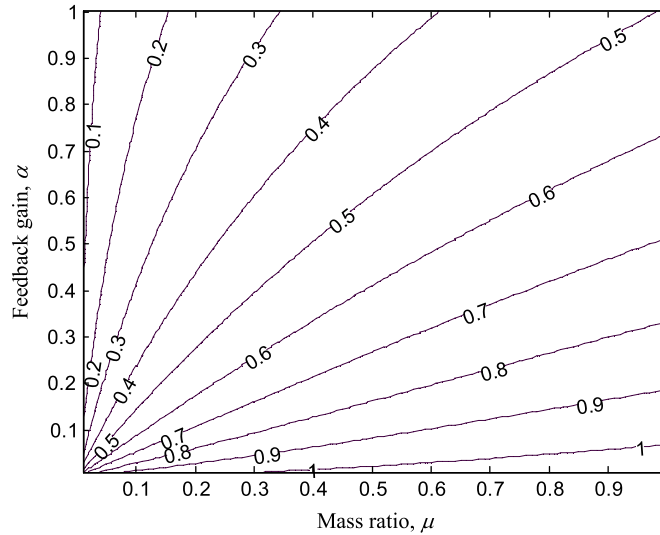


Fig. 8. Contour plot of $E[x^2]_{HVA}/E[x^2]_{PVA}$ (comparing Case A to Case C) with different mass ratio μ and displacement feedback gain α .

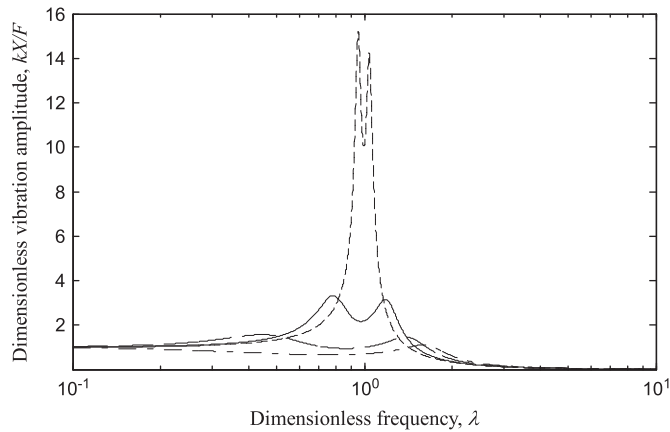


Fig. 9. Frequency response function of the primary structure (Case A) using Eq. (2) with $\alpha = 0.1$ (—), 0.5 (---), α_{max} (- · - · -), and that with PVA (Case C) with optimum tuning condition using Eq. (25) (···), $\mu = 0.01$.

$E[x^2]_{HVA_opt}/E[x^2]_{PVA_opt}$ reduces as the displacement feedback gain α increases. The frequency response functions of the mass M in Case A, $|H(\lambda)|$, is calculated using Eqs. (3), (16) and (17) with mass ratio $\mu = 0.01$ and feedback gain $\alpha = 0.1, 0.5$ and α_{max} and the results are plotted in Fig. 9 for illustration. The corresponding frequency response function of in Case C with an optimized PVA is calculated using Eqs. (25) and (26) and the result is also plotted as the dotted line in Fig. 9 for comparison. As shown in Fig. 9, the resonant vibration amplitude of the primary mass M in Case C using an optimized PVA is about fourteen times of that in Case A using the proposed HVA with $\alpha = \alpha_{max}$. Moreover, mean square vibration amplitudes of the primary mass M in Case C is 43% of that of Case A with $\alpha = \alpha_{max}$. These results show that the proposed HVA can significantly reduce both the resonant vibration amplitude and the mean square vibration amplitude of a SDOF vibrating system even when the mass ratio μ is small.

2.5. Comparison of the proposed design (Case A) with the design by Chatterjee [15]

H_∞ optimum control of the HVA was proposed by Chatterjee [15] to minimize the resonant vibration of the primary mass M of Case A as illustrated in Fig. 1a. Chatterjee derived the optimum α and β and then derived the optimum γ in terms of α and μ . The optimum tuning frequency γ and velocity feedback gain β of the HVA by [15] may be written in terms of μ and α as

$$\gamma_{opt. [15]} = \sqrt{\frac{4\alpha + 2\mu}{\mu(2 + \mu)}}, \text{ and} \tag{29a}$$

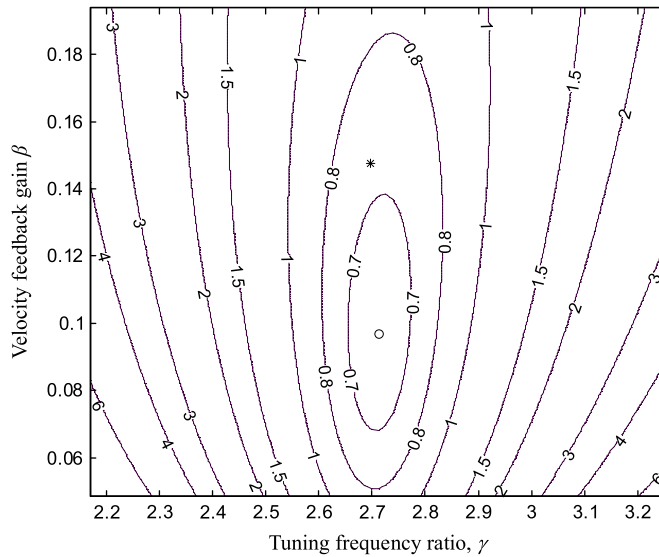


Fig. 10. Contour plot of $E[x^2]_{HVA}/\omega_n S_0$ of Case A with different tuning ratio γ and velocity feedback gain β . $\mu = 0.2$, $\alpha = 0.7$. \circ — γ_{opt} and β_{opt} proposed by the present method (Eqs. (16) and (17)); * — γ_{opt} and β_{opt} proposed by Chatterjee [15] (Eqs. (29a) and (29b)).

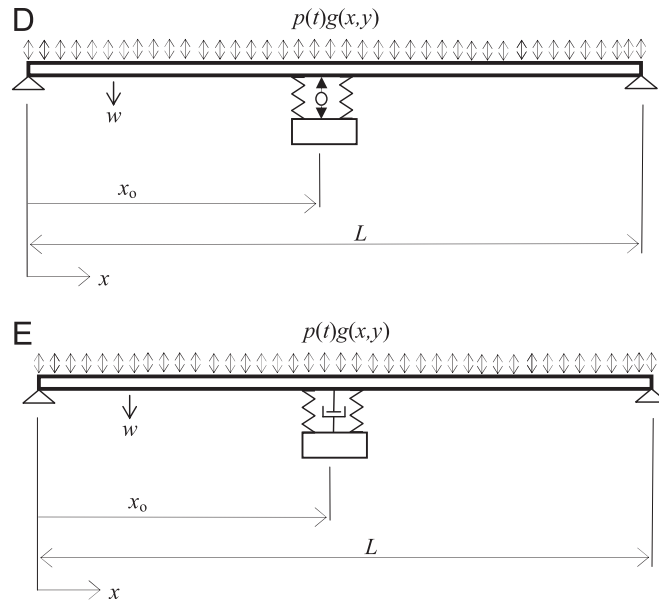


Fig. 11. Schematics of a simply supported beam with the proposed HVA or a PVA excited by a uniform disturbed force. Case D: Vibration control of a simply-supported beam using the proposed hybrid vibration absorber (HVA). Case E: Vibration control of a simply-supported beam using a passive vibration absorber (PVA).

$$\beta_{opt[15]} = \sqrt{\frac{3\mu^2(\mu + 2\alpha)}{4(2 + \mu)}} \quad \text{where } \alpha < 1 \tag{29b}$$

To illustrate the variations of $E[x^2]_{HVA_opt}/\omega_n S_0$ with the tuning ratio γ and velocity feedback gain β , $E[x^2]_{HVA_opt}/\omega_n S_0$ is calculated according to Eq. (18) with $\mu = 0.2$ and $\alpha = 0.7$ and then plotted in Fig. 10. It can be observed in Fig. 10 that $E[x^2]_{HVA_opt}/\omega_n S_0$ becomes a minimum at $\beta_{opt} = 0.097$ and $\gamma_{opt} = 2.712$. The optimum tuning frequency γ and velocity feedback gain β are calculated according to Eqs. (29a) and (29b) as $\beta_{opt[15]} = 0.148$ and $\gamma_{opt[15]} = 2.697$ and they are marked by the asterisk in Fig. 10 for comparison to the proposed set of optimum γ and β which marked by the small circle. The mean square motion of the primary mass M , $E[x^2]_{HVA_opt}/\omega_n S_0$ with HVA tuned according to the proposed γ and β is found to be 9.8% lower than that using the optimum parameters $\beta_{opt[15]}$ and $\gamma_{opt[15]}$ proposed by Chatterjee [15].

3. HVA applied to suppressing vibration of flexible beam structure

3.1. Mathematical model

In this section, Case A is compared with Case C but the primary vibrating system is changed from a SDOF system to a continuous beam structure as illustrated by Case D and Case E in Fig. 11. The mean square displacement response of a simply supported beam excited by an uniform distributed force as shown in Fig. 11 is to be suppressed using either the proposed HVA (Case D) or a PVA (Case E) attached on the beam at $x=x_0$.

Assuming the beam behaves like an Euler–Bernoulli beam, the equation of motion of the beam may be written as [16]

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = p(t)g(x) + F_h(t)\delta(x-x_0) \tag{30}$$

where $p(t)g(x)$ is the externally applied forcing function and F_h is the force acting by the HVA onto the beam. The eigenfunction and the eigenvalues of the beam may be written as

$$\varphi_i(x) = \frac{2}{L} \sin(\beta_i x) \quad \text{where} \quad \beta_i = \frac{i\pi}{L}, \quad i \in N \tag{31}$$

The spatial parts of the forcing functions and the Dirac delta function may be expanded in Fourier series and written as

$$g(x) = \sum_{i=1}^{\infty} a_i \varphi_i(x), \quad \text{and} \tag{32a}$$

$$\delta(x-x_0) = \sum_{i=1}^{\infty} b_i \varphi_i(x) \tag{32b}$$

where the Fourier coefficients a_i and b_i are, respectively

$$a_i = \frac{2}{n\pi L}, \quad i = 2n-1 \quad n \in N$$

$$\text{else } a_i = 0, \quad \text{and} \tag{33a}$$

$$b_i = \frac{2}{L} \sin\left(\frac{i\pi x_0}{L}\right) \tag{33b}$$

Assuming the HVA (Case D) is tuned to suppress the first vibration mode of the beam. The frequency response of the beam can be derived using a similar approach of Ref. [19] and written as

$$\frac{W_{HVA}(x, \lambda)}{P(\lambda)} = \frac{1}{\rho A \omega_n^2} \sum_{p=1}^{\infty} \frac{a_p - \frac{\mu L b_p \sum_{q=1}^{\infty} \frac{a_q \varphi_q(x_0)}{\gamma_q^2 - \lambda^2}}{\gamma_p^2 - \lambda^2 - \frac{2\alpha}{\gamma_p} + \mu L \sum_{r=1}^{\infty} \frac{b_r \varphi_r(x_0)}{\gamma_r^2 - \lambda^2}} \varphi_p(x) \tag{34}$$

where

$$\omega_n = \sqrt{\frac{EI\beta_1^4}{\rho A}}, \quad \omega_r = \sqrt{\frac{EI\beta_r^4}{\rho A}}, \quad \gamma_r = \frac{\omega_r}{\omega_n}, \quad \mu = \frac{m}{\rho AL}, \quad \varepsilon = \mu \varphi_1^2(x_0), \quad \alpha = \frac{a \varphi_1^2(x_0)}{2EIL\beta_1^4}$$

and

$$\beta = \frac{b \omega_n \varphi_1^2(x_0)}{2EIL\beta_1^4}$$

Using Eq. (34), the spatial average mean square vibration amplitude of the beam may be written as

$$\frac{\int_0^L \left| \frac{W_{HVA}(x, \lambda)}{P(\lambda)} \right|^2 dx}{L} = \left(\frac{1}{\rho A \omega_n^2} \right)^2 \sum_{p=1}^{\infty} \left| \frac{a_p - \frac{\mu L b_p \sum_{q=1}^{\infty} \frac{a_q \varphi_q(x_0)}{\gamma_q^2 - \lambda^2}}{\gamma_p^2 - \lambda^2 - \frac{2\alpha}{\gamma_p} + \mu L \sum_{r=1}^{\infty} \frac{b_r \varphi_r(x_0)}{\gamma_r^2 - \lambda^2}} \right|^2 \tag{35}$$

The optimum tuning frequency and velocity feedback gain can be derived using a similar approach of Ref. [19] and written similar to Eqs. (16) and (17), respectively as

$$\gamma_{HVA-b} = \sqrt{\frac{4\alpha^2 + 4\alpha\varepsilon + \mu^2}{\varepsilon(\varepsilon + \alpha\varepsilon + 2\alpha)}}, \quad \text{and} \tag{36a}$$

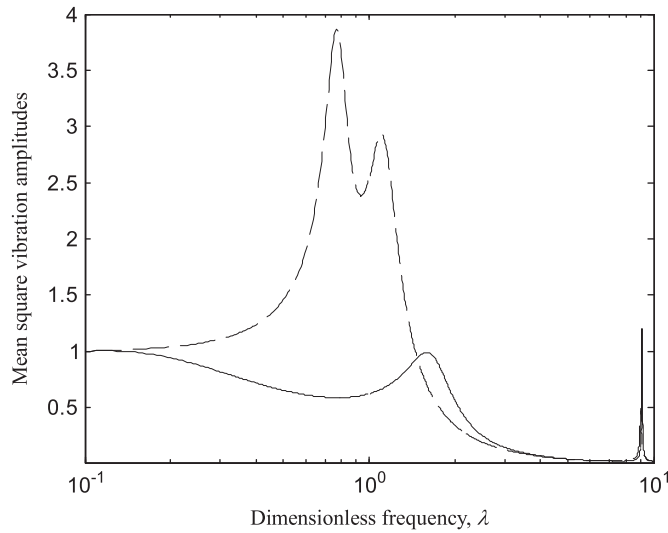


Fig. 12. Mean square vibration amplitude response of the whole beam with $\mu = 0.05$ and $\alpha = 1$. $-(1/L) \int_0^L W_{HVA}(x, \lambda)/P(\lambda)^2 dx$ of Case D; $-(1/L) \int_0^L W_{PVA}(x, \lambda)/P(\lambda)^2 dx$ of Case E in Fig. 11.

$$\beta_{HVA,b} = \frac{\varepsilon(2\alpha + \varepsilon)\sqrt{\varepsilon + 2\alpha - \alpha^2}}{2(\varepsilon + \alpha\varepsilon + 2\alpha)} \tag{36b}$$

The spatial average mean square vibration amplitude of the beam with the PVA (Case E) may be written as [16]

$$\frac{\int_0^L \left| \frac{W_{PVA}(x, \lambda)}{P(\lambda)} \right|^2 dx}{L} = \left(\frac{1}{\rho A \omega_n^2} \right)^2 \sum_{p=1}^{\infty} \left| \frac{a_p - \frac{\mu L b_p \sum_{q=1}^{\infty} \frac{a_q \phi_q(\alpha_0)}{\gamma_q^2 - \lambda^2}}{\frac{\gamma_p^2 - \lambda^2 + 2i\zeta_p \lambda}{\lambda^2 (\gamma_p^2 + 2i\zeta_p \lambda)} + \mu L \sum_{r=1}^{\infty} \frac{b_r \phi_r(\alpha_0)}{\gamma_r^2 - \lambda^2}}{\gamma_p^2 - \lambda^2} \right|^2 \tag{37}$$

The optimum tuning frequency and damping of the PVA may be written, respectively as [19]

$$\gamma_{PVA,b} = \sqrt{\frac{2 + \varepsilon}{2(1 + \varepsilon)^2}} \tag{38a}$$

and

$$\zeta_{a_PVA,b} = \frac{1}{2} \sqrt{\frac{\varepsilon(3\varepsilon + 4)}{2(2 + \varepsilon)(1 + \varepsilon)}} \tag{38b}$$

3.2. Numerical example

Assuming the length of the beam is $L = 1$ m and the absorber is attached at $x = x_0 = 0.5$ m of the beam. The dimensions of the cross section area are 0.025 m \times 0.025 m. The material of the beam is aluminium with density $\rho = 2710$ kg/m³ and modulus $E = 6.9$ GPa. The mass ratio μ is 0.05. The feedback gain of the displacement signal is assumed to be $\alpha = 1$. The optimum tuning ratio $\gamma_{HVA,b}$ and the optimum feedback gain of velocity $\beta_{HVA,b}$ are calculated according to Eqs. (36a) and (36b) to be 3.1754 and 0.1004, respectively. Similarly, the optimum tuning ratio $\gamma_{PVA,b}$ and the optimum damping $\zeta_{a_PVA,b}$ are calculated according to Eqs. (38a) and (38b) to be 0.8740 and 0.2087, respectively. The mean square vibration amplitude response of the whole beam, $(1/L) \int_0^L |W_{HVA}(x, \lambda)/P(\lambda)|^2 dx$ and $(1/L) \int_0^L |W_{PVA}(x, \lambda)/P(\lambda)|^2 dx$ with $\mu = 0.05$ and $\alpha = 1$ are calculated according to Eqs. (35) and (37), respectively and the results are plotted in Fig. 12 for comparison. As shown in Fig. 12, The maximum value of $(1/L) \int_0^L |W_{HVA}(x, \lambda)/P(\lambda)|^2 dx$ of Case D is about 4 times less than the maximum of $(1/L) \int_0^L |W_{PVA}(x, \lambda)/P(\lambda)|^2 dx$ of Case E. Moreover, the area under the curve of Case D is found to be about 40% less than that of Case E in Fig. 12. These results show that using the proposed optimized HVA can suppress the spatial average mean square vibration amplitude of the whole beam at resonance as well as over the entire frequency range of interest.

4. Conclusion

A hybrid vibration absorber (HVA) which is recently proposed and optimized by Chatterjee [15] for suppressing resonant vibration of a single degree-of-freedom (SDOF) system is re-optimized for suppressing wide frequency band vibration of SDOF system under stationary random force excitation. H_2 optimization is applied to the proposed absorber such that the mean square vibration amplitude of the vibrating structure over the entire frequency range is minimized, i.e., the total area under the power spectrum response curve is minimized. A methodology in the optimization to include damping in the primary vibrating structure is proposed and the effect of this damping on the optimization process is reported. It is found that optimum values of the system parameters do not exist if the damping in the primary system is relatively high.

The active element of the proposed HVA helps further reduce the vibration of the controlled structure and it can provide significant vibration absorption performance even at a low mass ratio between the absorber and the primary structure. Both the passive and active elements are optimized together for the minimization of the mean square vibration amplitude of the primary system. It has been proved analytically in Sections 2.3 and 2.4 that the minimum mean square vibration amplitude in Case C, $E[x^2]_{PVA_opt}$, tends to infinity as the mass ratio μ tends to zero while that in Case A, $E[x^2]_{HVA_opt_min}/\omega_n S_0$ tends to 0.5 as the mass ratio μ tends to zero. These results show that the proposed HVA but not the PVA can produce significant vibration suppression of the SDOF vibrating system if the mass ratio is low. The proposed HVA has been tested numerically on both SDOF and continuous vibrating structures and it provides much better suppression results than the passive vibration absorber.

Appendix A

According to Eq. (16),

$$\gamma_{opt} = \sqrt{\frac{4\alpha^2 + 4\alpha\mu + \mu^2}{\mu(\mu + \alpha\mu + 2\alpha)}} \tag{A1}$$

Substituting γ_{opt} into the inequality $\mu\gamma^2 - 2\alpha > 0$, we may write

$$\mu \left(\frac{4\alpha^2 + 4\alpha\mu + \mu^2}{\mu(\mu + \alpha\mu + 2\alpha)} \right) - 2\alpha > 0 \tag{A2}$$

$$\Rightarrow \frac{4\alpha^2 + 4\alpha\mu + \mu^2 - 2\alpha(\mu + \alpha\mu + 2\alpha)}{\mu + \alpha\mu + 2\alpha} > 0 \tag{A3}$$

$$\Rightarrow \frac{4\alpha^2 + 4\alpha\mu + \mu^2 - 2\alpha\mu - 2\alpha^2\mu - 4\alpha^2}{\mu + \alpha\mu + 2\alpha} > 0 \tag{A4}$$

$$\Rightarrow \frac{2\alpha\mu + \mu^2 - 2\alpha^2\mu}{\mu + \alpha\mu + 2\alpha} > 0 \tag{A5}$$

$$\Rightarrow 2\alpha\mu + \mu^2 - 2\alpha^2\mu > 0 \quad (\because \mu + \alpha\mu + 2\alpha > 0) \tag{A6}$$

$$\Rightarrow \mu + 2\alpha - 2\alpha^2 > 0 \tag{A7}$$

$$\Rightarrow \frac{1 + \sqrt{1 + 2\mu}}{2} > \alpha > \frac{1 - \sqrt{1 + 2\mu}}{2} \tag{A8}$$

$$\Rightarrow \frac{1 + \sqrt{1 + 2\mu}}{2} > \alpha > 0 \quad (\because \alpha > 0) \tag{A9}$$

$$\Rightarrow \frac{1 + \sqrt{1 + 2\mu}}{2} > \alpha \tag{A10}$$

Appendix B. B1: Searching for optimum α and γ

According to Eq. (13),

$$E[x^2] = \frac{\omega_n S_0}{4\beta\mu^2\gamma^2} \left(\mu^2(1 + \mu)\gamma^4 - 2\mu(\mu + \alpha\mu + 2\alpha)\gamma^2 + 4\alpha^2 + 4\beta^2 + 4\alpha\mu + \mu^2 \right) \tag{B1}$$

Consider the case if $\frac{\partial}{\partial \gamma} E[x^2] = \frac{\partial}{\partial \alpha} E[x^2] = 0$, the HVA will have an optimum set of tuning frequency γ and feedback gain α in term of β and μ . We may therefore write

$$\frac{\partial E[x^2]}{\partial \alpha} = \frac{\omega_n S_0}{4\beta\mu^2\gamma^2} (-2\mu\gamma^2(2 + \mu) + 8\alpha + 4\mu) = 0 \tag{B2a}$$

$$\frac{\partial E[x^2]}{\partial \gamma} = \frac{\omega_n S_0 \left(2\mu^2(1+\mu)\gamma^4 - 2(4\alpha^2 + 4\beta^2 + 4\alpha\mu + \mu^2) \right)}{4\beta\mu^2\gamma^3} = 0 \quad (\text{B2b})$$

Using Eq. (B2a), the displacement frequency feedback gain may be written as

$$\alpha = \frac{\mu\gamma^2(2+\mu) - 2\mu}{4} \quad (\text{B3})$$

Substituting Eq. (B3) into Eq. (B2b), we may write

$$\frac{1}{2}\mu^4\gamma^4 + 8\beta^2 = 0 \quad (\text{B4})$$

Eq. (B4) shows that there are no positive real root of mass ratio μ , γ and feedback gain β and hence no optimum values of μ , γ and feedback gain β exists in this case.

B2: Searching for optimum α and β

Consider the case if $\frac{\partial E[x^2]}{\partial \alpha} = \frac{\partial E[x^2]}{\partial \beta} = 0$, the HVA will have an optimum set of feedback gains α and β in term of γ and μ . We may therefore write

$$\frac{\partial E[x^2]}{\partial \alpha} = \frac{\omega_n S_0}{4\beta\mu^2\gamma^2} (-2\mu\gamma^2(2+\mu) + 8\alpha + 4\mu) = 0 \quad (\text{B5a})$$

$$\frac{\partial E[x^2]}{\partial \beta} = \frac{\omega_n S_0 \left(-\mu^2(1+\mu)\gamma^4 + 2\mu(\mu + \alpha\mu + 2\alpha)\gamma^2 + 4\beta^2 - 4\alpha^2 - 4\alpha\mu - \mu^2 \right)}{4\beta^2\mu^2\gamma^2} = 0 \quad (\text{B5b})$$

Using Eq. (B5a), the displacement frequency feedback gain may be written as

$$\alpha = \frac{\mu\gamma^2(2+\mu) - 2\mu}{4} \quad (\text{B6})$$

Substituting Eq. (B6) into Eq. (B5b), we may write

$$\beta = \frac{\mu\gamma\sqrt{4\mu - \mu^2\gamma^2}}{4} \quad (\text{B7})$$

Substituting Eqs. (B6) and (B7) into Eq. (B1), the mean square vibration amplitude of the primary mass M with the optimized HVA may be written as

$$\frac{E[x^2]_{\text{HVA_opt}}}{\omega_n S_0} = \frac{1}{2} \sqrt{\frac{4}{\mu\gamma^2} - 1} \quad (\text{B8})$$

As discussed in Section 2.2, a stable dynamic system of the HVA requires

$$\mu\gamma^2 - 2\alpha > 0 \quad (\text{B9})$$

Substituting Eq. (B6) into (B9), we may write

$$\mu\gamma^2 < 2 \quad (\text{B10})$$

Using Eqs. (B8) and (B10), we may write

$$\frac{E[x^2]_{\text{HVA_opt}}}{\omega_n S_0} > \frac{1}{2} \quad (\text{B11})$$

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