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Nonlinear Filtering with Fractional Brownian Motion Noise

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Abstract: In this paper, we consider the nonlinear filtering problem for the signal process corrupted by fractional Brownian motion noise. The signal process is assumed to be a Markov diffusion process. We obtain the Zakai equation and the Kushner-FKK equation in this setup. We also prove uniqueness of solution to these equations.

Keywords: Fractional Brownian motions; Kushner-FKK equation; Martingale problem; Nonlinear filtering; Zakai equation.

Mathematics Subject Classification: 60G35; 62M20; 93E11.

1. INTRODUCTION

The general filtering problem can be described as follows. The signal or system process $(X_t, 1 \leq t \leq T)$ is unobservable. Information about (X_t) is obtained by observing another process Y, which is a function of X

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corrupted by noise. The classic model for Y is

$$Y_t = \int_0^t h(X_s)ds + B_t, \quad 0 \le t \le T \tag{1}$$

where $(B_t, 0 \le t \le T)$ is assumed to be a Brownian motion (BM) and h is a measurable function. The observation σ-field $\mathcal{F}_t^Y = \sigma\{Y_s : 0 \le s \le t\}$ contains all the available information about X_t. The primary aim of filtering theory is to get an estimate of X_t based on the information $\mathcal{F}_t^Y$. This is given by the conditional distribution π_t, or, equivalently, the conditional expectation $E(f(X_t)|\mathcal{F}_t^Y)$ for a rich enough class of functions f. This estimate also minimizes the mean square error and, hence π is called the optimal filter.

Recently, the filtering problem for systems governed by fractional Brownian motions (FBM) has been studied by many authors. Kleptsyna et al. [11] consider the case that the signal process is driven by a FBM while the observation noise is still the usual BM. Kleptsyna et al. [10, 13], Kleptsyna and Le Breton [12], and Le Breton [16] studied the linear filtering problem with FBM as observation noise.

Nonlinear filtering problem with FBM observation noise has been studied by Coutin and Decreusefond [4], Gawarecki and Mandrekar [6], and Amirdjanova [1]. In these papers, the noise is modeled by a fractional Brownian motion B^H with Hurst parameter H (see next section for definition). Because of the well-known identity in Eq. (5), the above-mentioned authors modified the observation function to $\int_0^t \gamma_H(s,t)h(X_s)ds$ and their observation model becomes

$$Y_t = \int_0^t \gamma_H(s,t)(h(X_s)ds + dB_s). \tag{2}$$

Since $\int_0^\cdot h(X_S)ds$ is in the reproducing kernel Hilbert space (RKHS) of the Brownian motion, Kallianpur-Striebel formula can be applied, and Zakai equation and Kushner-FKK equation are then derived.

In this paper, we insist on using the original observation function and consider the following observation model:

$$Y_t = \int_0^t h(X_s)ds + B_s^H \tag{3}$$

where h is bounded and B_s^H is a FBM with Hurst parameter $H > \frac{1}{2}$. The signal process X_t is governed by the following stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \tag{4}$$

where W_t is a Brownian motion independent of B^H. Since the FBM is of long-term memory, the filtering must also be of long-term memory. To overcome this difficulty, we shall consider the filtering problem for the signal process, which is the corresponding path-valued process of the original signal. Here we want to mention that a Bayes formula, which is different from Eq. (15) in our paper, has been derived by Mandal and Mandrekar [17]. Since the terms there are not given explicitly and the adaptivity was not established, the authors did not develop a filtering theory out of that formula for the observation model in Eq. (3). Later on, jointly with Gawarecki, one of the authors studied the filtering problem in [6] for the model in Eq. (2), since the terms in the Bayes formula of [17] can be given explicitly in this case.

We now fix some notations. For a Polish space E, we denote the space of bounded continuous functions on E by $C_b(E)$. Let the bp-closure of a set V be the smallest set containing V, which is closed under bounded pointwise (bp) convergence of sequences. This set is denoted by $bp - closure(V)$. Let $D([0, T], E)$ denote the set of all cadlag (right continuous with left limits) functions from $[0, T]$ into E. Let $\mathscr{P}(E)$ and $\mathscr{M}_+(E)$ denote the spaces of probability measures and positive finite measures on E, respectively.

2. A TRANSFORMATION

In this section, we first recall the definition of the FBM and some of the basic properties. Then we introduce a transformation which will convert our model into the classical setup.

Let us fix a complete probability space $(\Omega, \mathscr{F}, P)$ on which all stochastic processes are defined.

Definition 2.1. For a given $H \in (1/2, 1)$, a stochastic process $B^H = (B_t^H, t \in [0, T])$ is a fractional Brownian motion with Hurst parameter H if:

(i) $B_0^H = 0$;
(ii) B^H is a zero-mean Gaussian process with continuous sample paths and stationary increments; and
(iii) The covariance function is given by

$$R_H(s, t) = \frac{V_H}{2}\left(|s|^{2H} + |t|^{2H} - |t - s|^{2H}\right)$$

where

$$V_H = Var\left(B_1^H\right) - \frac{-\Gamma(2 - 2H)\cos(\pi H)}{\pi H(2H - 1)}.$$

Now, from Theorem 4.5 in [2], we have that

$$B_t^H = \int_0^t \gamma_H(s,t)dB_s \tag{5}$$

where B_t is a Brownian motion and

$$\gamma_H(s,t) = \frac{s^{\frac{1}{2}-H}}{\Gamma\left(H-\frac{1}{2}\right)} \int_s^t \tau^{H-\frac{1}{2}}(\tau - s)^{H-\frac{3}{2}} d\tau.$$

Further

$$B_t = \int_0^t s^{H-\frac{1}{2}} d\left(\int_0^s k(s,\tau)dB_\tau^H\right) \tag{6}$$

where

$$k(s,\tau) = \frac{1}{\Gamma\left(H-\frac{1}{2}\right)}(s-\tau)^{\frac{1}{2}-H}\tau^{\frac{1}{2}-H}.$$

Define

$$Z_t = \int_0^t s^{H-\frac{1}{2}} d\left(\int_0^s k(s,\tau)dY_\tau\right) \tag{7}$$

and

$$S_t = \int_0^t s^{H-\frac{1}{2}} d\left(\int_0^s k(s,\tau)h(X_\tau)d\tau\right). \tag{8}$$

Note that Z_t is $\mathcal{F}_t^Y$-measurable. On the other hand, from [18], we have

$$Y_t = \int_0^t \gamma_H(s,t)dZ_s,$$

and, hence, $\mathcal{F}_t^Y = \mathcal{F}_t^Z$. So, Z_t can be regarded as the observation process. Then the observation model becomes

$$Z_t = S_t + B_t \tag{9}$$

We make the following boundedness assumption throughout this paper. *Condition* (BC) : $b, \sigma \in C_b^1(\mathbb{R})$ and $h \in C_b^2(\mathbb{R})$.

Let

$$\widehat{\mathbb{R}} = \{(t, x^t) : t \geq 0, x \in C([0,T],\mathbb{R})\}$$

where $x^t(S) = x(t \wedge s)$.

The following theorem, converting the filtering model to a classical one, is the key step in this paper.

Theorem 2.2. *The observation model is equivalent to*

$$Z_t = \int_0^t G(s, X^s)ds + B_t \tag{10}$$

where X^s is the path of X upto time s, G is a measurable map from $\widehat{\mathbb{R}}$ to $\mathbb{R}$ given by

$$\begin{aligned} G(s, X^s) = {} & \frac{h(X_0)\beta\left(\frac{3}{2} - H, \frac{3}{2} - H\right)(2 - 2H)}{\Gamma\left(H - \frac{1}{2}\right)} s^{\frac{1}{2}-H} \\ & + \frac{2 - 2H}{\Gamma\left(H - \frac{1}{2}\right)} s^{\frac{1}{2}-H} \int_0^1 (u(1-u))^{\frac{1}{2}-H} \int_0^{su} Lh(X_r)dr\,du \\ & + \frac{1}{\Gamma\left(H - \frac{1}{2}\right)} s^{\frac{3}{2}-H} \int_0^1 (u(1-u))^{\frac{1}{2}-H} Lh(X_{su})udu \\ & + \frac{2 - 2H}{\Gamma\left(H - \frac{1}{2}\right)} s^{\frac{1}{2}-H} \int_0^s \int_{\frac{r}{s}}^1 (u(1-u))^{\frac{1}{2}-H} du \\ & \times (\sigma h')(X_r)\left(\sigma(X_r)^{-1} dX_r - \left(b\sigma^{-1}\right)(X_r)dr\right) \\ & - \frac{1}{\Gamma\left(H - \frac{1}{2}\right)} S^{-1} \int_0^s r^{\frac{3}{2}-H} \left(1 - \frac{r}{s}\right)^{\frac{1}{2}-H} \\ & \times (\sigma h')(X_r)\left(\sigma(X_r)^{-1} dX_r - \left(b\sigma^{-1}\right)(X_r)dr\right). \end{aligned} \tag{11}$$

Proof. Applying Itô's formula to Eq. (4), we have

$$h(X_\tau) = h(X_0) + \int_0^\tau Lh(X_r)dr + \int_0^\tau (\sigma h')(X_r)dW_r$$

where

$$Lh(x) = \frac{1}{2}\sigma^2(x)h''(x) + b(x)h'(x).$$

Then, by Eq.(8),

$$S_t = S_t^1 + S_t^2 + S_t^3$$

where

$$\begin{aligned} S_t^1 &= \int_0^t s^{H-\frac{1}{2}} d\left(\int_0^s k(s, \tau)h(X_0)d\tau\right) \\ &= h(X_0)\int_0^t s^{H-\frac{1}{2}} d\left(\int_0^s \frac{1}{\Gamma\left(H - \frac{1}{2}\right)}(s-\tau)^{\frac{1}{2}-H}\tau^{\frac{1}{2}-H}d\tau\right) \\ &= \frac{h(X_0)\beta\left(\frac{3}{2} - H, \frac{3}{2} - H\right)(2 - 2H)}{\Gamma\left(H - \frac{1}{2}\right)\left(\frac{3}{2} - H\right)} t^{\frac{3}{2}-H}, \end{aligned}$$

$$
\begin{aligned}
S_t^2 &= \int_0^t S^{H-\frac{1}{2}} d\Big(\int_0^s k(s,\tau)\int_0^\tau Lh(X_r)dr\,d\tau\Big)\\
&= \frac{1}{\Gamma\big(H-\frac{1}{2}\big)}\int_0^t s^{H-\frac{1}{2}} d\Big(\int_0^1 (u(1-u))^{\frac{1}{2}-H} s^{2-2H}\int_0^{su} Lh(X_r)dr\,du\Big)\\
&= \frac{2-2H}{\Gamma\big(H-\frac{1}{2}\big)}\int_0^t s^{\frac{1}{2}-H}\int_0^1 (u(1-u))^{\frac{1}{2}-H}\int_0^{su} Lh(X_r)dr\,du\,ds\\
&\quad + \frac{1}{\Gamma\big(H-\frac{1}{2}\big)}\int_0^t s^{\frac{3}{2}-H}\int_0^1 (u(1-u))^{\frac{1}{2}-H} Lh(X_{su})u\,du\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
S_t^3 &= \int_0^t s^{H-\frac{1}{2}} d\Big(\int_0^s k(s,\tau)\int_0^\tau (\sigma h')(X_r)dW_r d\tau\Big)\\
&= \frac{1}{\Gamma\big(H-\frac{1}{2}\big)}\int_0^t s^{H-\frac{1}{2}} d\Big(\int_0^s (s-\tau)^{\frac{1}{2}-H}\tau^{\frac{1}{2}-H}\int_0^\tau (\sigma h')(X_r)dW_r d\tau\Big)\\
&= \frac{1}{\Gamma\big(H-\frac{1}{2}\big)}\int_0^t s^{H-\frac{1}{2}} d\Big(\int_0^s\int_\tau^s (s-\tau)^{\frac{1}{2}-H}\tau^{\frac{1}{2}-H} d\tau(\sigma h')(X_r)dW_r\Big)\\
&= \frac{1}{\Gamma\big(H-\frac{1}{2}\big)}\int_0^t s^{H-\frac{1}{2}}\int_0^s \frac{d}{ds}\int_\tau^s (s-\tau)^{\frac{1}{2}-H}\tau^{\frac{1}{2}-H} d\tau(\sigma h')(X_r)dW_r ds\\
&= \frac{2-2H}{\Gamma\big(H-\frac{1}{2}\big)}\int_0^t s^{\frac{1}{2}-H}\int_0^s\int_{\frac{r}{s}}^1 (u(1-u))^{\frac{1}{2}-H} du(\sigma h')(X_r)dW_r ds\\
&\quad - \frac{1}{\Gamma\big(H-\frac{1}{2}\big)}\int_0^t s^{-1}\int_0^s r^{\frac{3}{2}-H}\Big(1-\frac{r}{s}\Big)^{\frac{1}{2}-H}(\sigma h')(X_r)dW_r ds
\end{aligned}
$$

Note that

$$
dW_r = \sigma(X_r)^{-1}dX_r - \big(b\sigma^{-1}\big)(X_r)dr.
$$

Eq. (11) follows easily. □

3. BAYES FORMULA AND FILTERING EQUATIONS

From the observation model in Eq. (10), we see that we need to enrich the signal process. Let $\overline{X}_t = (t, X^t)$. Then (t, X^t) is a $\widehat{\mathbb{R}}$-valued Markov process. We now characterize its generator. Let

$$
\mathscr{D}(A_0) = \left\{ F \in C_b(\widehat{\mathbb{R}}) : \begin{array}{c} F(t,y) = \phi(t, y(t_1\wedge t), \cdots, y(t_n \wedge t)) \\ \text{for some } n \geq 1, 0 \leq t_1 < \cdots < t_n \leq T, \\ \phi \in C_b^2\big(\mathbb{R}^{n+1}\big) \end{array} \right\}. \tag{12}
$$

For any $F \in \mathcal{D}(A_0)$, define

$$A_0F(t, X^t) = \partial_0\phi(t, X_{t_1\wedge t,}, \cdots, X_{t_n\wedge t}) + \sum_{i=1}^{n} 1_{t<t_i}\partial_i\phi(t, X_{t_1\wedge t,}, \cdots, X_{t_n\wedge t})b(X_t) + \frac{1}{2}\sum_{i,j=1}^{n} 1_{t<t_i\wedge t_j}\partial_{ij}^2\phi(t, X_{t_1\wedge t,}, \cdots, X_{t_n\wedge t})\sigma^2(X_t) \qquad (13)$$

An application of Itô's formula to $F(t, X^t)$ for $F \in \mathcal{D}(A_0)$ implies that

$$F(\overline{X}_t) - \int_0^t A_0F(\overline{X}_s)ds$$

is a martingale. Hence, A_0 is a part of the generator of $\overline{X}_t$.

Define the optimal filter by

$$\pi_tF = \mathbb{E}(F(T, X^t)|\mathcal{F}_t^Y), \quad \forall F \in C_b(\widehat{\mathbb{R}}). \qquad (14)$$

Using the classical nonlinear filtering theory (see [9]), we obtain the following results.

Theorem 3.1. *Let the signal process (X_t) and the observation process (Y_t) be given by Eq.(4) and Eq.(5), respectively. Let P_x denote the probability distribution of the processes X^t. Then*

(i)

$$\pi_tF = \frac{\int F(t, x)e^{\int_0^t G(s,x^s)dZ_s - \frac{1}{2}\int_0^t G^2(s,x^s)ds}dP_X(x)}{\int e^{\int_0^t G(s,x^s)dZ_s - \frac{1}{2}\int_0^t G^2(s,x^s)ds}dP_X(x)}, \quad \forall F \in C_b(\widehat{\mathbb{R}}). \qquad (15)$$

(ii) *The process π_t satisfies the Kushner-FKK equation*

$$\pi_tF = \pi_0F + \int_0^t \pi_s(A_0F)ds + \int_0^t (\pi_s(GF) - \pi_sG\pi_sF)d\xi_s, \quad \forall F \in \mathcal{D}(A_0).$$

where $\xi_t = Z_t - \int_0^t \pi_s\, Gds$ is the innovation process.

(iii) *Define μ_t by*

$$\mu_tF = \pi_tF \exp\left(\int_0^t \pi_sGdZ_s - \frac{1}{2}\int_0^t |\pi_sG|^2ds\right) \qquad (16)$$

for any $F \in C_b(\widehat{\mathbb{R}})$. The process μ_t satisfies the Zakai equation:

$$\mu_tF = \pi_0F + \int_0^t \mu_s(A_0F)ds + \int_0^t \mu_s(GF)dZ_s, \quad \forall F \in \mathcal{D}(A_0) \qquad (17)$$

and

$$\pi_tF = \mu_tF/\mu_t1. \qquad (18)$$

Proof. We only need to verify that

$$\mathbb{E}\int_0^T G(s, X^s)^2 ds < \infty.$$

Here we only check the last term in Eq. (11) (denote it by G_5). Note that

$$\begin{aligned}\mathbb{E}\int_0^T G_5(s, X^s)^2 ds &\leq c\mathbb{E}\int_0^T s^{-2}\int_0^s r^{3-2H}\left(1-\frac{r}{s}\right)^{1-2H}(\sigma h')^2(X_r)dr\, ds\\ &\leq c\int_0^T s^{2-2H}ds\int_0^1 t^{3-2H}(1-t)^{1-2H}dt < \infty. \qquad \square\end{aligned}$$

4. UNIQUENESS OF μ_t AND π_t

The establishment of the uniqueness for the solution to the Kushner-FKK and the Zakai equation is very important for these equations to be useful to the filtering problem. Such problems have been studied by various authors using essentially operator techniques and via martingale problem. See [3] and the references therein.

In this section, we first recall some of the results in [3] and then apply them to our present setup.

Definition 4.1. Let B be an operator on $C(E)$ with domain $\mathscr{D}(B) \subset C_b(E)$. A process $(U_t, 0 \leq t \leq T)$ defined on some probability space $(\Omega, \mathscr{F}, P)$ is said to be a solution to the martingale problem for (B, v) if:

(i) $P \circ U_0^{-1} = v$;
(ii) $\int_0^t E|BF(U_s)|ds < \infty$, for every $t \leq T$, $F \in \mathscr{D}(B)$; and
(iii) for all $F \in \mathscr{D}(B)$, $F(U_t) - \int_0^t BF(U_s)ds$ is a martingale.

The martingale problem for (B, v) is said to be well-posed if there exists a solution U to the martingale problem for (B, v) and any two solutions have the same finite dimensional distributions.

The $D([0, T], E)$-martingale problem for (B, v) is said to be well-posed if there exists a cadlag solution $(U_t, 0 \leq t \leq T)$ to the martingale problem and for any two solutions with cadlag paths, their finite dimensional distributions are the same.

The following conditions were imposed by Bhatt et al. [3].

(C1) There exists $\Theta \in C(E)$, satisfying

$$|BF(x)| \leq C_F\Theta(x), \qquad F \in \mathscr{D}(B), \;\; x \in E.$$

(C2) There exists a countable subset $\{F_n\} \subset \mathscr{D}(B)$, such that

$$bp - closure\big(\{(F_n, \Theta^{-1}BF_n) : n \geq 1\}\big) \supset \big\{(F, \Theta^{-1}BF) : F \in \mathscr{D}(B)\big\}.$$

(C3) $\mathscr{D}(B)$ is an algebra that separates points in E and contains the constant functions

(C4) The $D([0,T], E)$ martingale problem for (B, δ_x) is well-posed for every $x \in E$.

(C5) For all $\nu \in \mathscr{P}(E)$, any progressively measurable solution to the martingale problem for (B, ν) admits a cadlag modification.

The next theorem gives the uniqueness for the solution to the Zakai Eq. (17).

Theorem 4.2. *If* (ρ_t) *is an* $\mathscr{F}_t^Y$*-adapted* $\mathscr{M}_+(\widehat{\mathbb{R}})$*-valued cadlag process satisfying*

$$\rho_t F = \pi_0 F + \int_0^t \rho_s(A_0 F)ds + \int_0^t \rho_s(GF)dZ_s, \quad \forall F \in \mathscr{D}(A_0) \tag{19}$$

and

$$\int_0^T \mathbb{E}\big[\rho_t(\widehat{\mathbb{R}})\big]dt < \infty, \tag{20}$$

then $\rho_t = \mu_t$, *for all* $0 \leq t \leq T$ *a.s., where* μ_t *is defined by Eq.* (16).

Proof. The conclusion follows from Theorem 4.1 in [3] if we verify the conditions (C1)–(C5) for A_0. It is easy to see that (C1) holds for A_0 with $\Theta(x) \equiv 1$. (C2) and (C3) can be verified easily. Now we check (C4) and (C5).

Let $\overline{X}_t$ be a $\widehat{\mathbb{R}}$-valued measurable solution to the matingale problem for A_0. It is clear that $\overline{X}_t = (t, \widetilde{X}_t)$ where $\widetilde{X}_t(\cdot) \in C([0,T], \mathbb{R})$ and stopped at t. Take $F(t, \widetilde{X}_t) = \phi(\widetilde{X}_t(t \wedge t_1))$. Then for $t \geq t_1$,

$$\phi(\widetilde{X}_t(t_1)) - \int_0^{t_1} \left(\phi'(\widetilde{X}_s(s))b(\widetilde{X}_s(s)) + \frac{1}{2}\phi''(\widetilde{X}_s(s))\sigma^2(\widetilde{X}_s(s))\right)ds$$

is a martingale. This implies that $\phi(\widetilde{X}_t(t_1))$, $t \geq t_1$ is a martingale. We may replace ϕ by ϕ^2, so that $\phi^2(\widetilde{X}_t(t_1))$, $t \geq t_1$ is also a martingale. This implies that $\forall t, t' \geq t_1$, we have $\widetilde{X}_t(t_1) = (\widetilde{X}_{t'}(t_1))$, a.s.

Take $F(t, (\widetilde{X}_t) = \phi(\widetilde{X}_t(t))$ (namely, take $t_1 = T$ in previous choice of F). Then

$$\phi(\widetilde{X}_t(t)) - \int_0^t L\phi(\widetilde{X}_s(s))ds$$

is a martingale. Hence, the process $t \to \widetilde{X}_t(t)$ has a continuous version X_t. Therefore, for fixed t and s, $(t \geq s)$, we have $\widetilde{X}_t(s) = X_S$ a.s. Since $\widetilde{X}_t(s)$ and X_s are continuous in s, we have for fixed t, $\widetilde{X}_t(s) = X_s$ a.s. for all $s \leq t$. This proves that for fixed t, $\widetilde{X}_t = X^t$ a.s. Namely, X^t is a continuous version of $\widetilde{X}_t$. Since X^t is continuous, this proves (C5). This also shows the uniqueness of the $D([0,T], \widehat{\mathbb{R}})$-martingale problem for A_0 and, hence, (C4) holds. □

Theorem 4.3. *Let $(v_t) \subset \mathcal{P}(\widehat{\mathbb{R}})$ be an F_t^Y-adapted cadlag process, which is a solution of the Kushner-FKK equation*

$$v_t F = v_0 F + \int_0^t v_s(A_0 F) ds + \int_0^t [v_s(GF) - (v_s G)(v_s F)] d\xi_s^v, \quad \forall F \in \mathcal{D}(A_0),$$

where

$$\xi_s^v = Z_t - \int_0^t v_s G ds.$$

If $\{\rho_t\} \subset M_+(\widehat{\mathbb{R}})$ defined by

$$\rho_t F = v_t f \exp\left(\int_0^t v_s G dZ_s - \frac{1}{2} \int_0^t |v_s G|^2 ds \right), \quad \forall F \in C_b(\widehat{\mathbb{R}}), \tag{21}$$

satisfies Eq. (20) *then $v_t = \pi_t$, for all $0 \leq t \leq T$ a.s.*

Proof. An application of Itô formula shows that ρ defined by Eq. (21) satisfies the Zakai Eq. (19). Theorem 4.2 is then applicable and it implies that $\rho_t = \mu_t$, for all $t \leq T$ a.s. Since

$$v_t F = \frac{\rho_t F}{\rho_t 1}, \quad \pi_t F = \frac{\mu_t F}{\mu_t 1},$$

the result is proved. □

5. ROBUSTNESS

In this section, we consider a sequence of functions $(b^n(x), \sigma^n(x), h^n(x))$, which converges to $(b(x), \sigma(x), h(x))$ uniformly in x as $n \to \infty$. Further, we assume that (b^n, σ^n, h^n) satisfies the condition (BC) uniformly in n, and assume $\sigma(x)$ and $\sigma^n(x)$ have lower bounds, different from 0 uniformly in n. Let X^n and Z^n be defined on a probability space $(\Omega^n, \mathcal{F}^n, P^n)$ by

$$dX_t^n = b^n(X_t^n) dt + \sigma^n(X_t^n) dW_t^n$$

and

$$Z_t^n = \int_0^t G^n(s, (X^n)^s)ds + B_t^n.$$

We consider the convergence of the optimal filter π_t^n.

Theorem 5.1. *As $n \to \infty$, we have $\pi_t^n F \to \pi_t F$ in distribution.*

Proof. Define probabilities Q^n and Q by

$$\begin{aligned}\frac{dP^n}{dQ^n} &= \exp\left(\int_0^T G^n(s, (X^n)^s)dZ_s^n - \frac{1}{2}\int_0^T G^n(s, (X^n)^s)^2 ds\right)\\ &\equiv L_T^n(X^n, Z^n)\end{aligned}$$

and

$$\begin{aligned}\frac{dP}{dQ} &= \exp\left(\int_0^T G(s, X^s)dZ_s - \frac{1}{2}\int_0^T G(s, X^s)^2 ds\right)\\ &\equiv L_T(X, Z).\end{aligned}$$

Then, under Q_n, Z^n is a Brownian motion independent of X^n; under Q, Z is a Brownian motion independent of X. We now verify that

$$Q^n \circ (X^n, Z_T^n, L_T^n(X^n, Z^n))^{-1} \to Q \circ (X, Z, L_T, (X, Z))^{-1}.$$

From the theory of SDE, it is easy to show that

$$Q^n \circ (X^n)^{-1} = P^n \circ (X^n)^{-1} \to P \circ X^{-1}.$$

Let $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{P})$ be a probability space on which $(\widetilde{X}^n, \widetilde{X}, \widetilde{Z})$ are defined such that $\widetilde{X}^n \to \widetilde{X}$ a.s., $Q^n \circ (X^n, Z^n)^{-1} = \widetilde{P} \circ (\widetilde{X}^n, \widetilde{Z})^{-1}$, and $\widetilde{Z}$ is a Brownian motion independent of $\widetilde{X}^n$ under $\widetilde{P}$. Then

$$\mathbb{E}^{Q^n} f(X^n, Z^n, L_T^n, (X^n, Z^n)) = \mathbb{E}^{\widetilde{P}} f(\widetilde{X}^n, \widetilde{Z}, L_T^n(\widetilde{X}^n, \widetilde{Z})).$$

By Theorem 2.2 in [14], we get

$$L_T^n(\widetilde{X}^n, \widetilde{Z}) \to L_T(\widetilde{X}, \widetilde{Z}) \text{ in probability.}$$

Thus, we have that

$$\begin{aligned}\mathbb{E}^{Q^n} f(X^n, Z^n, L_T^n(X^n, Z^n)) &\to \mathbb{E}^{\widetilde{P}} f(\widetilde{X}, \widetilde{Z}, L_T(\widetilde{X}, \widetilde{Z}))\\ &= \mathbb{E}^{Q} f(X, Z, L_T(X, Z)).\end{aligned}$$

By Theorem 2.1 in [7], we have $\pi_t^n F \to \pi_t F$ in distribution. □

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