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A subsystem approach for analysis of dynamic vibration absorbers suppressing broadband vibration

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ABSTRACT

Dynamic vibration absorbers are commonly designed and tuned to suppress vibrations of one vibration mode of a vibrating structure even it is a multi degree-of-freedom (MDOF) or continuous structure. Resonance at other vibration modes of the structure may still occur if the exciting force has a wide frequency band. A subsystem approach is proposed for analysis of the added stiffness and damping of vibration absorbers to the primary structure to which they are attached. The transfer function between the counteracting force from the vibration absorber and the vibration amplitude can then be derived for the comparison of their counteracting forces to the primary system. The major advantage of using the proposed method is that different designs of vibration absorber can be analysed separately from the primary system and therefore the dynamics characteristics of different designs of vibration absorber can be compared efficiently.

1. Introduction

Forced vibration of machines and civil structures under broadband excitation is commonly seen. It can cause the problems of structural damage and reduce the accuracy in machine control. Therefore, various vibration suppression devices were proposed for different mechanical systems for reducing the vibration problems [1–17]. Performance indices were also proposed in different applications to measure the vibration suppression performance of these vibration suppression devices.

Design of vibration absorbers can be optimized according to the optimization criterion chosen to suit the requirements of a given application. One of the most common performance indices is H_∞ optimization criterion [1,4–8,10,13,14], which is to minimize the maximum amplitude magnification factor, i.e. H_∞ norm, of the primary system. The method can best handle a vibrating system under sinusoidal excitations. Another commonly used performance index is H_2 optimization criterion [2–4,9,11,12,14,15], which aims at reducing the total vibration energy of the vibrating system under white noise excitation. In this optimization criterion, the area under the frequency response curve of the primary system, i.e. H_2 norm, is minimized. However, the analytical derivations of both H_∞ and H_2 optimum parameters of the vibration absorbers found in literature are made for only one vibration mode of the primary vibrating system because of the complication in the derivation if other vibration modes are considered as well. If the primary vibrating system is a MDOF or continuous system, then the vibration responses of those untargeted modes will be neglected. However, the responses of these untargeted modes can be significant if the exciting force has a wide frequency band such that many vibration modes of the primary system are excited.

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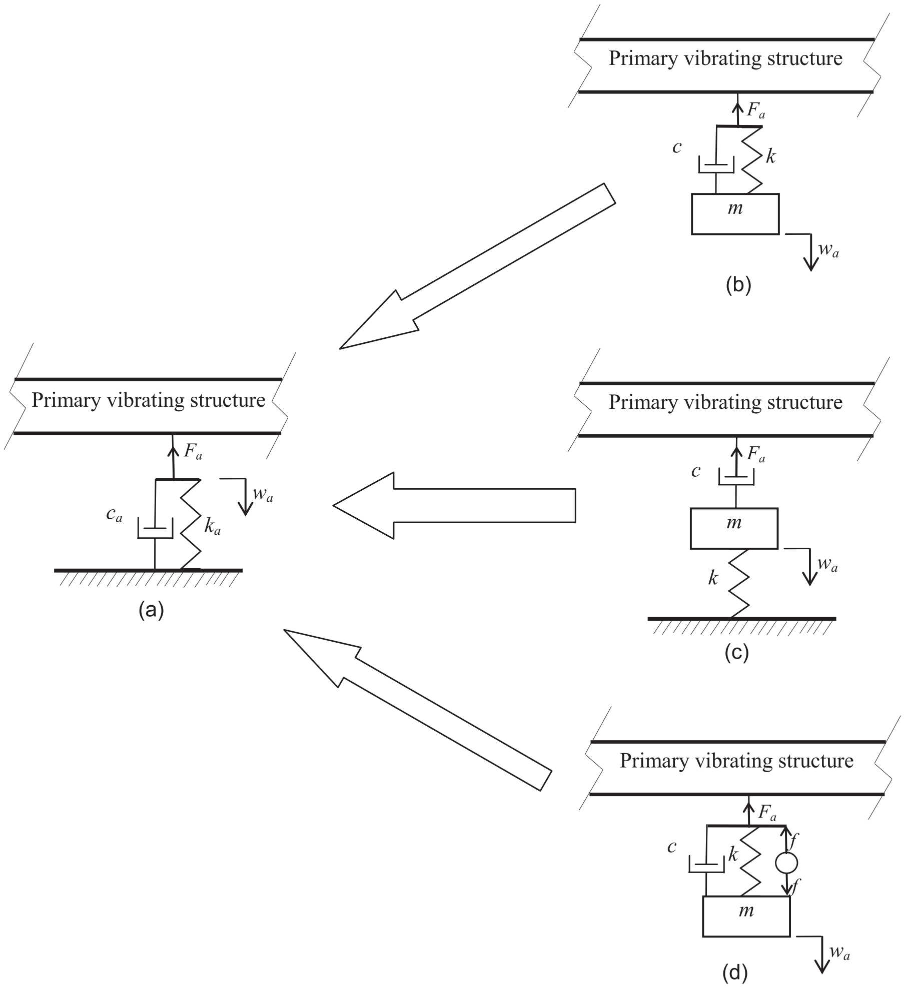


Fig. 1. Schematics of a vibrating structure attached with a (a) ground-hooked spring and damper (Case 1), (b) standard dynamic vibration absorber (Case 2), (c) variant dynamic vibration absorber (Case 3), and (d) active–passive vibration absorber (Cases 4–7).

In this article, a subsystem approach is proposed for analysis of the added stiffness and damping to the primary vibrating structure to which they are attached. The novelty in this work is the new approach of comparing the added stiffness and damping to the primary structure of different designs of dynamic vibration absorber (DVA) by "transforming" the DVA to the equivalent ground-hooked spring and damper as illustrated in Fig. 1. The added stiffness and damping to the primary vibrating structure of six different designs of dynamic vibration absorbers found in literature are derived and compared. The major advantage of the proposed analysis method is that the added stiffness and damping to the primary vibrating structure from the vibration absorber is modelled independent of the primary structure. The suppression capability of different absorbers can then be determined and compared by calculating the vibration response of the coupled system by using the mathematical model of the baseline model. Since there is no need to derive the individual vibration response expression of the coupled system in each of the cases being considered, the proposed method provides a more efficient way for the calculation of the vibration response of the structure after an absorber is attached. To the knowledge of the authors, the proposed approach of comparing different DVAs is original and never be reported before in the literature. As illustrated in the response spectra in Figs. 2 and 3, the added stiffness and damping due to the absorber will in general shift the resonant frequencies and damp down the peak responses of the vibrating structure. As illustrated in Fig. 1, the added stiffness and damping due to the absorber generate the counteracting force to the vibrating structure at the attachment point. While the traditional methods such as poles analysis [16,17] and the norms of the response transfer function of the primary structure [10,22] measure the vibrations of the primary structure together with the vibration absorber as a whole dynamic system, the proposed analysis method provides a simpler modelling and a more efficient way for estimating the added stiffness and damping from the absorber to the primary structure. Since the added damping effect from the absorbers to the primary system can be analysed separately from the primary system, different designs of vibration absorber can be compared more efficiently.

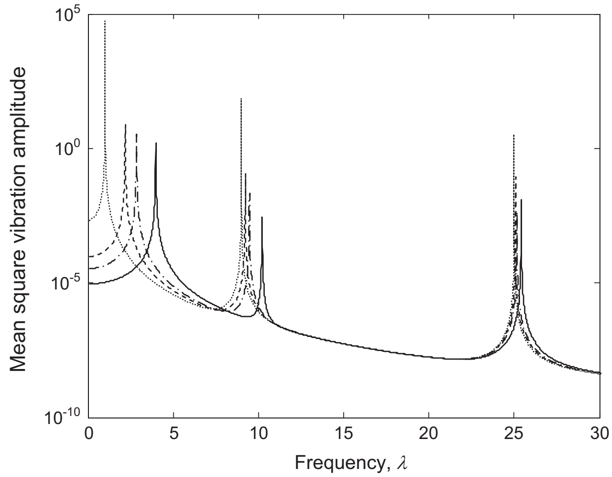


Fig. 2. Mean square vibration amplitude, $(\rho A\omega_1{}^2/L)\int_0^\infty |W(x,s)/P(s)|^2 dx$, of the simply supported beam in Fig. 1 with $c_a=0$ and stiffness $k_a/\rho A\omega_n^2=0$ ( ), 1 (——), 2 (- - -) and 5 (––––).

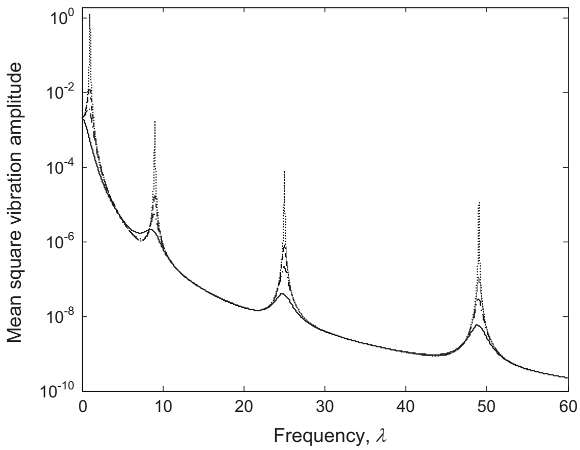


Fig. 3. Mean square vibration amplitude, $(\rho A\omega_1^2/L)\int_0^L |W(x,s)/P(s)|^2 dx$, of the simply supported beam in Fig. 1 with $k_a=0$ and dimensionless damping coefficient $c_a/\rho A\omega_1=0.01$ ( ), 0.1 (——), 0.2 (- - -) and 0.5 (––––).

Six common vibration absorber designs listed as Cases 2–7 in Table 1, including two passive and four active–passive absorbers, are analysed using the proposed subsystem analysis method. In comparison to the baseline ground hooked spring and damper referred to Case 1 in Table 1, some important characteristics of typical dynamic vibration absorbers on the untargeted resonances of the primary structures are revealed.

2. Multi degree-of-freedom vibrating system with a single degree-of-freedom vibration suppression device

Seven common types of vibration suppression devices as listed in Table 1 are considered in the following and their added stiffness and damping effect on the untargeted vibration modes are analysed and compared.

2.1. Vibration suppression with a ground hooked spring and damper (case 1)

Without loss of generality, consider a simply supported beam excited by an external force $p(t)g(x)$ within a domain $x\in(0,L)$ with a ground hooked spring and damper at $x=x_a$ as shown in Fig. 1. Assuming an Euler–Bernoulli beam, the equation of motion of the beam may be written as

$$\rho A\frac{\partial^2 w}{\partial t^2}+EI\frac{\partial^4 w}{\partial x^4}=p(t)g(x)+f_a(t)\delta(x-x_a) \tag{1a}$$

$$f_a(t)=-k_a w(x_a,t)-c_a\dot{w}(x_a,t) \tag{1b}$$

Table 1
Vibration suppression devices and their transfer functions.

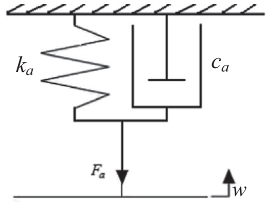
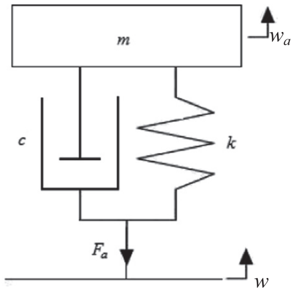
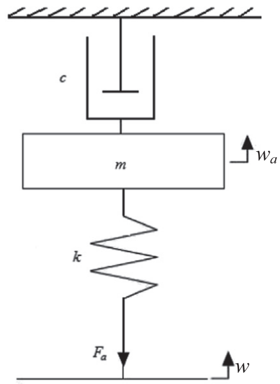
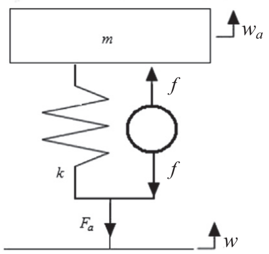
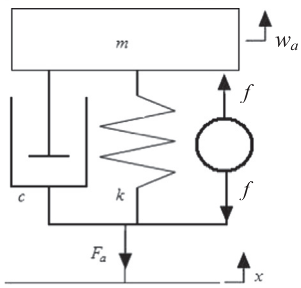
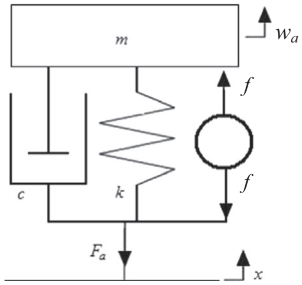
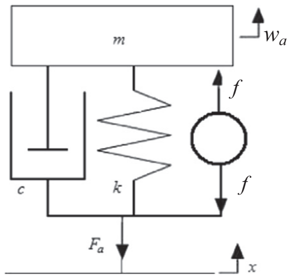
Case	Vibration suppression device	Refs.	Schematic diagrams	Transfer function of F_a/W
1	Spring and damper	[1]		$k+cs$
2	Standard dynamic vibration absorber	[1–4, 14, 18–21]		$\frac{ms^2(cs+k)}{ms^2+cs+k}$
3	Variant vibration absorber	[5–9]		$\frac{k(ms^2+cs)}{ms^2+cs+k}$
4	Undamped hybrid vibration absorber with displacement and velocity feedback	[10,11]		$\frac{mks^2}{ms^2+bs+k-a}$ where $f=aw_a-b\dot{w}_a$
5	Damped hybrid vibration absorber with displacement feedback	[13]		$\frac{ms^2(cs+k+a)}{ms^2+cs+k}$ where $f=a\,w$

Table 1 (*continued*)

Case	Vibration suppression device	Refs.	Schematic diagrams	Transfer function of F_a/W
6	Damped hybrid vibration absorber with displacement and velocity feedback	[12]		$\frac{ms^2((b+c)s+k+a)}{ms^2+cs+k}$ where $f=aw+b\dot{w}$
7	Damped hybrid vibration absorber with acceleration feedback	[15]		$\frac{ms^2(as^2+cs+k)}{ms^2+cs+k}$ where $f=a\ddot{w}$

where $f_a(t)$ is the force due to the spring and damper acting on the beam; E the modulus of elasticity; A the cross section area; I the second moment of area and ρ the density of the beam. $g(x)$ and $\delta(x-x_a)$ may be expanded as

$$g(x)=\sum_{i=1}^{\infty} a_i\varphi_i(x) \tag{1c}$$

$$\delta(x-x_a)=\sum_{i=1}^{\infty} b_i\varphi_i(x) \tag{1d}$$

where $\varphi_i(x)$ is the eigenfunction of the beam without any other devices and the Fourier coefficients a_i and b_i may be written, respectively, as

$$a_i=\frac{\int_0^L g(x)\varphi_i(x)dx}{L} \quad \text{and} \quad b_i=\frac{\varphi_i(x_a)}{L}. \tag{1e}$$

Performing Laplace transformation in Eq. (1b), the force acting on the beam due to the ground hooked spring and damper in the steady state may be written as

$$F_a=-k_aW(x_a)-\mathrm{j}\omega c_aW(x_a) \tag{2a}$$

where $\mathrm{j}^2=-1$. The mean square vibration amplitude of the beam can be derived and written as [21]

$$\frac{1}{L}\int_0^L\left|\frac{W(x,\mathrm{j}\omega)}{P(\mathrm{j}\omega)}\right|^2dx=\left(\frac{1}{\rho A\omega_n^2}\right)^2\sum_{u=1}^{\infty}\left|\frac{a_u-b_u\left(\frac{\sum_{v=1}^{\infty}a_v/(\gamma_v^2-\lambda^2)}{(1/(\eta+\mathrm{j}\zeta\lambda))+\sum_{v=1}^{\infty}b_v/(\gamma_v^2-\lambda^2)}\right)}{\gamma_u^2-\lambda^2}\right|^2 \tag{2b}$$

where

$$\omega_n={\beta_n}^2\sqrt{\frac{EI}{\rho A}},\quad \beta_n=\frac{n\pi}{L},\quad \gamma_v=\frac{\omega_v}{\omega_n},\quad \lambda=\frac{\omega}{\omega_n},\quad \eta=\frac{k_a}{\rho A{\omega_n}^2}\quad\text{and}\quad \zeta=\frac{c_a}{\rho A\omega_n}$$

A numerical example is presented in the following for illustration of the effect of this vibration suppression device (Case 1) to the untargeted vibration modes of the primary beam structure. Assuming the length of the beam is $L=1$ m with a cross section 0.025 m × 0.025 m. The spring and damper are attached at $x_a=0.5$ m of the beam as shown in Fig. 1. The material of the beam is aluminium with $\rho=2710$ kg m^{-3} and $E=6.9$ GPa. The Fourier coefficients a_i and b_i can be written, respectively, as

$$a_i=\tfrac{2}{n\pi L},\quad i=2n-1\quad n\in N$$
$$\text{else}\quad a_i=0,\ \text{and} \tag{3a}$$

$$b_i = \frac{2}{L}\sin\left(\frac{i\pi x_0}{L}\right) \tag{3b}$$

The beam is assumed to be excited by a white noise. The frequency responses of the beam in terms of mean square vibration amplitude with only spring support ($k_a \neq 0$, $c_a=0$) and only damper support ($k_a=0$, $c_a \neq 0$) are calculated using Eq. (2b) with results plotted in Figs. 2 and 3 respectively. The absorber is tuned at the first natural vibration mode of the beam and therefore ω_n is replaced by ω_1 in the calculations. The resonance frequencies are increased when k_a increases as shown in Fig. 2 and the mean square vibration amplitude at resonances are reduced when c_a increases as shown in Fig. 3. So the stiffness k_a creates a shift of the resonance frequencies of the beam while the damping c_a has the effect of changing the mean square vibration amplitude of the beam at resonances.

Using Case 1 as the baseline design model, the equivalent stiffness k_a and damping coefficient c_a of two passive dynamic vibration absorbers (Cases 2 and 3) and four passive–active vibration absorbers (Cases 4–7) are derived in the following and compared to Case 1. The comparisons give new insights into the dynamic effect of different designs of vibration absorbers applied to multi degree-of-freedom vibrating systems with excitation of wide frequency band.

2.2. Passive dynamic vibration absorber (Cases 2 and 3)

Consider the standard dynamic vibration absorber [1–3] (Case 2) as shown in Fig. 4. The equation of the motion can be expressed as

$$m\ddot{w}_a = -k(w_a - w(x_a)) - c(\dot{w}_a - \dot{w}(x_a)) \tag{4a}$$

$$f_a = k(w_a - w(x_a)) + c(\dot{w}_a - \dot{w}(x_a)) \tag{4b}$$

where f_a is the force generated by the vibration suppression device onto the primary mass. Eliminating w_a using Eqs. (4a) and (4b) and taking Laplace transformation, the transfer function of $F_a/W(x_a)$ can be derived as

$$\frac{F_a}{W(x_a)} = -\frac{ms^2(cs+k)}{ms^2+cs+k} \tag{5a}$$

where F_a and $W(x_a)$ are the Laplace transform of f_a and $w(x_a)$, respectively.

At steady-state vibration, Eq. (5a) may be rewritten with $s = \mathrm{j}\omega$ as

$$\begin{aligned} F_a &= \frac{m\omega^2(k+\mathrm{j}c\omega)}{k-m\omega^2+\mathrm{j}c\omega}W(x_a) \\ &= \frac{\left(mk\omega^2(k-m\omega^2)+m(c\omega^2)^2\right)-\mathrm{j}\omega^5 m^2 c}{(k-m\omega^2)^2+(c\omega)^2}W(x_a) \end{aligned} \tag{5b}$$

Comparing Eq. (2a) with Eq. (5b) above, the equivalent stiffness $k_a(\omega)$ and $c_a(\omega)$ of the absorber Case 2 to Case 1 can be written respectively as

$$k_a(\omega) = -\frac{mk\omega^2(k-m\omega^2)+m(c\omega^2)^2}{(k-m\omega^2)^2+(c\omega)^2}, \quad \text{and} \tag{6a}$$

$$c_a(\omega) = \frac{m^2 c\omega^4}{(k-m\omega^2)^2+(c\omega)^2}. \tag{6b}$$

Eqs. (6a) and (6b) show that $k_a(\omega) \approx 0$ and $c_a(\omega) \approx 0$ at low excitation frequencies ($\omega \approx 0$), suggesting no counteracting force and hence no vibration suppression effect of the dynamic vibration absorber. Eqs. (6a) and (6b) also show that when the excitation frequency is very large, $k_a(\omega)$ and $c_a(\omega)$ become respectively,

$$k_a(\infty) = k - \frac{c^2}{m} \tag{7a}$$

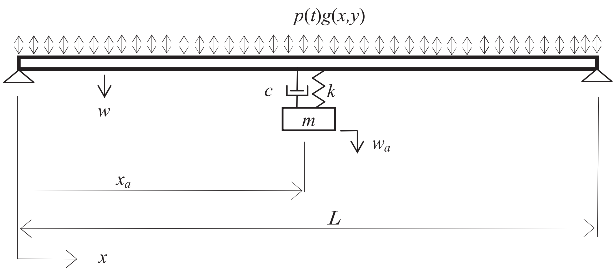


Fig. 4. Schematics of a simply supported beam with a standard dynamic vibration absorber (Case 2) excited by a disturbed force.

$$c_a(\infty) = c \tag{7b}$$

According to the discussion in Section 2.1, a change in the stiffness k_a in Case 1 would alter the resonant frequencies of the beam (Fig. 1) while an increase of damping c_a will reduce the mean square vibration amplitude at resonances of the beam. Eq. (7a) shows that the high order resonant frequencies of the beam with the absorber will be changed while Eq. (7b) shows that Case 2 design can provide damping effect to the high order resonances of the beam with the absorber.

Three common ways for tuning the standard vibration absorber Case 2, including the undamped classical tuning [1] ($c=0$ and $k/\rho A{\omega_1}^2 = 1$), H_∞ optimum tuning [1,4] and H_2 optimum tuning [2–4], are considered. The corresponding frequency spectra of the mean square vibration amplitude of the beam are calculated according to Eq. (2b) and plotted in Fig. 5. The mass ratio between the absorber mass and the beam mass μ is 0.05 in the calculations. The absorber is tuned to the first mode of the beam and a significant reduction of the mean square vibration amplitude of the whole beam can be observed at the first resonance of the beam for all three types of tuning of the absorber (Fig. 5). However, only H_∞ and H_2 optimum tunings provide some damping at other resonances of the beam as shown in Fig. 5. The solid line and the dotted line in Fig. 5 overlapped because they are too close to each other. This figure shows that both H_2 and H_∞ optimum tuning of the absorber produce similar mean square vibration amplitude of the beam considered. The frequency responses at resonances from mode 2 to mode 5 by H_∞ optimum tuning and H_2 optimum tuning, the equivalent stiffness and damping coefficient are calculated and listed in Table 2 for illustration of the dynamic effects of the absorber to the untargeted vibration modes of the beam. As shown in Table 2, both H_∞ optimum tuning and H_2 optimum tuning provide stiffness and damping effects on the high frequency modes of the beam with the absorber.

The equivalent stiffness and damping of absorber Case 2 to those of Case 1, i.e. $k_a(\omega)$ and $c_a(\omega)$, are calculated according to Eqs. (6a) and (6b) respectively and plotted in Fig. 6 to show the stiffness and damping effects of this absorber to the beam at different vibrating frequencies. Fig. 6 shows that this type of absorber provides both stiffness and damping to the beam at high frequencies but no effect at low frequencies. It reveals that if the absorber is tuned at the second or higher modal frequency of the beam, there will be a strong resonant response at the first or lower vibration modes. This prediction is confirmed by Fig. 7 with the absorber Case 2 tuned at the second vibration mode of the beam, i.e. $\omega_n = \omega_2$ and the mean square vibration amplitude of the beam is calculated according to Eq. (2b). As shown in Fig. 7, the absorber leads to strong suppression of the second resonance of the beam and some damping to the higher order resonances but no effect to the first resonance of the beam. This observation agrees with the calculation of Dayou [22] for this type of vibration absorber.

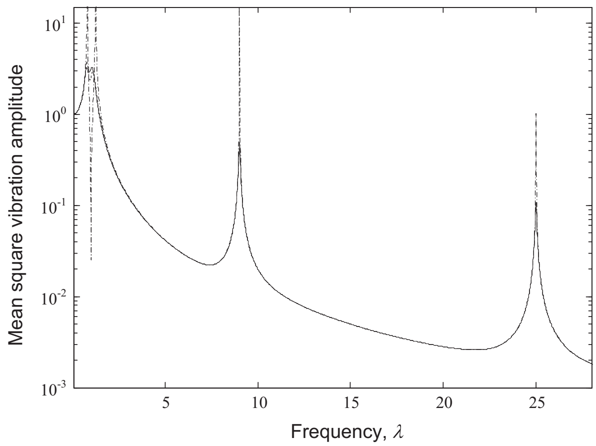


Fig. 5. Mean square vibration amplitude, $(\rho A{\omega_1}^2/L)\int_0^L |W(x,j\omega)/P(j\omega)|^2 dx$, of the simply support beam with a dynamic vibration absorber (Case 2) in Fig. 4. $\mu=0.05$. ---- H_∞ optimum tuning ($k_a = \rho A{\omega_1}^2/(1+\mu)$, $c_a = \rho A\omega_1\sqrt{3\mu/(8(1+\mu))}$), ——$H_2$ optimum tuning ($k_a = \rho A{\omega_1}^2(1/(1+\mu))\sqrt{(\mu+2)/2}$, $c_a = \rho A\omega_1\sqrt{\mu(3\mu+4)/(8(1+\mu)(2+\mu))}$), – – – –Undamped tuning ($k_a/\rho A{\omega_1}^2 = 1$, $c_a=0$).

Table 2
Vibration amplitude, equivalent stiffness and damping coefficient of the standard dynamic vibration absorber (Case 2) from modes 2 to 5 with H_∞ and H_2 optimum tuning conditions [14].

	DVA (Case 2) H_∞ optimum tuning			DVA (Case 2) H_2 optimum tuning		
	$\frac{\rho A{\omega_1}^2}{L}\int_0^L \lvert\frac{W}{P}\rvert^2 dx$	$\frac{k_a}{\rho A\omega_1^2}$	$\frac{c_a}{\rho A\omega_1}$	$\frac{\rho A{\omega_1}^2}{L}\int_0^L \lvert\frac{W}{P}\rvert^2 dx$	$\frac{k_a}{\rho A\omega_1^2}$	$\frac{c_a}{\rho A\omega_1}$
Mode 2 at 9 Hz	0.427	0.523	0.423	0.482	0.635	0.371
Mode 3 at 25 Hz	0.095	0.521	0.418	0.108	0.631	0.366
Mode 4 at 49 Hz	0.035	0.521	0.417	0.039	0.631	0.365
Mode 5 at 81 Hz	0.016	0.521	0.417	0.019	0.631	0.365

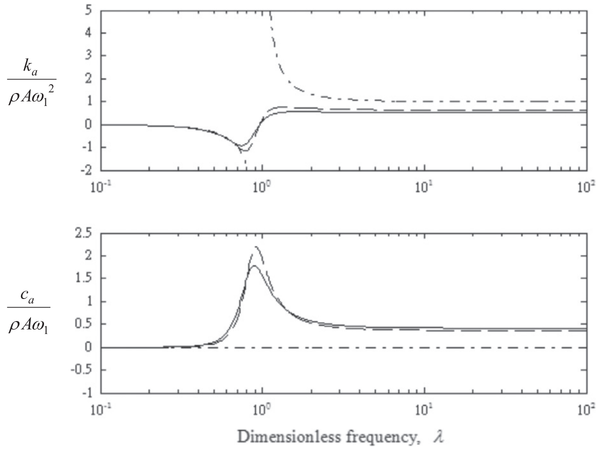


Fig. 6. Frequency responses of dimensionless equivalent stiffness $k_a/\rho A\omega_1^2$ and damping coefficient $c_a/\rho A\omega_1$ of the standard dynamic vibration absorber (Case 2) according to Eqs. (6a) and (6b), respectively. Figure legends are same as those in Fig. 5.

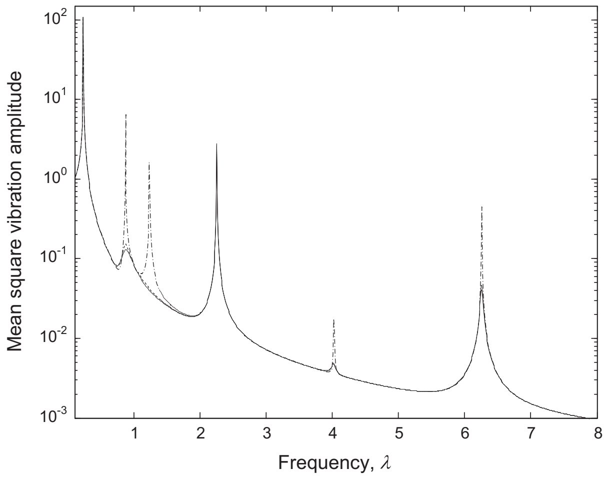


Fig. 7. Mean square vibration amplitude, $(\rho A\omega_2{}^2/L)\int_0^L |W(x,\mathrm{j}\omega)/P(\mathrm{j}\omega)|^2 dx$, of the simply-support beam with a dynamic vibration absorber (Case 2) in Fig. 4. μ=0.05, $x_a=0.35$. ----H_∞ optimum tuning ($k_a=\rho A\omega_2{}^2/(1+\mu)$, $c_a=\rho A\omega_2\sqrt{3\mu/(8(1+\mu))}$), ——$H_2$ optimum tuning ($k_a=\rho A\omega_2{}^2(1/(1+\mu))\sqrt{(\mu+2)/2}$, $c_a=\rho A\omega_2\sqrt{\mu(3\mu+4)/(8(1+\mu)(2+\mu))}$), - - - - Undamped tuning ($k_a/\rho A\omega_2{}^2=1$, c_a=0).

A variant form of the vibration absorber (Case 3) as shown in Table 1 was reported recently [5–9] and found to be more effective for vibration suppression than the standard DVA (Case 2). Its counteracting force F_a/W and its equivalent stiffness $k_a(\omega)$ and damping $c_a(\omega)$ are derived in Appendix A. Eqs. (A8) and (A9) show that for absorber Case 3, $k_a(0)=0$ and $c_a(0)=0$ so this type of absorber has no effect to the stiffness and damping of the beam at low excitation frequencies. Eqs. (A8) and (A9) also show that for absorber Case 3, $k_a(\infty)=k$ and $c_a(\infty)=0$, so this type of absorber has stiffness effect but no damping effect to the high frequency resonances of the beam with absorber. An interpretation of the little effect of the damper at high frequency in this case can also be deduced from Eq. (A1) in Appendix A of the manuscript, the equation of motion of the absorber mass m can be written as $(k-m\omega^2+\mathrm{j}c\omega)w_a=kw(x_a)$. When frequency $\omega\to\infty$, $\omega w_a\to 0$ and hence the damper force $\mathrm{j}c\omega w_a\to 0$. The vibration amplitude as well as the velocity of the absorber mass of Case 3 approach to zero at high frequency and therefore the damper has practically little effect at high frequency vibration.

2.3. Passive–active vibration absorber (Cases 4–7)

The same analysis method is also applied to four passive–active or hybrid vibration absorbers found in literature (Cases 4–7 listed in Table 1). Consider the hybrid absorber proposed by Chatterjee [10] (Case 4) with the active force $f=aw_a-b\dot{w}_a$. The counteracting force F_a, $k_a(\omega)$ and $c_a(\omega)$ are derived in Appendix B and written as

$$F_a=-\frac{mks^2}{ms^2+bs+k-a}W(x_a), \tag{8a}$$

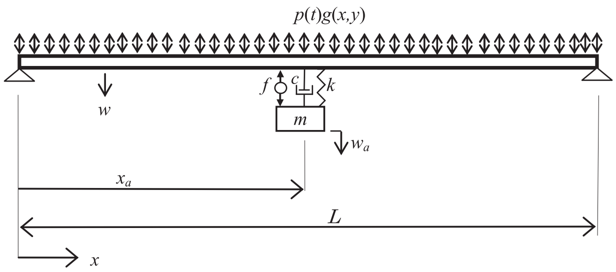


Fig. 8. Schematics of a simply supported beam with an active–passive vibration absorber (Cases 4–7) excited by a disturbed force. f is the active force of the absorber.

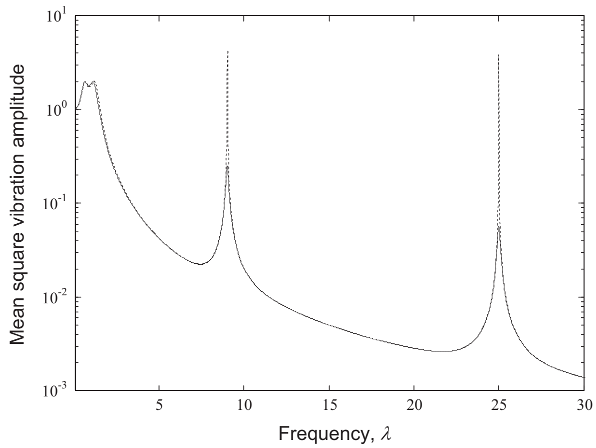


Fig. 9. Mean square vibration amplitude, $(\rho A\omega_1^2/L)\int_0^L |W(x,j\omega)/P(j\omega)|^2 dx$, of the simply-support beam with a dynamic vibration absorber in Fig. 4. μ=0.2. —— Case 4 [10]; ----Case 5 [13].

$$k_a = -\frac{mk\omega^2(k-a-m\omega^2)}{(k-a-m\omega^2)^2+(b\omega)^2} \quad \text{and} \quad c_a = -\frac{mkb\omega^2}{(k-a-m\omega^2)^2+(b\omega)^2} \tag{8b}$$

Eq. (8b) shows that for absorber Case 4, $k_a(0)=0$ and $c_a(0)=0$. Therefore, this type of absorber has no effect to the stiffness and damping of the beam at low excitation frequencies. Moreover, $k_a(\infty)=k$ and $c_a(\infty)=0$, indicating that this type of absorber affects the stiffness but has no impact on the damping for the high frequency modes of the beam with the absorber.

Absorber Case 5 proposed by Cheung et al. [13] with active force $f=aw(x_a)$ is considered and compared to Case 4 in the following. The counteracting force F_a, $k_a(\omega)$ and $c_a(\omega)$ of Case 5 are derived in Appendix C written as

$$F_a = -\frac{ms^2(cs+k+a)}{ms^2+cs+k}W(x_a), \tag{9a}$$

$$k_a = -\frac{m\omega^2(k+a)(k-m\omega^2)+(c\omega)^2}{(k-m\omega^2)^2+(c\omega)^2} \quad \text{and} \quad c_a = \frac{mc\omega^2(m\omega^2+a)}{(k-m\omega^2)^2+(c\omega)^2} \tag{9b}$$

Eq. (9b) shows that for absorber Case 5, $k_a(0)=0$ and $c_a(0)=0$, so this type of absorber has no effect on the stiffness and damping of the beam at low excitation frequency. Moreover, $k_a(\infty)=k+a-(c^2/m)$ and $c_a(\infty)=c$, suggesting that the absorber brings about stiffness as well as damping effects to the higher order resonances of the beam with the absorber.

The numerical examples in the previous section are used for illustrating the effect of the hybrid absorber Cases 4 and 5 to the untargeted vibration modes of the beam in the following. The hybrid absorber Case 4 or 5 is attached at $x_a=0.5$ m of the beam as shown in Fig. 8. The frequency spectra of the mean square vibration amplitude of the beam with absorber Cases 4 and 5 are calculated according to the theories derived in Refs. [10,13] and plotted in Fig. 9. The absorbers are optimized according to H_∞ criteria and tuned at the first natural frequency of the beam. Fig. 9 shows that both Cases 4 and 5 designs provide good vibration suppression at the tuned vibration mode of the beam. However, only Case 5, but not Case 4, has observable damping effect at the higher order resonances of the beam with the absorber.

The mean square responses at resonances from mode 2 to mode 5 of the beam, the equivalent stiffness and damping coefficient are calculated according to Eqs. (2b), (8b) and (9b) respectively and listed in Table 3 for illustration of the dynamic effects of the two absorbers to the untargeted vibration modes of the beam. As shown in Table 3, absorber Case 4 creates very little damping effect to the beam at high frequencies and therefore the resonant response at higher order

Table 3
Vibration amplitude, equivalent stiffness and damping coefficient of hybrid vibration absorbers (Cases 4 and 5) from modes 2 to 5 with H_∞ optimum tuning [10,13].

	Hybrid vibration absorber (Case 4)			Hybrid vibration absorber (Case 5)		
	$\frac{\rho A\omega_1{}^2}{L}\int_0^L \left\|\frac{W}{P}\right\|^2 dx$	$\frac{k_a}{\rho A\omega_1^2}$	$\frac{c_a}{\rho A\omega_1}$	$\frac{\rho A\omega_1{}^2}{L}\int_0^L \left\|\frac{W}{P}\right\|^2 dx$	$\frac{k_a}{\rho A\omega_1^2}$	$\frac{c_a}{\rho A\omega_1}$
Mode 2 at 9 Hz	4.380	2.704	0.066	0.254	5.921	0.755
Mode 3 at 25 Hz	3.826	2.526	0.007	0.057	5.919	0.706
Mode 4 at 49 Hz	0.355	2.507	0.002	0.021	5.919	0.700
Mode 5 at 81 Hz	0.329	2.503	0.001	0.010	5.919	0.699

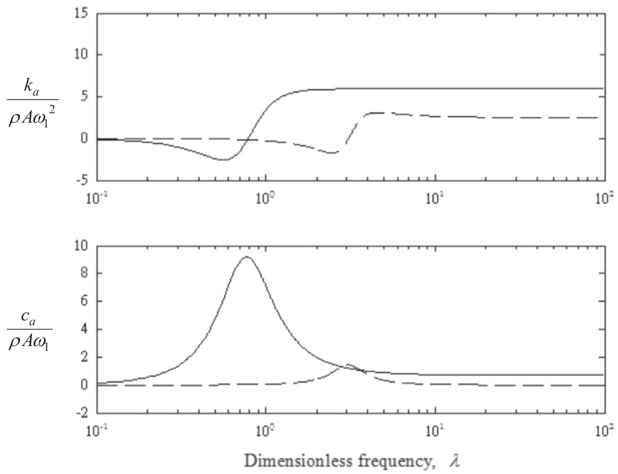


Fig. 10. Frequency responses of dimensionless equivalent stiffness $k_a/\rho A\omega_1^2$ and damping coefficient $c_a/\rho A\omega_1$ of (Case 4 ———) according to Eq. (8b) and (Case 5– – – –) according to Eq. (9c), respectively.

modes is much higher than that using absorber Case 5. This comparison shows that absorber Case 5 would be more suitable than Case 4 to damp down high frequency vibration of a multi degree-of-freedom vibrating structure under broadband excitation.

The equivalent stiffness and damping of absorber Cases 4 and 5 to those of Case 1, i.e. $k_a(\omega)$ and $c_a(\omega)$, are calculated using Eqs. (8b) and (9b) respectively and plotted in Fig. 10 to show the stiffness and damping effects of these absorbers on the beam at different vibrating frequencies. Fig. 10 shows that absorber Case 5 imposes more significant stiffness and damping effects than Case 4 absorber does at both high frequencies and low frequencies.

Two other reported hybrid vibration absorbers (Cases 6 and 7 in Table 1) are analysed using the same approach. The counteracting force F_a, equivalent stiffness $k_a(\omega)$ and damping coefficient $c_a(\omega)$ of these two types of absorber are derived in Appendices D and E, respectively. Based on the derived $k_a(\omega)$ and $c_a(\omega)$, it is found that both Cases 6 and 7 provide stiffness and damping effects to the primary structure at high vibration frequency but no effect at low frequency. A summary of these analysis results is listed in Table 4 for comparison.

Table 4 shows that both $k_a(0)$ and $c_a(0)$ of all considered absorbers (Cases 2–7) are zeroes. It means all these six absorber designs have no or little stiffness and damping effects on the primary structure at low frequencies. On the other hand, all these absorbers have $k_a(\infty)\neq 0$ and therefore can shift the high order resonant frequencies of the primary structure. Absorber Cases 2, 5–7 have $c_a(\infty)\neq 0$ so these absorbers can damp down the high order resonant responses of the primary structure. The hybrid absorber Cases 4–7 may provide negative restoring forces to the primary structure if $k_a(\omega)$ or $c_a(\omega)$ is negative and may cause unstable vibration of the primary structure. As shown in Table 4, $k_a(\omega)$ of all hybrid absorber Cases 4–7 may become negative depending on the excitation frequency ω and other parameters of the absorber. $c_a(\omega)$ of absorbers Cases 6 and 7 may become negative depending on the excitation frequency ω and other parameters of the absorber. The parameter values of these absorbers must be chosen carefully to prevent unstable vibration of the primary structure. The proposed method therefore provides an easy way for checking of the stability in the hybrid vibration absorber design.

3. Conclusion

A subsystem approach is proposed for the analysis of the added stiffness and damping effects of different types of dynamic vibration absorbers to a MDOF primary structure to which they are attached. Six common designs, including two

Table 4
Equivalent stiffness and damping coefficient of the vibration suppression device at $\omega \to 0$ and $\omega \to \infty$.

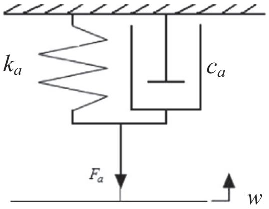
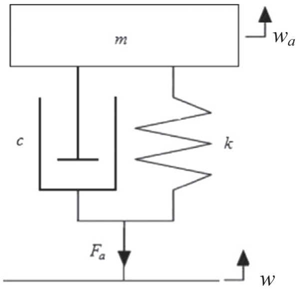
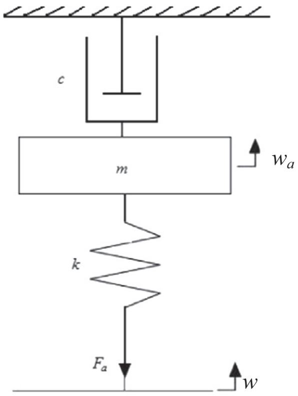
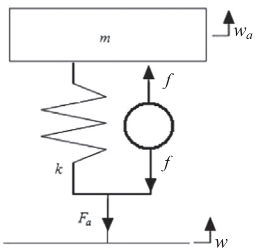
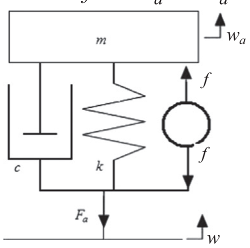
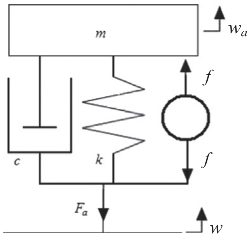
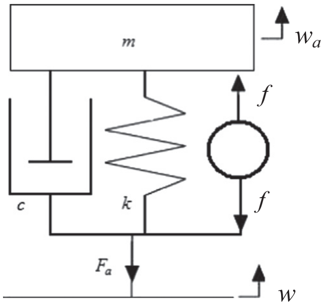
Case	Vibration suppression device	Refs.	Equivalent k_a and c_a to Case 1	$k_a(0)$	$c_a(0)$	$k_a(\infty)$	$c_a(\infty)$
1		[1]		k	c	k	c
2		[1–4,14,19–21]	$k_a(\omega) = -\frac{mk\omega^2(k-m\omega^2)+m(c\omega^2)^2}{(k-m\omega^2)^2+(c\omega)^2}$ $c_a(\omega) = \frac{m^2 c\omega^4}{(k-m\omega^2)^2+(c\omega)^2}$	0	0	$k-\frac{c^2}{m}$	c
3		[5–9]	$k_a(\omega) = \frac{-mk\omega^2(k-m\omega^2)+k(c\omega)^2}{(k-m\omega^2)^2+(c\omega)^2}$ $c_a(\omega) = \frac{k^2 c}{(k-m\omega^2)^2+(c\omega)^2}$	0	0	k	0
4	 where $f = aw_a - b\dot{w}_a$	[10,11]	$k_a = -\frac{mk\omega^2(k-a-m\omega^2)}{(k-a-m\omega^2)^2+(b\omega)^2}$ $c_a = \frac{mkb\omega^2}{(k-a-m\omega^2)^2+(b\omega)^2}$	0	0	k	0
5	 where $f = aw$	[13]	$k_a = -\frac{m\omega^2(k+a)(k-m\omega^2)+mc^2\omega^4}{(k-m\omega^2)^2+(c\omega)^2}$ $c_a = \frac{mc\omega^2(m\omega^2+a)}{(k-m\omega^2)^2+(c\omega)^2}$	0	0	$k+a-\frac{c^2}{m}$	c

Table 4 (*continued*)

Case	Vibration suppression device	Refs.	Equivalent k_a and c_a to Case 1	$k_a(0)$	$c_a(0)$	$k_a(\infty)$	$c_a(\infty)$
6	 where $f = aw - b\dot{w}$	[12]	$k_a = -\frac{m\omega^2(k+a)(k-m\omega^2)+mc\omega^4(b+c)}{(k-m\omega^2)^2+(c\omega)^2}$ $c_a = \frac{mc\omega^2(m\omega^2+a)-mb\omega^2(k-m\omega^2)}{(k-m\omega^2)^2+(c\omega)^2}$	0	0	$k+a-\frac{c(b+c)}{m}$	$b+c$
7	 where $f = a\ddot{w}$	[15]	$k_a = -\frac{m\omega^2(k-a\omega^2)(k-m\omega^2)+mc^2\omega^4}{(k-m\omega^2)^2+(c\omega)^2}$ $c_a = \frac{mc\omega^4(m-a)}{(k-m\omega^2)^2+(c\omega)^2}$	0	0	∞	$\frac{c(m-a)}{m}$

passive and four active–passive absorbers found in literature are analysed. Based on the proposed analysis method, some new characteristics of the absorbers are revealed. The first one is that all absorbers considered have zero or very little stiffness and damping effects to the primary system at vibrating frequencies lower to the tuned frequency of the absorber. This finding shows that the absorbers considered only suitable to be used to suppress the first or low order vibration mode of the primary system or else the primary system will have strong resonant vibration at those low order mode. This limitation of the absorbers concerned has not been pointed out explicitly in other research reported found in the literature. This finding provides new insights to common vibration absorbers reported in literature and it helps engineers understand the limitation of common DVAs when they want to apply them for solving vibration problems.

The second finding based on the proposed comparison is the damping effect to the primary system generated from the DVAs concerned at frequency higher than the tuned frequency of the DVA. Analyses show that the standard design of passive absorber (Case 2) can provide additional damping to the primary structure at high frequency but the variant design of passive absorber (Case 3) cannot. In the analysis of the four active–passive absorbers (Cases 4–7), it is found that three designs of active–passive absorbers (Cases 5–7) can provide additional damping to the primary structure at high frequency but the design of Case 4 cannot. The proposed analysis approach provides new insight to common designs of DVA found in the relevant literature.

The third finding based on the proposed comparison is the added negative stiffness and damping effects of active–passive vibration absorbers (Cases 4–7) leading to unstable vibration of the primary system. As shown in Table 4, the added stiffness, $k_a(\omega)$ of all hybrid absorber Cases 4–7 may become negative depending on the excitation frequency ω and other parameters of the absorber. The added damping, $c_a(\omega)$ of absorbers Cases 6 and 7 may become negative depending on the excitation frequency ω and other parameters of the absorber. The proposed analysis approach provides a tool to help the design of the active element in the active–passive vibration absorbers.

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Appendix A. (Case 3)

Case 3 Variant vibration absorber [5–9]

The equation of the motion can be written as

$$m\ddot{w}_a = -k(w_a - w(x_a)) - c\dot{w}_a \tag{A1}$$

$$f_a = k(w_a - w(x_a)) \tag{A2}$$

Performing Laplace Transformation on Eqs. (A1) and (A2), we may write

$$\frac{W_a}{W(x_a)} = \frac{k}{ms^2 + cs + k} \tag{A3}$$

$$F_a = k(W_a - W(x_a)) \tag{A4}$$

Using Eqs. (A3) and (A4) we may write

$$\frac{F_a}{W(x_a)} = -\frac{k(ms^2 + cs)}{ms^2 + cs + k} \tag{A5}$$

At steady state, we may replace s in Eq. (A5) by jω and write

$$\frac{F_a}{W(x_a)} = -\frac{k(-m\omega^2 + \mathrm{j}c\omega)}{k - m\omega^2 + \mathrm{j}c\omega} \tag{A6}$$

$$\begin{aligned} F_a = &-\frac{-mk\omega^2(k - m\omega^2) + k(c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2} W(x_a) \\ &-\mathrm{j}\omega \frac{k^2 c}{(k - m\omega^2)^2 + (c\omega)^2} W(x_a) \end{aligned} \tag{A7}$$

Comparing Eq. (2a) with (A7) above, the stiffness and damping of absorber Case 3 equivalent to Case 1 may be written respectively as

$$k_a = \frac{-mk\omega^2(k - m\omega^2) + k(c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2} \tag{A8}$$

$$c_a = \frac{k^2 c}{(k - m\omega^2)^2 + (c\omega)^2} \tag{A9}$$

Appendix B. (Case 4)

Hybrid dynamic vibration absorber [10] with active force $f = aw_a - b\dot{w}_a$ and $c = 0$. The equations of motion may be written as

$$m\ddot{w}_a = -k(w_a - w(x_a)) + aw_a - b\dot{w}_a \tag{B1}$$

$$f_a = k(w_a - w(x_a)) - aw_a + b\dot{w}_a \tag{B2}$$

Performing Laplace Transformation on Eqs. (B1) and (B2), we may write

$$(ms^2 + bs + k - a)W_a = kW(x_a) \tag{B3}$$

$$F_a = -ms^2 W_a \tag{B4}$$

Using Eqs. (B3) and (B4) we may write

$$\frac{F_a}{W(x_a)} = -\frac{mks^2}{ms^2 + bs + k - a} \tag{B5}$$

At steady state, we may replace s in Eq. (B5) by jω and write

$$\begin{aligned} F_a &= \frac{mk\omega^2}{k - a - m\omega^2 + \mathrm{j}b\omega} W(x_a) \\ &= \frac{mk\omega^2(k - a - m\omega^2)}{(k - a - m\omega^2)^2 + (b\omega)^2} W(x_a) - \mathrm{j}\omega \frac{mkb\omega^2}{(k - a - m\omega^2)^2 + (b\omega)^2} W(x_a) \end{aligned} \tag{B6}$$

Comparing Eq. (2a) with (B6) above, the stiffness and damping of absorber Case 4 equivalent to Case 1 may be written respectively as

$$k_a = -\frac{mk\omega^2(k - a - m\omega^2)}{(k - a - m\omega^2)^2 + (b\omega)^2} \tag{B7}$$

$$c_a = \frac{mkb\omega^2}{\left(k-a-m\omega^2\right)^2+(b\omega)^2} \tag{B8}$$

Appendix C. (Case 5)

Hybrid dynamic vibration absorber [13] with active force $f = aw(x_a)$. The equations of motion may be written as

$$m\ddot{w}_a = -k(w_a - w(x_a)) - c(\dot{w}_a - \dot{w}(x_a)) + f \tag{C1}$$

$$f_a = k(w_a - w(x_a)) + c(\dot{w}_a - \dot{w}(x_a)) - f \tag{C2}$$

Performing Laplace Transformation on Eqs. (C1) and (C2), we may write

$$F_a = -\frac{ms^2(cs+k+a)}{ms^2+cs+k} W(x_a) \tag{C3}$$

At steady state, we may replace s in Eq. (C3) by jω and write

$$\begin{aligned} F_a = &\frac{m\omega^2(k+a)\left(k-m\omega^2\right)+(c\omega)^2}{\left(k-m\omega^2\right)^2+(c\omega)^2} W(x_a) \\ &-\mathrm{j}\omega \frac{mc\omega^2\left(m\omega^2+a\right)}{\left(k-m\omega^2\right)^2+(c\omega)^2} W(x_a) \end{aligned} \tag{C4}$$

Comparing Eq. (2a) with (C4) above, the stiffness and damping of absorber Case 5 equivalent to Case 1 may be written respectively as

$$k_a = -\frac{m\omega^2(k+a)\left(k-m\omega^2\right)+(c\omega)^2}{\left(k-m\omega^2\right)^2+(c\omega)^2} \tag{C5}$$

$$c_a = \frac{mc\omega^2\left(m\omega^2+a\right)}{\left(k-m\omega^2\right)^2+(c\omega)^2} \tag{C6}$$

Appendix D. (Case 6)

Hybrid dynamic vibration absorber [12] with active force $f = aw(x_a) + b\dot{w}(x_a)$. The equations of motion may be written as

$$m\ddot{w}_a = -k(w_a - w(x_a)) - c(\dot{w}_a - \dot{w}(x_a)) + f \tag{D1}$$

$$f_a = k(w_a - w(x_a)) + c(\dot{w}_a - \dot{w}(x_a)) - f \tag{D2}$$

Performing Laplace Transformation on Eqs. (D1) and (D2), we may write

$$F_a = -\frac{ms^2((b+c)s+k+a)}{ms^2+cs+k} W(x_a) \tag{D3}$$

At steady state, we may replace s in Eq. (D5) by jω and write

$$\begin{aligned} F_a = &\frac{m\omega^2(k+a)\left(k-m\omega^2\right)+mc(b+c)\omega^4}{\left(k-m\omega^2\right)^2+(c\omega)^2} W(x_a) \\ &+\mathrm{j}\omega \frac{m\omega^2\left(b\left(k-m\omega^2\right)-c\left(m\omega^2+a\right)\right)}{\left(k-m\omega^2\right)^2+(c\omega)^2} W(x_a) \end{aligned} \tag{D4}$$

Comparing Eq. (2a) with (D4) above, the stiffness and damping of absorber Case 6 equivalent to Case 1 may be written respectively as

$$k_a = -\frac{m\omega^2(k+a)\left(k-m\omega^2\right)+mc(b+c)\omega^4}{\left(k-m\omega^2\right)^2+(c\omega)^2} \tag{D5}$$

$$c_a = \frac{m\omega^2\left(c\left(m\omega^2+a\right)-b\left(k-m\omega^2\right)\right)}{\left(k-m\omega^2\right)^2+(c\omega)^2} \tag{D6}$$

Appendix E. (Case 7)

Hybrid dynamic vibration absorber [15] with active force $f = a\ddot{w}(x_a)$. The equations of motion may be written as

$$m\ddot{w}_a = -k(w_a - w(x_a)) - c(\dot{w}_a - \dot{w}(x_a)) + f \tag{E1}$$

$$f_a = k(w_a - w(x_a)) + c(\dot{w}_a - \dot{w}(x_a)) - f \tag{E2}$$

Performing Laplace Transformation on Eqs. (E1) and (E2), we may write

$$F_a = -\frac{ms^2(as^2 + cs + k)}{ms^2 + cs + k} W(x_a) \tag{E3}$$

At steady state, we may replace s in Eq. (E3) by jω and write

$$F_a = \frac{m\omega^2(k - a\omega^2)(k - m\omega^2) + mc^2\omega^4}{(k - m\omega^2)^2 + (c\omega)^2} W(x_a) + \mathrm{j}\omega \frac{mc\omega^4(a - m)}{(k - m\omega^2)^2 + (c\omega)^2} W(x_a) \tag{E4}$$

Comparing Eq. (2a) with (E4) above, the stiffness and damping of absorber Case 7 equivalent to Case 1 may be written respectively as

$$k_a = -\frac{m\omega^2(k - a\omega^2)(k - m\omega^2) + mc^2\omega^4}{(k - m\omega^2)^2 + (c\omega)^2} \tag{E5}$$

$$c_a = \frac{mc\omega^4(m - a)}{(k - m\omega^2)^2 + (c\omega)^2} \tag{E6}$$

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