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The transmissibility of vibration isolators with cubic nonlinear damping under both force and base excitations

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ABSTRACT

The influence of a nonlinear damping which is a function of both the velocity and displacement is investigated for a single degree of freedom (sdof) isolator. The analytical relationships between the force or displacement transmissibility and the nonlinear damping coefficient are developed in the frequency domain for the isolator systems subjected to both force and base excitation. It is theoretically shown that the cubic order nonlinear damping can produce much better isolation performance, i.e., obvious peak suppression at resonant frequency and very close transmissibility to system linear dampingover non-resonant frequencies under both force and base displacement excitations. Moreover, when only the pure cubic order nonlinear damping is used without linear damping, the force or displacement transmissibility is even better. The results are compared with the other nonlinear damping terms previously studied in the literature. Numerical studies are presented to illustrate the results.

1. Introduction

Inserting the vibration isolator between the vibration source and the vibration receiver is one of the fundamental ways to reduce the unwanted vibrations and to protect the equipments from disturbance. The basic concept of the vibration isolator is that, when the frequency of the excitation Ω is larger than $\sqrt{2}\omega_0$, where ω_0 is the undamped natural frequency of the isolator, the transmitted force, F_t (or the transmitted displacement, X_t) reaches a value less than the excitation force, F_i (or the excitation displacement, X_i) [1]. The ratio F_t/F_i and X_t/X_i are denoted as force transmissibility and displacement transmissibility respectively. There is a well-known dilemma associated with linear viscous damping systems. That is when the linear damping coefficient is increased, the force transmissibility under both base excitation and force excitation when excitation frequency $\Omega < \sqrt{2}\omega_0$ is further reduced, but the performance when excitation frequency $\Omega > \sqrt{2}\omega_0$ is contrarily deteriorated [2,3]. In order to overcome this dilemma, isolators with nonlinear stiffness and nonlinear damping have been studied by many authors in exploring the potential nonlinear benefits in vibration control [1,4–9]. Another reason of the study on nonlinear stiffness and nonlinear damping is that almost all the isolators in practical vibration systems are inherently nonlinear [10,11]. Therefore, it is important to take into account the existence of the nonlinearity in order to reach a better isolation performance.

Ravindra and Mallik [4] analyzed the vibration isolators having nonlinearity in both stiffness and damping terms under both force and base excitations. The transmissibility was obtained by the method of harmonic balance, and the effects of various types of damping to the transmissibility were also studied. The jump phenomenon was observed in the

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transmissibility curve when nonlinear stiffness was introduced in the isolator. Based on nonlinear output frequency response of the Volterra-class nonlinear systems [5,12,13], nonlinear dampings (which usually are pure functions of velocity) under force excitation are studied in [6,14,15] for vibration isolators. It was shown that the cubic nonlinear viscous damping can produce an ideal vibration isolation that only the force transmissibility over the resonant region is modified but it remains almost unaffected over the non-resonant regions. Milovanovic et al. [16] studied the vibration isolators with linear and cubic nonlinearities in stiffness and damping terms under based excitation. The influence of the nonlinear parameters on the displacement transmissibility was studied, and they presented that the absolute displacement transmissibility of the isolator with cubic damping tends to unity as $\Omega \to \infty$, which corresponds to a rigidly connected system.

In the present study, a cubic nonlinear damping (i.e., $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$), which is a function of both the displacement and velocity, is investigated in vibration isolators. Although the nonlinear damping which is usually a pure function of velocity such as $(\mathrm{d}(\cdot)/\mathrm{d}t)^3$ has been studied in the literatureas mentioned before [6,14,15], the nonlinear damping relating to both the displacement and velocity are not well developed and understood [17]. By using the concept of output frequency response function, the analytical relationship between the force and absolute displacement transmissibility and the nonlinear damping coefficient of the vibration isolator are derived. It is theoretically shown that, the introduction of $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ can produce much better vibration isolation performance for the isolator under both force excitation and base displacement excitation. System equivalent damping can be very high around the resonant frequency but would be similar to system linear dampingover non-resonant frequencies. Therefore, the transmissibility is significantly suppressed around the resonant frequency but remain almost the same as when only the linear damping is used. Moreover, when only the pure cubic order nonlinear damping term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ is used in the system with the linear damping coefficient $\xi_1=0$, both the force and displacement transmissibilities are even better. This may provide an ideal damping characteristic in practical applications. Simulation results are provided to illustrate the results.

2. Nonlinear vibration isolators and transmissibility functions

In this section, nonlinear isolators subjected to force excitation and base excitation, are investigated. The nonlinear isolator is modeled as a parallel combination of a linear spring with stiffness k and a nonlinear damper. The nonlinear damping force is given as

$$F_{nd} = c\frac{\mathrm{d}(\cdot)}{\mathrm{d}t} + c_2(\cdot)^2\frac{\mathrm{d}(\cdot)}{\mathrm{d}t} + c_4\left[\frac{\mathrm{d}(\cdot)}{\mathrm{d}t}\right]^3 \tag{1}$$

where c is the linear damping coefficient, and c_2, c_4 are the cubic order nonlinear damping characteristic parameters.

In Fig. 1, the force excitation

$$F(t) = \overline{A}\sin(\omega t) \tag{2}$$

is directly exerted on the mass M with amplitude $\overline{A}$ and frequency ω. $F_{\mathrm{out}}(t)$ is the force transmitted to the base, and $x_1(t)$ is the absolute displacement of the mass M.

In Fig. 2, the input base excitation is

$$u(t) = \overline{A}\sin(\omega t) \tag{3}$$

where $\overline{A}$ is the amplitude and ω is the excitation frequency.

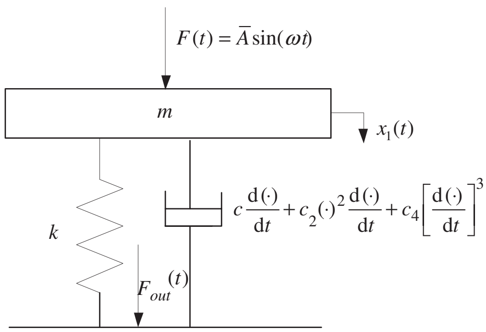

Fig. 1. Isolator subjected to force excitation.

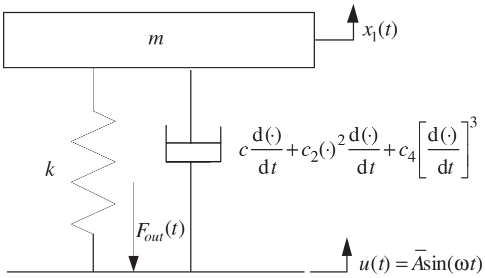

Fig. 2. Isolator subjected to base excitation.

The governing equations and the non-dimensional transmissibility expressions are derived separately in what follows.

2.1. The isolator subjected to force excitation and the force transmissibility

From Fig. 1, the governing equation of the isolator under force excitation can be given as

$$m\ddot{x}_1=-kx_1-c\dot{x}_1-c_2x_1^2\dot{x}_1-c_4\dot{x}_1^3+\bar{A}\sin(\omega t) \tag{4}$$

The force ratio $T_{\text{fr}}(t)$ is defined as

$$T_{\text{fr}}(t)=\frac{F_{\text{out}}}{\bar{A}}=\frac{kx_1}{\bar{A}}+\frac{c\dot{x}_1}{\bar{A}}+\frac{c_2x_1^2\dot{x}_1}{\bar{A}}+\frac{c_4\dot{x}_1^3}{\bar{A}} \tag{5}$$

By defining the following non-dimensional parameters:

$$\begin{aligned}
&\omega_0 t=\tau,\quad \omega_0=\sqrt{\tfrac{k}{m}},\quad \Omega=\tfrac{\omega}{\omega_0}\\
&z_1(\tau)=x_1(t)=x_1\left(\tfrac{\tau}{\omega_0}\right),\quad \dot{x}_1(t)=\omega_0\dot{z}_1(\tau),\quad \ddot{x}_1(t)=\omega_0^2\ddot{z}_1(\tau)\\
&y_1(\tau)=\tfrac{kz_1(\tau)}{\bar{A}},\quad y_2(\tau)=T_f(\tau),\\
&\xi_1=\tfrac{c}{\sqrt{km}},\quad \beta_2=\tfrac{c_2\bar{A}^2}{\sqrt{k^5m}},\quad \beta_4=\tfrac{c_4\bar{A}^2}{\sqrt{k^3m^3}}
\end{aligned} \tag{6}$$

the governing equation (4) and the force ratio (5) can be expressed as the following non-dimensional form:

$$\begin{cases}\ddot{y}_1+y_1+\xi_1\dot{y}_1+\beta_2y_1^2\dot{y}_1+\beta_4\dot{y}_1^3=\sin(\Omega\tau)\\ y_2=y_1+\xi_1\dot{y}_1+\beta_2y_1^2\dot{y}_1+\beta_4\dot{y}_1^3\end{cases} \tag{7}$$

Denote $T_f(\Omega)$ as the force transmissibility of the vibration isolator in terms of the normalized frequency Ω, it can be expressed as

$$T_f(\Omega)=\left|Y_2(j\Omega)\right| \tag{8}$$

where $Y_2(j\Omega)=Y_2(j\omega)\big|_{\omega=\Omega}$, the output spectrum of the second output of system (7).

2.2. The isolator subjected to base excitation, and the force and displacement transmissibility

From Fig. 2, the isolator model under base excitation can be written as

$$m\ddot{x}_1=k(u-x_1)+c(\dot{u}-\dot{x}_1)+c_2(u-x_1)^2(\dot{u}-\dot{x}_1)+c_4(\dot{u}-\dot{x}_1)^3 \tag{9}$$

The force ratio $T_{\text{fr}}(t)$ in this case is denoted by

$$T_{\text{fr}}(t)=\frac{-F_{\text{out}}}{k\bar{A}}=\frac{-1}{\bar{A}}(u-x_1)-\frac{c}{k\bar{A}}(\dot{u}-\dot{x}_1)-\frac{c_2}{k\bar{A}}(u-x_1)^2(\dot{u}-\dot{x}_1)-\frac{c_4}{k\bar{A}}(\dot{u}-\dot{x}_1)^3 \tag{10}$$

Denote the relative displacement x of the isolator as

$$x=x_1-u \tag{11}$$

Then Eqs. (9,10) can be rewritten as

$$\begin{cases}m\ddot{x}+kx+c\dot{x}+c_2x^2\dot{x}+c_4\dot{x}^3=m\bar{A}\omega^2\sin(\omega t)\\ T_{fr}=\frac{1}{\bar{A}}x+\frac{c}{k\bar{A}}\dot{x}+\frac{c_2}{k\bar{A}}x^2\dot{x}+\frac{c_4}{k\bar{A}}\dot{x}^3\end{cases} \tag{12}$$

Define the following non-dimensional parameters:

$$\begin{aligned} &\omega_0 t=\tau, \omega_0=\sqrt{\tfrac{k}{m}}, \quad \Omega=\tfrac{\omega}{\omega_0} \\ &z(\tau)=x(t)=x\left(\tfrac{\tau}{\omega_0}\right), \quad \dot{x}(t)=\omega_0\dot{z}(\tau), \quad \ddot{x}(t)=\omega_0^2\ddot{z}(\tau) \\ &y_1(\tau)=\tfrac{z(\tau)}{\overline{A}}, \quad y_2(\tau)=T_f(\tau), \\ &\xi_1=\tfrac{c}{\sqrt{km}}, \quad \beta_2=\tfrac{c_2\overline{A}^2}{\sqrt{km}}, \quad \beta_4=\tfrac{c_4\sqrt{k}\overline{A}^2}{\sqrt{m^3}} \end{aligned} \tag{13}$$

Then Eq. (12) can be rewritten as the following non-dimensional form:

$$\begin{cases} \ddot{y}_1+y_1+\xi_1\dot{y}_1+\beta_2 y_1^2\dot{y}_1+\beta_4\dot{y}_1^3=\Omega^2\sin(\Omega\tau) \\ y_2=y_1+\xi_1\dot{y}_1+\beta_2 y_1^2\dot{y}_1+\beta_4\dot{y}_1^3 \end{cases} \tag{14}$$

Similar to Section 2.1, the force transmissibility of the vibration isolator in terms of the normalized frequency Ω, $T_f(\Omega)$, can also be expressed as

$$T_f(\Omega)=\left|Y_2(j\Omega)\right| \tag{15}$$

where $Y_2(j\Omega)$ is the output spectrum $Y_2(j\omega)$ of the second output of system (14) evaluated at frequency $\omega=\Omega$.

While comparing the equations in (7) with the equations in (14), it can be seen that the base displacement excitation is equivalent to a force excitation when the strength of disturbing force is proportional to the square of exciting frequency, and the force transmissibility under the force excitation and that under the base displacement excitation have the same expression.

In the following, the absolute displacement transmissibility is derived. In this case, the governing equation is given in Eq. (9), and the displacement ratio is defined by

$$T_{dr}(t)=x=\frac{x_1}{\overline{A}} \tag{16}$$

Then Eq. (9) can be rewritten as

$$m\ddot{x}+kx+c\dot{x}-ku_1-c\dot{u}_1=c_2\overline{A}^2(u_1-x)^2(\dot{u}_1-\dot{x})+c_4\overline{A}^2(\dot{u}_1-\dot{x})^3 \tag{17}$$

where x is the displacement ratio and u_1 is given by

$$u_1=\frac{u}{\overline{A}}=\sin(\omega t) \tag{18}$$

Using the following non-dimensional variables:

$$\begin{aligned} &\omega_0 t=\tau, \quad \omega_0=\sqrt{\tfrac{k}{m}}, \quad \Omega=\tfrac{\omega}{\omega_0} \\ &y(\tau)=x(t)=x\left(\tfrac{\tau}{\omega_0}\right), \quad \dot{x}(t)=\omega_0\dot{y}(\tau), \quad \ddot{x}(t)=\omega_0^2\ddot{y}(\tau) \\ &u_2(\tau)=u_1(t)=u_1\left(\tfrac{\tau}{\omega_0}\right)=\sin(\Omega\tau), \quad \dot{u}_1(t)=\omega_0\dot{u}_2(\tau) \\ &\xi_1=\tfrac{c}{\sqrt{km}}, \quad \beta_2=\tfrac{c_2\overline{A}^2}{\sqrt{km}}, \quad \beta_4=\tfrac{c_4\sqrt{k}\overline{A}^2}{\sqrt{m^3}} \end{aligned} \tag{19}$$

Eq. (17) can be written into the following non-dimensional form:

$$\ddot{y}+y+\xi_1\dot{y}-u_2-\xi_1\dot{u}_2=\beta_2(u_2-y)^2(\dot{u}_2-\dot{y})+\beta_4(\dot{u}_2-\dot{y})^3 \tag{20}$$

Thus, the displacement transmissibility of the vibration isolator in terms of the normalized frequency Ω, $T_d(\Omega)$, can be expressed as

$$T_d(\Omega)=\left|X(j\Omega\omega_0)\right|=\left|Y(j\Omega)\right| \tag{21}$$

where $Y(j\Omega)$ is the output spectrum $Y(j\omega)$ of system (20) evaluated at frequency $\omega=\Omega$.

In the next section, an explicit and analytical relationship between the force or displacement transmissibility and the nonlinear damping coefficients β_2 and β_4 will be developed in the frequency domain for the nonlinear isolators.

3. The force and displacement transmissibility in the frequency domain

The nonlinear output frequency response concept is recently proposed [5,12,13] for the frequency domain study of the nonlinear Volterra systems, which represent a wide classes of nonlinear systems whose input and output can be expressed as the Volterra series around the equilibrium. One of the advantages of this concept is that it can give an explicit analytical relationship between the output frequency response and the parameters of the nonlinear systems which can be described by differential equation models. Therefore, the analytical relationships between the force transmissibility $T_f(\Omega)$or displacement transmissibility $T_d(\Omega)$ and the nonlinear damping coefficients β_2 and β_4 are established in this section.

3.1. The force transmissibility $T_f(\Omega)$

From systems (7) and (14), the base displacement excitation is basically equivalent to a force excitation, and the force ratio under base displacement excitation has the same form as that under force excitation. In this section, the force transmissibility in the frequency domain under both two excitation types will be developed.

According to [15,18,19], systems (7) and (14) can be regarded as a one-input-two-output system, the output spectra can be obtained as

$$Y_J(j\omega)=\sum_{n=1}^{N}\frac{1}{2^{n-1}}\sum\nolimits_{\omega_1+\cdots+\omega_n=\omega}H_n^J(j\omega_1,\ldots,j\omega_n)U(\omega_1)\cdots U(\omega_n),\quad J=1,2 \tag{22}$$

where $H_n^J(j\omega_1,\ldots,j\omega_n)$ is the nth order generalized frequency response function (GFRF) between the input and the Jth output of the system, N is the maximum order of nonlinearity in the Volterra series expansion of the system outputs. $U(\omega_i)$ is the input Fourier transform. For system (7),

$$U(\omega_i)=\begin{cases}-j & \text{when } \omega_i=\Omega,\quad i=1,\ldots,n\\ 0 & \text{otherwise}\\ j & \text{when } \omega_i=-\Omega,\quad i=1,\ldots,n\end{cases} \tag{23a}$$

For system (14),

$$U(\omega_i)=\begin{cases}-j\Omega^2 & \text{when } \omega_i=\Omega,\quad i=1,\ldots,n\\ 0 & \text{otherwise}\\ j\Omega^2 & \text{when } \omega_i=-\Omega,\quad i=1,\ldots,n\end{cases} \tag{23-b}$$

Eq. (22) involves the computation of the nth order GFRFs $H_n^J(j\omega_1,\ldots,j\omega_n)$. The explicit expression and its derivation for $H_n^2(j\omega_1,\ldots,j\omega_n)$ can be referred to Appendix A. With this result, the output spectrum $Y_2(j\omega)$ of the second output of system (7) and system (14) can be written as

$$Y_2(j\omega)=P_0(j\omega)+P_{10}(j\omega)\beta_2+P_{11}(j\omega)\beta_4+\cdots+\sum_{m=0}^{n}P_{nm}(j\omega)\beta_2^{n-m}\beta_4^m+\cdots+\sum_{m=0}^{[N/2]}P_{[N/2]m}(j\omega)\beta_2^{[N/2]-m}\beta_4^m \tag{24}$$

where

$$P_0(j\omega)=H_1^2(j\omega)U(j\omega) \tag{25}$$

$$P_{nm}(j\omega)=\frac{1}{2^{2n}}\frac{-(j\omega)^2}{L(j\omega)}\sum\nolimits_{\omega_1+\cdots+\omega_{2n+1}=\omega}\prod_{i=1}^{2n+1}H_1^1(j\omega_i)U(\omega_i)\sum_{z=1}^{N_{\bar{n}}}\frac{\prod_{k=1}^{(n\text{-}m)+3m}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j_k^n)}]}{\prod_{i=1}^{n\text{-}1}L[j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i(j_i^n)}]} \tag{26}$$

The definition of $L(j\omega)$ is given in Appendix A. Eq. (24) presents an analytical relationship between the second output spectrum and the nonlinear characteristic parameters β_2 and β_4. According to Eqs. (8) and (15), the force transmissibility can be given by

$$T_f(\Omega)=\left|Y_2(j\Omega)\right|=\left|P_0(j\Omega)+\sum_{n=1}^{[N/2]}\sum_{m=0}^{n}P_{nm}(j\Omega)\beta_2^{n-m}\beta_4^m\right| \tag{27}$$

It can be seen that Eqs. (25)–(27) are explicit functions of the input and first-order GFRF. Substituting Eq. (23a) into Eqs. (25) and (26), for system (7) (nonlinear isolator subjected to force excitation) it can be obtained that

$$P_0(j\Omega)=\frac{j(1+j\xi_1\Omega)}{L(j\Omega)} \tag{28}$$

$$P_{nm}(j\Omega)=\frac{1}{2^{2n}}\frac{j\Omega^2}{\left|L(j\Omega)\right|^{2n}[L(j\Omega)]^2}\sum\nolimits_{\omega_1+\cdots+\omega_{2n+1}=\Omega}\sum_{z=1}^{N_{\bar{n}}}\frac{\prod_{k=1}^{(n\text{-}m)+3m}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j_k^n)}]}{\prod_{i=1}^{n-1}L[j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i(j_i^n)}]} \tag{29}$$

where $\omega_i\in\{-\Omega,\Omega\},i=1,\ldots,2n+1$.

Similarly for system (14) (the nonlinear isolator subjected to base displacement excitation), it can be obtained that

$$P_0(j\Omega)=\frac{j\Omega^2(1+j\xi_1\Omega)}{L(j\Omega)} \tag{30}$$

$$P_{nm}(j\Omega)=\frac{1}{2^{2n}}\frac{j\Omega^{4n+4}}{|L(j\Omega)|^{2n}[L(j\Omega)]^2}\sum\nolimits_{\omega_1+\cdots+\omega_{2n+1}=\Omega}\sum_{z=1}^{N_{\bar{n}}}\frac{\prod_{k=1}^{(n\text{-}m)+3m}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\prod_{i=1}^{n\text{-}1}L[j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i(j^n_i)}]} \tag{31}$$

where $\omega_i\in\{-\Omega,\Omega\}, i=1,\ldots,2n+1$.

From Eq. (27), when there is no nonlinear damping coefficients, i.e., $\beta_2=0$ and $\beta_4=0$, the nonlinear sdof vibration isolatorbecomes a linear isolator. The force transmissibility will be easy to obtain.

For system (7), the linear isolator is under force excitation,

$$T_f(\Omega)=|Y_2(j\Omega)|=|P_0(j\Omega)|=\left|\frac{j(1+j\xi_1\Omega)}{L(j\Omega)}\right|=\sqrt{\frac{1+(\xi_1\Omega)^2}{\left(1-\Omega^2\right)^2+(\xi_1\Omega)^2}} \tag{32}$$

For system (14), the linear isolator is under base displacement excitation,

$$T_f(\Omega)=|Y_2(j\Omega)|=|P_0(j\Omega)|=\left|\frac{j\Omega^2(1+j\xi_1\Omega)}{L(j\Omega)}\right|=\Omega^2\sqrt{\frac{1+(\xi_1\Omega)^2}{\left(1-\Omega^2\right)^2+(\xi_1\Omega)^2}} \tag{33}$$

3.2. The displacement transmissibility $T_d(\Omega)$

In this case, the displacement transmissibility is given by Eq. (21).In order to obtain the output spectrum $Y(j\omega)$ of system (20), Eq. (20) is expanded as

$$\begin{aligned}&\ddot{y}+y+\xi_1\dot{y}-u_2-\xi_1\dot{u}_2-\beta_2u_2^2\dot{u}_2-\beta_4\dot{u}_2^3+2\beta_2yu_2\dot{u}_2+\beta_2\dot{y}u_2^2\\&+3\beta_4\dot{y}\dot{u}_2^2+\beta_2y^2\dot{u}_2-2\beta_2y\dot{y}u_2-3\beta_4\dot{y}^2\dot{u}_2-\beta_2y^2\dot{y}+\beta_4\dot{y}^3=0\end{aligned} \tag{34}$$

According to Ref. [20], the coefficients of the nonlinear differential Eq. (34) can be expressed in the general form as

$$\begin{aligned}&C_{1,0}(2)=1, C_{1,0}(1)=\xi_1, C_{1,0}(0)=1, C_{0,1}(1)=-\xi_1, C_{0,1}(0)=-1\\&C_{0,3}(0,0,1)=-\beta_2, C_{0,3}(1,1,1)=-\beta_4\\&C_{1,2}(0,0,1)=2\beta_2, C_{1,2}(1,0,0)=\beta_2, C_{1,2}(1,1,1)=3\beta_4 \ \textit{else}\ C_{p,q}(\cdot)=0\\&C_{2,1}(0,0,1)=\beta_2, C_{2,1}(0,1,0)=-2\beta_2, C_{2,1}(1,1,1)=-3\beta_4\\&C_{3,0}(0,0,1)=-\beta_2, C_{3,0}(1,1,1)=\beta_4,\end{aligned} \tag{35}$$

In order to obtain the output spectrum $Y(j\omega)$ of system (34), the following propositions are given.

Proposition 1. *The (2n+1)th GFRF $H_{2n+1}(j\omega_1,\ldots,j\omega_{2n+1})$ of system (34) when there exists only one nonlinear term with nonlinear coefficient $C_{p,q}(l_1,l_2,l_3)$, where, $p+q=3, p,q=0,\cdots,3, l_1,l_2,l_3=0,1$, under the assumption that there is m terms of 1, i.e. (3-m) terns of 0, in l_1,l_2,l_3, can be* determined *as*

$$H^1_{2n+1}(j\omega_1,\ldots,j\omega_{2n+1})=\frac{C^n_{p,q}(l_1,l_2,l_3)}{L(j\omega_1+\cdots+j\omega_{2n+1})}\sum_{z=1}^{N_{n,p}}\prod_{i=1}^{(p\text{-}1)n+1}H^1_1(j\omega_{j_i})\frac{\prod_{k=1}^{mn}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\prod_{i=1}^{n\text{-}1}L[j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i(j^n_i)}]} \tag{36}$$

Proof. See Appendix B □.

When only one nonlinear term is introduced in Eq. (34), Proposition 1 states that the high order GFRF can be represented as a straightforward function of the nonlinear coefficient and the linear transfer function. So when the linear transfer function and the nonlinear coefficient introduced are known, the high order GFRF can be obtained directly according to Eq. (36). Proposition 1 also shows how the output nonlinear degree p and the order of the derivative of both input and output affect the high order GFRF.

Proposition 2. *The (2n+1)th GFRF $H_{2n+1}(j\omega_1,\ldots,j\omega_{2n+1})$ of system (34) when there exists $_L$ nonlinear terms with nonlinear coefficients $C_{p_i,q_i}(l_1,l_2,l_3)$, where $p_i+q_i=3, p_i,q_i=0,\ldots,3, i=1,\ldots,n_l$, $l_1,l_2,l_3=0,1$, under the assumption that there is m_i terms of 1, i.e. (3-m_i) terms of 0, in l_1,l_2,l_3 for $C_{p_i,q_i}(l_1,l_2,l_3)$, can be* determined *as*:

$$\begin{aligned}H^1_{2n+1}(j\omega_1,\cdots,j\omega_{2n+1})&=\frac{\sum\limits_{j=1}^{n_L}C_{p_j,q_j}}{L(j\omega_1+\cdots+j\omega_{2n+1})}\sum\nolimits^{1}_{n_{\sum_{i=1}^{n_L}}\, n_{ji}=n\text{-}1}, L, n_{jn_L}\\&=0,^{n-1}\left(\prod_{i=1}^{n_L}C^{n_{ji}}_{p_i,q_i}(l_{i1},l_{i2},l_{i3})\sum_{z=1}^{N_{n,p_j}}\prod_{k=1}^{(p_j-1)+\sum_{i=1}^{n_L}(p_i-1)n_{ji}+1}H^1_1(j\omega_{k_i})\frac{\prod_{k=1}^{\sum_{i=1}^{n_L}n_{jmi}m_i}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\prod_{i=1}^{n-1}L[j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i(j^n_i)}]}\right)\end{aligned} \tag{37}$$

where $n_{ji}=n_{jmi}$, $i\neq j$, $n_{jj}+1=n_{jmj}$, $\sum_{i=1}^{n_L}n_{ji}=n-1$, $\sum_{i=1}^{n_L}n_{jmi}=n$.

Proof. See Appendix C □.

Proposition 2 considers the case that more than one nonlinear term exist in Eq. (34), which is an extension of Proposition 1. It implies how the nonlinear coefficient $C_{p_i q_i}$, the output nonlinear degrees p_i and the order of the derivate of both input and output l_{i1}, l_{i2}, l_{i3} influence the high order GFRF analytically.

System (20) or (34) can be seen as a single-input-single-output system, so the output spectrum can be obtained via Eq. (22) with $J=1$ using the input stated in Eq. (19) or its Fourier transformin Eq. (23a). The $(2n+1)$th order GFRF can be obtained from Proposition 2. Therefore the output spectrum can be written as

$$Y(j\omega) = P_0(j\omega) + \sum_{n=1}^{[N/2]}\left(\sum_{n_{jm_1},\dots,n_{jmn_L}=0,\ \sum_{i=1}^{n_L} n_{jmi}=n}^{n}\left(\prod_{i=1}^{n_L} C_{p_i,q_i}^{n_{jmi}}(l_{i1},l_{i2},l_{i3}) P_{nk}(j\omega)\right)\right) \tag{38}$$

$$P_0(j\omega) = \frac{1+j\omega\xi_1}{L(j\omega)} U[j\omega] \tag{39}$$

$$P_{nk}(j\omega) = \frac{1}{2^{2n}} \frac{\prod_{i=1}^{2n+1} U(j\omega)}{L(j\omega)} \sum\nolimits_{\omega_1+\cdots+\omega_{2n+1}=\omega} \sum_{z=1}^{N_{n,p_j}} \prod_{k=1}^{\sum_{i=1}^{n_L}(p_i-1)n_{jmi}+1} H_1^1(j\omega_{k_i}) \frac{\prod_{k=1}^{\sum_{i=1}^{n_L} n_{jmi} m_i}[j\omega_{l_k(1)}^z+\cdots+j\omega_{l_k(j_k^n)}^z]}{\prod_{i=1}^{n-1} L[j\omega_{l_i(1)}^z+\cdots+j\omega_{l_i(j_i^n)}^z]} \tag{40}$$

According to Eq. (21), the displacement transmissibility can be obtained by substituting Eq. (23a) into Eqs. (39) and (40):

$$T_d(\Omega) = |Y(j\Omega)| = \left|P_0(j\Omega) + \sum_{n=1}^{[N/2]}\left(\sum_{n_{jm_1},\dots,n_{jmn_L}=0,\ \sum_{i=1}^{n_L} n_{jmi}=n}^{n}\left(\prod_{i=1}^{n_L} C_{p_i,q_i}^{n_{jmi}}(l_{i1},l_{i2},l_{i3}) P_{nk}(j\Omega)\right)\right)\right| \tag{41}$$

where

$$P_0(j\Omega) = \frac{(-j)(1+j\Omega\xi_1)}{L(j\Omega)} \tag{42}$$

$$P_{nk}(j\Omega) = \tfrac{1}{2^{2n}} \tfrac{(-j)}{L(j\Omega)} \sum\nolimits_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{n,p_j}} \prod_{k=1}^{\sum_{i=1}^{n_L}(p_i-1)n_{jmi}+1} H_1^1(j\omega_{k_i}) \frac{\prod_{k=1}^{\sum_{i=1}^{n_L} n_{jmi} m_i}[j\omega_{l_k(1)}^z+\cdots+j\omega_{l_k(j_k^n)}^z]}{\prod_{i=1}^{n-1} L[j\omega_{l_i(1)}^z+\cdots+j\omega_{l_i(j_i^n)}^z]} \omega_{k_i}, \omega_{l_k(i)}^z, \omega_{l_i(i)}^z \in \{\Omega, -\Omega\} \tag{43}$$

4. The effects of nonlinear damping coefficients on vibration isolation

In Section 3, the force transmissibility under both force and base excitations and the absolute displacement transmissibility under base displacement excitation are derived. All of them have an explicit analytical relationship with the nonlinear damping coefficients introduced. The following results can be obtained.

Proposition 3. *For force excitation, the nonlinear* damping *$(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ can produce the following performance in force transmissibility*:

(I) *When $\Omega \gg 1$ or $\Omega \ll 1$,*

$$T(\Omega) \approx |P_0(j\Omega)| = \sqrt{\frac{1+(\xi_1\Omega)^2}{(1-\Omega^2)^2+(\xi_1\Omega)^2}} \tag{44}$$

(II) *When $\Omega \approx 1$, there exists a β such that the force transmissibility can be expressed in an alternating series with respect to the nonlinear coefficient β_2 if $0<\beta_2<\beta$. The force transmissibility can therefore be suppressed by exploiting the properties of alternating series.*

Proof. See Appendix D □.

Proposition 3 indicates that the nonlinear damping term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ has almost no effect on the force transmissibility over the non-resonant frequency regions where the frequency is much lower or much higher than the resonant frequency, while the force transmissibility is obviously suppressed at the resonant frequency due to the introduction of the nonlinear damping term under force excitation.

Proposition 4. *When the isolator is under base* displacement *excitation, the force transmissibility with the nonlinear damping $(d(\cdot)/dt)^3$ is* dramatically deteriorated *thigh frequency, while the nonlinear* damping *$(\cdot)^2(d(\cdot)/dt)$ can make the force* *transmissibility very close to the low-damping linear referenced case at the same frequency. The introduction of any one of these two cubic degree nonlinear terms can both make the force transmissibility very close to the low-damping linear referenced case at low frequency*:

(I) *When $\Omega \ll 1$, both $(\cdot)^2(d(\cdot)/dt)$ and $(d(\cdot)/dt)^3$ can make*

$$T(\Omega) \approx |P_0(j\Omega)| = \Omega^2 \sqrt{\frac{1+(\xi_1\Omega)^2}{(1-\Omega^2)^2+(\xi_1\Omega)^2}} \tag{45}$$

(II) *When $\Omega \gg 1$, only $(\cdot)^2(d(\cdot)/dt)$ term can make*

$$T(\Omega) \approx |P_0(j\Omega)| = \Omega^2 \sqrt{\frac{1+(\xi_1\Omega)^2}{(1-\Omega^2)^2+(\xi_1\Omega)^2}} \tag{46}$$

While the nonlinear term $(d(\cdot)/dt)^3$ will make

$$T(\Omega) \gg |P_0(j\Omega)| = \Omega^2 \sqrt{\frac{1+(\xi_1\Omega)^2}{\left(1-\Omega^2\right)^2+(\xi_1\Omega)^2}} \tag{47}$$

(III) *When $\Omega \approx 1$, there exists a β such that the force transmissibility can be expressed in an alternating series with respect to the nonlinear coefficient $\beta_2(\beta_4)$ if $0<\beta_2<\beta(0<\beta_4<\beta)$. The force transmissibility can therefore be suppressed by exploiting the properties of alternating series.*

Proof. See Appendix E □.

Proposition 4 shows that both the two nonlinear damping terms can significantly reduce the force transmissibility over the resonant frequency and remain the force transmissibility almost unaffected at low frequency. However, at high frequency the nonlinear term $(d(\cdot)/dt)^3$ dramatically increases the force transmissibility while the nonlinear damping term $(\cdot)^2(d(\cdot)/dt)$ keeps the force transmissibility very close to the low-damping linear referenced case. These indicates that the nonlinear damping term which is a function of both displacement and velocity produce much better force transmissibility performance than the nonlinear damping term which is only velocity-dependent under base displacement excitation.

For the displacement transmissibility under base displacement excitation, it is very similar to the force transmissibility discussed above.

Proposition 5. *Consider the displacement transmissibility under base displacement excitation. The performance at high frequency with the nonlinear damping $(d(\cdot)/dt)^3$ is dramatically deteriorated, while the nonlinear damping $(\cdot)^2(d(\cdot)/dt)$ can make the displacement transmissibility very close to the low-damping linear referenced case over this frequency region. Both of these two cubic order nonlinearities can make the displacement transmissibility very close to the low-damping linear case aglow frequency:*

(I) *When $\Omega \ll 1$, both $(\cdot)^2(d(\cdot)/dt)$ and $(d(\cdot)/dt)^3$ can make*

$$T(\Omega) \approx |P_0(j\Omega)| = \sqrt{\frac{1+(\xi_1\Omega)^2}{(1-\Omega^2)^2+(\xi_1\Omega)^2}} \tag{48}$$

(II) *When $\Omega \gg 1$, only $(\cdot)^2(d(\cdot)/dt)$ term can make*

$$T(\Omega) \approx |P_0(j\Omega)| = \sqrt{\frac{1+(\xi_1\Omega)^2}{(1-\Omega^2)^2+(\xi_1\Omega)^2}} \tag{49}$$

while the nonlinear term $(d(\cdot)/dt)^3$ will make

$$T(\Omega) \gg |P_0(j\Omega)| = \frac{\sqrt{1+(\xi_1\Omega)^2}}{\sqrt{\left(1-\Omega^2\right)^2+(\xi_1\Omega)^2}} \tag{50}$$

(III) *When $\Omega \approx 1$, there exists a $\beta > 0$ such that*

$$\frac{d[T(\Omega)]^2}{d\beta_2} < 0 \left(\text{or } \frac{d[T(\Omega)]^2}{d\beta_4} < 0\right) \tag{51}$$

if $0<\beta_2<\beta$ (or $0<\beta_4<\beta$)

Proof. See Appendix F □.

5. Simulation studies and discussions

The following simulations using the Runge–Kutta method are given to verify the theoretical results above.

Fig. 3 represents the force transmissibility for the isolator subjected to force excitation under different linear damping coefficients and different cubic order nonlinear damping terms. The solid line and dash line represent the force transmissibility when only linear damping coefficient is introduced in the isolator where $\xi_1=0.1$ and $\xi_1=0.325$ respectively. The dot line shows the performance when the linear damping coefficient and cubic order damping term $(\mathrm{d}(\cdot)/\mathrm{d}t)^3$ are introduced in the isolator where $\xi_1=0.1$ and $\beta_4=0.028$, while the star line represents the performance when linear damping coefficient and cubic order nonlinear term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ are introduced in the isolator where $\xi_1=0.1$ and $\beta_2=0.4$. In Fig. 3, it is shown that the star line and the dot line are almost superimposed on the solid line at both low frequency and high frequency, and have the force transmissibility much smaller than the solid line around the resonant frequency. So it is clear that the cubic order nonlinear damping terms $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ and $(\mathrm{d}(\cdot)/\mathrm{d}t)^3$ can both produce the ideal isolation performance, that is, the force transmissibility over the resonant frequency is obviously suppressed while keeping almost unaffected over the non-resonant regions. The dash line is presented as a reference, from which it can be seen that in order to get the same force transmissibility in the resonant frequency with that when the cubic order nonlinear term is introduced in the isolator, the linear damping coefficient ξ_1 is increased from 0.1 to 0.325, and then the force transmissibility at high frequency increases obviously.

In Fig. 4 the force transmissibility under base displacement excitation produced by the cubic order nonlinear term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ (where the linear damping coefficient $\xi_1=0.1$ and the cubic order nonlinear coefficient $\beta_2=0.1$) is presented in star line. The force transmissibility has been significantly suppressed over the resonant frequency and over the non-resonant regions remains very close to the performance presented in solid line when only linear damping coefficient is introduced where $\xi_1=0.1$. The performance when cubic order damping term $(\mathrm{d}(\cdot)/\mathrm{d}t)^3$ is introduced in the isolator where $\xi_1=0.1$ and $\beta_4=0.03$ is shown in dot line, from which it can be seen that the force transmissibility at high frequency is dramatically deteriorated compared with that when there only exists the linear damping coefficient where $\xi_1=0.1$. The performance deterioration at high frequency limits the practical use of the cubic order damping term $(\mathrm{d}(\cdot)/\mathrm{d}t)^3$ although the force transmissibility is also suppressed over the resonant frequency. The dash line is also presented as a reference where only the linear damping term with the coefficient $\xi_1=0.306$ is introduced, from which it can be seen that in order to reach the similar force transmissibility to the case that the cubic order nonlinear damping term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ is introduced at resonant frequency, the force transmissibility increases obviously at high frequency.

In Fig. 5 the star line stands for the absolute displacement transmissibility when the cubic order nonlinear term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ is introduced where the linear damping coefficient $\xi_1=0.1$ and the cubic order nonlinear damping coefficient $\beta_2=0.1$. The transmissibility is significantly suppressed at the resonant frequency and over the non-resonant frequency it remains very close to the solid line which represents the linear damping case with linear damping coefficient $\xi_1=0.1$. The absolute displacement transmissibility produced by the cubic order damping term $(\mathrm{d}(\cdot)/\mathrm{d}t)^3$ with $\xi_1=0.1$ and $\beta_4=0.03$ is shown in dot line, which tends to be 0 dB as Ω tends to be infinity, and this corresponds to a rigidly-connected system. The dash solid line shows that when the linear damping coefficient is increased from 0.1 to 0.306 in order to achieve the similar absolute displacement transmissibility over the resonant frequency to the case that the cubic order nonlinear terms are introduced in the isolator, the performance at high frequency increases obviously.

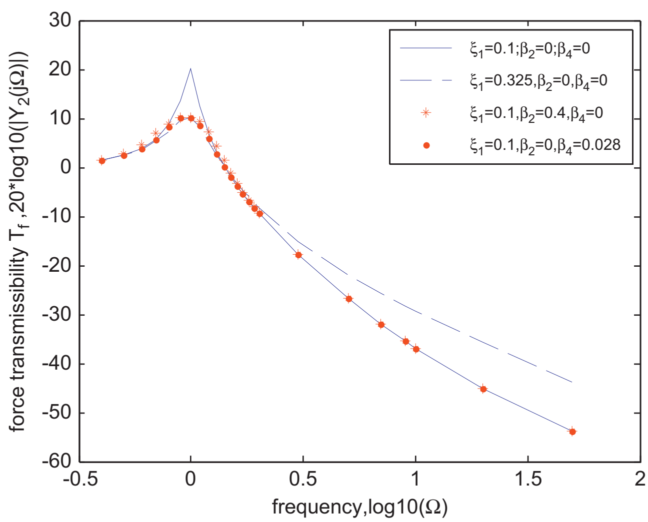

Fig. 3. The force transmissibility for an sdof isolator subjected to force excitation with different linear damping coefficients and different cubic order nonlinear terms.

From Figs. 3 to 5, the cubic order nonlinear term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ can produce ideal isolation performance, that is, significant vibration suppression over the resonant frequency and low damping effect over non-resonant frequencies, under both force and base displacement excitations. In order to have a more straightforward insight into the nonlinear mechanism in vibration suppression, the equivalent damping coefficients of the vibration isolator system under different cases are provided in Fig. 6. As shown in Fig. 6, the cubic order nonlinear terms $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ and $(\mathrm{d}(\cdot)/\mathrm{d}t)^3$ have equivalent linear damping coefficient very close to 0.306 at resonant frequency. Therefore, these two nonlinear terms have the transmissibility very close to the case when the linear coefficient $\xi_1=0.306$ at resonant frequency as presented in Figs. 4 and 5. At high frequency, the nonlinear term $(\mathrm{d}(\cdot)/\mathrm{d}t)^3$ has a very large equivalent linear damping coefficient, which corresponds to the deteriorated isolation performance in Figs. 4 and 5, but the equivalent linear damping coefficient of the nonlinear term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ remains very small, which is close to 0.1. The better isolation performance of the nonlinear term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ at high frequency presented in Figs. 4 and 5 is therefore produced by this small equivalent damping effect. It is known that the ideal isolation performance requires the damping coefficient to be larger at resonant frequency but smaller at high frequency. Fig. 6 shows that the nonlinear damping characteristic $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ can achieve this objective much better than the other cases.

In what follows, the case when only the cubic order nonlinear damping term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ is introduced in the isolator with the linear damping coefficient $\xi_1$beingzero, is studied since it has much better damping effect over all frequencies as discussed above.

In Fig. 7 the star line represents the force transmissibility for the isolator with only the cubic order nonlinear term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ and $\beta_2=0.1$subjected to the base displacement excitation. The dash line indicates the case that the linear

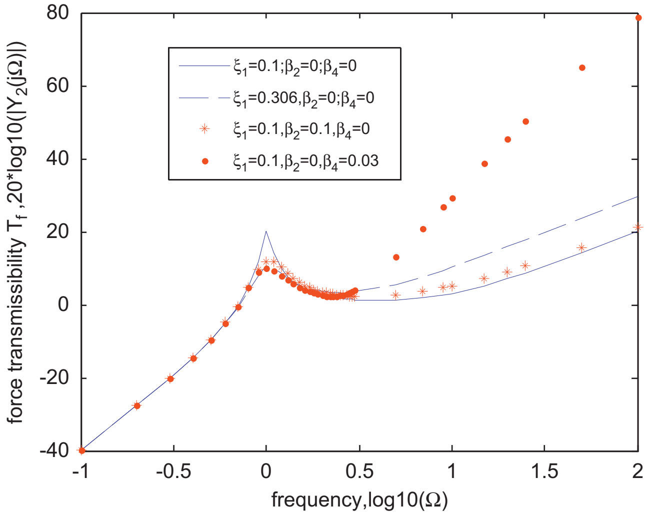

Fig. 4. The force transmissibility for the isolator subjected to base displacement excitation with different linear damping coefficients and different cubic order nonlinear terms.

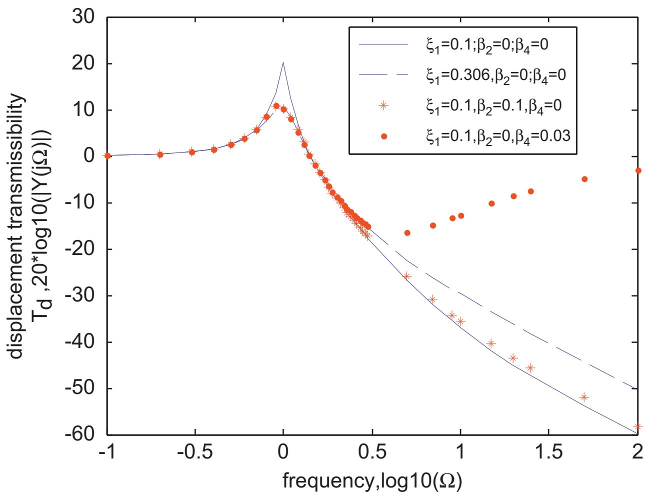

Fig. 5. The absolute displacement transmissibility for the isolator subjected to base displacement excitation with different linear damping coefficients and under different cubic order nonlinear terms.

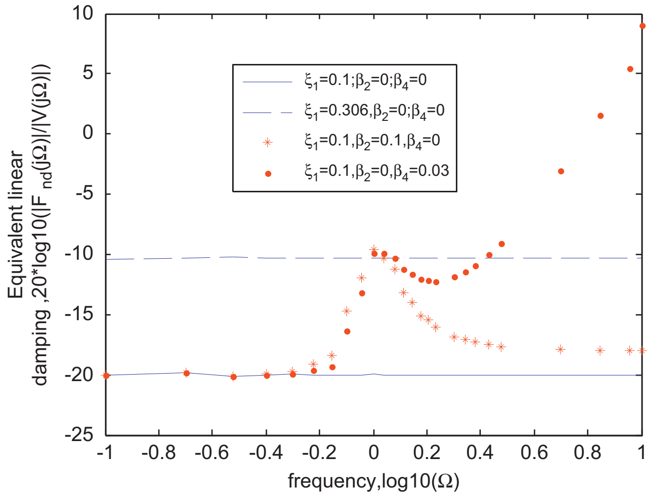

Fig. 6. The equivalent linear damping coefficient of the isolator subjected to base displacement excitation with different linear damping and different cubic order nonlinear damping terms.

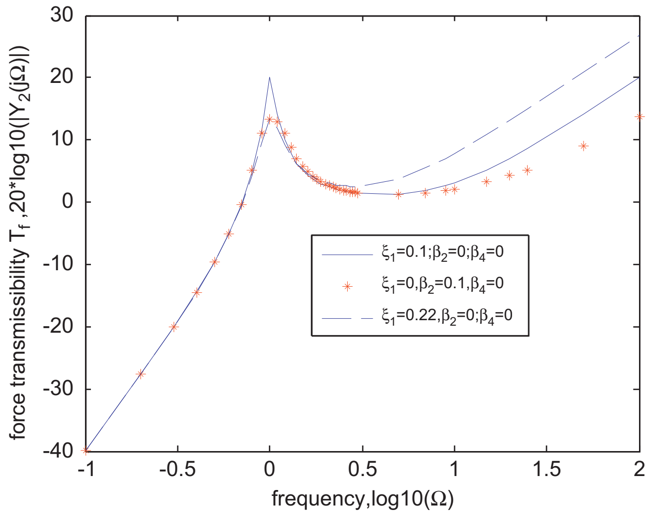

Fig. 7. The force transmissibility for an isolator subjected to base displacement excitation with the pure cubic order nonlinear damping term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$.

damping coefficient is increased from 0.1 to 0.22 in order to obtain the same force transmissibility as that when only the cubic order nonlinear term is introduced in the isolator. It can be seen that when the linear damping coefficient is increased to suppress the transmissibility around the resonant frequency, the transmissibility at high frequency increases obviously. This is the famous dilemma in vibration isolations. Comparing the star line with the solid line, it is evidently that the cubic order nonlinear term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ can overcome the dilemma and produce an ideal isolation performance that the transmissibility is suppressed at both resonant frequency and high frequency and keep almost unaffected at low frequency.

In Fig. 8 the absolute displacement transmissibility for the isolator subjected to base displacement excitation when only the cubic order nonlinear term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ is introduced in the isolator with $\beta_2=0.1$ is presented in star line, compared with the cases with linear damping coefficient $\xi_1=0.1$ and $\xi_1=0.306$ respectively. Similarly, the cubic order nonlinear term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ can also overcome the dilemma in absolute displacement transmissibility and produce a much better isolation performance.

6. Conclusions

In this paper, the influence of a cubic order nonlinear damping term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ is studied for an sdof isolator system. It is shown that the proposed nonlinear damping can overcome the dilemma in vibration isolation that when the linear

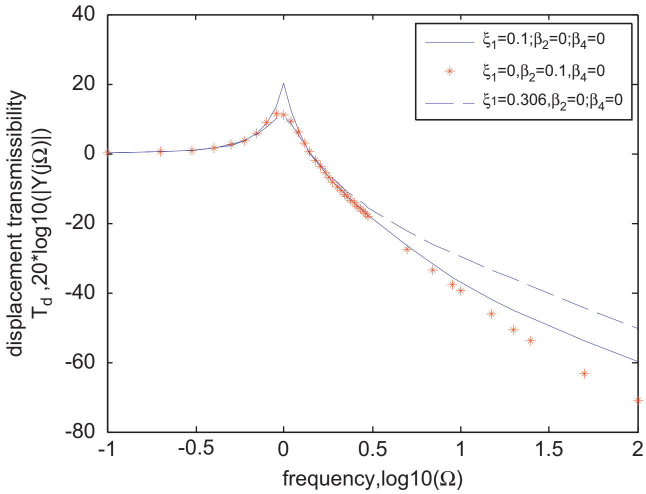

Fig. 8. The absolute displacement transmissibility for an isolator subjected to base displacement excitation with pure cubic order nonlinear term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ and with pure linear damping.

damping coefficient is increased to reduce the transmissibility over the resonant frequency it at the same time increases the transmissibility at high frequency. The force transmissibility and the absolute displacement transmissibility for the isolator subjected to both force excitation and base displacement excitation when cubic order nonlinear terms are introduced are established, from which the analytical and explicit relationship between the transmissibility and the cubic order nonlinear coefficients can be obtained. Much better isolation performance is produced with the cubic order nonlinear damping term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$, and numerical studies are given to verify the theoretical results. The following conclusions can be made:

(I) The cubic order nonlinear damping term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ can produce better isolation performance for an isolator under both force excitation and base displacement excitations. While the known cubic nonlinear damping which is only a pure function of velocity is limited in vibration control subjected to base excitations. This may imply that the optimal nonlinear damping could be dependent not only on velocity but also on displacement.
(II) The proposed cubic order nonlinear damping term can obviously suppress the transmissibility over the resonant frequency and remains very close to the low-damping linear referenced case over the non-resonant frequency regions. It can demonstrate even better performance when the system linear damping is zero. This provides a fairly ideal damping characteristic in practical applications.
(III) The nonlinear frequency domain method adopted in this study provides a powerful tool for the analysis and design of nonlinear damping systems. It can provide a straightforward expression for the relationship between nonlinear output spectrum and any characteristic parameters which define the nonlinearity of the system and thus facilitate the nonlinear analysis and design. Further study will focus on more general optimal analysis and design of nonlinear stiffness and damping characteristics in vibration control.

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Appendix A

The specific expression of $H_n^l(j\omega_1,\ldots,j\omega_n)$ can be obtained according to [15] considering the one-input-two-output nonlinear differential model as follows:

$$H_1^2(j\omega_1)=(1+j\omega_1\xi_1)H_1^1(j\omega_1) \tag{A-1}$$

$$H_n^2(j\omega_1,\ldots,j\omega_n)=-(j\omega_1+\cdots+j\omega_n)^2H_n^1(j\omega_1,\ldots,j\omega_n)\ n=2,\ldots,N \tag{A-2}$$

$$H_1^1(j\omega_1)=\frac{-1}{L(j\omega_1)} \tag{A-3}$$

$$H_3^1(j\omega_1,j\omega_2,j\omega_3)=\beta_2\frac{\prod_{i=1}^{3}H_1^1(j\omega_i)}{L(j\omega_1+j\omega_2+j\omega_3)}(j\omega_1)+\beta_4\frac{\prod_{i=1}^{3}H_1^1(j\omega_i)}{L(j\omega_1+j\omega_2+j\omega_3)}\prod_{i=1}^{3}(j\omega_i) \quad \text{(A-4)}$$

$$H_{2n}^1(j\omega_1,\ldots,j\omega_{2n})=0,\ \ n=1,\ldots,\lfloor N/2\rfloor \quad \text{(A-5)}$$

$$H_{2n+1}^1(j\omega_1,\cdots j\omega_{2n+1})=\sum_{m=0}^{n}\beta_2^{n-m}\beta_4^{m}\frac{\prod_{i=1}^{2n+1}H_1^1(j\omega_i)}{L(j\omega_1+\cdots+j\omega_{2n+1})}\sum_{z=1}^{N_{\bar{n}}}\frac{\prod_{k=1}^{(n-m)+3m}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j_k^n)}]}{\prod_{i=1}^{n-1}L[j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i(j_i^n)}]}\ \ n=1,\ldots,\lfloor N/2\rfloor \quad \text{(A-6)}$$

where

$$\begin{array}{ll} j_i^n\in\{3,5,\ldots,2n-1\} & i=1,\ldots,n-1,n\geq 2\\ j_k^n\in\{1,3,\ldots,2n-1\} & k=1,\ldots,n-1,n\geq 2\\ \omega^Z_{l_i(\bar{j})}\in\{\omega_1,\ldots,\omega_{2n+1}\}, & i=1,\ldots,n-1,\bar{j}=1,\ldots,j_i^n,n\geq 2\\ \omega^Z_{l_k(\bar{j})}\in\{\omega_1,\ldots,\omega_{2n+1}\}, & k=1,\ldots,n-1,\bar{j}=1,\ldots,j_k^n,n\geq 2\\ \prod_{i=1}^{n-1}L[j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i(j_i^n)}]=1 & \text{for } n=1 \end{array} \quad \text{(A-7)}$$

N_n is an n dependent integer:

$$L[(j\omega_1+\cdots+j\omega_n)]=-[1+\xi_1(j\omega_1+\cdots+j\omega_n)+(j\omega_1+\cdots+j\omega_n)^2] \quad \text{(A-8)}$$

Appendix B. Proof of Proposition 1

According to Ref. [12], $C_{p,q}(l_1,l_2,l_3),p+q=3,p,q=0,\ldots,3,l_1,l_2,l_3=0,1$ will only exist in the $(2n+1)$th GFRF (generalize frequency response function) $H_{2n+1}(j\omega_1,\ldots,j\omega_{2n+1})$, where $n=1,2,\ldots$, this means that the even order GFRF $H_{2n}(j\omega_1,\ldots,j\omega_{2n})=0$. Assume that all the GFRF whose order less than $(2n+1)$ all satisfy proposition 1. The $(2n+1)$th GFRF can be obtained [13]:

$$H_{2n+1}^1(j\omega_1,\ldots,j\omega_{2n+1})=\frac{C_{p,q}(l_1,l_2,l_3)}{L(j\omega_1+\cdots+j\omega_{2n+1})}(j\omega_{2n+1-q+1})^{l_{p+1}}\cdots(j\omega_{2n+1})^{l_{p+q=3}}H^1_{2n+1-q,p}(j\omega_1,\cdots j\omega_{2n+1-q}) \quad \text{(B-1)}$$

$$H^1_{2n+1\text{-}q,p}(j\omega_1,\ldots,j\omega_{2n+1-q})=\sum_{r_1\cdots r_p=1,\sum r_i=2n+1-q}^{2n-1}\prod_{i=1}^{p}H^1_{r_i}(j\omega_{X+1},\ldots,j\omega_{X+r_i})(j\omega_{X+1}+\cdots+j\omega_{X+r_i})^{l_i}$$

where $X=\sum_{x=1}^{i-1}r_x$.

According to the assumption, $H_{r_i}(j\omega_{X+1},\ldots,j\omega_{X+r_i})$ can be expressed as

$$H^1_{r_i}(j\omega_1,\ldots,j\omega_{n_{r_i}})=\frac{C^{n_{r_1}}_{p,q}(l_1,l_2,l_3)}{L(j\omega_1+\cdots+j\omega_{r_i})}\sum_{z=1}^{N_{n_{r_1},p}}\prod_{i=1}^{(p-1)n_{r_1}+1}H_1^1(j\omega_{j_i})\frac{\prod_{k=1}^{mn_{r_i}}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j_k^{n_{r_i}})}]}{\prod_{i=1}^{n_{r_i}-1}L[j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i(j_i^{n_{r_i}})}]}$$

where $r_i=2n_{r_i}+1$. Then,

$$\prod_{i=1}^{p}H^1_{r_i}(j\omega_{X+1},\ldots,j\omega_{X+r_i})(j\omega_{X+1}+\cdots+j\omega_{X+r_i})^{l_i}$$

$$=\prod_{i=1}^{p}\frac{C^{n_{r_i}}_{p,q}(l_1,l_2,l_3)(j\omega_{X+1}+\cdots+j\omega_{X+r_i})^{l_i}}{L(j\omega_{X+1}+\cdots+j\omega_{X+r_i})}\sum_{z=1}^{N_{n_{r_i},p}}\prod_{i=1}^{(p\text{-}1)n_{r_i}+1}H_1^1(j\omega_{j_i})\frac{\prod_{k=1}^{mn_{r_i}}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j_k^{n_{r_i}})}]}{\prod_{i=1}^{n_{r_i}\text{-}1}L[j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i(j_i^{n_{r_i}})}]}$$

$$=\frac{C_{p,q}^{\sum_{i=1}^{p}n_{r_i}}(l_1,l_2,l_3)\prod_{i=1}^{p}(j\omega_{X+1}+\cdots+j\omega_{X+r_i})^{l_i}}{\prod_{i=1}^{p}L(j\omega_{X+1}+\cdots+j\omega_{X+r_i})}\sum_{z=1}^{N_{\sum_{i=1}^{p}n_{r_i},p}}\prod_{i=1}^{(p-1)\sum_{i=1}^{p}n_{r_i}+p}H_1^1(j\omega_{j_i})\frac{\prod_{k=1}^{m\sum_{i=1}^{p}n_{r_i}}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k\left(j_k^{\sum_{i=1}^{p}n_{r_i}}\right)}]}{\prod_{i=1}^{\sum_{i=1}^{p}n_{r_i}-p}L[j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i\left(j_i^{\sum_{i=1}^{p}n_{r_i}}\right)}]}$$

Because

$$\sum_{i=1}^{p} r_i = 2n+1-q = 2\sum_{i=1}^{p} n_{r_i} + p$$

then,

$$\sum_{i=1}^{p} n_{r_i} = n-1$$

$$\prod_{i=1}^{p} H_{r_i}^{1}(j\omega_{X+1},\ldots,j\omega_{X+r_i})(j\omega_{X+1}+\cdots+j\omega_{X+r_i})^{l_i}$$

$$= C_{p,q}^{n\text{-}1}(l_1,l_2,l_3)\sum_{z=1}^{N_{n,p}}\prod_{i=1}^{(p-1)n+1} H_1^1(j\omega_i)\frac{\prod_{i=1}^{p}(j\omega_{X+1}+\cdots+j\omega_{X+r_i})^{l_i}}{\prod_{i=1}^{p} L(j\omega_{X+1}+\cdots+j\omega_{X+r_i})}\frac{\prod_{k=1}^{m(n-1)}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k\left(j_k^{\sum_{i=1}^{p} n_{r_i}}\right)}]}{\prod_{i=1}^{n-1-p} L[j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i\left(j_i^{\sum_{i=1}^{p} n_{r_i}}\right)}]} \quad \text{(B-2)}$$

Substitute Eq. (B-2) into Eq. (B-1), then the Proposition 1 holds.

Appendix C. Proof of Proposition 2

Similar to the Proof of Proposition 1,

$$H_{2n+1}^1(j\omega_1,\ldots,j\omega_{2n+1})$$
$$= \sum_{j=1}^{n_L}\frac{C_{p_j,q_j}(l_1,l_2,l_3)}{L(j\omega_1+\cdots+j\omega_{2n+1})}\left(j\omega_{2n+1-q_j+1}\right)^{l_{p_j+1}}\cdots(j\omega_{2n+1})^{l_{p_j+q_j=3}}H^1_{2n+1-q_j,p_j}\left(j\omega_1,\ldots,j\omega_{2n+1\text{-}q_j}\right)$$

$$H^1_{2n+1-q_j,p_j}\left(j\omega_1,\ldots,j\omega_{2n+1-q_j}\right) = \sum_{r_1\cdots r_{p_j}=1,\ \sum r_k=2n+1-q_j}^{2n-1}\prod_{k=1}^{p_j} H^1_{r_k}(j\omega_{X+1},\ldots,j\omega_{X+r_k})(j\omega_{X+1}+\cdots+j\omega_{X+r_k})^{l_k}$$

where $X=\sum_{x=1}^{k-1} r_x . H_{r_k}(j\omega_{X+1},\ldots,j\omega_{X+r_k})$ can be expressed as

$$H^1_{r_k}(j\omega_{X+1},\ldots,j\omega_{X+r_k})$$
$$= \sum_{kj=1}^{n_L} C_{p_{kj},q_{kj}}\frac{\sum_{\substack{n_{rk1,\ldots,n_{rkn_L}}=0\\ \sum_{i=1}^{n_L} n_{rki}=n_{rk}-1}}^{n_{rk}-1}\left(\prod_{i=1}^{n_L} C^{n_{rki}}_{p_i,q_i}(l_{i1},l_{i2},l_{i3})\sum_{z=1}^{N_{n_{rk},p_{kj}}}\prod_{k=1}^{(p_{kj}-1)+\sum_{i=1}^{n_L}(p_{ki}-1)n_{rki}+1} H_1^1(j\omega_{k_i})\frac{\prod_{k=1}^{\sum_{i=1}^{n_L} n_{rkmi}m_i}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j_k^n)}]}{\prod_{i=1}^{n_{rk}-1} L[j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i(j_i^n)}]}\right)}{L(j\omega_{X+1}+\cdots+j\omega_{X+r_k})}$$

where $r_k=2n_{rk}+1$, $\sum_{i=1}^{n_L} n_{rki}+1=n_{rk}$, $\sum_{i=1}^{n_L} n_{rkmi}=n_{rk}$, $n_{rkmj}=n_{rkj}+1$, $n_{rkmi}=n_{rki}$, $i\neq j$. Then,

$$\prod_{k=1}^{p_j} H^1_{r_k}(j\omega_{X+1},\ldots,j\omega_{X+r_k})$$
$$= \prod_{k=1}^{p_j} C_{p_{kj_k},q_{kj_k}}\frac{\sum\prod_{i=1}^{n_L} C_{p_i,q_i}^{\sum_{k=1}^{p_j} n_{rki}}(l_{i1},l_{i2},l_{i3})}{\prod_{k=1}^{p_j} L(j\omega_{X+1}+\cdots+j\omega_{X+r_k})}$$
$$\times\sum_{z=1}^{N_{\sum n_{rk},\sum p_{kj}}}\prod_{k=1}^{\sum_{k=1}^{p_j}\left(p_{kj_k}-1\right)+\sum_{k=1}^{p_j}\sum_{i=1}^{n_L}(p_i-1)n_{rki}+p_j} H_1^1(j\omega_{k_i})\frac{\prod_{k=1}^{\sum_{k=1}^{p_j}\sum_{i=1}^{n_L} n_{rkmi}m_i}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j_k^n)}]}{\prod_{i=1}^{\sum_{k=1}^{p_j} n_{rk}-p_j} L[j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i(j_i^n)}]}$$

Because

$$\sum_{k=1}^{p_j} r_k = \sum_{k=1}^{p_j}(2n_{rk}+1) = 2\sum_{k=1}^{p_j} n_{rk}+p_j = 2n+1-q_j$$

so

$$\sum_{k=1}^{p_j} n_{rk} = n-1$$

Denote

$$\sum_{k=1}^{p_j} n_{rki} = n'_{ji} \sum_{k=1}^{p_j} n_{rkmi} = n'_{jmi}$$

$$\prod_{k=1}^{p_j} H^1_{r_k}(j\omega_{X+1},\ldots,j\omega_{X+r_k})$$

$$= \frac{\prod_{k=1}^{p_j} C_{p_{kj_k},q_{kj_k}} \sum \prod_{i=1}^{n_L} C_{p_i,q_i}^{n'_{ji}}(l_{i1},l_{i2},l_{i3})}{\prod_{k=1}^{p_j} L(j\omega_{X+1}+\cdots+j\omega_{X+r_k})} \sum_{z=1}^{N_{n,p_j}} \prod_{k=1}^{\sum_{k=1}^{p_j}\left(p_{kj_k}-1\right)+\sum_{i=1}^{n_L}(p_i-1)n'_{ji}+p_j} H_1^1(j\omega_{k_i}) \frac{\prod_{k=1}^{\sum_{i=1}^{n_L} n'_{jmi}m_i}[j\omega^z_{l_{k(1)}}+\cdots+j\omega^z_{l_k(j_k^n)}]}{\prod_{i=1}^{n-1-p_j} L[j\omega^z_{l_{i(1)}}+\cdots+j\omega^z_{l_i(j_i^n)}]}$$

Assume that $j_k = j$, then

$$C_{p_{kj_k},q_{kj_k}} \prod_{i=1}^{n_L} C_{p_i,q_i}^{n'_{ji}}(l_{i1},l_{i2},l_{i3}) = C_{p_j,q_j} \prod_{i=1}^{n_L} C_{p_i,q_i}^{n'_{ji}}(l_{i1},l_{i2},l_{i3}) = \prod_{i=1}^{n_L} C_{p_i,q_i}^{n''_{ji}}(l_{i1},l_{i2},l_{i3})$$

$$\left(p_{kj_k}-1\right)+\sum_{i=1}^{n_L}(p_i-1)n'_{ji}+p_j = \left(p_j-1\right)+\sum_{i=1}^{n_L}(p_i-1)n'_{ji}+p_j = \sum_{i=1}^{n_L}(p_i-1)n''_{ji}+p_j$$

where $n''_{jj} = n'_{jj}+1$, $n''_{ji} = n''_{ji}$, $i \neq j$. Because $n_{rkmj} = n_{rkj}+1$, $n_{rkmi} = n_{rki}$, $i \neq j$, it can be obtained that

$$\sum_{i=1}^{n_L} n'_{ji} + p_j = \sum_{i=1}^{n_L} n'_{jmi}$$

Assume that there are k_i terms of C_{p_i,q_i} in $\prod_{k=1}^{p_j} C_{p_{kj_k},q_{kj_k}}$, $\sum_{i=1}^{n_L} k_i = p_j$, then

$$n'_{ji}+k_i = \sum_{k=1}^{p_j} n_{rki}+k_i = \sum_{k=1}^{p_j} n_{rkmi} = n'_{jmi}$$

so,

$$\prod_{k=1}^{p_j} H^1_{r_k}(j\omega_{X+1},\ldots,j\omega_{X+r_k})$$

$$= \sum_{n_{j1}=0,\ldots,n_{jn_L}=0;\sum_{i-1}^{n_L} n_{ji}=n-1}^{n-1} \left(\prod_{i=1}^{n_L} C_{p_i,q_i}^{n_{ji}}(l_{i1},l_{i2},l_{i3}) \sum_{z=1}^{N_{n,p_j}} \prod_{k=1}^{\sum_{i=1}^{n_L}(p_i-1)n_{ji}+p_j} H_1^1(j\omega_{k_i}) \frac{\prod_{k=1}^{\sum_{i=1}^{n_L} n'_{jmi}m_i}[j\omega^z_{l_{k(1)}}+\cdots+j\omega^z_{l_k(j_k^n)}]}{\prod_{i=1}^{n-1} L[j\omega^z_{l_{i(1)}}+\cdots+j\omega^z_{l_i(j_i^n)}]} \right)$$

where $n_{ji} = n'_{ji}+k_i = n'_{jmi}$, then,

$$\left(j\omega_{2n+1-q_j+1}\right)^{l_{p_j+1}} \cdots (j\omega_{2n+1})^{l_{p_j+q_j=3}} \prod_{k=1}^{p_j} H^1_{r_k}(j\omega_{X+1},\ldots,j\omega_{X+r_k})(j\omega_{X+1}+\cdots+j\omega_{X+r_k})^{l_k}$$

$$= \sum_{n_{j1}=0,\ldots,n_{jn_L}=0;\sum_{i-1}^{n_L} n_{ji}=n-1}^{n-1} \left(\prod_{i=1}^{n_L} C_{p_i,q_i}^{n_{ji}}(l_{i1},l_{i2},l_{i3}) \sum_{z=1}^{N_{n,p_j}} \prod_{k=1}^{\sum_{i=1}^{n_L}(p_i-1)n_{ji}+p_j} H_1^1(j\omega_{k_i}) \frac{\prod_{k=1}^{\sum_{i=1}^{n_L} n'_{jmi}m_i}[j\omega^z_{l_{k(1)}}+\cdots+j\omega^z_{l_k(j_k^n)}]}{\prod_{i=1}^{n-1} L[j\omega^z_{l_{i(1)}}+\cdots+j\omega^z_{l_i(j_i^n)}]} \right.$$
$$\left. \times \prod_{k=1}^{p_j} (j\omega_{X+1}+\cdots+j\omega_{X+r_k})^{l_k} \left(j\omega_{2n+1-q_j+1}\right)^{l_{p_j+1}} \cdots (j\omega_{2n+1})^{l_{p_j+q_j=3}} \right)$$

$$= \sum_{n_{j1}=0,\ldots,n_{jn_L}=0;\sum_{i-1}^{n_L} n_{ji}=n-1}^{n-1} \left(\prod_{i=1}^{n_L} C_{p_i,q_i}^{n_{ji}}(l_{i1},l_{i2},l_{i3}) \sum_{z=1}^{N_{n,p_j}} \prod_{k=1}^{\sum_{i=1}^{n_L}(p_i-1)n_{ji}+p_j} H_1^1(j\omega_{k_i}) \frac{\prod_{k=1}^{\sum_{i=1}^{n_L} n'_{jmi}m_i+m_j}[j\omega^z_{l_{k(1)}}+\cdots+j\omega^z_{l_k(j_k^n)}]}{\prod_{i=1}^{n-1} L[j\omega^z_{l_{i(1)}}+\cdots+j\omega^z_{l_i(j_i^n)}]} \right)$$

$$= \sum_{n_{j1}=0,\ldots,n_{jn_L}=0;\sum_{i-1}^{n_L} n_{ji}=n-1}^{n-1} \left(\prod_{i=1}^{n_L} C_{p_i,q_i}^{n_{ji}}(l_{i1},l_{i2},l_{i3}) \sum_{z=1}^{N_{n,p_j}} \prod_{k=1}^{\sum_{i=1}^{n_L}(p_i-1)n_{ji}+p_j} H_1^1(j\omega_{k_i}) \frac{\prod_{k=1}^{\sum_{i=1}^{n_L} n'_{jmi}m_i}[j\omega^z_{l_{k(1)}}+\cdots+j\omega^z_{l_k(j_k^n)}]}{\prod_{i=1}^{n-1} L[j\omega^z_{l_{i(1)}}+\cdots+j\omega^z_{l_i(j_i^n)}]} \right)$$

where $n_{jmi} = n'_{jmi} = n_{ji}$, $mi \neq mj$, $n_{jmj} = n'_{jmj}+1 = n_{jj}+1$:

$$\sum_{k=1}^{p_j} n_{rk} = \sum_{k=1}^{p_j}\sum_{i=1}^{n_L} n_{rkmi} = \sum_{i=1}^{n_L}\sum_{k=1}^{p_j} n_{rkmi} = \sum_{i=1}^{n_L} n'_{jmi} = \sum_{i=1}^{n_L} n_{ji} = n-1$$

so,

$$\sum_{i=1}^{n_L} n_{jmi} = \sum_{i=1}^{n_L} n'_{jmi} + 1 = n$$

This completes the proof □.

Appendix D. Proof of Proposition 3

(I) When only the nonlinear term $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ is introduced in the isolation system, i.e. $\beta_2 \neq 0$ and $\beta_4 = 0$, only the term $P_{n0}(j\Omega)$ is needed to be considered in the force transmissibility under force excitation, Eq. (26):

$$|P_{n0}(j\Omega)| = \left| \frac{1}{2^{2n}} \frac{-j\Omega^2}{|L(j\Omega)|^{2n}[L(j\Omega)]^2} \sum\nolimits_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\bar n}} \frac{\prod_{k=1}^{n}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\prod_{i=1}^{n-1} L[j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i(j^n_k)}]} \right|$$
$$\le \frac{1}{2^{2n}} \frac{\Omega^2}{\left(\sqrt{\left(1-\Omega^2\right)^2+(\xi_1\Omega)^2}\right)^{2n+2}}$$
$$\sum\nolimits_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\bar n}} \frac{\prod_{k=1}^{n}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\prod_{i=1}^{n-1} L[1+\left(j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i(j^n_k)}\right)^2+\xi_1\left(j\omega^z_{l_i(1)}+\cdots+j\omega^z_{l_i(j^n_k)}\right)]}$$

when $\Omega \ll 1$,

$$|P_{n0}(j\Omega)| \le \frac{1}{2^{2n}}\Omega^2 \sum\nolimits_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\bar n}} \prod_{k=1}^{n} |[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]| = \frac{1}{2^{2n}}\Omega^{n+2} c1(n)$$

where

$$c1(n) = \sum\nolimits_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\bar n}} \prod_{k=1}^{n} \left| \frac{[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\Omega} \right|$$

is a bounded constant which is an n dependent but independent of Ω. So when Ωo1,

$$|P_{n0}(j\Omega)| \le \frac{1}{2^{2n}}\Omega^{n+2} c1(n) \approx 0 \quad \text{for } n = 1,2,\ldots,\lfloor N/2 \rfloor$$

when $\Omega \gg 1$

$$|P_{n0}(j\Omega)| \le \frac{1}{2^{2n}} \frac{\Omega^2}{\left(\sqrt{\left(1-\Omega^2\right)^2+(\xi_1\Omega)^2}\right)^{2n+2}}$$
$$\times \sum\nolimits_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\bar n}} \frac{\prod_{k=1}^{n}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\prod_{i=1}^{n-1} |L[1+\left(j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}\right)^2+\xi_1\left(j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}\right)]|}$$
$$\approx \frac{1}{2^{2n}} \frac{\Omega^2}{\Omega^{4n+4}} \frac{\Omega^n}{\Omega^{2n-2}} \sum\nolimits_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\bar n}} \prod_{k=1}^{n} \left| \frac{[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\Omega} \right| = \frac{1}{2^{2n}} \frac{1}{\Omega^{5n}} c2(n)$$

where

$$c2(n) = \sum\nolimits_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\bar n}} \prod_{k=1}^{n} \left| \frac{[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\Omega} \right|$$

is a bounded constant which is a n dependent but independent of Ω. So when $\Omega \gg 1$,

$$|P_{n0}(j\Omega)| \le \frac{1}{2^{2n}} \frac{1}{\Omega^{5n}} c2(n) \approx 0 \quad \text{for } n = 1,2,\ldots,\lfloor N/2 \rfloor$$

So conclusion (I) of Proposition 3 holds.

(II) The proof is given in Theorem 3 of Ref. [17].

Appendix E. Proof of Proposition 4

When only $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ term is introduced in the isolator, only $P_{n0}(j\Omega)$ is needed to be considered in the force transmissibility under base displacement excitation. Similarly, only $P_{0n}(j\Omega)$ is needed to be consider when only $(\mathrm{d}(\cdot)/\mathrm{d}t)^3$ is introduced:

$$|P_{n0}(j\Omega)| = \left| \frac{1}{2^{2n}} \frac{-j\Omega^{4n+4}}{|L(j\Omega)|^{2n}[L(j\Omega)]^2} \sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\overline{n}}} \frac{\prod_{k=1}^{n}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\prod_{i=1}^{n-1} L[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]} \right| \le \frac{1}{2^{2n}} \frac{\Omega^{4n+4}}{\left(\sqrt{\left(1-\Omega^2\right)^2+(\xi_1\Omega)^2}\right)^{2n+2}}$$

$$\times \sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\overline{n}}} \frac{\prod_{k=1}^{n}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\prod_{i=1}^{n-1} L[1+\left(j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}\right)^2+\xi_1\left(j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}\right)]}$$

$$|P_{0n}(j\Omega)| = \left| \frac{1}{2^{2n}} \frac{-j\Omega^{4n+4}}{|L(j\Omega)|^{2n}[L(j\Omega)]^2} \sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\overline{n}}} \frac{\prod_{k=1}^{3n}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\prod_{i=1}^{n-1} L[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]} \right|$$

$$\le \frac{1}{2^{2n}} \frac{\Omega^{4n+4}}{\left(\sqrt{\left(1-\Omega^2\right)^2+(\xi_1\Omega)^2}\right)^{2n+2}}$$

$$\times \sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\overline{n}}} \frac{\prod_{k=1}^{3n}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\prod_{i=1}^{n-1} L[1+\left(j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}\right)^2+\xi_1\left(j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}\right)]}$$

(I) When $\Omega \ll 1$

$$|P_{n0}(j\Omega)| \le \frac{1}{2^{2n}} \Omega^{4n+4} \sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\overline{n}}} \prod_{k=1}^{n} |[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]| = \frac{1}{2^{2n}} \Omega^{5n+4} c1(n)$$

where

$$c1(n) = \sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\overline{n}}} \prod_{k=1}^{n} \left| \frac{[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\Omega} \right|$$

is a bounded constant which is a n dependent but independent of Ω. So when $\Omega \ll 1$,

$$|P_{n0}(j\Omega)| \le \frac{1}{2^{2n}} \Omega^{5n+4} c1(n) \approx 0 \quad \text{for } n = 1,2,\ldots,\lfloor N/2 \rfloor$$

and

$$|P_{0n}(j\Omega)| \le \frac{1}{2^{2n}} \Omega^{4n+4} \sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\overline{n}}} \prod_{k=1}^{3n} |[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]| = \frac{1}{2^{2n}} \Omega^{7n+4} c1'(n)$$

where

$$c1'(n) = \sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\overline{n}}} \prod_{k=1}^{3n} \left| \frac{[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\Omega} \right|$$

is a bounded constant which is a n dependent but independent of Ω. So when Ωo1,

$$|P_{0n}(j\Omega)| \le \frac{1}{2^{2n}} \Omega^{7n+4} c1(n) \approx 0 \quad \text{for } n = 1,2,\ldots,\lfloor N/2 \rfloor$$

(II) When $\Omega \gg 1$,

$$|P_{n0}(j\Omega)|$$

$$\le \frac{1}{2^{2n}} \frac{\Omega^{4n+4}}{\Omega^{4n+4}} \frac{\Omega^n}{\Omega^{2n-2}} \sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega} \sum_{z=1}^{N_{\overline{n}}} \prod_{k=1}^{n} \left| \frac{[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\Omega} \right|$$

$$= \frac{1}{2^{2n}} \frac{1}{\Omega^{n-2}} c1(n)$$

where

$$c1(n)=\sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega}\sum_{z=1}^{N_{\bar{n}}}\prod_{k=1}^{n}\left|\frac{[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\Omega}\right|$$

is a bounded constant which is an n dependent but independent Ω So when $\Omega\gg1$,
While

$$|P_{0n}(j\Omega)|\le\frac{1}{2^{2n}}\frac{\Omega^{4n+4}}{\Omega^{4n+4}}\frac{\Omega^{3n}}{\Omega^{2n-2}}\sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega}\sum_{z=1}^{N_{\bar{n}}}\prod_{k=1}^{3n}\left|\frac{[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\Omega}\right|=\frac{1}{2^{2n}}\Omega^{n+2}c1'(n)$$

where

$$c1'(n)=\sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega}\sum_{z=1}^{N_{\bar{n}}}\prod_{k=1}^{3n}\left|\frac{[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\Omega}\right|$$

is a bounded constant which is dependent on n but independent of Ω. So when $\Omega\gg1$, the upper limit of $|P_{0n}(j\Omega)|$ is proportional to the square of the exciting frequency, and then the force transmissibility when $(\mathrm{d}(\cdot)/\mathrm{d}t)^3$ term is introduced in the isolator under base excitation in high frequency is much larger than that in linear case.

(III) The proof is given in Ref. [17].

Appendix F. Proof of Proposition 5

$$|P_{nk}(j\Omega)|=\left|\frac{1}{2^{2n}}\frac{(-j)}{L(j\Omega)}\sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega}\sum_{z=1}^{N_{n,p_j}}\prod_{k=1}^{\sum_{i=1}^{n_L}(p_i-1)n_{jmi}+1}H_1^1(j\omega_{k_i})\frac{\prod_{k=1}^{\sum_{i=1}^{n_L}n_{jmi}m_i}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\prod_{i=1}^{n-1}L[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}\right|$$

$$\le\frac{1}{2^{2n}}\frac{\prod_{k=1}^{\sum_{i=1}^{n_L}(p_i-1)n_{jmi}+1}|H_1^1(j\Omega)|}{|L(j\Omega)|}\sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega}\sum_{z=1}^{N_{n,p_j}}\frac{\prod_{k=1}^{\sum_{i=1}^{n_L}n_{jmi}m_i}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\prod_{i=1}^{n-1}L[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}$$

$$=\frac{1}{2^{2n}}\frac{\left(\sqrt{1+(\Omega\xi_1)^2}\right)^{\sum_{i=1}^{n_L}(p_i-1)n_{jmi}+1}}{\left(\sqrt{\left(1-\Omega^2\right)^2+(\Omega\xi_1)^2}\right)^{\sum_{i=1}^{n_L}(p_i-1)n_{jmi}+2}}\sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega}\sum_{z=1}^{N_{n,p_j}}\frac{\prod_{k=1}^{\sum_{i=1}^{n_L}n_{jmi}m_i}[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\prod_{i=1}^{n-1}L[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}$$

(I) When $\Omega\ll1$

$$|P_{nk}(j\Omega)|\le\frac{1}{2^{2n}}\Omega^{\sum_{i=1}^{n_L}n_{jmi}m_i}\sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega}\sum_{z=1}^{N_{n,p_j}}\prod_{k=1}^{\sum_{i=1}^{n_L}n_{jmi}m_i}\left|\frac{[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\Omega}\right|=\frac{1}{2^{2n}}\Omega^{\sum_{i=1}^{n_L}n_{jmi}m_i}c1(n)$$

where

$$c1(n)=\sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega}\sum_{z=1}^{N_{n,p_j}}\prod_{k=1}^{\sum_{i=1}^{n_L}n_{jmi}m_i}\left|\frac{[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\Omega}\right|$$

is a bounded constant which is dependent on n but independent of Ω. So when $\Omega\ll1$,

$$|P_{nk}(j\Omega)|\le\frac{1}{2^{2n}}\Omega^{\sum_{i=1}^{n_L}n_{jmi}m_i}c1(n)\approx0\quad\text{for } n=1,2,\ldots,\lfloor N/2\rfloor$$

(II) When $\Omega\gg1$, $|P_{nk}(j\Omega)|$ is considered when only $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ term or only $(\mathrm{d}(\cdot)/\mathrm{d}t)^3$ is introduced in the isolator respectively. The upper limits of $|P_{nk}(j\Omega)|$ under different conditions are presented in Table F1 where

$$C\_\text{const}(n)=\frac{1}{2^{2n}}\sum_{\omega_1+\cdots+\omega_{2n+1}=\Omega}\sum_{z=1}^{N_{n,p_j}}\prod_{k=1}^{\sum_{i=1}^{n_L}n_{jmi}m_i}\left|\frac{[j\omega^z_{l_k(1)}+\cdots+j\omega^z_{l_k(j^n_k)}]}{\Omega}\right|$$

is a bounded constant which is dependent on n but independent of Ω.

Table F1
Upper limit.

	$(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$	$(\mathrm{d}(\cdot)/\mathrm{d}t)^3$
$p_i=0,\ n=1$	$(1/\Omega)C\_\mathrm{const}(1)$	$\Omega C\_\mathrm{const}(1)$
$p_i=1$	$[1/(\Omega^{n+1})]C\_\mathrm{const}(n)$	$[\Omega^{n-1}]C\_\mathrm{const}(n)$
$p_i=2$	$[1/(\Omega^{2n+1})]C\_\mathrm{const}(n)$	$[1/\Omega]C\_\mathrm{const}(n)$
$p_i=3$	$[1/(\Omega^{3n+1})]C\_\mathrm{const}(n)$	$[1/(\Omega^{n-1})]C\_\mathrm{const}(n)$

In the table above, the second column is the upper limit when only $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ is introduced, and the third column is the upper limit when only $(\mathrm{d}(\cdot)/\mathrm{d}t)^3$ is introduced. The second row represents the upper limit of the pure cubic input nonlinearity, and the third to fifth rows represent the upper limit when all the n terms of nonlinear coefficients having the same degree of output nonlinearity p_i. It can be seen from the second column that the upper limit of the pure cubic input nonlinearity is proportional to$1/\Omega$, which tends to zero the slowest while comparing to the third to fifth row. So the absolute displacement transmissibility when only $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ is introduced in the isolator is very close to that in the linear case, i.e., only a little larger than the linear absolute displacement transmissibility.

In the third column, the upper limit of the cubic pure input nonlinearity is proportional to the frequency Ω, and the upper limit when all the nonlinear coefficients are composed of $C_{1,2}(1,1,1)$is proportional to Ω^{n-1}, so the absolute displacement transmissibility in high frequency when only $(\mathrm{d}(\cdot)/\mathrm{d}t)^3$ term is introduced in the isolator under base excitation is much larger than that in linear case.

(III) Consider firstly that only $(\cdot)^2(\mathrm{d}(\cdot)/\mathrm{d}t)$ is introduced in the isolator:

$$[T(\Omega)]^2=\left[P_0(j\Omega)+\sum_{n=1}^{\lfloor N/2\rfloor}\left(\sum_{n_{jm_1},\ldots,n_{jmn_L}=0,\ \sum_{i=1}^{n_L}n_{jmi}=n}^{n}\left(\prod_{i=1}^{n_L}C_{p_i,q_i}^{n_{jmi}}(l_{i1},l_{i2},l_{i3})P_{nk}(j\Omega)\right)\right)\right]$$
$$\times\left[P_0(-j\Omega)+\sum_{n=1}^{\lfloor N/2\rfloor}\left(\sum_{n_{jm_1},\ldots,n_{jmn_L}=0,\ \sum_{i=1}^{n_L}n_{jmi}=n}^{n}\left(\prod_{i=1}^{n_L}C_{p_i,q_i}^{n_{jmi}}(l_{i1},l_{i2},l_{i3})P_{nk}(-j\Omega)\right)\right)\right]$$
$$=\sum_{n=0}^{2\lfloor N/2\rfloor}\sum_{q=0}^{n}\sum_{\substack{jm_1,\ldots,n_{jmn_L}=0\\ \sum_{i=1}^{n_L}n_{jmi}=q,\ \sum_{i=1}^{n_L}n'_{jmi}=n-q}}^{n}\prod_{i=1}^{n_L}C_{p_i,q_i}^{n_{jmi}+n'_{jmi}}(l_{i1},l_{i2},l_{i3})P_{qk_1}(j\Omega)P_{(n-q)k_2}(-j\Omega)$$

and

$$\frac{\mathrm{d}[T(\Omega)]^2}{\mathrm{d}\beta_2}=\mathrm{Re}(P_0(j\Omega)P_1(-j\Omega))$$
$$+\sum_{n=2}^{2\lfloor N/2\rfloor}\sum_{q=0}^{n}\sum_{\substack{jm_1,\ldots,n_{jmn_L}=0\\ \sum_{i=1}^{n_L}n_{jmi}=q,\ \sum_{i=1}^{n_L}n'_{jmi}=n-q}}^{n}\left[\sum_{i=1}^{n_L}\left(\frac{n_{jmi}+n'_{jmi}}{C_{p_i,q_i}}\right)\frac{\mathrm{d}C_{p_i,q_i}}{\mathrm{d}\beta_2}\right]\prod_{i=1}^{n_L}C_{p_i,q_i}^{n_{jmi}+n'_{jmi}}(l_{i1},l_{i2},l_{i3})P_{qk_1}(j\Omega)P_{(n-q)k_2}(-j\Omega)$$

when $\Omega\approx 1$, $P_0(j)=-(1+j\xi_1)/\xi_1$, $P_1(-j)=[\left(-1/\xi_1^2+1/4\xi_1^4\right)-j\left(-1/2\xi_1+3/2\xi_1^3\right)]$

$$\mathrm{Re}[P_0(j)P_1(-j)]=-\left(\frac{1}{2\xi_1}+\frac{1}{2\xi_1^3}+\frac{1}{4\xi_1^5}\right)<0$$

Therefore, when $\Omega\approx 1$, there must exist a $\beta>0$ such that

$$\frac{d[T(\Omega)]^2}{d\beta_2}<0\quad \text{if } 0<\beta_2<\beta$$

When only the nonlinear term $(\mathrm{d}(\cdot)/\mathrm{d}t)^3$ is introduced in the isolator, the proof can be done by following the same procedure. Then Proposition 5 holds.

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