MODERATE DEVIATIONS AND NONPARAMETRIC INFERENCE FOR MONOTONE FUNCTIONS

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This paper considers self-normalized limits and moderate deviations of nonparametric maximum likelihood estimators for monotone functions. We obtain their self-normalized Cramér-type moderate deviations and limit distribution theorems for the nonparametric maximum likelihood estimator in the current status model and the Grenander-type estimator. As applications of the results, we present a new procedure to construct asymptotical confidence intervals and asymptotical rejection regions of hypothesis testing for monotone functions. The theoretical results can guarantee that the new test has the probability of type II error tending to 0 exponentially. Simulation studies also show that the new nonparametric test works well for the most commonly used parametric survival functions such as exponential and Weibull survival distributions.

1. Introduction. The maximum likelihood estimator (MLE) of a monotone density function is the Grenander estimator [see Grenander (1956), Prakasa Rao (1969), and Groeneboom (1985)]. The asymptotic distribution of the Grenander estimator was obtained in Prakasa Rao (1969). Groeneboom and Wellner (1992) derived the limit distribution of the nonparametric maximum likelihood estimator (NPMLE) for current status data. Banerjee and Wellner (2001) proposed a likelihood ratio (LR) test for monotone functions to construct pointwise confidence intervals for the monotone functions and obtained its limit distribution. The method was studied further in Banerjee and Wellner (2005a, 2005b) and Banerjee (2007). Banerjee and Wellner (2001, 2005a, 2005b) investigated the power behavior of the LR test, and demonstrated the superior performance of the LR test over several existing methods in the current status model. Recently,

Groeneboom, Jongbloed and Witte (2010) proposed two estimators: the maximum smoothed likelihood and the smoothed maximum likelihood estimators and established their asymptotic normality, but the asymptotic variances depend on the derivative of the density function. Moreover, Groeneboom (2014) studied the maximum smoothed likelihood estimator for the interval censoring model, Groeneboom and Jongbloed (2014) discussed these nonparametric estimators, and Groeneboom and Jongbloed (2015) provided an alternative way to deduce the distribution of the likelihood ratio and extended it to the Grenander estimator.

Asymptotic behavior for Grenander's estimator and its extensions have been studied widely in the last 30 years. The asymptotic normality of the L_1-distance between the decreasing density and its Grenander estimator was obtained by Groeneboom, Hooghiemstra and Lopuhaä (1999), and a similar result for monotone regression can be found in Durot (2002). Huang and Wellner (1995) studied the limit distribution of a Grenander-type estimator for a monotone hazard rate in a right-censoring model. Durot (2007) gave the limit distribution for the L_p-error of a Grenander-type estimator. Recently, the limiting distribution of the supremum distance was studied by Durot, Kulikov and Lopuhaä (2012), and a law of the iterated logarithm for the Grenander's estimator was established by Dümbgen, Wellner and Wolff (2016).

Recently, self-normalized Cramér-type moderate deviations in statistics have attracted wide attention; see, for example, Liu and Shao (2010, 2013), Chang, Shao and Zhou (2016). In these papers, the statistics can be represented analytically by samples. The NPMLE of current status data and Grenander-type estimator are inverse statistical problems, and their weak limits are the maximum point of a Brownian motion minus a parabola. In this paper, we study self-normalized limit distributions and Cramér-type moderate deviations of the NPMLE for monotone functions. The self-normalized method is based on the self-normalized Cramér-type moderate deviations, and the self-normalized limit distributions for the NPMLE in the current status model and Grenander-type estimator are obtained. As applications, we use these results to construct asymptotical confidence intervals and asymptotical rejection regions of hypothesis testing for monotone functions. The new procedure for statistical inference is different from the LR method in Banerjee and Wellner (2001) and Groeneboom and Jongbloed (2015) and the smoothed maximum likelihood method in Groeneboom, Jongbloed and Witte (2010). Theoretically, the proposed test has the probability of type II error tending to 0 with exponential decay by moderate deviations. Such theoretical properties of the type 2 error are not generally well studied for other competing methods.

The rest of the paper is organized as follows. Section 2 presents the main results including the self-normalized Cramér-type moderate deviations, and the self-normalized limit distributions for the NPMLE in the current status model and Grenander-type estimator. In Section 3, the main results in the current status model are proved using a general Cramér-type moderate deviation theorem and a general

limit distribution theorem. The proofs of these general results are postponed to Sections 5. Our proofs are based on the strong approximation, Talagrand deviation inequality and small ball estimates. In Section 4, we use the theoretical results to construct statistical hypothesis tests and pointwise confidence intervals. In addition, we conduct simulation studies to evaluate the finite-sample performance of the proposed method and compare it with the LR test in Section 4. The proofs of some technical lemmas and the results for Grenander-type estimator are given in Appendix A of the Supplementary Material [Gao, Xiong and Zhao (2017)], and some useful known results are presented in Appendix B of the Supplementary Material [Gao, Xiong and Zhao (2017)] for the convenience of the reader.

2. Main results. We consider two types of problems: the NPMLE in the current status model and the Grenander-type estimator.

2.1. *Current status data.* Consider a study that concerns the time to occurrence of a certain event of interest such as failure or onset of a disease. In many cases, the event time cannot be observed completely since it is often too expensive or even impossible to observe the exact event occurrence time. However, the current status of the subjects at a random examination time can be much more easily observed. This means that for each subject, the event occurrence time is unknown, but it is known whether the event has occurred before the examination time. Such data are called current status data or case 1 interval censored data in survival analysis [Groeneboom and Wellner (1992), Huang and Wellner (1995), Groeneboom (1996), Sun (2006), Groeneboom, Jongbloed and Witte (2010)].

Let $X_1, \ldots, X_n$ and $T_1, \ldots, T_n$ be independent i.i.d. samples from unknown distribution functions F_0 and G on the nonnegative half-line, respectively. Set $\delta_i = I_{\{X_i \le T_i\}}$. Here, X_i is interpreted as the (unobserved) time of onset of a disease; T_i is a check-up time at which patient i is observed to be ill or not: $\delta_i = 1$ or 0. The observations consist of the pairs $(\delta_1, T_1), \ldots, (\delta_n, T_n)$ in the current status model. The NPMLE $\hat{F}_n$ for F_0 is defined to be the one that maximizes the log-likelihood function

$$F \mapsto \sum_{i=1}^{n} \big(\delta_i \log F(T_i) + (1 - \delta_i) \log\big(1 - F(T_i)\big) \big),$$

where F is a right-continuous distribution function. Since the likelihood function depends on F only through the values $F(T_i)$, the NPMLE is not unique. To avoid this ambiguity, $\hat{F}_n$ is assumed to be a constant on each interval $[T_{(i-1)}, T_{(i)})$, where $T_{(i)}$ is the ith order statistic of $T_1, \ldots, T_n$. A basic assumption is as follows.

ASSUMPTION (A1). Let $t_0 > 0$ be a fixed point such that $0 < F_0(t_0)$, $G(t_0) < 1$. Let F_0 and G be continuous on $[0, \infty)$ and differentiable on a closed neighborhood $[t_0 - \epsilon_0, t_0 + \epsilon_0]$ of the point t_0, with strictly positive derivatives $f_0(t)$ and $g(t)$, $t \in [t_0 - \epsilon_0, t_0 + \epsilon_0]$, respectively, where $0 < \epsilon_0 < t_0$.

Throughout the paper, the notation $\xi_n \xrightarrow{d} \xi$ means that the random sequence ξ_n converges to a random variable ξ in distribution. The notation $\operatorname{argmin}_{t \in U} f(t)$ [resp., $\operatorname{argmax}_{t \in U} f(t)$] stands for the point $t \in U$ at which $f(t)$ is minimized (resp., maximized), where the largest value is chosen when multiple minimizers (resp., maximizers) exist.

Groeneboom and Wellner (1992) established the strong consistency of the NPMLE, and derived the limit distribution of the NPMLE $\hat{F}_n(t_0)$ for a fixed point t_0 as follows: under Assumption (A1) stated above,

$$\frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\sigma_F^2(t_0))^{1/3}} \xrightarrow{d} \tau,$$

where

$$\sigma_F^2(t_0) := \frac{4 F_0(t_0)(1 - F_0(t_0)) f_0(t_0)}{g(t_0)}, \qquad \tau := \operatorname*{argmin}_{t \in \mathbb{R}} \{W(t) + t^2\},$$

and $\{W(t), t \in \mathbb{R}\}$ is a two-sided Wiener process with $W(0) = 0$.

By Corollaries 3.3 and 3.4 in Groeneboom (1989), the density function f_τ of τ is symmetric and bounded continuous, and has the tail behavior

$$(2.1) \qquad f_\tau(x) \sim \frac{2}{\mathrm{Ai}'(a_1)} 4^{1/3} |x| \exp\left\{-\frac{2}{3}|x|^3 + 2^{1/3} a_1 |x|\right\}, \qquad |x| \to \infty$$

where $a_1 \approx -2.3381$ is the largest zero of the Airy function Ai and $\mathrm{Ai}'(a_1) \approx 0.7002$. By Lemma 2.1 in Hooghiemstra and Lopuhaä (1998), the distribution function F_τ of τ satisfies

$$(2.2) \qquad 1 - F_\tau(x) \sim \frac{1}{\mathrm{Ai}'(a_1)} 4^{1/3} \frac{1}{x} \exp\left\{-\frac{2}{3}x^3 + 2^{1/3} a_1 x\right\}, \qquad x \to +\infty.$$

In order to use Groeneboom and Wellner's (1992) result for statistical inference, we need to construct a consistent estimator of f_0 or to obtain a self-normalized limit distribution of the NPMLE. This is one of our motivations. Another motivation of the paper is to study the tail probabilities of $\hat{F}_n(t_0) - F_0(t_0)$ and to construct confidence intervals and rejection regions for hypothesis testing for $F_0(t_0)$. We consider the following asymptotic behavior of $\hat{F}_n(t_0) - F_0(t_0)$: Cramér-type moderate deviation and moderate deviation principle (MDP).

The Cramér-type moderate deviation is to find $b_n \to \infty$ such that

$$\sup_{x \in [0, b_n]} \left| \frac{1}{1 - F_\tau(x)} P\left(\frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\sigma_F^2(t_0))^{1/3}} \geq x\right) - 1 \right| \to 0.$$

The moderate deviation principle (MDP) is to find two suitable sequences of positive numbers $b_n \to \infty$ and $\beta_n \to \infty$ with $\frac{n^{1/3}}{b_n} \to \infty$ such that

$$P\left(\left| \frac{1}{b_n} \frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\sigma_F^2(t_0))^{1/3}} \right| > x\right) \approx e^{-I(x)\beta_n},$$

where $I(x)$ is a suitable *rate* function and β_n is called the *speed*. If one replaces $\frac{n^{1/3}}{b_n} \to \infty$ by $\frac{n^{1/3}}{b_n} \to c > 0$, this result is called the large deviation principle (LDP). Generally, the rate function I of the MDP can be easily calculated, and provides a useful method for statistical inference. Some notion and properties in large deviation are given in Appendix B of the Supplementary Material [Gao, Xiong and Zhao (2017)].

Generally, the bigger b_n is, the faster the convergence rate is. In the i.i.d. case, $b_n = O(n^{1/6})$ is the best for the Cramér-type moderate deviation [see Petrov (1975), de la Peña, Lai and Shao (2009)]. For the NPMLE, due to a technical reason, we assume that b_n satisfies, as $n \to \infty$,

$$(2.3) \qquad b_n \to \infty \quad \text{and} \quad \frac{b_n^{10} \log n}{n^{1/3}} \to 0.$$

That is, $b_n \to \infty$ and $b_n = o(n^{1/30}/(\log n)^{1/10})$.

To study moderate deviations of the NPMLE, we need the following assumption.

ASSUMPTION (A2).   The functions $f_0(t)$ and $g(t)$ are Lipschitz continuous on $[t_0 - \epsilon_0, t_0 + \epsilon_0]$.

Let $\hat{g}_n$ be a classical kernel density estimator of g_0 defined by

$$(2.4) \qquad \hat{g}_n(t) = \frac{1}{nh_n} \sum_{k=1}^{n} I_{[0,1]}\left(\frac{t - T_k}{h_n}\right),$$

where $h_n \to 0$ and $\frac{nh_n}{b_n^3} \to \infty$. Then by the large deviation principle for the classical kernel density estimator [see the proof of Proposition 3.1 in Gao (2003)], for any $r > 0$,

$$(2.5) \qquad \lim_{n\to\infty} \frac{1}{b_n^3} \log P\left(|\hat{g}_n(t_0) - g_0(t_0)| > r\right) = -\infty,$$

since $\lim_{n\to\infty} \frac{nh_n}{b_n^3} = \infty$. In order to obtain a faster speed, one can take $h_n = \frac{1}{\log n}$. We can also replace $\log n$ by a sequence r_n with $\log n \le r_n = o(n^{1/3}/b_n)$ to obtain a faster convergence speed.

Let us introduce two estimators of f_0 as follows:

$$(2.6) \qquad \hat{f}_n(t) = \frac{n^{1/3}}{b_n \log n}\left(\hat{F}_n(t + b_n \log n/n^{1/3}) - \hat{F}_n(t)\right)$$

and

$$(2.7) \qquad \tilde{f}_n(t) = \frac{n^{1/3}}{\log n}\left(\hat{F}_n(t + \log n/n^{1/3}) - \hat{F}_n(t)\right).$$

For the estimators $\hat{f}_n(t)$ and $\tilde{f}_n(t)$, we also have the following exponential convergence.

PROPOSITION 2.1.   (i) *Assume that Assumptions* (A1) *and* (A2) *hold. Then for any $r > 0$,*

$$(2.8) \qquad \lim_{n \to \infty} \frac{1}{b_n^3} \log P\big(\big|\hat{F}_n(t_0) - F_0(t_0)\big| > r\big) = -\infty$$

and

$$(2.9) \qquad \lim_{n \to \infty} \frac{1}{b_n^3} \log P\big(\big|\hat{f}_n(t_0) - f_0(t_0)\big| > r\big) = -\infty.$$

(ii) *Assume that Assumption* (A1) *holds. Then for any $r > 0$,*

$$(2.10) \qquad \lim_{n \to \infty} P\big(\big|\tilde{f}_n(t_0) - f_0(t_0)\big| > r\big) = 0.$$

Now, we state the self-normalized Cramér-type moderate deviations and the self-normalized limit distribution theorem for the NPMLE in the current status model.

THEOREM 2.1.   *Assume that Assumptions* (A1) *and* (A2) *hold. Then for each $\rho > 0$, uniformly for $0 \le x \le \rho b_n$,*

$$(2.11) \qquad \frac{1}{1 - F_\tau(x)} P\bigg(\pm \frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\hat{\sigma}_n^2(t_0))^{1/3}} \ge x\bigg) \to 1,$$

that is,

$$(2.12) \qquad \frac{1}{1 - F_\tau(x)} P\bigg(\frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\hat{\sigma}_n^2(t_0))^{1/3}} \ge x\bigg) \to 1$$

and

$$(2.13) \qquad \frac{1}{1 - F_\tau(x)} P\bigg(\frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\hat{\sigma}_n^2(t_0))^{1/3}} \le -x\bigg) \to 1,$$

where $\hat{\sigma}_n^2(t_0) := 4\hat{F}_n(t_0)(1 - \hat{F}_n(t_0))\hat{f}_n(t_0)/\hat{g}_n(t_0)$.

In particular, the sequence $\{\frac{n^{1/3}}{b_n} \frac{\hat{F}_n(t_0) - F_0(t_0)}{(\hat{\sigma}_n^2(t_0))^{1/3}}, n \ge 1\}$ satisfies the MDP in $\mathbb{R}$ with speed b_n^3 and rate function $I(x) = \frac{2}{3}|x|^3$, that is, for any open subset A of $\mathbb{R}$,

$$\liminf_{n \to \infty} \frac{1}{b_n^3} \log P\bigg(\frac{n^{1/3}}{b_n} \frac{\hat{F}_n(t_0) - F_0(t_0)}{(\hat{\sigma}_n^2(t_0))^{1/3}} \in A\bigg) \ge -\inf_{x \in A} I(x),$$

and for any closed subset B of $\mathbb{R}$,

$$\limsup_{n \to \infty} \frac{1}{b_n^3} \log P\bigg(\frac{n^{1/3}}{b_n} \frac{\hat{F}_n(t_0) - F_0(t_0)}{(\hat{\sigma}_n^2(t_0))^{1/3}} \in B\bigg) \le -\inf_{x \in B} I(x).$$

THEOREM 2.2.   *Assume that Assumption* (A1) *holds. Then*

(2.14)
$$\frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\tilde{\sigma}_n^2(t_0))^{1/3}} \xrightarrow{d} \tau,$$

where $\tilde{\sigma}_n^2(t_0) := 4\hat{F}_n(t_0)(1 - \hat{F}_n(t_0))\tilde{f}_n(t_0)/\hat{g}_n(t_0).$

2.2. *Grenander-type estimator.*   Let λ be a decreasing (or increasing) function on $[0, 1]$ and $\Lambda(t) = \int_0^t \lambda(u)\,du$, $t \in [0, 1]$. Let $\Lambda_n(t)$ be a cadlag (right continuous with finite left-hand limits at every point) step process and let $\hat{\Lambda}_n$ be the least concave majorant (resp., greatest convex majorant) of Λ_n. The Grenander-type estimator $\hat{\lambda}_n$ of λ is defined as the left derivative of $\hat{\Lambda}_n$. Then $\hat{\lambda}_n$ is monotone, left-continuous and piecewise constant.

We introduce the following assumptions.

(GA1) The function λ is monotone and differentiable on $[0, 1]$,

$$\inf_{t\in[0,1]} |\lambda'(t)| > 0 \quad \text{and} \quad \sup_{t\in[0,1]} |\lambda'(t)| < \infty.$$

(GA2) Let W be either a Brownian bridge or a Brownian motion. There exist positive absolute constants C, K, κ, $L : [0, 1] \to \mathbb{R}$ and versions of Λ_n and W such that for all $r \in [0, n]$,

(2.15)
$$P\Big(\sup_{u\in[0,1]} |n(\Lambda_n(u) - \Lambda(u)) - n^{1/2}W(L(u))| \geq C\log n + r \Big)$$
$$\leq K\exp\{-\kappa r\}.$$

Moreover, L is increasing and twice differentiable in $(0, 1)$, and $L'(s) > 0$ for any $s \in (0, 1)$.

We define an estimator of λ' as follows:

(2.16)
$$\hat{\lambda}_n^{(1)}(t) = \frac{n^{1/3}}{b_n\log n}\big(\hat{\lambda}_n(t + b_n\log n/n^{1/3}) - \hat{\lambda}_n(t)\big).$$

The estimators $\hat{\lambda}_n(t)$ and $\hat{\lambda}_n^{(1)}(t)$ also have exponential convergence.

PROPOSITION 2.2.   *Assume that* (GA1) *and* (GA2) *hold.*
(i) *For any* $r > 0$, *we have*

(2.17) $\displaystyle \lim_{n\to\infty} \frac{1}{b_n^3} \log\max\{P(|\hat{\lambda}_n(t) - \lambda(t)| > r), P(|\hat{\lambda}_n^{(1)}(t) - \lambda'(t)| > r)\} = -\infty.$

(ii) *If* $\liminf_{n\to\infty} \frac{b_n}{(\log n)^3} > 1$, *then*

(2.18) $\displaystyle P\Big(\lim_{n\to\infty} \hat{\lambda}_n(t) = \lambda(t) \Big) = 1 \quad \text{and} \quad P\Big(\lim_{n\to\infty} \hat{\lambda}_n^{(1)}(t) = \lambda'(t) \Big) = 1.$

The following result is the self-normalized Cramér-type moderate deviations for the Grenander-type estimator.

THEOREM 2.3.   *Assume that* (GA1) *and* (GA2) *hold. Then for* $t \in (0, 1)$ *and* $\rho > 0$, *uniformly for* $0 \leq x \leq \rho b_n$,

$$(2.19) \qquad \frac{1}{1 - F_\tau(x)} P\left(\pm \frac{n^{1/3}(\hat{\lambda}_n(t) - \lambda(t))}{(4L'(t)|\hat{\lambda}_n^{(1)}(t)|)^{1/3}} \geq x\right) \to 1.$$

In particular, the sequence $\{\frac{n^{1/3}}{b_n} \frac{\hat{\lambda}_n(t) - \lambda(t)}{(4L'(t)|\hat{\lambda}_n^{(1)}(t)|)^{1/3}}, n \geq 1\}$ *satisfies the MDP in* $\mathbb{R}$ *with speed* b_n^3 *and rate function* $I(x) = \frac{2}{3}|x|^3$.

The next subsections give several corollaries of Theorem 2.3.

2.2.1. *The monotone density function.*   Let $X_1, \ldots, X_n$ be independent random variables taking values in [0, 1] with the common distribution function F and the monotone density function f. Then the monotone estimator based on the empirical distribution function of $X_1, \ldots, X_n$ is the Grenander estimator $\hat{f}_n$.

COROLLARY 2.1.   *Assume that* (GA1) *holds for* f. *Then for* $t \in (0, 1)$ *and* $\rho > 0$, *uniformly for* $0 \leq x \leq \rho b_n$,

$$\frac{1}{1 - F_\tau(x)} P\left(\pm \frac{n^{1/3}(\hat{f}_n(t) - f(t))}{(4\hat{f}_n(t)|\hat{f}_n^{(1)}(t)|)^{1/3}} \geq x\right) \to 1.$$

2.2.2. *Monotone hazard function with right-censored data.*   Let $(X_1, \delta_1), \ldots, (X_n, \delta_n)$ be a right-censored sample, where $X_i = \min\{T_i, Y_i\}$, $\delta_i = I_{\{T_i \leq Y_i\}}$, T_i, $i = 1, \ldots, n$ are nonnegative i.i.d. failure times, and $Y_i, i = 1, \ldots, n$ are i.i.d. censoring times independent of $T_i, i = 1, \ldots, n$. Assume that the common distribution function F of $T_i, i = 1, \ldots, n$ is absolutely continuous with density function f. Assume that F has a monotone hazard rate $\lambda = f/(1 - F)$. Let N_n be the Nelson–Aalen estimator, defined as follows: if $t_1 < \cdots < t_k$ are the distinct uncensored times and n_j is the number of X_i's that are greater than or equal to t_j ($j = 1, \ldots, k$), then

$$N_n(t) = \sum_{i=1}^{k-1} I_{[t_i, t_{i+1})}(t) \sum_{j \leq i} \frac{1}{n_j} + I_{[t_k, 1]}(t) \sum_{j \leq k} \frac{1}{n_j}.$$

Let Λ_n be the restriction of N_n to [0, 1] and let G be the common distribution function of $Y_i, i = 1, \ldots, n$. Let $\hat{\lambda}_n$ be the monotone estimator based on Λ_n [see Huang and Wellner (1995)]. Let F_n an G_n be the Kaplan–Meier estimators of F and G, respectively.

COROLLARY 2.2. *Suppose that* (GA1) *holds,* $F(1) < 1$, $\lim_{t \to 1} G(t) < 1$, *and G has a bounded continuous first derivative on* $(0, 1)$. *Then for $t \in (0, 1)$ and* $\rho > 0$, *uniformly for* $0 \le x \le \rho b_n$,

$$\frac{1}{1 - F_\tau(x)} P\left(\pm \frac{n^{1/3}(\hat{\lambda}_n(t) - \lambda(t))(1 - F_n(t))(1 - G_n(t))}{(4\hat{\lambda}_n(t)|\hat{\lambda}_n^{(1)}(t)|)^{1/3}} \ge x\right) \to 1.$$

2.2.3. *The monotone regression model.*   Consider the model

$$Y_{i,n} = \lambda(i/n) + \varepsilon_{i,n}, \qquad i = 1, \ldots, n,$$

where $\varepsilon_{i,n}, i = 1, \ldots, n$, are independent random variables with mean zero. Let

$$\Lambda_n(t) = \frac{1}{n} \sum_{i \le nt} Y_{i,n}, \qquad t \in [0, 1].$$

Then the monotone estimator $\hat{\lambda}_n$ based on Λ_n is a slight modification of the Brunk estimator.

COROLLARY 2.3.   *Suppose that* (GA1) *holds and there exists $\delta > 0$ such that* $\sup_{i,n} E(\exp\{\delta|\varepsilon_{i,n}|\}) < \infty$. *Assume, moreover, that* $\text{var}(\varepsilon_{i,n}) = \sigma^2(i/n)$ *for some* $\sigma^2 : [0, 1] \to \mathbb{R}_+$. *If σ^2 has a bounded first derivative and satisfies* $\inf_t \sigma^2(t) > 0$, *then for $t \in (0, 1)$ and $\rho > 0$, uniformly for* $0 \le x \le \rho b_n$,

$$\frac{1}{1 - F_\tau(x)} P\left(\pm \frac{n^{1/3}(\hat{\lambda}_n(t) - \lambda(t))}{(4\sigma^2(t)|\hat{\lambda}_n^{(1)}(t)|)^{1/3}} \ge x\right) \to 1.$$

3. Proofs of the main results.   In this section, we only prove Proposition 2.1, and Theorems 2.1 and 2.2. The proofs of the results on the Grenander-type estimator are similar and will be given in Appendix A of the Supplementary Material [Gao, Xiong and Zhao (2017)]. The following two results, Theorems 3.1 and 3.2, play an important role in our proofs. Their proofs will be given in Section 5.

THEOREM 3.1.   *Assume that Assumptions* (A1) *and* (A2) *hold. Let χ_n be a sequence of nonnegative numbers such that $b_n^{4.5} \chi_n \to 0$ as $n \to \infty$. Then for each $\rho > 0$, uniformly for* $0 \le x \le \rho b_n$,

$$(3.1) \qquad \frac{1}{1 - F_\tau(x)} P\left(\pm \frac{n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n))}{(\sigma_F^2(t_0))^{1/3}} \ge x\right) \to 1.$$

In particular, the sequence $\{\frac{n^{1/3}}{b_n} \frac{\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)}{(\sigma_F^2(t_0))^{1/3}}, n \ge 1\}$ *satisfies the MDP in* $\mathbb{R}$ *with speed b_n^3 and rate function* $I(x) = \frac{2}{3}|x|^3$.

The next result is an extension of Groeneboom and Wellner's limit distribution result.

THEOREM 3.2. *Assume that Assumption* (A1) *holds. Let* χ_n *be a sequence of nonnegative numbers such that* $\chi_n \to 0$ *as* $n \to \infty$.*Then*

$$\frac{n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n))}{(\sigma_F^2(t_0))^{1/3}} \xrightarrow{d} \tau.$$

PROOF OF PROPOSITION 2.1. Equation (2.8) is obvious. When n is large enough,

$$\begin{aligned}
\big|\hat{f}_n(t_0) - f_0(t_0)\big| &\leq \frac{1}{\log n}\frac{n^{1/3}}{b_n}\big\{\big|\hat{F}_n\big(t_0 + b_n \log n/n^{1/3}\big) - F_0\big(t_0 + b_n \log n/n^{1/3}\big)\big| \\
&\quad + \big|\hat{F}_n(t_0) - F_0(t_0)\big|\big\} \\
&\quad + \left|\frac{F(t_0 + b_n \log n/n^{1/3}) - F(t_0)}{b_n \log n/n^{1/3}} - f_0(t_0)\right|.
\end{aligned}$$

Since

$$\left|\frac{F(t_0 + b_n \log n/n^{1/3}) - F(t_0)}{b_n \log n/n^{1/3}} - f_0(t_0)\right| \to 0,$$

then by Theorems 3.1, (2.9) holds. The proof of (2.10) is similar to that of (2.9). $\square$

PROOF OF THEOREM 2.1. By Proposition 2.1(i) and the Delta method in large deviations [Gao and Zhao (2011)], for any $\epsilon > 0$,

$$(3.2) \qquad \lim_{n\to\infty} \frac{1}{b_n^3} \log P\left(\frac{n^{1/3}}{b_n \log n}\left|\frac{(\sigma_F^2(t_0))^{1/3}}{(\hat{\sigma}_n^2(t_0))^{1/3}} - 1\right| > \epsilon\right) = -\infty.$$

For each $\rho > 0$ and $0 \leq x \leq \rho b_n$, we have

$$\begin{aligned}
&\frac{1}{1 - F_\tau(x)} P\left(\pm\frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\hat{\sigma}_n^2(t_0))^{1/3}} \geq x\right) \\
&\quad \leq \frac{1}{P(\tau \geq x)} P\left(\frac{n^{1/3}}{b_n \log n}\left|\frac{(\sigma_F^2(t_0))^{1/3}}{(\hat{\sigma}_n^2(t_0))^{1/3}} - 1\right| > 1\right) \\
&\qquad + \frac{P(\pm\frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\sigma_F^2(t_0))^{1/3}} \geq x(1 - \frac{b_n \log n}{n^{1/3}}))}{P(\tau \geq x(1 - \frac{b_n \log n}{n^{1/3}}))} \frac{P(\tau \geq x(1 - \frac{b_n \log n}{n^{1/3}}))}{1 - F_\tau(x)}
\end{aligned}$$

and

$$\frac{1}{1 - F_\tau(x)} P\left(\pm \frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\hat{\sigma}_n^2(t_0))^{1/3}} \geq x\right)$$

$$\geq -\frac{1}{P(\tau \geq x)} P\left(\frac{n^{1/3}}{b_n \log n} \left|\frac{(\sigma_F^2(t_0))^{1/3}}{(\hat{\sigma}_n^2(t_0))^{1/3}} - 1\right| > 1\right)$$

$$+ \frac{P(\pm \frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{(\sigma_F^2(t_0))^{1/3}} \geq x(1 + \frac{b_n \log n}{n^{1/3}}))}{P(\tau \geq x(1 + \frac{b_n \log n}{n^{1/3}}))} \frac{P(\tau \geq x(1 + \frac{b_n \log n}{n^{1/3}}))}{1 - F_\tau(x)}.$$

Note that by (2.2) and (3.2),

$$\sup_{x \in [0, \rho b_n]} \frac{1}{P(\tau \geq x)} P\left(\frac{n^{1/3}}{b_n \log n} \left|\frac{(\sigma_F^2(t_0))^{1/3}}{(\hat{\sigma}_n^2(t_0))^{1/3}} - 1\right| > 1\right) \to 0$$

and

$$\sup_{0 \leq x \leq \rho b_n} \left|\frac{1}{1 - F_\tau(x)} P\left(\tau \geq x\left(1 + \frac{b_n \log n}{n^{1/3}}\right)\right) - 1\right| \to 0.$$

Then (2.12) follows from (3.1) with $\chi_n = 0$. $\square$

PROOF OF THEOREM 2.2.   By Proposition 2.1(ii) and the continuous mapping theorem [van der Vaart and Wellner (1996), page 20], for any $\epsilon > 0$,

$$\lim_{n \to \infty} P\left(\left|\frac{(\sigma_F^2(t_0))^{1/3}}{(\tilde{\sigma}_n^2(t_0))^{1/3}} - 1\right| > \epsilon\right) = 0.$$

Again, the conclusion follows from Theorem 3.1 with $\chi_n = 0$. $\square$

4. Applications. In this section, we focus on applications of the self-normalized Cramér-type MDP to statistical inference for the current status model. We also will present some simulation results to compare the finite-sample performance of the new test constructed by using the self-normalized Cramér-type MDP and the LR test [Banerjee and Wellner (2001, 2005a, 2005b), Groeneboom and Jongbloed (2015)].

4.1. *Hypothesis testing*.   In this subsection, we apply the results in Section 2 to hypothesis testing problems for unknown distribution function F in the current status model. Let F_0 and F_1 be two distributions on $[0, \infty)$ and let $t_0 \in (0, \infty)$. Assume that $F_0(t_0) \neq F_1(t_0)$. Consider testing

$$H_0 : F(t_0) = F_0(t_0) \quad \text{versus} \quad H_1 : F(t_0) = F_1(t_0).$$

We can use Theorem 2.1 to construct a rejection region and to prove the probability of type II error tending to 0 with exponential decay. Consider the test statistic

$$T_n := \frac{n^{1/3}(\hat{F}_n(t_0) - F_0(t_0))}{\hat{\sigma}_n},$$

where $\hat{\sigma}_n = (\hat{\sigma}_n^2(t_0))^{1/3}$.

Given $0 < \alpha < 1$, by Theorem 2.1, under H_0,

$$\lim_{n \to \infty} \sup_{x \in [0, \rho b_n]} \left| \frac{P(|T_n| \geq x)}{2(1 - F_\tau(x))} - 1 \right| = 0.$$

Take the rejection region as $\{|T_n| \geq c(\alpha)\}$, where $c(\alpha)$ is a positive constant such that $\alpha/2 = 1 - F_\tau(c(\alpha))$. Then the probability β_n of type II error is

$$\beta_n = P(|T_n| < c(\alpha)|F_1(t_0)).$$

Next, we establish the exponential convergence of β_n.

THEOREM 4.1. *Suppose that Assumptions* (A1) *and* (A2) *hold for* F_0 *and* F_1. *Then the probability of type II error tends to* 0 *with exponential decay speed* $\exp\{-rb_n^3\}$ *for all* $r > 0$, *that is,*

$$\lim_{n \to \infty} \frac{1}{b_n^3} \log \beta_n = -\infty.$$

PROOF. The probability β_n of type II error is

$$\beta_n = P(|T_n| < c(\alpha)|F_1(t_0))$$

$$\leq P\left(\left| \frac{n^{1/3}(\hat{F}_n(t_0) - F_1(t_0))}{b_n \hat{\sigma}_n} \right| \geq \frac{n^{1/3}}{b_n \hat{\sigma}_n} |F_0(t_0) - F_1(t_0)| - \frac{c(\alpha)}{b_n} \Big| F_1(t_0) \right).$$

By Proposition 2.1(i) and the Delta method in large deviations [Gao and Zhao (2011)], for any $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{b_n^3} \log P\left(\frac{n^{1/3}}{b_n \log n} \left| \frac{(\sigma_{F_1}^2(t_0))^{1/3}}{\hat{\sigma}_n} - 1 \right| > \epsilon \Big| F_1(t_0) \right) = -\infty.$$

Thus, by Theorem 2.1, we have $\lim_{n \to \infty} \frac{1}{b_n^3} \log \beta_n = -\infty$. □

4.2. *Pointwise confidence interval.* For given confidence level $1 - \alpha$, let $\tau_{\alpha/2}$ be a critical value such that $P(\tau > \tau_{\alpha/2}) = \alpha/2$.

A direct application of Theorem 2.1 yields an approximate $(1 - \alpha)$ confidence interval for $F_0(t_0)$ as

$$\left(\hat{F}_n(t_0) - \frac{\hat{\sigma}_n}{n^{1/3}} \tau_{\alpha/2}, \hat{F}_n(t_0) + \frac{\hat{\sigma}_n}{n^{1/3}} \tau_{\alpha/2} \right).$$

4.3. *Simulation study.* We conducted a simulation study to assess the finite-sample performance of the proposed test by rejecting $H_0 : F(t_0) = F_0(t_0)$ if

$$\left| \hat{F}_n(t_0) - F_0(t_0) \right| \geq \frac{\hat{\sigma}_n}{n^{1/3}} \tau_{\alpha/2},$$

where $\hat{F}_n(t_0)$ is the NPMLE of $F_0(t_0)$. For $\alpha = 0.05$, one can find $\tau_{0.025} = 0.9982$ from Table 2 in Groeneboom and Wellner (2001) or Table I in Abrevaya and Huang (2005).

For the testing problem, other available methods include the LR, MLE-based, score and Wald test statistics [see Banerjee and Wellner (2001, 2005a, 2005b)]. Based on simulations, Banerjee and Wellner (2005a) concluded that the LR-based method is better than the MLE-based methods for the situations considered, while Banerjee and Wellner (2005b) showed that the LR and score statistics are comparable and are more powerful than the Wald statistic. Note that these so-called score test statistics are based on the difference between the unrestricted and the restricted MLEs, which can be considered as extensions of the LR test statistic. Here, we focus on comparison of the new test and the LR test. To compute the LR test statistic, we also need to find the unrestricted MLE. For this, we used the new method presented in Section 2 of Groeneboom and Jongbloed (2015).

To generate current status data, suppose that failure time X follows an exponential distribution or a Weibull distribution, and examination time T follows a uniform distribution over i an interval $[0, T_0]$. Let $\{x_i, i = 1, \ldots, n\}$ and $\{t_i, i = 1, \ldots, n\}$ be independent samples from the distributions of X and T, respectively. The generated current status data consist of $\{(t_i, \delta_i), i = 1, \ldots, n\}$ with $\delta_i = I_{(x_i \leq t_i)}$.

Consider two cases: (i) $F_0(t) = 1 - \exp(-0.5t)$ and $F_1(t) = 1 - \exp(\lambda t)$ with $\lambda \neq 0.5$; and (ii) $F_0(t) = 1 - \exp(-(0.5t)^{0.5})$ and $F_1(t) = 1 - \exp(-(\lambda t)^{0.5})$ with $\lambda \neq 0.5$. We set $T_0 = 2$ under H_0 and 3 or 5 under H_1.

To apply the proposed test, we need to choose b_n. Since b_n is assumed to satisfy the following condition:

$$b_n \to \infty \quad \text{and} \quad b_n = o\big(n^{1/30}/(\log n)^{1/10}\big),$$

we can simply take $b_n = n^{1/(30+\varepsilon)}$ with $\varepsilon > 0$. According to the general principle "the bigger b_n is, the faster the convergence rate is", we only considered values of b_n close to $n^{1/30}$ in our simulations. In addition, we need to choose h_n for estimation of g_0. As pointed out in Section 2.1, we prefer to take $h_n = 1/r_n$ where $r_n = n^{1/(3+\delta)}/b_n$ with $\delta > 0$. Here, we took values of r_n close to $n^{1/3}/b_n$ to obtain a faster convergence speed. All the simulation results reported here are based on 1000 Monte Carlo replications using the R software, while the corresponding R code is available via the website http://www.mypolyuweb.hk/~mazhao.

Tables 1 and 2 report the estimated sizes and powers of the proposed test with $b_n = n^{1/30.1}$ and $r_n = n^{1/(3.1)}/b_n$ and LR test at nominal level 0.05 based on simulated current status data for two cases with different values of λ. In each setting,

TABLE 1

Estimated sizes and powers of the proposed and LR tests at nominal level 0.05 *for the case of exponential distribution*

		n = 100		n = 200		n = 300	
λ	T_0	Proposed	LR	Proposed	LR	Proposed	LR
0.5	2	0.058	0.049	0.058	0.063	0.048	0.055
0.7	3	0.261	0.157	0.309	0.227	0.408	0.324
0.9		0.513	0.427	0.681	0.645	0.737	0.732
0.3		0.517	0.386	0.641	0.558	0.727	0.682
0.1		1.000	1.000	1.000	1.000	1.000	1.000
0.7	5	0.262	0.124	0.278	0.190	0.320	0.232
0.9		0.446	0.339	0.520	0.446	0.636	0.592
0.3		0.543	0.331	0.568	0.434	0.637	0.542
0.1		1.000	0.997	1.000	1.000	1.000	1.000

we take $t_0 = 1$ and considered three sample sizes $n = 100, 200, 300$. It can be seen from the tables that the estimated sizes using both tests are close to nominal level 0.05, and the estimated powers of the proposed test are higher than those of the LR test for the situations considered here. Very similar results were obtained for the proposed test with other values of b_n close to $n^{1/30}$ and other values of r_n close to $n^{1/3}/b_n$ as well.

In addition, we conducted simulations to compare the power behavior of the proposed test and the LR test for the contiguous alternatives with the exponential and Weibull distributions, respectively. Let f_0 be the density function of F_0.

TABLE 2

Estimated sizes and powers of the proposed and LR tests at nominal level 0.05 *for the case of Weibull distribution*

		n = 100		n = 200		n = 300	
λ	T_0	Proposed	LR	Proposed	LR	Proposed	LR
0.5	2	0.064	0.038	0.054	0.046	0.041	0.062
0.7	3	0.273	0.079	0.263	0.116	0.270	0.138
0.9		0.405	0.215	0.431	0.290	0.498	0.389
0.3		0.390	0.239	0.419	0.320	0.480	0.390
0.1		0.980	0.964	1.000	0.999	0.999	0.999
0.7	5	0.302	0.065	0.262	0.083	0.284	0.113
0.9		0.376	0.133	0.366	0.194	0.412	0.270
0.3		0.454	0.216	0.432	0.251	0.446	0.309
0.1		0.965	0.916	0.994	0.992	1.000	0.999

Following Banerjee and Wellner (2005b), we define local alternatives $\{F_n\}$ as

$$F_n(t) = F_0(t) + n^{-1/3}B(n^{1/3}(t - t_0)),$$

where $B(t) = \eta f_0(t_0)(c - |t|)1_{[-c,c]}(t)$, and t_0, c, η are fixed positive numbers. Take $\eta = 0.9$. The failure times can be generated by using the method given in Banerjee and Wellner (2005b). Again suppose examination time T follows a uniform distribution over interval $[0, 3]$. The simulation results are shown in Tables 3 and 4. The tables include the estimated powers (the rejection percentages of the null hypothesis over 1000 Monte Carlo replicates) of the proposed test and the LR test for the contiguous alternatives $H_1 : F(t_0) = F_n(t_0)$ with different values of t_0 in two cases of F_0, respectively. The results indicate that the proposed test outperforms the LR test for the contiguous alternatives considered here.

We also carried out a simulation study using the *Simulation Setting 3* designed by Banerjee and Wellner (2005a) with t_0 as chosen there. The results of estimated confidence intervals indicate that our method does not perform as well as the LR method for this case where the function F_0 behaves capriciously.

TABLE 3

Estimated powers of the proposed and LR tests at nominal level 0.05 *for the contiguous alternatives with the exponential distribution*

t_0	c	$n = 100$		$n = 200$		$n = 300$	
		Proposed	**LR**	**Proposed**	**LR**	**Proposed**	**LR**
0.5	1	0.225	0.060	0.167	0.071	0.157	0.059
	2	0.284	0.110	0.230	0.125	0.232	0.137
	3	0.506	0.361	0.427	0.325	0.438	0.342
	4	0.734	0.633	0.705	0.635	0.689	0.618
	5	0.888	0.837	0.884	0.870	0.893	0.870
1.0	1	0.152	0.052	0.156	0.053	0.141	0.075
	2	0.225	0.073	0.191	0.090	0.177	0.091
	3	0.356	0.192	0.354	0.189	0.350	0.188
	4	0.559	0.372	0.532	0.351	0.546	0.377
	5	0.762	0.606	0.754	0.580	0.747	0.600
1.5	1	0.148	0.048	0.139	0.054	0.133	0.055
	2	0.225	0.093	0.195	0.067	0.173	0.074
	3	0.323	0.121	0.299	0.132	0.267	0.124
	4	0.491	0.253	0.486	0.253	0.463	0.273
	5	0.660	0.432	0.662	0.466	0.637	0.435
2.5	1	0.218	0.062	0.139	0.059	0.102	0.056
	2	0.288	0.085	0.197	0.083	0.158	0.074
	3	0.351	0.142	0.274	0.141	0.254	0.128
	4	0.441	0.215	0.337	0.208	0.299	0.178
	5	0.537	0.353	0.472	0.327	0.417	0.328

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TABLE 4
Estimated powers of the proposed and LR tests at nominal level 0.05 *for the contiguous alternatives with Weibull distribution*

t_0	c	$n = 100$		$n = 200$		$n = 300$	
		Proposed	**LR**	**Proposed**	**LR**	**Proposed**	**LR**
0.5	1	0.208	0.048	0.182	0.049	0.171	0.042
	2	0.286	0.068	0.258	0.075	0.246	0.075
	3	0.448	0.136	0.382	0.105	0.369	0.142
	4	0.624	0.324	0.581	0.275	0.552	0.245
	5	0.752	0.535	0.759	0.546	0.756	0.521
1.0	1	0.212	0.060	0.212	0.055	0.184	0.059
	2	0.262	0.058	0.231	0.057	0.188	0.054
	3	0.343	0.085	0.291	0.076	0.283	0.074
	4	0.432	0.112	0.392	0.125	0.387	0.138
	5	0.536	0.180	0.541	0.215	0.523	0.227
1.5	1	0.198	0.054	0.192	0.046	0.187	0.059
	2	0.256	0.061	0.233	0.040	0.218	0.050
	3	0.294	0.058	0.258	0.062	0.247	0.064
	4	0.351	0.091	0.348	0.094	0.331	0.113
	5	0.445	0.155	0.440	0.150	0.422	0.153
2.5	1	0.223	0.070	0.142	0.066	0.100	0.060
	2	0.269	0.082	0.168	0.079	0.127	0.063
	3	0.272	0.109	0.210	0.097	0.183	0.101
	4	0.301	0.124	0.214	0.122	0.187	0.109
	5	0.310	0.157	0.245	0.169	0.208	0.160

4.4. *Conclusions.* We have proposed a new test for current status data using the self-normalized Cramér-type MDP. When the size is fixed, the new test has the probability of a type II error tending to 0 with exponential decay, and thus it has good power. Our simulation study has demonstrated that the null distribution of the proposed test statistic has a reasonable approximation and that the new test is more powerful than the LR test for most of the settings considered here. However, when the true distribution function behaves capriciously, the proposed approach does not work as well as the LR method. Compared with the LR test, the new test enjoys the following advantages:

(i) The probability of type II error tends to 0 with exponential decay, which implies that the power behavior of the new test is very good.

(ii) The rejection region of the proposed test has a simple analytical expression, and thus is easy to implement. In contrast, the rejection region of the LR test must be obtained numerically.

(iii) Computationally, conducting the proposed test procedure is quite simple, but performing the LR test procedure is time-consuming.

5. Proofs of Theorems 3.1 and 3.2. In this section, we propose a new approach to study the moderate deviation problem. Let us first give an outline of the proof. In Section 5.1, we convert the moderate deviation estimates of $\hat{F}_n(t_0 + \chi_n)$ to those for the locations τ_n of minimums of a sequence of processes $Z_n(t), n \geq 1$. Then, in Section 5.2, we show that the processes $\{Z_n(t), n \geq 1\}$ can be strongly approximated by the processes $\{\hat{Z}_n(t), n \geq 1\}$, where $\hat{Z}_n(t)$ can be represented by a two-sided Wiener process plus a parabolic drift. After that, in Section 5.3, we show that τ_n is exponentially equivalent to the locations $\hat{\tau}_n$ of minimums of the processes $\hat{Z}_n(t)$ by the Talagrand deviation inequality and small ball estimates. The proofs of Theorems 3.1 and 3.2 are given respectively in Sections 5.4 and 5.5 using the asymptotic properties of $\hat{\tau}_n$.

5.1. *An equivalent representation of the moderate deviations.* In this subsection, we convert the moderate deviation estimates of $\hat{F}_n(t_0)$ to those of the minima of a sequence of stochastic processes.

Let $\mathbb{P}_n$ be the empirical measure of $(X_1, T_1), \ldots, (X_n, T_n)$, that is,

$$\mathbb{P}_n(A) = \mathbb{P}_n I_A = \frac{1}{n} \sum_{i=1}^{n} I_A(X_i, T_i), \qquad A \in \mathcal{B}(\mathbb{R}_+^2).$$

Set $A = \{(x, t) : x \leq t\}$ and define

$$V_n(t) = \frac{1}{n} \sum_{i=1}^{n} \delta_i I_{[0,t]}(T_i) = \mathbb{P}_n I_{A \cap (\mathbb{R} \times [0,t])},$$

$$G_n(t) = \frac{1}{n} \sum_{i=1}^{n} I_{[0,t]}(T_i) = \mathbb{P}_n I_{\mathbb{R} \times [0,t]}.$$

The function $s \mapsto V_n \circ G_n^{-1}(s)$ equals the cumulative sum diagram [van der Vaart and Wellner (1996)], and hence, for every observation point t_i,

$$\hat{F}_n(t_i) \leq x \quad \text{if and only if} \quad \operatorname*{argmin}_s \{V_n(s) - x G_n(s)\} \geq t_i.$$

Thus, the limit distribution of $\hat{F}_n(t)$ can be derived by studying the locations of minima of the sequence of processes $s \mapsto V_n(s) - x G_n(s)$. For each t, the value $\hat{F}_n(t)$ equals $\hat{F}_n(T_i)$ for some i, and hence, $\hat{F}_n(t) \leq x$ if and only if the argmin appearing in the display is larger than the observation time T_i that is just left of t. Define $\kappa_n(t) := \sup\{1 \leq k \leq n, T_k \leq t\}$. Then

$$\hat{F}_n(t) \leq x \quad \text{if and only if} \quad \operatorname*{argmin}_s \{V_n(s) - x G_n(s)\} \geq T_{(\kappa_n(t))}.$$

Let $\{\ell_n, n \geq 1\}$ be a sequence of positive numbers satisfying that

$$1 \leq \ell_n \quad \text{and} \quad \lim_{n \to \infty} \frac{\ell_n^{10} \log n}{n^{1/3}} = 0.$$

Let χ_n be nonnegative such that $\ell_n^{4.5}\chi_n \to 0$. Note that ℓ_n will be taken as b_n and 1 in the proofs of Theorems 3.1 and 3.2, respectively.

Then, for any $x \in \mathbb{R}$,

$$P\left(\frac{n^{1/3}}{\ell_n}\big(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)\big) > x\right)$$

$$= P\left(\operatorname*{argmin}_{t \in [(t_0 + \chi_n)n^{1/3}, \infty)} Y_n^x\big(t_0 + \chi_n + n^{-1/3}t\big) < n^{1/3}\big(T_{(\bar{\kappa}_n)} - (t_0 + \chi_n)\big)\right),$$

where $\bar{\kappa}_n = \kappa_n(t_0 + \chi_n)$ and $Y_n^x(s) = V_n(s) - (F_0(t_0 + \chi_n) + n^{-1/3}\ell_n x)G_n(s)$.

Note that the decomposition of $Y_n^x(t_0 + \chi_n + n^{-1/3}t)$ on page 299 of van der Vaart and Wellner (1996) is not suitable for the application of strong approximation. In order to apply the strong approximation, we write $Y_n^x(t_0 + \chi_n + n^{-1/3}t)$ as

$$Z_n^x(t) = U_n(t) + r_n(t) + x\ell_n\gamma_n(t) + R_n(t) + x\ell_n\Gamma_n(t)$$

with

$$\begin{cases}
U_n(t) := \dfrac{1}{n^{1/3}}\sum_{i=1}^n \big(I_{\{X_i \leq t_0 + \chi_n\}} - F_0(t_0 + \chi_n)\big)\big(I_{\{T_i \leq t_0 + \chi_n + n^{-1/3}t\}} - I_{\{T_i \leq t_0 + \chi_n\}}\big), \\[2mm]
r_n(t) := n^{2/3}E\big((I_{\{X_1 \leq T_1\}} - I_{\{X_1 \leq t_0 + \chi_n\}})(I_{\{T_1 \leq t_0 + \chi_n + n^{-1/3}t\}} - I_{\{T_1 \leq t_0 + \chi_n\}})\big), \\[2mm]
\gamma_n(t) := n^{1/3}\big(G\big(t_0 + \chi_n + n^{-1/3}t\big) - G(t_0 + \chi_n)\big), \\[2mm]
R_n(t) := \dfrac{1}{n^{1/3}}\sum_{i=1}^n (I_{\{X_i \leq T_i\}} - I_{\{X_i \leq t_0 + \chi_n\}})(I_{\{T_i \leq t_0 + \chi_n + n^{-1/3}t\}} - I_{\{T_i \leq t_0 + \chi_n\}}) - r_n(t), \\[2mm]
\Gamma_n(t) := \dfrac{1}{n^{2/3}}\sum_{i=1}^n (I_{\{T_i \leq t_0 + \chi_n + n^{-1/3}t\}} - I_{\{T_i \leq t_0 + \chi_n\}}) - \gamma_n(t).
\end{cases}$$

This decomposition plays a major role in the proof of our main result. Therefore,

$$P\left(\frac{n^{1/3}}{\ell_n}\big(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)\big) > x\right)$$

$$= P\left(\operatorname*{argmin}_{t \in [-(t_0 + \chi_n)n^{1/3}, \infty)} Z_n^{-x}(t) < n^{1/3}\big(T_{(\bar{\kappa}_n)} - (t_0 + \chi_n)\big)\right).$$

Similarly, for any $x \in \mathbb{R}$,

$$P\left(\frac{n^{1/3}}{\ell_n}\big(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)\big) \leq -x\right)$$

$$= P\left(\operatorname*{argmin}_{t \in [-(t_0 + \chi_n)n^{1/3}, \infty)} Z_n^x(t) \geq n^{1/3}\big(T_{(\bar{\kappa}_n)} - (t_0 + \chi_n)\big)\right).$$

LEMMA 5.1.   *For any $\delta > 0$,*

(5.1) $$\limsup_{n\to\infty} \frac{1}{\ell_n^3} \log P\big(n^{1/3}(t_0 + \chi_n - T_{(\bar{\kappa}_n)}) \geq \ell_n^{-2}\delta\big) = -\infty.$$

PROOF.   For any $\delta > 0$,

$$P\big(n^{1/3}(t_0 + \chi_n - T_{(\bar{\kappa}_n)}) \geq \ell_n^{-2}\delta\big)$$
$$= P\big(T_i \notin \big(t_0 + \chi_n - n^{-1/3}\ell_n^{-2}\delta, t_0 + \chi_n\big) \text{ for all } i = 1,\ldots,n\big)$$
$$= \left(1 - \int_{t_0+\chi_n-n^{-1/3}\ell_n^{-2}\delta}^{t_0+\chi_n} g(t)\, dt\right)^n.$$

Thus,

$$\limsup_{n\to\infty} \frac{1}{\ell_n^3} \log P\big(n^{1/3}(t_0 + \chi_n - T_{(\bar{\kappa}_n)}) \geq \ell_n^{-2}\delta\big) = -\infty. \qquad \square$$

PROPOSITION 5.1.   *For each $\rho > 0$, we have*

(5.2)
$$\limsup_{n\to\infty} \sup_{x\in[0,\rho]} \frac{P(n^{1/3}(\hat{F}_n(t_0+\chi_n) - F_0(t_0+\chi_n)) > x\ell_n)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}$$
$$\leq \limsup_{n\to\infty} \sup_{x\in[0,\rho]} \frac{P(\operatorname{argmin}_{t\in[-(t_0+\chi_n)n^{1/3},\infty)} Z_n^{-x}(t) \leq 0)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)},$$

(5.3)
$$\liminf_{n\to\infty} \inf_{x\in[0,\rho]} \frac{P(n^{1/3}(\hat{F}_n(t_0+\chi_n) - F_0(t_0+\chi_n)) > x\ell_n)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}$$
$$\geq \lim_{\delta\downarrow 0} \liminf_{n\to\infty} \inf_{x\in[0,\rho]} \frac{P(\operatorname{argmin}_{t\in[-(t_0+\chi_n)n^{1/3},\infty)} Z_n^{-x}(t) \leq -\ell_n^{-2}\delta)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)},$$

(5.4)
$$\liminf_{n\to\infty} \inf_{x\in[0,\rho]} \frac{P(n^{1/3}(\hat{F}_n(t_0+\chi_n) - F_0(t_0+\chi_n)) \leq -x\ell_n)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}$$
$$\geq \liminf_{n\to\infty} \inf_{x\in[0,\rho]} \frac{P(\operatorname{argmin}_{t\in[-(t_0+\chi_n)n^{1/3},\infty)} Z_n^{x}(t) \geq 0)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}$$

and

(5.5)
$$\limsup_{n\to\infty} \sup_{x\in[0,\rho]} \frac{P(n^{1/3}(\hat{F}_n(t_0+\chi_n) - F_0(t_0+\chi_n)) \leq -x\ell_n)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}$$
$$\leq \lim_{\delta\downarrow 0} \limsup_{n\to\infty} \sup_{x\in[0,\rho]} \frac{P(\operatorname{argmin}_{t\in[-(t_0+\chi_n)n^{1/3},\infty)} Z_n^{x}(t) \geq -\ell_n^{-2}\delta)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}.$$

PROOF.   For each $\rho > 0$ and $0 \leq x \leq \rho$, we have

$$\frac{P(\operatorname{argmin}_{t \in [-(t_0+\chi_n)n^{1/3}, \infty)} Z_n^{-x}(t) \leq -\ell_n^{-2}\delta) - P(n^{1/3}(t_0 + \chi_n - T_{(\bar{\kappa}_n)}) \geq \ell_n^{-2}\delta)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}$$

$$\leq \frac{P(n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)) > x\ell_n)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}$$

$$\leq \frac{P(\operatorname{argmin}_{t \in [-(t_0+\chi_n)n^{1/3}, \infty)} Z_n^{-x}(t) \leq 0)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}$$

and

$$\frac{P(\operatorname{argmin}_{t \in [-(t_0+\chi_n)n^{1/3}, \infty)} Z_n^{x}(t) \geq 0)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}$$

$$\leq \frac{P(n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)) \leq -x\ell_n)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}$$

$$\leq \frac{P(\operatorname{argmin}_{t \in [-(t_0+\chi_n)n^{1/3}, \infty)} Z_n^{x}(t) \geq -\ell_n^{-2}\delta) + P(n^{1/3}(t_0 + \chi_n - T_{(\bar{\kappa}_n)}) \geq \ell_n^{-2}\delta)}{1 - F_\tau(x(\sigma_F^2(t_0))^{-1/3}\ell_n)}.$$

Therefore, by Lemma 5.1, we obtain the conclusions of the proposition.   □

5.2. *A strong approximation theorem associated with the NPMLE.*   In this subsection, we present an exponential approximation for a version of $U_n(t)$ by a two-sided Wiener process. A similar approximation is also derived for the process $Z_n^x(t)$. This estimate will play a pivotal role in our proof of the moderate deviation result.

LEMMA 5.2.   *Let $Y_i = I_{\{X_i \leq t_0 + \chi_n\}} - F_0(t_0 + \chi_n)$. There exist a version $(\{Y_i^*, i \geq 1\}, \{T_i^*, i \geq 1\})$ of $(\{Y_i, i \geq 1\}, \{T_i, i \geq 1\})$, a two-sided Wiener process $W_n(t)$ with $W_n(0) = 0$, $n \geq 1$ and two positive absolute constants C, K such that for any $\delta > 0$,*

(5.6)
$$P\Big(\sup_{t \in [-(t_0+\chi_n)n^{1/3}, \infty)} |U_n^*(t) - n^{1/6}\sigma_U W_n(G_n^*(t))| \geq \delta \Big)$$
$$\leq \frac{n^{KC}}{K} \exp\{-K\delta n^{1/3}\},$$

where

$$U_n^*(t) := \frac{1}{n^{1/3}} \sum_{i=1}^{n} Y_i^* I_{\{t_0+\chi_n < T_i^* \leq t_0+\chi_n+n^{-1/3}t\}},$$

$$G_n^*(t) = G(t_0 + \chi_n + n^{-1/3}t) - G(t_0 + \chi_n)$$

and

$$\sigma_U^2 = E\big((I_{\{X_i \leq t_0+\chi_n\}} - F_0(t_0 + \chi_n))^2\big) = F_0(t_0 + \chi_n)(1 - F_0(t_0 + \chi_n)).$$

PROOF.   Since for any two-sided Wiener process W_n with $W_n(0) = 0$,

$$\{W_n(G(t_0 + \chi_n + n^{-1/3}t)) - W_n(G(t_0 + \chi_n), t \in \mathbb{R}\} \overset{d}{=} \{W_n(G_n^*(t)), t \in \mathbb{R}\},$$

applying Theorem 5 in Maumy (2004) or Theorem B.3 in Appendix B of the Supplementary Material [Gao, Xiong and Zhao (2017)] to Y_i and $Z_i = G(T_i)$, there exist a version $(\{Y_i^*, i \geq 1\}, \{T_i^*, i \geq 1\})$ of $(\{Y_i, i \geq 1\}, \{T_i, i \geq 1\})$, a Wiener process $W_n(t)$, $n \geq 1$ and two positive absolute constants C, K such that for all $\delta > 0$, (5.6) holds.   □

We can write

$$\begin{cases} R_n(t) = \sum_{i=1}^{n} \frac{1}{n}(g_n(X_i, T_i, t) - r_n(t)) =: \sum_{i=1}^{n} Z_{ni}(t), \\ \Gamma_n(t) = \sum_{i=1}^{n} \frac{1}{n}(h_n(T_i, t) - \gamma_n(t)) =: \sum_{i=1}^{n} \xi_{ni}(t), \end{cases}$$

where

$$\begin{cases} g_n(y, u, t) = n^{2/3}(I_{\{y \leq u\}} - I_{\{y \leq t_0 + \chi_n\}})(I_{\{u \leq t_0 + \chi_n + n^{-1/3}t\}} - I_{\{u \leq t_0 + \chi_n\}}), \\ h_n(u, t) = n^{1/3}(I_{\{u \leq t_0 + \chi_n + n^{-1/3}t\}} - I_{\{u \leq t_0 + \chi_n\}}). \end{cases}$$

When $t \in [-\epsilon_0 n^{1/3}/2, \epsilon_0 n^{1/3}/2]$,

$$\begin{cases} r_n(t) = \int_0^t \int_0^u f_0(t_0 + \chi_n + n^{-1/3}s)\, ds\, g(t_0 + \chi_n + n^{-1/3}u)\, du, \\ \gamma_n(t) = \int_0^t g(t_0 + \chi_n + n^{-1/3}u)\, du. \end{cases}$$

The following lemma is straightforward.

LEMMA 5.3.   (i) If $\ell_n = 1$ for all $n \geq 1$, then, under Assumption (A1),

(5.7)
$$\begin{cases} \lim_{n \to \infty} \sup_{|t| \leq N}\left|r_n(t) - \frac{f_0(t_0)g(t_0)t^2}{2}\right| = 0, \\ \lim_{n \to \infty} \sup_{|t| \leq N}\left|\gamma_n(t) - g(t_0)t\right| = 0. \end{cases}$$

(ii) If Assumptions (A1) and (A2) hold, then there exists a positive constant L such that, for any $N \geq 1$,

$$\begin{cases} \sup_{|t| \leq N\ell_n}\left|r_n(t) - \frac{f_0(t_0)g(t_0)t^2}{2}\right| \leq L(n^{-1/3}N^3\ell_n^3 + n^{-2/3}N^4\ell_n^4 + N^2\ell_n^2\chi_n), \\ \sup_{|t| \leq N\ell_n}\left|\gamma_n(t) - g(t_0)t\right| \leq L(N^2\ell_n^2 n^{-1/3} + N\ell_n\chi_n). \end{cases}$$

In particular, (5.7) holds.

Set

$$
\begin{cases}
\sigma_{R,n,\lambda}^2 = E\left(\sup_{t\in[-1,1]}\sum_{i=1}^n |Z_{ni}(\lambda t)|^2\right), & \mu_{R,n,\lambda} = E\left(\sup_{t\in[-1,1]}|R_n(\lambda t)|\right), \\[2ex]
\sigma_{\Gamma,n,\lambda}^2 = E\left(\sup_{t\in[-1,1]}\sum_{i=1}^n |\xi_{ni}(\lambda t)|^2\right), & \mu_{\Gamma,n,\lambda} = E\left(\sup_{t\in[-1,1]}|\Gamma_n(\lambda t)|\right).
\end{cases}
$$

Then by Proposition 2.1 in Giné and Guillou (2001) or Lemma B.1 in Appendix B of the Supplementary Material [Gao, Xiong and Zhao (2017)], there exists a positive constant L such that

$$
(5.8) \quad
\begin{cases}
\sigma_{R,n,N\ell_n}^2 \le LN^2\ell_n^2 n^{-1/3}, & \mu_{R,n,N\ell_n} \le LN\ell_n n^{-1/6}(\log n)^{1/2}, \\[2ex]
\sigma_{\Gamma,n,N\ell_n}^2 \le L\ell_n Nn^{-2/3}, & \mu_{\Gamma,n,N\ell_n} \le LN^{1/2}\ell_n^{1/2}n^{-1/3}(\log n)^{1/2}.
\end{cases}
$$

Now, define

$$
\tilde{Z}_n^x(t) = U_n(t) + \frac{f_0(t_0)g(t_0)t^2}{2} + x\ell_n g(t_0)t,
$$

$$
\tilde{Z}_n^{x,*}(t) = U_n^*(t) + \frac{f_0(t_0)g(t_0)t^2}{2} + x\ell_n g(t_0)t
$$

and

$$
\hat{Z}_n^x(t) = n^{1/6}\sigma_U W_n\big(G_n^*(t)\big) + \frac{f_0(t_0)g(t_0)t^2}{2} + x\ell_n g(t_0)t.
$$

Then the processes $\tilde{Z}_n^x$ and $\tilde{Z}_n^{x,*}$ have the same distribution.

PROPOSITION 5.2. *Suppose Assumption (A1) holds. If $\ell_n = 1$ for all $n \ge 1$, or Assumption (A2) holds, then for any $\delta > 0$ and $\rho > 0$,*

$$
(5.9) \quad \limsup_{n\to\infty}\frac{1}{\ell_n^3}\log\sup_{|x|\in[0,\rho]}P\left(\ell_n^{5/2}\sup_{|t|\le N\ell_n}\big|Z_n^x(t)-\tilde{Z}_n^x(t)\big|\ge\delta\right) = -\infty
$$

and

$$
(5.10) \quad \limsup_{n\to\infty}\frac{1}{\ell_n^3}\log\sup_{|x|\in[0,\rho]}P\left(\ell_n^{5/2}\sup_{|t|\le N\ell_n}\big|\tilde{Z}_n^{x,*}(t)-\hat{Z}_n^x(t)\big|\ge\delta\right) = -\infty.
$$

PROOF.   First, by (5.6),

$$
\limsup_{n\to\infty}\frac{1}{\ell_n^3}\log P\left(\ell_n^{5/2}\sup_{t\in[-(t_0+\chi_n)n^{1/3},\infty)}\big|U_n^*(t)-n^{1/6}\sigma_U W_n\big(G_n^*(t)\big)\big|\ge\delta\right) = -\infty.
$$

Thus, (5.10) is valid.

Applying Talagrand's inequality (see Theorem 1.4 in Talagrand (1996), or Theorem B.1 in Appendix B of the Supplementary Material [Gao, Xiong and Zhao

(2017)]) to $\{n^{1/3}R_n(N\ell_n t); t \in [-1, 1]\}$ and $\{n^{1/3}\ell_n\Gamma_n(N\ell_n t); t \in [-1, 1]\}$, there exists a constant $L \in (0, \infty)$ such that for all $n \geq 1$ and any $\delta > 0$,

$$P\left(\ell_n^{5/2}\left|\sup_{|t|\leq N\ell_n}|R_n(t)| - E\left(\sup_{|t|\leq N\ell_n}|R_n(t)|\right)\right| \geq \delta\right)$$

$$\leq L\exp\left\{-\frac{\delta n^{1/3}/\ell_n^{5/2}}{L}\log\left(1 + \frac{L\delta/\ell_n^{5/2}}{n^{1/3}\sigma_{R,n,N\ell_n}^2}\right)\right\}$$

and

$$P\left(\ell_n^{7/2}\left|\sup_{|t|\leq N\ell_n}|\Gamma_n(t)| - E\left(\sup_{|t|\leq N\ell_n}|\Gamma_n(t)|\right)\right| \geq \delta\right)$$

$$= P\left(\ell_n\left|\sup_{t\in[-1,1]}|n^{1/3}\Gamma_n(N\ell_n t)| - E\left(\sup_{t\in[-1,1]}|n^{1/3}\Gamma_n(N\ell_n t)|\right)\right| \geq \delta n^{1/3}/\ell_n^{5/2}\right)$$

$$\leq L\exp\left\{-\frac{\delta n^{1/3}/\ell_n^{5/2}}{L}\log\left(1 + \frac{L\delta/\ell_n^{5/2}}{n^{1/3}\ell_n^2\sigma_{\Gamma,n,N\ell_n}^2}\right)\right\}.$$

Therefore, by (5.8), if $\ell_n = 1$ for all $n \geq 1$, or Assumption (A2) holds, then

$$\limsup_{n\to\infty}\frac{1}{\ell_n^3}\log P\left(\ell_n^{5/2}\sup_{|t|\leq N\ell_n}|R_n(t)| \geq \delta\right) = -\infty$$

and

$$\limsup_{n\to\infty}\frac{1}{\ell_n^3}\log P\left(\ell_n^{7/2}\sup_{|t|\leq N\ell_n}|\Gamma_n(t)| \geq \delta\right) = -\infty.$$

These yield (5.9). □

5.3. *Exponential equivalence.* For each $x \in \mathbb{R}$, let

$$\begin{cases} \tau_n^x = \underset{t\in[-(t_0+\chi_n)n^{1/3},\infty)}{\operatorname{argmin}} Z_n^x(t), \\[2ex] \hat{\tau}_n^x = \underset{t\in[-(t_0+\chi_n)n^{1/3},\infty)}{\operatorname{argmin}} \hat{Z}_n^x(t), \\[2ex] \tilde{\tau}_n^x = \underset{t\in[-(t_0+\chi_n)n^{1/3},\infty)}{\operatorname{argmin}} \tilde{Z}_n^x(t), \\[2ex] \tilde{\tau}_n^{x,*} = \underset{t\in[-(t_0+\chi_n)n^{1/3},\infty)}{\operatorname{argmin}} \tilde{Z}_n^{x,*}(t). \end{cases}$$

Then the processes $\tilde{\tau}_n^x$ and $\tilde{\tau}_n^{x,*}$ have the same distribution.

The main result in this subsection is the following exponential equivalence theorem.

PROPOSITION 5.3.   *Suppose Assumption* (A1) *holds. If* $\ell_n = 1$ *for all* $n \geq 1$, *or Assumption* (A2) *holds with* $\ell_n \to \infty$, *then for any* $\delta > 0$ *and* $\rho > 0$,

$$(5.11) \qquad \limsup_{n \to \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P\big(|\tilde{\tau}_n^{x,*} - \hat{\tau}_n^x| \geq \ell_n^{-2} \delta\big) = -\infty$$

and

$$(5.12) \qquad \limsup_{n \to \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P\big(|\tau_n^x - \tilde{\tau}_n^x| \geq \ell_n^{-2} \delta\big) = -\infty.$$

Proposition 5.3 will be proved through the following two lemmas. Their proofs are based on the Talagrand deviation inequality and small ball estimates, and will be given in Appendix A of the Supplementary Material [Gao, Xiong and Zhao (2017)].

LEMMA 5.4.   *For each* $\rho > 0$,

$$(5.13) \qquad \limsup_{N \to \infty} \limsup_{n \to \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P\big(\max\{|\tilde{\tau}_n^x|, |\tau_n^x|\} \geq \ell_n N\big) = -\infty$$

and

$$(5.14) \qquad \limsup_{N \to \infty} \limsup_{n \to \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P\big(\max\{|\hat{\tau}_n^x|, |\tilde{\tau}_n^{x,*}|\} \geq \ell_n N\big) = -\infty.$$

LEMMA 5.5.   *For each* $\rho > 0$, $\delta > 0$ *and* $N \geq 1$,

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P\bigg(|\hat{\tau}_n^x| \leq N\ell_n, \hat{Z}_n^x(\hat{\tau}_n^x) - \inf_{\substack{|s| \leq N\ell_n, \\ |s - \hat{\tau}_n^x| \geq \delta \ell_n^{-2}}} \hat{Z}_n^x(s) \geq \frac{-\epsilon}{\ell_n^{5/2}}\bigg)$$

$$(5.15) \qquad = -\infty$$

and

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P\bigg(|\tilde{\tau}_n^x| \leq N\ell_n, \tilde{Z}_n^x(\tilde{\tau}_n^x) - \inf_{\substack{|s| \leq N\ell_n, \\ |s - \tilde{\tau}_n^x| \geq \delta \ell_n^{-2}}} \tilde{Z}_n^x(s) \geq \frac{-\epsilon}{\ell_n^{5/2}}\bigg)$$

$$(5.16) \qquad = -\infty.$$

Finally, we complete the proof of Proposition 5.3.

PROOF OF PROPOSITION 5.3.   We only show (5.11). For any $\delta > 0$, $N \geq 1$, and $\epsilon \in (0, 1/2)$, set

$$\Delta_n = \hat{Z}_n^x(\hat{\tau}_n^x) - \inf_{|s| \leq N\ell_n, |s - \hat{\tau}_n^x| \geq \delta \ell_n^{-2}} \hat{Z}_n^x(s).$$

Note that on $\{|\tilde{\tau}_n^{x,*}| \le \ell_n N\}$, for any $|s| \le \ell_n N$,

$$
\begin{aligned}
\hat{Z}_n^x(s) - \hat{Z}_n^x(\tilde{\tau}_n^{x,*}) &= \hat{Z}_n^x(s) - \tilde{Z}_n^{x,*}(\tilde{\tau}_n^{x,*}) + \tilde{Z}_n^{x,*}(\tilde{\tau}_n^{x,*}) - \hat{Z}_n^x(\tilde{\tau}_n^{x,*}) \\
&\ge \hat{Z}_n^x(s) - \tilde{Z}_n^{x,*}(s) + \tilde{Z}_n^{x,*}(\tilde{\tau}_n^{x,*}) - \hat{Z}_n^x(\tilde{\tau}_n^{x,*}) \\
&\ge -2 \sup_{|s| \le \ell_n N} |\hat{Z}_n^x(s) - \tilde{Z}_n^{x,*}(s)|.
\end{aligned}
$$

Then

$$
\begin{aligned}
P\big(|\tilde{\tau}_n^{x,*} &- \hat{\tau}_n^x| \ge \ell_n^{-2}\delta\big) \\
&\le P\big(\max\{|\tilde{\tau}_n^{x,*}|, |\hat{\tau}_n^x|\} \ge N\ell_n\big) + P\big(|\hat{\tau}_n^x| \le N\ell_n, \Delta_n \ge -\epsilon/\ell_n^{5/2}\big) \\
&\quad + P\big(\max\{|\tilde{\tau}_n^{x,*}|, |\hat{\tau}_n^x|\} \le N\ell_n, \ell_n^{5/2}\Delta_n < -\epsilon, |\tilde{\tau}_n^{x,*} - \hat{\tau}_n^x| \ge \ell_n^{-2}\delta\big) \\
&\le P\big(\max\{|\tilde{\tau}_n^{x,*}|, |\hat{\tau}_n^x|\} \ge N\ell_n\big) + P\big(|\hat{\tau}_n| \le N\ell_n, \Delta_n \ge -\epsilon/\ell_n^{5/2}\big) \\
&\quad + P\big(\max\{|\tilde{\tau}_n^{x,*}|, |\hat{\tau}_n^x|\} \le N\ell_n, \ell_n^{5/2}\big(\hat{Z}_n^x(\hat{\tau}_n^x) - \hat{Z}_n^x(\tilde{\tau}_n^{x,*})\big) < -\epsilon\big) \\
&\le P\big(\max\{|\tilde{\tau}_n^{x,*}|, |\hat{\tau}_n^x|\} \ge N\ell_n\big) + P\big(|\hat{\tau}_n| \le N\ell_n, \Delta_n \ge -\epsilon/\ell_n^{5/2}\big) \\
&\quad + P\Big(2\ell_n^{5/2} \sup_{|s| \le \ell_n N} |\hat{Z}_n^x(s) - \tilde{Z}_n^{x,*}(s)| \ge \epsilon\Big).
\end{aligned}
$$

Thus, by Lemmas 5.4 and 5.5 and Proposition 5.2, (5.11) holds. $\quad\square$

5.4. *Proof of Theorem* 3.1.   In this subsection we present the proof of Theorem 3.1. For $c > 0$, define $\tau(c) = \operatorname{argmin}_s\{W(s) + cs^2\}$ and $\tau := \tau(1)$. Noting that $c^{1/3}(W(s) + cs^2)$ and $W(c^{2/3}s) + (c^{2/3}s)^2$ have identical distributions, it then follows from $\tau(c) = \operatorname{argmin}_s\{c^{1/3}(W(s) + cs^2)\}$ that $\tau(c) \stackrel{d}{=} c^{-2/3}\tau$. Define

$$
(5.17) \quad
\begin{cases}
\hat{Y}_n^x(s) = W_n(s) + \dfrac{f_0(t_0)g(t_0)}{2\sigma_U}p_n^2\big(n^{-1/3}s\big) + \dfrac{g(t_0)}{\sigma_U}x\ell_n p_n\big(n^{-1/3}s\big), \\[2mm]
\qquad n^{-1/3}s \in \big(-G(t_0 + \chi_n), 1 - G(t_0 + \chi_n)\big), \\[2mm]
Y_n^x(s) = W_n(s) + \dfrac{f_0(t_0)}{2\sigma_U g(t_0)}s^2 + \dfrac{1}{\sigma_U}x\ell_n s, \qquad s \in \mathbb{R},
\end{cases}
$$

where $p_n(s)$ is the inverse function of $q_n(u) := G(t_0 + \chi_n + n^{-1/3}u) - G(t_0 + \chi_n)$. Set $\hat{T}_n^x = \operatorname{argmin}_s \hat{Y}_n^x(s)$ and $T_n^x = \operatorname{argmin}_s Y_n^x(s)$. Then $\hat{T}_n^x = n^{1/3}q_n(\hat{\tau}_n^x)$, and

$$
(5.18) \quad \lim_{n \to \infty} \ell_n^3 \sup_{|s| \le N\ell_n} |n^{1/3}q_n(s) - g(t_0)s| = 0.
$$

When $s \in [-\epsilon_0 n^{1/3}, \epsilon_0 n^{1/3}]$, we can write

$$
p_n\big(n^{-1/3}s\big) = \int_0^{n^{-1/3}s} \frac{n^{1/3}\,du}{g(t_0 + \chi_n + n^{-1/3}u)} = \int_0^{n^{-2/3}s} \frac{n^{2/3}\,dv}{g(t_0 + \chi_n + v)}.
$$

Thus,

$$\sup_{|s| \le N \ell_n} \left| p_n(n^{-1/3}s) - \frac{s}{g(t_0 + \chi_n)} \right|$$

$$\le \int_0^{n^{-2/3} N \ell_n} \frac{n^{2/3} |g(t_0 + \chi_n + v) - g(t_0 + \chi_n)| \, dv}{g(t_0 + \chi_n + v) g(t_0 + \chi_n)},$$

which implies that when $\ell_n = 1$, $\lim_{n \to \infty} \sup_{|s| \le N \ell_n} |p_n(n^{-1/3}s) - \frac{s}{g(t_0)}| = 0$, and that when Assumptions (A1) and (A2) hold,

$$(5.19) \qquad \lim_{n \to \infty} \ell_n^{3.5} \sup_{|s| \le N \ell_n} \left| p_n(n^{-1/3}s) - \frac{s}{g(t_0)} \right| = 0.$$

Therefore, if Assumption (A1) holds and $\ell_n = 1$ for all $n \ge 1$, or Assumption (A2) also holds, then for each $\rho > 0$ and any $\delta > 0$,

$$(5.20) \quad \limsup_{N \to \infty} \limsup_{n \to \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P\left(\ell_n^{5/2} \sup_{|s| \le \ell_n N} |\hat{Y}_n^x(s) - Y_n^x(s)| \ge \delta \right) = -\infty.$$

By the same arguments as used in the proofs of Lemma 5.4 and Proposition 5.3, the following result holds.

LEMMA 5.6.   (i) *For each $\rho > 0$,*

$$(5.21) \qquad \limsup_{N \to \infty} \limsup_{n \to \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P\left(\max\{|\hat{T}_n^x|, |T_n^x|\} \ge \ell_n N \right) = -\infty.$$

(ii) *If Assumption (A1) holds and $\ell_n = 1$ for all $n \ge 1$, or Assumption (A2) also holds, then for each $\rho > 0$ and any $\delta > 0$,*

$$(5.22) \qquad \limsup_{n \to \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0, \rho]} P\left(|\hat{T}_n^x - T_n^x| \ge \delta \ell_n^{-2} \right) = -\infty.$$

Write

$$T_n^x = \operatorname*{argmin}_s \left\{ W_n\left(s - \frac{x g(t_0) \ell_n}{f_0(t_0)} \right) - W_n\left(-\frac{x g(t_0) \ell_n}{f_0(t_0)} \right) + \frac{f_0(t_0)}{2 \sigma_U g(t_0)} s^2 \right\}$$

$$- \frac{x g(t_0) \ell_n}{f_0(t_0)}.$$

Thus, T_n^x and $-\frac{x g(t_0) \ell_n}{f_0(t_0)} + \left(\frac{f_0^2(t_0)}{4 F_0(t_0)(1 - F_0(t_0)) g^2(t_0)} \right)^{-1/3} \tau$ are identical in distribution.

Now we are ready to complete the proof of the theorem.

PROOF OF THEOREM 3.1.   We only show (3.1). Take $\ell_n = b_n$. Then for each $\rho > 0$, by (5.18), Lemma 5.6 and (2.1), we have

$$\liminf_{\delta \downarrow 0} \liminf_{n \to \infty} \inf_{x \in [0, \rho]} \frac{1}{1 - F_\tau(b_n(\sigma_F^2(t_0))^{-1/3}x)} P(\hat{\tau}_n^{-x} < -b_n^{-2}\delta)$$

$$= \lim_{\delta \downarrow 0} \liminf_{n \to \infty} \inf_{x \in [0, \rho]} \frac{1}{1 - F_\tau(b_n(\sigma_F^2(t_0))^{-1/3}x)}$$
$$\times P(\hat{T}_n^{-x} < n^{1/3} q_n(-b_n^{-2}\delta))$$

$$= \lim_{\delta \downarrow 0} \liminf_{n \to \infty} \inf_{x \in [0, \rho]} \frac{1}{1 - F_\tau(b_n(\sigma_F^2(t_0))^{-1/3}x)} P(T_n^{-x} < -b_n^{-2}\delta)$$

$$= \lim_{\delta \downarrow 0} \liminf_{n \to \infty} \inf_{x \in [0, \rho]} \frac{1}{1 - F_\tau(b_n(\sigma_F^2(t_0))^{-1/3}x)}$$
$$\times P(\tau < -b_n^{-2}\delta - b_n(\sigma_F^2(t_0))^{-1/3}x)$$

$$= 1$$

and

$$\limsup_{n \to \infty} \sup_{x \in [0, \rho]} \frac{1}{1 - F_\tau(b_n(\sigma_F^2(t_0))^{-1/3}x)} P(\hat{\tau}_n^{-x} \le 0) = 1.$$

Thus, by Propositions 5.3 and 5.1, this yields

$$\lim_{n \to \infty} \sup_{x \in [0, \rho b_n]} \left| \frac{1}{1 - F_\tau(x)} P\left(\frac{n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n))}{(\sigma_F^2(t_0))^{1/3}} \ge x \right) - 1 \right| = 0.$$

In particular, we obtain that for any $x > 0$,

$$(5.23) \quad \lim_{n \to \infty} \frac{1}{b_n^3} \log P\left(\pm \frac{1}{b_n} \frac{n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n))}{(\sigma_F^2(t_0))^{1/3}} > x \right) = -\frac{2}{3}x^3.$$

Equation (5.23) is equivalent to the statement that the sequence $\{\frac{n^{1/3}}{b_n}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)), n \ge 1\}$ satisfies the MDP in $\mathbb{R}$ with speed b_n^3 and rate function $I(x)$ [cf. Dembo and Zeitouni (1998), page 7].   □

5.5. *Proof of Theorem 3.2.*   The proof of Theorem 3.2 is similar to that of Theorem 3.1. Take $\ell_n \equiv 1$. Then for $x \in \mathbb{R}$, by Propositions 5.1 and 5.3, Lemma 5.6

and (5.18), we have

$$
\limsup_{n \to \infty} P\big(n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)) \leq x\big)
$$

$$
\leq \lim_{\delta \downarrow 0} \limsup_{n \to \infty} P\big(\tau_n^{-x} > -\delta\big)
$$

$$
= \lim_{\delta \downarrow 0} P\left(\frac{xg(t_0)}{f_0(t_0)} + \left(\frac{f_0^2(t_0)}{4F_0(t_0)(1 - F_0(t_0))g^2(t_0)}\right)^{-1/3} \tau > -\delta\right)
$$

$$
= P\left(\left(\frac{4 f_0(t_0) F_0(t_0)(1 - F_0(t_0))}{g(t_0)}\right)^{1/3} \tau \leq x\right)
$$

and

$$
\liminf_{n \to \infty} P\big(n^{1/3}(\hat{F}_n(t_0 + \chi_n) - F_0(t_0 + \chi_n)) \leq x\big)
$$

$$
\geq \liminf_{n \to \infty} P\big(\tau_n^{-x} \geq 0\big)
$$

$$
= P\left(\left(\frac{4 f_0(t_0) F_0(t_0)(1 - F_0(t_0))}{g(t_0)}\right)^{1/3} \tau \leq x\right).
$$

SUPPLEMENTARY MATERIAL

Supplement to "Moderate deviations and nonparametric inference for monotone functions" (DOI: 10.1214/17-AOS1583SUPP; .pdf). The supplement [Gao, Xiong and Zhao (2017)] contains all remaining technical proofs omitted from the main text due to space constraints, in which we prove Lemmas 5.4 and 5.5, Proposition 2.2, Theorem 2.3 and its Corollaries 2.1–2.3.

REFERENCES

ABREVAYA, J. and HUANG, J. (2005). On the bootstrap of the maximum score estimator. *Econometrica* **73** 1175–1204. MR2149245

BANERJEE, M. (2007). Likelihood based inference for monotone response models. *Ann. Statist.* **35** 931–956. MR2341693

BANERJEE, M. and WELLNER, J. A. (2001). Likelihood ratio tests for monotone functions. *Ann. Statist.* **29** 1699–1731. MR1891743

BANERJEE, M. and WELLNER, J. A. (2005a). Confidence intervals for current status data. *Scand. J. Stat.* **32** 405–424. MR2204627

BANERJEE, M. and WELLNER, J. A. (2005b). Score statistics for current status data: Comparisons with likelihood ratio and Wald statistics. *Int. J. Biostat.* **1** Article ID 3. MR2232228

CHANG, J., SHAO, Q.-M. and ZHOU, W.-X. (2016). Cramér-type moderate deviations for Studentized two-sample U-statistics with applications. *Ann. Statist.* **44** 1931–1956. MR3546439

DEMBO, A. and ZEITOUNI, O. (1998). *Large Deviations Techniques and Applications*, 2nd ed. *Applications of Mathematics* (*New York*) **38**. Springer, New York. MR1619036

DE LA PEÑA, V. H., LAI, T. L. and SHAO, Q.-M. (2009). *Self-Normalized Processes*: *Limit Theory and Statistical Applications*. Springer, Berlin. MR2488094

DÜMBGEN, L., WELLNER, J. A. and WOLFF, M. (2016). A law of the iterated logarithm for Grenander's estimator. *Stochastic Process*. *Appl*. **126** 3854–3864. MR3565482

DUROT, C. (2002). Sharp asymptotics for isotonic regression. *Probab. Theory Related Fields* **122** 222–240. MR1894068

DUROT, C. (2007). On the $\mathbb{L}_p$-error of monotonicity constrained estimators. *Ann*. *Statist*. **35** 1080–1104. MR2341699

DUROT, C., KULIKOV, V. N. and LOPUHAÄ, H. P. (2012). The limit distribution of the L_∞-error of Grenander-type estimators. *Ann*. *Statist*. **40** 1578–1608. MR3015036

GAO, F. (2003). Moderate deviations and large deviations for kernel density estimators. *J. Theoret*. *Probab.* **16** 401–418. MR1982035

GAO, F., XIONG, J. and ZHAO, X. (2018). Supplement to "Moderate deviations and nonparametric inference for monotone functions." DOI:10.1214/17-AOS1583SUPP.

GAO, F. and ZHAO, X. (2011). Delta method in large deviations and moderate deviations for estimators. *Ann*. *Statist*. **39** 1211–1240. MR2816352

GINÉ, E. and GUILLOU, A. (2001). On consistency of kernel density estimators for randomly censored data: Rates holding uniformly over adaptive intervals. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **37** 503–522. MR1876841

GRENANDER, U. (1956). On the theory of mortality measurement. II. *Skand*. *Aktuarietidskr*. **39** 125–153. MR0093415

GROENEBOOM, P. (1985). Estimating a monotone density. In *Proceedings of the Berkeley Conference in Honor of Jerzy Neyman and Jack Kiefer*, *Vol. II* (*Berkeley*, *Calif.*, 1983) 539–555. Wadsworth, Belmont, CA. MR0822052

GROENEBOOM, P. (1989). Brownian motion with a parabolic drift and Airy functions. *Probab. Theory Related Fields* **81** 79–109. MR0981568

GROENEBOOM, P. (1996). Lectures on inverse problems. In *Lectures on Probability Theory and Statistics* (*Saint-Flour*, 1994). *Lecture Notes in Math*. **1648** 67–164. Springer, Berlin. MR1600884

GROENEBOOM, P. (2014). Maximum smoothed likelihood estimators for the interval censoring model. *Ann*. *Statist*. **42** 2092–2137. MR3262478

GROENEBOOM, P., HOOGHIEMSTRA, G. and LOPUHAÄ, H. P. (1999). Asymptotic normality of the L_1 error of the Grenander estimator. *Ann*. *Statist*. **27** 1316–1347. MR1740109

GROENEBOOM, P. and JONGBLOED, G. (2014). *Nonparametric Estimation Under Shape Constraints*: *Estimators*, *Algorithms and Asymptotics*. *Cambridge Series in Statistical and Probabilistic Mathematics* **38**. Cambridge Univ. Press, New York. MR3445293

GROENEBOOM, P. and JONGBLOED, G. (2015). Nonparametric confidence intervals for monotone functions. *Ann*. *Statist*. **43** 2019–2054. MR3375875

GROENEBOOM, P., JONGBLOED, G. and WITTE, B. I. (2010). Maximum smoothed likelihood estimation and smoothed maximum likelihood estimation in the current status model. *Ann*. *Statist*. **38** 352–387. MR2589325

GROENEBOOM, P. and WELLNER, J. A. (1992). *Information Bounds and Nonparametric Maximum Likelihood Estimation*. *DMV Seminar* **19**. Birkhäuser, Basel. MR1180321

GROENEBOOM, P. and WELLNER, J. A. (2001). Computing Chernoff's distribution. *J. Comput*. *Graph*. *Statist*. **10** 388–400. MR1939706

HOOGHIEMSTRA, G. and LOPUHAÄ, H. P. (1998). An extremal limit theorem for the argmax process of Brownian motion minus a parabolic drift. *Extremes* **1** 215–240. MR1814624

HUANG, J. and WELLNER, J. A. (1995). Estimation of a monotone density or monotone hazard under random censoring. *Scand*. *J*. *Stat*. **22** 3–33. MR1334065

LIU, W. and SHAO, Q.-M. (2010). Cramér-type moderate deviation for the maximum of the periodogram with application to simultaneous tests in gene expression time series. *Ann. Statist.* **38** 1913–1935. MR2662363

LIU, W. and SHAO, Q.-M. (2013). A Carmér moderate deviation theorem for Hotelling's T^2-statistic with applications to global tests. *Ann. Statist.* **41** 296–322. MR3059419

MAUMY, M. (2004). Strong approximations for the compound empirical process. *Ann. I.S.U.P.* **48** 69–83. MR2083781

PETROV, V. V. (1975). *Sums of Independent Random Variables. Ergebnisse der Mathematik und ihrer Grenzgebiete* **82**. Springer, New York. Translated from the Russian by A. A. Brown. MR0388499

PRAKASA RAO, B. L. S. (1969). Estimation of a unimodal density. *Sankhyā Ser. A* **31** 23–36. MR0267677

SUN, J. (2006). *The Statistical Analysis of Interval-Censored Failure Time Data*. Springer, New York. MR2287318

TALAGRAND, M. (1996). New concentration inequalities in product spaces. *Invent. Math.* **126** 505–563. MR1419006

VAN DER VAART, A. W. and WELLNER, J. A. (1996). *Weak Convergence and Empirical Processes: With Applications to Statistics*. Springer, New York. MR1385671

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SUPPLEMENT TO "MODERATE DEVIATIONS AND NONPARAMETRIC INFERENCE FOR MONOTONE FUNCTIONS"

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APPENDIX A

In this Appendix, we prove Lemmas 5.4 and 5.5, and the results for Grenander-type estimator, including Proposition 2.2, Theorem 2.3 and Corollaries 2.1-2.3.

A.1. The proofs of Lemmas 5.4 and 5.5. First, let us give several elementary lemmas.

LEMMA A.1. *Let $\{W(t), t \in \mathbb{R}\}$ be a two-side Wiener process with $W(0) = 0$. Then for any $\delta > 0$,*

$$(A.1) \qquad \limsup_{N \to \infty} \limsup_{n \to \infty} \frac{1}{\ell_n^3} \log P \left(\sup_{|s| \geq \ell_n N} \frac{|W(s)|}{\ell_n |s|} \geq \delta \right) = -\infty.$$

In particular,

$$(A.2) \qquad \limsup_{N \to \infty} \limsup_{n \to \infty} \frac{1}{\ell_n^3} \log P \left(\sup_{|s| \geq \ell_n N} \frac{|W(s)|}{s^2} \geq \delta \right) = -\infty.$$

PROOF. Note that the case of $\ell_n = 1$ is easy. So we consider the case of $\ell_n \to \infty$ only. For any $N \geq 2$, we have

$$
P \left(\sup_{|s| \geq \ell_n N} \frac{|W(s)|}{\ell_n |s|} \geq \delta \right) \leq P \left(\sup_{k \geq \ell_n N} \sup_{|s| \in [k, k+1]} \frac{|W(s)|}{k} \geq \ell_n \delta \right)
$$

$$
\leq 4 \sum_{k \geq \ell_n N} P \left(|W(1)| \geq k^{1/2} \ell_n \delta / 2 \right)
$$

$$
\leq 4 \sum_{k \geq \ell_n N} \exp \left\{ -\frac{k \ell_n^2 \delta^2}{16} \right\} E \left(\exp\{ |W(1)|^2 / 16 \} \right)
$$

$$
\leq \frac{4 E \left(\exp\{ |W(1)|^2 / 16 \} \right) \exp \left\{ -\frac{\ell_n^3 N \delta^2}{16} \right\}}{1 - \exp \left\{ -\frac{\ell_n^2 \delta^2}{16} \right\}},
$$

1

which implies that

$$\limsup_{n\to\infty} \frac{1}{\ell_n^3} \log P\left(\sup_{|s|\geq \ell_n N} \frac{|W(s)|}{\ell_n |s|} \geq \delta \right) \leq -\frac{\delta^2 N}{16}.$$

Therefore, (A.1) holds.

$\square$

LEMMA A.2.   *For each $\rho > 0$, when n and N are large enough, we have*

$$(A.3) \qquad \sup_{\substack{|t|\geq \ell_n N, \\ t\geq -(t_0+\chi_n)n^{1/3}}} \sup_{|x|\in[0,\rho]} \frac{|x|\ell_n|\gamma_n(t)|}{r_n(t)} \leq \delta/4.$$

PROOF. Set

$$m_f = \inf_{s\in[t_0-\epsilon_0,t_0+\epsilon_0]} f_0(s),\ m_g = \inf_{s\in[t_0-\epsilon_0,t_0+\epsilon_0]} g(s), M_g = \sup_{s\in[t_0-\epsilon_0,t_0+\epsilon_0]} g(s).$$

Then, for all $t\in[-\epsilon_0 n^{1/3}, \epsilon_0 n^{1/3}]$,

$$r_n(t) \geq m_f m_g n^{2/3} \int_{t_0+\chi_n}^{t_0+\chi_n+n^{-1/3}t} (u-(t_0+\chi_n))du \geq \frac{1}{2}m_f m_g t^2,$$

and $|\gamma_n(t)| \leq M_g|t|$. Thus, for all $t\in[-\epsilon_0 n^{1/3}, \epsilon_0 n^{1/3}]$,

$$r_n(t) \geq \frac{1}{2}m_f m_g M_g^{-2}(G_n^*(t))^2 n^{2/3},$$

and for all $|t|\geq \epsilon_0 n^{1/3}$

$$r_n(t) \geq \frac{1}{2}m_f m_g \epsilon_0^2 n^{2/3} \geq \frac{1}{2}m_f m_g \epsilon_0^2 n^{1/3}|\gamma_n(t)|, \quad |\gamma_n(t)| \leq n^{1/3}.$$

Thus, (A.3) holds when n and N are large enough. $\square$

LEMMA A.3.
(A.4)

$$\limsup_{N\to\infty}\limsup_{n\to\infty} \frac{1}{\ell_n^3} \log P\left(\sup_{\substack{|t|\geq \ell_n N, \\ t\geq -(t_0+\chi_n)n^{1/3}}} \frac{n^{1/6}\sigma_U|W_n(G_n^*(t))|}{r_n(t)} \geq \frac{\delta}{4} \right) = -\infty,$$

and

$$(A.5) \qquad \limsup_{N\to\infty}\limsup_{n\to\infty} \frac{1}{\ell_n^3} \log P\left(\sup_{\substack{|t|\geq \ell_n N, \\ t\geq -(t_0+\chi_n)n^{1/3}}} \frac{|U_n(t)|}{r_n(t)} \geq \frac{\delta}{4} \right) = -\infty.$$

PROOF. We write

$$P\left(\sup_{\substack{|t|\geq\ell_n N,\\ t\geq-(t_0+\chi_n)n^{1/3}}}\frac{n^{1/6}\sigma_U|W_n(G_n^*(t))|}{r_n(t)}\geq\delta/4\right)$$

$$\leq P\left(\sup_{\ell_n N\leq|t|\leq\epsilon_0 n^{1/3}}\frac{n^{1/6}\sigma_U|W_n(G_n^*(t))|}{\frac{1}{2}m_f m_g M_g^{-2}(G_n^*(t))^2 n^{2/3}}\geq\delta/4\right)$$

(A.6)
$$+P\left(\sup_{\substack{|t|\geq\epsilon_0 n^{1/3},\\ t\geq-(t_0+\chi_n)n^{1/3}}}\frac{n^{1/6}\sigma_U|W_n(G_n^*(t))|}{\frac{1}{2}m_f m_g\epsilon_0^2 n^{2/3}}\geq\delta/4\right)$$

$$\leq P\left(\sup_{|t|\geq m_g N\ell_n}\frac{|W_n(t)|}{t^2}\geq\frac{\delta}{8}m_f m_g M_g^{-2}\sigma_U^{-1}\right)$$

$$+P\left(\sup_{|t|\leq 1}|W_n(t)|\geq\frac{\delta}{8}m_f m_g\sigma_U^{-1}\epsilon_0^2 n^{1/2}\right).$$

By Lemma A.1,

$$\limsup_{N\to\infty}\limsup_{n\to\infty}\frac{1}{\ell_n^3}\log P\left(\sup_{|t|\geq m_g N\ell_n}\frac{|W_n(t)|}{t^2}\geq\frac{\delta}{8}m_f m_g M_g^{-2}\sigma_U^{-1}\right)=-\infty.$$

By

$$P\left(\sup_{|t|\leq 1}|W_n(t)|\geq\frac{\delta}{8}m_f m_g\epsilon_0^2\sigma_U^{-1}n^{1/2}\right)\leq 2P\left(|W_n(1)|\geq\frac{\delta}{8}m_f m_g\epsilon_0^2\sigma_U^{-1}n^{1/2}\right),$$

we have

$$\limsup_{n\to\infty}\frac{1}{n^{1/3}}\log P\left(\sup_{|t|\leq 1}|W_n(t)|\geq\frac{\delta}{8}m_f m_g\epsilon_0^2\sigma_U^{-1}n^{1/2}\right)=-\infty.$$

Therefore, (A.4) follows. □

LEMMA A.4. *For each $\rho>0$,*
(A.7)
$$\limsup_{n\to\infty}\frac{1}{\ell_n^3}\log\sup_{|x|\in[0,\rho]}P\left(\sup_{\substack{|t|\geq\ell_n N,\\ t\geq-(t_0+\chi_n)n^{1/3}}}\frac{|R_n(t)|+|x|\ell_n|\Gamma_n(t)|}{r_n(t)}\geq 1-\frac{3\delta}{4}\right)=-\infty.$$

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PROOF. Since there exists a positive constant L such that

$$E(|Z_{n1}(t)|^2) \leq Ln^{-2/3}\min\{t^2 n^{-2/3}, \epsilon_0^2\}$$

and

$$E(|\xi_{n1}(t)|^2) \leq Ln^{-4/3}\min\{|t|n^{-1/3}, \epsilon_0\},$$

then, by Lemma B.1, there exists a positive constant M such that

$$E\left(\sup_{\substack{|t|\geq \ell_n N, \\ t\geq -(t_0+\chi_n)n^{1/3}}} \frac{|R_n(t)|}{r_n(t)}\right)$$

$$\leq E\left(\sup_{\substack{\ell_n N \leq |t| \leq \epsilon_0 n^{1/3}, \\ t\geq -(t_0+\chi_n)n^{1/3}}} \frac{|R_n(t)|}{\frac{1}{2}m_f m_g \epsilon_0^2 t^2}\right) + E\left(\sup_{\substack{|t|\geq \epsilon_0 n^{1/3}, \\ t\geq -(t_0+\chi_n)n^{1/3}}} \frac{|R_n(t)|}{\frac{1}{2}m_f m_g \epsilon_0^2 n^{2/3}}\right)$$

$$\leq Mn^{-1/6}(\log n)^{1/2},$$

$$E\left(\sup_{\substack{|t|\geq \ell_n N, \\ t\geq -(t_0+\chi_n)n^{1/3}}} \frac{\sum_{i=1}^n |Z_{ni}(t)|^2}{(r_n(t))^2}\right) \leq Mn^{-1/3},$$

$$E\left(\sup_{\substack{|t|\geq \ell_n N, \\ t\geq -(t_0+\chi_n)n^{1/3}}} \frac{\ell_n|\Gamma_n(t)|}{r_n(t)}\right) \leq Mn^{-1/3}(\log n)^{1/2},$$

and

$$E\left(\sup_{\substack{|t|\geq \ell_n N, \\ t\geq -(t_0+\chi_n)n^{1/3}}} \frac{\ell_n^2 \sum_{i=1}^n |\xi_{ni}(t)|^2}{(r_n(t))^2}\right) \leq Mn^{-2/3}.$$

Therefore, for any $r > 0$, when n is large enough,

$$P\left(\sup_{\substack{|t|\geq \ell_n N, \\ t\geq -(t_0+\chi_n)n^{1/3}}} \frac{|R_n(t)| + |x|\ell_n|\Gamma_n(t)|}{r_n(t)} \geq r\right)$$

$$\leq P\left(\left|\sup_{\substack{|t|\geq \ell_n N, \\ t\geq -(t_0+\chi_n)n^{1/3}}} \frac{|R_n(t)|}{r_n(t)} - E\left(\sup_{\substack{|t|\geq \ell_n N, \\ t\geq -(t_0+\chi_n)n^{1/3}}} \frac{|R_n(t)|}{r_n(t)}\right)\right| \geq \frac{r}{4}\right)$$

$$+ P\left(|x|\left|\sup_{\substack{|t|\geq \ell_n N, \\ t\geq -(t_0+\chi_n)n^{1/3}}} \frac{|\Gamma_n(t)|}{r_n(t)} - E\left(\sup_{\substack{|t|\geq \ell_n N, \\ t\geq -(t_0+\chi_n)n^{1/3}}} \frac{|\Gamma_n(t)|}{r_n(t)}\right)\right| \geq \frac{r}{4}\right).$$

Then, applying Talagrand's inequality (see Theorem B.1) to

$$\left\{ \frac{n^{1/3}|R_n(t)|}{r_n(t)}; |t| \geq \ell_n N, t \geq -(t_0 + \chi_n)n^{1/3} \right\}$$

and

$$\left\{ \frac{n^{1/3}|\Gamma_n(t)|}{r_n(t)}; |t| \geq \ell_n N, t \geq -(t_0 + \chi_n)n^{1/3} \right\},$$

there exists a constant $L \in (0, \infty)$ such that for all $n \geq 1$,

$$P\left(\left| \sup_{\substack{|t| \geq \ell_n N, \\ t \geq -(t_0+\chi_n)n^{1/3}}} \frac{|R_n(t)|}{r_n(t)} - E\left(\sup_{\substack{|t| \geq \ell_n N, \\ t \geq -(t_0+\chi_n)n^{1/3}}} \frac{|R_n(t)|}{r_n(t)} \right) \right| \geq \frac{r}{4} \right)$$
$$\leq L \exp\left\{ -\frac{n^{1/3}r/4}{L} \log\left(1 + \frac{Lr/4}{M} \right) \right\},$$

and

$$P\left(|x| \left| \sup_{\substack{|t| \geq \ell_n N, \\ t \geq -(t_0+\chi_n)n^{1/3}}} \frac{|\Gamma_n(t)|}{r_n(t)} - E\left(\sup_{\substack{|t| \geq \ell_n N, \\ t \geq -(t_0+\chi_n)n^{1/3}}} \frac{|\Gamma_n(t)|}{r_n(t)} \right) \right| \geq \frac{r}{4} \right)$$
$$\leq L \exp\left\{ -\frac{n^{1/3}r/(4|x|)}{L} \log\left(1 + \frac{Ln^{1/3}r/(4|x|)}{M} \right) \right\}.$$

Therefore,

$$\limsup_{n \to \infty} \frac{1}{\ell_n^3} \log \sup_{x \in [0,\rho]} P\left(\sup_{\substack{|t| \geq \epsilon_0 n^{1/3}, \\ t \geq -(t_0+\chi_n)n^{1/3}}} \frac{|R_n(t)| + |x|\ell_n|\Gamma_n(t)|}{r_n(t)} \geq r \right) = -\infty.$$

Thus, (A.7) is proved. $\qquad \square$

LEMMA A.5. *For each $\rho > 0$, and $\delta \in (0, 1/2)$,*

$$(A.8) \quad \limsup_{N \to \infty} \limsup_{n \to \infty} \frac{1}{\ell_n^3} \log \sup_{|x| \in [0,\rho]} P\left(\inf_{\substack{|t| \geq \ell_n N, \\ t \geq -(t_0+\chi_n)n^{1/3}}} \frac{Z_n^x(t)}{r_n(t)} \leq \delta \right) = -\infty,$$

and

(A.9) $\limsup\limits_{N\to\infty}\limsup\limits_{n\to\infty}\dfrac{1}{\ell_n^3}\log\sup\limits_{|x|\in[0,\rho]}P\left(\inf\limits_{\substack{|t|\geq\ell_nN,\\ t\geq-(t_0+\chi_n)n^{1/3}}}\dfrac{\hat{Z}_n^x(t)}{t^2}\leq\delta\right)=-\infty.$

PROOF. We only show (A.8). The proof of (A.9) is the same. It is obvious that

$$Z_n^x(t)\geq r_n(t)-|x|\ell_n|\gamma_n(t)|-|R_n(t)|-|x|\ell_n|\Gamma_n(t)|-|U_n(t)|.$$

By Lemma A.2, we then have

(A.10)
$$P\left(\inf\limits_{\substack{|t|\geq\ell_nN,\\ t\geq-(t_0+\chi_n)n^{1/3}}}\dfrac{Z_n^x(t)}{r_n(t)}\leq\delta\right)$$
$$\leq P\left(\sup\limits_{\substack{|t|\geq\ell_nN,\\ t\geq-(t_0+\chi_n)n^{1/3}}}\dfrac{|U_n(t)|}{r_n(t)}\geq\delta/2\right)$$
$$+P\left(\sup\limits_{\substack{|t|\geq\ell_nN,\\ t\geq-(t_0+\chi_n)n^{1/3}}}\dfrac{|R_n(t)|+|x|\ell_n|\Gamma_n(t)|}{r_n(t)}\geq1-7\delta/4\right).$$

The conclusion then follows from (A.10), Lemmas A.3 and A.4.   □

PROOF OF LEMMA 5.4. We only show (5.13). By $\tilde{Z}_n^x(0)=0$ and $Z_n^x(0)=0$, we have

$$P\left(|\tau_n^x|\geq\ell_nN,\ \inf\limits_{\substack{|t|\geq\ell_nN,\\ t\geq-(t_0+\chi_n)n^{1/3}}}Z_n^x(t)>0\right)=0,$$

and

$$P\left(|\tilde{\tau}_n^x|\geq\ell_nN,\ \inf\limits_{\substack{|t|\geq\ell_nN,\\ t\geq-(t_0+\chi_n)n^{1/3}}}\tilde{Z}_n^x(t)>0\right)=0.$$

Therefore,

$$P\left(\max\{|\tilde{\tau}_n^x|, |\tau_n^x|\} \geq \ell_n N\right)$$

$$\leq P\left(\inf_{\substack{|t| \geq \ell_n N, \\ t \geq -(t_0 + \chi_n)n^{1/3}}} \frac{\tilde{Z}_n^x(t)}{r_n(t)} \leq \delta\right) + P\left(|\tau_n^x| \geq \ell_n N, \inf_{\substack{|t| \geq \ell_n N, \\ t \geq -(t_0 + \chi_n)n^{1/3}}} \frac{Z_n^x(t)}{r_n(t)} \geq \delta\right)$$

$$+ P\left(\inf_{\substack{|t| \geq \ell_n N, \\ t \geq -(t_0 + \chi_n)n^{1/3}}} \frac{Z_n^x(t)}{r_n(t)} \leq \delta\right) + P\left(|\tilde{\tau}_n^x| \geq \ell_n N, \inf_{\substack{|t| \geq \ell_n N, \\ t \geq -(t_0 + \chi_n)n^{1/3}}} \frac{\tilde{Z}_n^x(t)}{r_n(t)} \geq \delta\right)$$

$$= P\left(\inf_{\substack{|t| \geq \ell_n N, \\ t \geq -(t_0 + \chi_n)n^{1/3}}} \frac{Z_n^x(t)}{r_n(t)} \leq \delta\right) + P\left(\inf_{\substack{|t| \geq \ell_n N, \\ t \geq -(t_0 + \chi_n)n^{1/3}}} \frac{\tilde{Z}_n^x(t)}{r_n(t)} \leq \delta\right),$$

and so, Lemma A.5 yields (5.13). □

PROOF OF LEMMA 5.5. We first show (5.15). Let u_0 be some real number and set $I_n^0 = [u_0, u_0 + \ell_n^{-2}\delta] \cap [-N\ell_n, N\ell_n]$. Since the increments of $\hat{Z}_n^x$ are independent, we have

$$P\left(\left|\inf_{s \in I_n^0} \hat{Z}_n^x(s) - \inf_{|s| \leq N\ell_n, s < u_0} \hat{Z}_n^x(s)\right| \leq \epsilon/\ell_n^{5/2}\right)$$

$$= P\left(\left|\inf_{s \in I_n^0} \hat{Z}_n^x(s) - \hat{Z}_n^x(u_0) - \left(\inf_{|s| \leq N\ell_n, s \leq u_0} \hat{Z}_n^x(s) - \hat{Z}_n^x(u_0)\right)\right| \leq \epsilon/\ell_n^{5/2}\right)$$

$$\leq \sup_{c \in \mathbb{R}} P\left(\left|\inf_{s \in I_n^0} \hat{Z}_n^x(s) - \hat{Z}_n^x(u_0) - c/\ell_n^{5/2}\right| \leq \epsilon/\ell_n^{5/2}\right).$$

We can write

$$P\left(\left|\inf_{s \in I_n^0} \hat{Z}_n^x(s) - \hat{Z}_n^x(u_0) - c/\ell_n^{5/2}\right| \leq \epsilon/\ell_n^{5/2}\right)$$

$$= P\left(\inf_{u \in [0,1]}\left(W(u) - \frac{\sigma_U^{-1} D_n(u_0 + h_n(l_n u))}{n^{1/6} l_n^{1/2}}\right) \in \left[\frac{(c-\epsilon)\sigma_U^{-1}}{l_n^{1/2} n^{1/6} \ell_n^{5/2}}, \frac{(c+\epsilon)\sigma_U^{-1}}{l_n^{1/2} n^{1/6} \ell_n^{5/2}}\right]\right),$$

where

$$l_n = \left(G\left(t_0 + \chi_n + n^{-1/3}(u_0 + \ell_n^{-2}\delta)\right) - G\left(t_0 + \chi_n + n^{-1/3}u_0\right)\right),$$

$h_n(u)$ is the inverse function of the function

$$u_n(s) = \left(G\left(t_0 + \chi_n + n^{-1/3}(u_0 + s)\right) - G\left(t_0 + \chi_n + n^{-1/3}u_0\right)\right),$$

and
$$D_n(u) = \frac{f_0(t_0)g(t_0)}{2}(u^2 - u_0^2) + x\ell_n g(t_0)(u - u_0).$$

By Girsanov's theorem (see Revuz and Yor (1998)) and Hölder's inequality,

$$P\left(\inf_{s\in[0,1]}\left(W(s) - \frac{\sigma_U^{-1}D_n(u_0 + h_n(l_ns))}{n^{1/6}l_n^{1/2}}\right) \in \left[\frac{(c-\epsilon)\sigma_U^{-1}}{l_n^{1/2}n^{1/6}\ell_n^{5/2}}, \frac{(c+\epsilon)\sigma_U^{-1}}{l_n^{1/2}n^{1/6}\ell_n^{5/2}}\right]\right)$$

$$=E\left(M_n(1)I_{\left[\frac{(c-\epsilon)\sigma_U^{-1}}{l_n^{1/2}n^{1/6}\ell_n^{5/2}}, \frac{(c+\epsilon)\sigma_U^{-1}}{l_n^{1/2}n^{1/6}\ell_n^{5/2}}\right]}\left(\inf_{s\in[0,1]}W(s)\right)\right)$$

$$\leq \left(E\left((M_n(1))^2\right)P\left(\inf_{s\in[0,1]}W(s) \in \left[\frac{(c-\epsilon)\sigma_U^{-1}}{l_n^{1/2}n^{1/6}\ell_n^{5/2}}, \frac{(c+\epsilon)\sigma_U^{-1}}{l_n^{1/2}n^{1/6}\ell_n^{5/2}}\right]\right)\right)^{1/2}$$

$$=\exp\left\{\frac{1}{2}\int_0^1\left|\frac{\sigma_U^{-1}\frac{d}{ds}D_n(u_0 + h_n(l_ns))}{n^{1/6}l_n^{1/2}}\right|^2 ds\right\}$$

$$\times\left(P\left(\inf_{s\in[0,1]}W(s) \in \left[\frac{(c-\epsilon)\sigma_U^{-1}}{l_n^{1/2}n^{1/6}\ell_n^{5/2}}, \frac{(c+\epsilon)\sigma_U^{-1}}{l_n^{1/2}n^{1/6}\ell_n^{5/2}}\right]\right)\right)^{1/2},$$

where

$$M_n(t) = \exp\left\{\int_0^t \frac{\sigma_U^{-1}\frac{d}{ds}D_n(u_0 + h_n(l_ns))}{n^{1/6}l_n^{1/2}}dW(s) - \frac{1}{2}\int_0^t\left|\frac{\sigma_U^{-1}\frac{d}{ds}D_n(u_0 + h_n(l_ns))}{n^{1/6}l_n^{1/2}}\right|^2 ds\right\}$$

is a martingale.

It is obvious that for any $u_0 \in [-N\ell_n, N\ell_n]$, and $|x| \in [0, \rho]$,

$$\left|\frac{\frac{d}{ds}D_n(u_0 + h_n(l_ns))}{n^{1/6}l_n^{1/2}}\right|^2$$

$$=\frac{(f_0(t_0)g(t_0)(u_0 + h_n(l_ns)) + x\ell_n g(t_0))^2 n^{1/3}l_n}{(g_0(t_0 + \chi_n + n^{-1/3}(u_0 + l_ns)))^2}$$

$$\leq L\ell_n^2 n^{1/3}l_n \leq L$$

where $L = L(N, \rho) > 0$ is a constant which only depends on ρ and N.

By the small ball estimates of Brownian motion and a property of Gaus-

sian measures [Li and Shao (2001)],

$$P\left(\inf_{s\in[0,1]} W(s) \in \left[\frac{(c-\epsilon)\sigma_U^{-1}}{l_n^{1/2}n^{1/6}\ell_n^{5/2}}, \frac{(c+\epsilon)\sigma_U^{-1}}{l_n^{1/2}n^{1/6}\ell_n^{5/2}}\right]\right)$$

$$\leq P\left(\sup_{s\in[0,1]} |W(s)| \leq \frac{\epsilon\sigma_U^{-1}}{l_n^{1/2}n^{1/6}\ell_n^{5/2}}\right)$$

$$\leq \exp\left\{-\frac{K\ell_n^3}{\epsilon^2\sigma_U^{-2}}\right\},$$

where $K > 0$ is a constant. Thus

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{\ell_n^3}\log\sup_{c\in\mathbb{R}} P\left(\inf_{s\in[0,1]}\left(W(s) - \frac{\sigma_U^{-1}D_n(u_0+h_n(l_ns))}{n^{1/6}l_n^{1/2}}\right)\right.$$

$$\left.\in\left[\frac{(c-\epsilon)\sigma_U^{-1}}{l_n^{1/2}n^{1/6}\ell_n^{5/2}}, \frac{(c+\epsilon)\sigma_U^{-1}}{l_n^{1/2}n^{1/6}\ell_n^{5/2}}\right]\right) = -\infty,$$

and so

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{\ell_n^3}\log\sup_{|x|\in[0,\rho]} P\left(\left|\inf_{s\in I_n^0}\hat{Z}_n^x(s) - \inf_{|s|\leq N\ell_n, s<u_0}\hat{Z}_n^x(s)\right| \leq \epsilon/\ell_n^{5/2}\right) = -\infty.$$

Similarly, we have

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{\ell_n^3}\log\sup_{|x|\in[0,\rho]} P\left(\left|\inf_{s\in I_n^0}\hat{Z}_n^x(s) - \inf_{|s|\leq N\ell_n, s>u_0+\delta}\hat{Z}_n^x(s)\right| \leq \epsilon/\ell_n^{5/2}\right) = -\infty.$$

For each integer k such that $-N\ell_n^3\delta^{-1} - 1 \leq k \leq N\ell_n^3\delta^{-1} + 1$, set

$$I_n^k = [k\delta, (k+1)\delta] \cap [-N\ell_n, N\ell_n].$$

Then

$$P\left(|\hat{\tau}_n^x| \leq N\ell_n, \hat{Z}_n^x(\hat{\tau}_n^x) - \inf_{|s|\leq N\ell_n, |s-\hat{\tau}_n^x|\geq \ell_n^{-2}\delta}\hat{Z}_n^x(s) \geq -\epsilon/\ell_n^{5/2}\right)$$

$$\leq \sum_{-N\ell_n^3\delta^{-1}-1\leq k\leq N\ell_n^3\delta^{-1}+1} P\left(\left|\inf_{s\in I_n^k}\hat{Z}_n^x(s) - \inf_{|s|\leq N\ell_n, s\notin I_n^k}\hat{Z}_n^x(s)\right| \leq \epsilon/\ell_n^{5/2}\right),$$

and so, (5.15) holds.

Finally, since

$$P\left(\left|\inf_{s\in I_n^0}\tilde{Z}_n^{x,*}(s)-\tilde{Z}_n^{x,*}(u_0)-c/\ell_n\right|\le\epsilon/\ell_n^{5/2}\right)$$

$$\le P\left(\inf_{u\in[0,1]}\left(W(u)-\frac{\sigma_U^{-1}D_n(u_0+h_n(l_nu))}{n^{1/6}l_n^{1/2}}\right)\in\left[\frac{(c-2\epsilon)\sigma_U^{-1}}{l_n^{1/2}n^{1/6}\ell_n^{5/2}},\frac{(c+2\epsilon)\sigma_U^{-1}}{l_n^{1/2}n^{1/6}\ell_n^{5/2}}\right]\right)$$

$$+P\left(\ell_n^{5/2}\sup_{s\in I_n^0}\left|\tilde{Z}_n^{x,*}(s)-\hat{Z}_n^x(s)\right|\ge\epsilon\right),$$

then by Proposition 5.2 ,

$$\limsup_{N\to\infty}\limsup_{n\to\infty}\frac{1}{\ell_n^3}\log\sup_{|x|\in[0,\rho]}P\left(\ell_n^{5/2}\sup_{s\in I_n^0}\left|\tilde{Z}_n^{x,*}(s)-\hat{Z}_n^x(s)\right|\ge\epsilon\right)=-\infty.$$

Thus, (5.15) implies

$$\lim_{\epsilon\to0}\limsup_{n\to\infty}\frac{1}{\ell_n^3}\log\sup_{|x|\in[0,\rho]}P\left(|\tilde{\tau}_n^{x,*}|\le N\ell_n,\tilde{Z}_n^{x,*}(\tilde{\tau}_n^{x,*})-\inf_{\substack{|s|\le N\ell_n,\\|s-\tilde{\tau}_n^{x,*}|\ge\delta}}\tilde{Z}_n^{x,*}(s)\ge\frac{-\epsilon}{\ell_n^{5/2}}\right)=-\infty.$$

This yields (5.16). $\square$

A.2. The proofs of the results for Grenander-type estimator.
Similar to the proof of Theorem 2.1, we first show a general result.

THEOREM A.1.   *Suppose (GA1) and (GA2) hold. Let χ_n be a sequence of non-negative numbers such that*

$$b_n^4\chi_n\to0.$$

Then for each $t\in(0,1)$ and $\rho>0$, uniformly for $0\le x\le\rho b_n$,

(A.11) $$\frac{1}{1-F_\tau(x)}P\left(\pm\frac{n^{1/3}\left(\hat{\lambda}_n(t+\chi_n)-\lambda(t+\chi_n)\right)}{|4\lambda'(t)L'(t)|^{1/3}}\ge x\right)\to1.$$

In particular, $\left\{\frac{n^{1/3}}{b_n}\frac{n^{1/3}(\hat{\lambda}_n(t+\chi_n)-\lambda(t+\chi_n))}{|4\lambda'(t)L'(t)|^{1/3}},n\ge1\right\}$ satisfies the MDP in $\mathbb{R}$ with speed b_n^3 and rate function

(A.12) $$J(x)=\frac{2}{3}|x|^3.$$

PROOF OF THEOREM A.1. We assume here that λ is decreasing. Since the proof is similar to that of Theorem 3.1, we omit some details. Let $\{\ell_n, n \geq 1\}$ be a sequence of positive numbers satisfying that

$$(A.13) \qquad 1 \leq \ell_n \text{ and } \lim_{n \to \infty} \frac{\ell_n^{10}}{n^{1/3}} = 0.$$

Let χ_n be nonnegative such that $\ell_n^4 \chi_n \to 0$. Let Λ_n^+ denote the upper version of Λ_n defined by

$$\Lambda_n^+(0) = \Lambda_n(0) \text{ and } \Lambda_n^+(t) = \max\{\Lambda_n(t), \lim_{u \uparrow t} \Lambda_n(u)\}.$$

Define a stochastic process $\{\hat{U}_n(s), s \in \mathbb{R}\}$ by

$$(A.14) \qquad \hat{U}_n(s) := \operatorname*{argmax}_{v \in [0,1]} \left\{\Lambda_n^+(v) - sv\right\}.$$

Then the function $\hat{U}_n(s) = \sup\{u \in [0,1]; \hat{\lambda}_n(u) \geq s\}, s \in \mathbb{R}$, is the inverse of the function $\hat{\lambda}_n$ in the sense that

$$\hat{\lambda}_n(u) < s \text{ if and only if } \hat{U}_n(s) < u \text{ for every } s \text{ and } u.$$

Define

$$(A.15) \qquad Z_n^x(s) = n^{2/3} M_n \left(n^{-1/3}s\right) + n^{2/3} m_n \left(n^{-1/3}s\right) + xs\ell_n,$$

and

$$(A.16) \qquad \tau_n^x = \operatorname*{argmax}_{s \in [-(t+\chi_n)n^{1/3}, n^{1/3}(1-(t+\chi_n))]} \left\{Z_n^x(s)\right\}$$

where

$$(A.17) \quad M_n(u) = (\Lambda_n^+(t+\chi_n+u) - \Lambda_n^+(t+\chi_n) - (\Lambda(t+\chi_n+u) - \Lambda(t+\chi_n))),$$

and

$$(A.18) \qquad m_n(u) = (\Lambda(t+\chi_n+u) - \Lambda(t+\chi_n) - \lambda(t+\chi_n)u)).$$

Then, we can write that for any $x > 0$,

$$(A.19) \qquad \begin{aligned} & P\left(\frac{n^{1/3}}{\ell_n}\left(\hat{\lambda}_n(t+\chi_n) - \lambda(t+\chi_n)\right) < -x\right) \\ = & P\left(\hat{U}_n\left(\lambda(t+\chi_n) - \ell_n n^{-1/3}x\right) < t+\chi_n\right) = P\left(\tau_n^x < 0\right). \end{aligned}$$

Similarly, we can show that

(A.20) $$P\left(\frac{n^{1/3}}{\ell_n}\left(\hat{\lambda}_n(t+\chi_n) - \lambda(t+\chi_n)\right) \geq x\right) = P\left(\tau_n^x \geq 0\right).$$

Without lost of generality, we assume that $B(s) = W(s) - sW(1), s \in [0,1]$ is a Brownian bridge in (GA2). Then for any $\delta > 0$,

(A.21)
$$\limsup_{n\to\infty} \frac{1}{\ell_n^3} \log P\left(\ell_n^{5/2} \sup_{s\in[-(t+\chi_n)n^{1/3},n^{1/3}(1-(t+\chi_n))]} \left|n^{2/3}M_n\left(n^{-1/3}s\right) - n^{1/6}\hat{B}_n(s)\right| \geq \delta\right)$$
$$= -\infty,$$

where

$$\begin{cases} \hat{W}_n(s) = \left(W\left(L\left(t+\chi_n+n^{-1/3}s\right)\right) - W(L(t+\chi_n))\right), s \in \mathbb{R}, \\ \hat{B}_n(s) = \hat{W}_n(s) - \left(L\left(t+\chi_n+n^{-1/3}s\right) - L(t+\chi_n)\right)W(1), s \in \mathbb{R}. \end{cases}$$

For each $x \in \mathbb{R}$, define

(A.22)
$$\begin{cases} \hat{Z}_n^x(s) = n^{1/6}\hat{W}_n(s) + \frac{1}{2}\lambda'(t+\chi_n)s^2 + xs\ell_n, \\ \tilde{Z}_n^x(s) = n^{1/6}\hat{B}_n(s) + n^{2/3}m_n\left(n^{-1/3}s\right) + xs\ell_n, \end{cases}$$

and set

(A.23)
$$\begin{cases} \hat{\tau}_n^x = \underset{s\in[-(t+\chi_n)n^{1/3},n^{1/3}(1-(t+\chi_n))]}{\operatorname{argmax}} \left\{\hat{Z}_n^x(s)\right\}, \\ \tilde{\tau}_n^x = \underset{s\in[-(t+\chi_n)n^{1/3},n^{1/3}(1-(t+\chi_n))]}{\operatorname{argmax}} \left\{\tilde{Z}_n^x(s)\right\}. \end{cases}$$

Then, the following lammas are analogous to Proposition 5.2, Lemma 5.4 and Proposition 5.3.

LEMMA A.6.   *Assume that (GA1) and (GA2) hold. Then for each $\rho > 0$ and $\delta > 0$,*
(A.24)
$$\limsup_{n\to\infty} \frac{1}{\ell_n^3} \log \sup_{|x|\in[0,\rho]} P\left(\ell_n^{5/2} \sup_{|s|\leq N\ell_n} \left|Z_n^x(s) - \tilde{Z}_n^x(s)\right| \geq \delta\right) = -\infty,$$

and
(A.25)
$$\limsup_{n\to\infty} \frac{1}{\ell_n^3} \log \sup_{|x|\in[0,\rho]} P\left(\ell_n^{5/2} \sup_{|s|\leq N\ell_n} \left|\tilde{Z}_n^x(s) - \hat{Z}_n^x(s)\right| \geq \delta\right) = -\infty.$$

LEMMA A.7. *Assume that (GA1) and (GA2) hold. Then for each $\rho > 0$,*
(A.26)
$$\limsup_{N\to\infty} \limsup_{n\to\infty} \frac{1}{\ell_n^3} \log \sup_{|x|\in[0,\rho]} P\left(\max\{|\hat{\tau}_n^x|, |\tilde{\tau}_n^x|, |\tau_n^x|\} \geq \ell_n N\right) = -\infty.$$

LEMMA A.8. *Assume that (GA1) and (GA2) hold. Then for each $\rho > 0$ and $\delta > 0$,*

(A.27)
$$\limsup_{n\to\infty} \frac{1}{\ell_n^3} \log \sup_{|x|\in[0,\rho]} P\left(|\tilde{\tau}_n^x - \hat{\tau}_n^x| \geq \ell_n^{-2}\delta\right) = -\infty$$

and

(A.28)
$$\limsup_{n\to\infty} \frac{1}{\ell_n^3} \log \sup_{|x|\in[0,\rho]} P\left(|\tau_n^x - \tilde{\tau}_n^x| \geq \ell_n^{-2}\delta\right) = -\infty.$$

Define
(A.29)
$$\begin{cases} \hat{Y}_n^x(s) = n^{1/6}W\left(L\left(t+\chi_n+n^{-1/3}s\right) - L(t+\chi_n)\right) + \frac{1}{2}\lambda'(t+\chi_n)s^2 + xs\ell_n, \\ Y_n^x(s) = \sqrt{L'(t+\chi_n)}W(s) + \frac{1}{2}\lambda'(t+\chi_n)s^2 + xs\ell_n, \end{cases}$$

and

(A.30)
$$\hat{T}_n^x = \operatorname*{argmax}_{s\in[-tn^{1/3}, n^{1/3}(1-(t+\chi_n))]} \left\{\hat{Y}_n^x(s)\right\},$$
$$T_n^x = \operatorname*{argmax}_{s\in[-tn^{1/3}, n^{1/3}(1-(t+\chi_n))]} \left\{Y_n^x(s)\right\}.$$

Then

(A.31)
$$\limsup_{n\to\infty} \frac{1}{\ell_n^3} \log \sup_{|x|\in[0,\rho]} P\left(|\hat{T}_n^x - T_n^x| \geq \ell_n^{-2}\delta\right) = -\infty.$$

Then, T_n^x and

$$-\frac{x\ell_n}{\lambda'(t+\chi_n)} + \left(\frac{(\lambda'(t+\chi_n))^2}{4L'(t+\chi_n)}\right)^{-1/3}\tau$$

are identical in distribution.

Thus, one can complete the proof of Theorem A.1 in a similar way to the proof of Theorem 3.1.

$\square$

PROOF OF PROPOSITION 2.2. By Theorem A.1, (i) is similar to Proposition 2.1 . (ii) is a consequence of (i) and Borel-Cantelli lemma.

□

PROOF OF THEOREM 2.3. By Proposition 2.2 (i) and the Delta method in large deviations [Gao and Zhao (2011)], for any $\epsilon > 0$,

$$(A.32) \qquad \lim_{n\to\infty} \frac{1}{b_n^3} \log P\left(\left| \frac{(4L'(t)|\lambda'(t)|)^{1/3}}{\left(4L'(t)|\hat{\lambda}_n^{(1)}(t)|\right)^{1/3}} 1 \right| > \epsilon \right) = -\infty.$$

Theorem 2.3 follows from Theorem A.1 with $\chi_n = 0$.

□

PROOF OF COROLLARY 2.1. By the strong approximation theorem [Komlós, Major and Tusnády (1975), Theorem 3], under (GA1), (GA2) holds with $L = F$ and a Brownian bridge W. Thus, the result follows from Theorem 2.3 and Proposition 2.2.

□

PROOF OF COROLLARY 2.2. Assume that (GA1) holds, $F(1) < 1$, $\lim_{t\to 1} G(t) < 1$, and G has a bounded continuous first derivative on $(0, 1)$. Then by (36) in Durot (2007), (GA2) holds with

$$L(t) = \int_0^t \frac{\lambda(u)}{(1 - F(u))(1 - G(u))} du, t \in [0, 1].$$

Note that in the case of nonrandom censoring times with $Y_i = 1$, one has $G(u) = 0$ for all $u < 1$. In this case, $L = (1 - F)^{-1} - 1$.

By the exponential inequality of the Kaplan-Meier estimator [Bitouze, Laurent, and Massart (1999)], for any $r > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n^3} \log P\left(\sup_{s\in[0,1]} |G_n(s) - G(s)| > r \right) = -\infty,$$

and

$$\lim_{n\to\infty} \frac{1}{b_n^3} \log P\left(\sup_{s\in[0,1]} |F_n(s) - F(s)| > r \right) = -\infty.$$

Thus, by Theorem 2.3 and Proposition 2.2, we obtain the conclusion.

□

PROOF OF COROLLARY 2.3. From the result in Sakhanenko (1985), there exist positive constants C and K, versions of $\{\varepsilon_{i,n}\}$ and the standard Brownian motion B_n such that

$$E\left(\exp\left\{C\sup_{t\in[0,1]}\left|\sum_{i\leq nt}\varepsilon_{i,n}-n^{1/2}B_n\left(\frac{1}{n}\sum_{i\leq nt}\sigma^2(i/n)\right)\right|\right\}\right)\leq 1+K\sum_{i\leq n}\sigma^2(i/n).$$

Thus, by Chebyshev's inequality, (GA2) holds with

$$L(t)=\int_0^t\sigma^2(u)du.$$

In particular, if $\varepsilon_{i,n}, i=1,\cdots,n, n\geq 1$ are i.i.d. with variance $\sigma^2\in(0,\infty)$, then L reduces to $L(t)=t\sigma^2$. □

APPENDIX B: LARGE DEVIATIONS, TALAGRAND INEQUALITY AND KMT APPROXIMATION

B.1. Large deviations. First, let us introduce some notion in large deviations [see Dembo and Zeitouni (1998)]. For a metric space $\mathcal{X}$, $\mathcal{B}(\mathcal{X})$ is the Borel σ-algebra of $\mathcal{X}$. Let $(\Omega_n,\mathcal{F}_n,P_n), n\geq 1$ be a sequence of probability spaces and let $\{X_n, n\geq 1\}$ be a sequence of measurable maps from Ω_n to $\mathcal{X}$. Let $\{\lambda(n), n\geq 1\}$ be a sequence of positive numbers tending to $+\infty$ and let $I:\mathcal{X}\to[0,+\infty]$ be inf-compact; that is, $[I\leq L]$ is compact for any $L\in\mathbb{R}$. Then $\{X_n, n\geq 1\}$ is said to satisfy the lower bound of large deviation (LLD) with speed $\lambda(n)$ and rate function I, if for any open measurable subset G of $\mathcal{X}$,

$$(B.1) \qquad l(G):=\liminf_{n\to\infty}\frac{1}{\lambda(n)}\log P_n(X_n\in G)\geq-\inf_{x\in G}I(x).$$

$\{X_n, n\geq 1\}$ is said to satisfy the upper bound of large deviation (ULD) with speed $\lambda(n)$ and rate function I, if for any closed measurable subset F of $\mathcal{X}$,

$$(B.2) \qquad \mathcal{U}(F):=\limsup_{n\to\infty}\frac{1}{\lambda(n)}\log P_n(X_n\in F)\leq-\inf_{x\in F}I(x).$$

We say that $\{X_n, n\geq 1\}$ satisfies the large deviation principle (LDP) with speed $\lambda(n)$ and rate function I, if both LLD and ULD hold.

B.2. Talagrand inequality. For each $T \in (0, \infty)$, let $l_\infty([0, T])$ be the Banach space of all bounded real functions x on $[0, T]$, equipped with the sup-norm $\|x\| = \sup_{t \in [0, T]} |x(t)|$. The following lemma is a consequence of Talagrand inequality [see Theorem 1.4 in Talagrand (1996)].

THEOREM B.1.   *Let $\{Z_1(t), t \in [0, T]\}, \cdots , \{Z_n(t), t \in [0, T]\}$ be independent right continuous stochastic processes. Suppose that $E(Z_i) = 0$ for $1 \le i \le n$ and there exists constant $C \in (0, \infty)$ such that $\max_{1 \le i \le n} \|Z_i\| \le C$. Then, there exists a universal constant K such that for any $r \ge 0$,*

$$P\left(\left|\left\|\sum_{i=1}^{n} Z_i\right\| - E\left(\left\|\sum_{i=1}^{n} Z_i\right\|\right)\right| \ge r\right) \le K \exp\left\{-\frac{r}{CK}\log\left(1 + \frac{Cr}{\sigma_n^2}\right)\right\},$$

where $\sigma_n^2 = E\left(\|\sum_{i=1}^{n} Z_i^2\|\right)$.

Let $\mathbb{F}$ be a uniformly bounded collection of measurable functions on $\mathbb{R}^d$. The class $\mathbb{F}$ is called to be a bounded measurable VC (Vapnik-Červonenkis) class of functions if there exist positive numbers A and v such that, for every probability measure μ on $\mathbb{R}^d$ and every $0 < \tau < 1$,

$$N(\mathbb{F}, \|\cdot\|_{L_2(\mu)}, \tau\|F\|_{L_2(\mu)}) \le \left(\frac{A}{\tau}\right)^v,$$

where $F = \sup\{|g|; g \in \mathbb{F}\}$ and $N(\mathbb{F}, \|\cdot\|_{L_2(\mu)}, \tau)$ denotes the τ-covering number of the metric space $(\mathbb{F}, \|\cdot\|_{L_2(\mu)})$, that is, the smallest number of balls of radius not larger than τ and centers in $\mathbb{F}$ needed to cover $\mathbb{F}$. The pair (A, v) is called the characteristic of the class $\mathbb{F}$. For any map Φ from $\mathbb{F}$ to $\mathbb{R}$, denote by $\|\Phi\|_\mathbb{F} = \sup\{|\Phi(f)|; f \in \mathbb{F}\}$.

It is known that $\mathbb{F}_1 = \{I_{(a,b]}, a \le b\}$ is a VC class on $\mathbb{R}$ and $\mathbb{F}_2 = \{I_{a < x \le t \le b}, a \le b\}$ is a VC class on $\mathbb{R}^2$ (see Van der Vaart and Wellner (1996)).

The following lemma is taken from Giné and Guillou(2001).

LEMMA B.1.   *Let $\mathbb{F}$ be a measurable uniformly bounded VC class of functions, and let $\{X_n, n \ge 1\}$ be a sequence of i.i.d. random variables taking their values in $\mathbb{R}^d$ on probability space $(\Omega, \mathcal{F}, P)$. Let σ^2 and U be any numbers such that $\sigma^2 \ge \sup_{f \in \mathbb{F}} E((f(X_1) - E(f(X_1)))^2)$, $U \ge \sup_{f \in \mathbb{F}} \sup_{x \in \mathbb{R}^d} |f(x)|$ and $0 < \sigma \le U/2$. Then, there exists constant L depending only on the characteristic (A, v) of the class $\mathbb{F}$, such that*

$$E\left(\sup_{f \in \mathbb{F}}\left|\sum_{i=1}^{n}(f(X_i) - Ef(X_i))\right|\right) \le L\left(U\log\frac{AU}{\sigma} + \sqrt{n}\sigma\sqrt{\log\frac{AU}{\sigma}}\right),$$

and

$$E\left(\sup_{f\in\mathbb{F}}\left|\sum_{i=1}^{n}(f(X_i)-E(f(X_1)))^2\right|\right) \leq L\left(\sqrt{n}\sigma + U\sqrt{\log\frac{AU}{\sigma}}\right)^2.$$

B.3. KMT approximation theorem.

THEOREM B.2 (Strong approximation theorem). *[Komlós, Major and Tusn'ady (1975), Theorem 3]. For each $n \geq 1$, let $\{Z_i, i \geq 1\}$ be a sequence of i.i.d. random variables with uniform distribution on $[0,1]$. Then there are a version $\{Z_i^*, i \geq 1\}$ of $\{Z_i, i \geq 1\}$ and Brownian bridges $B_n(t)$, $n \geq 1$ such that for all $x > 0$,*
(B.3)

$$P\left(\sup_{t\in[0,1]}\left|n\left(F_n^*(t)-t\right)-n^{1/2}B_n(t)\right| \geq C\log n + x\right) \leq K\exp\{-\lambda x\},$$

where $F_n^*(t)$ be the empirical distribution function based on the sample Z_1^*, $\cdots$, Z_n^*; and C, K, λ are positive absolute constants.

THEOREM B.3. *[Theorem 1 in Diebolt (1995), and Theorem 5 in Maumy (2004)]. Let $\{Y_i, i \geq 1\}$ and $\{Z_i, i \geq 1\}$ be two independent sequences of i.i.d. random variables and let Z_1 obey uniform distribution on $[0,1]$. Assume that $E(Y_1) = 0$, $\sigma^2 = E(Y_1^2) < \infty$ and*

$$E\left(\exp\{\kappa|Y_1|\}\right) < \infty \text{ for some } \kappa > 0.$$

Set

$$H_n(t) = \frac{1}{n}\sum_{i=1}^{n}Y_iI_{\{Z_i\leq t\}}, \quad t \in [0,1].$$

Then there are a version $(\{Y_i^*, i \geq 1\}, \{Z_i^*, i \geq 1\})$ of $(\{Y_i, i \geq 1\}, \{Z_i, i \geq 1\})$ and Wiener processes $W_n(t)$, $n \geq 1$ such that for all $x > 0$,

$$\text{(B.4)} \quad P\left(\sup_{t\in[0,1]}\left|nH_n^*(t)-n^{1/2}\sigma W_n(t)\right| \geq C\log n + x\right) \leq K\exp\{-\lambda x\},$$

where

$$H_n^*(t) = \frac{1}{n}\sum_{i=1}^{n}Y_i^*I_{\{Z_i^*\leq t\}}, \quad t \in [0,1],$$

and C, K, λ are positive absolute constants.

REFERENCES

[1] BITOUZÉ, D., LAURENT, B. AND MASSART, P. (1999). A Dvoretzky-Kiefer-Wolfowitz type inequality for the Kaplan-Meier estimator. *Ann. Inst. H. Poincare Probab. Statist.* **35** 735–763.

[2] DEMBO, A. AND ZEITOUNI, O. (1998). *Large deviations techniques and applications.* Second edition. Springer, Berlin.

[3] DIEBOLT, J. (1995). A nonparametric test for the regression function: Asymptotic theory. *J. Statist. Plann. Inference* **44** 1–17.

[4] GAO, F. Q. AND ZHAO, X. (2011). Delta Method in Large Deviations and Moderate Deviations for Estimators. *Ann. Statist.* **39** 1211–1240.

[5] GINÉ, E. AND GUILLOU, A. (2001). On consistency of kernel density estimators for randomly censored data: rates holding uniformly over adaptive intervals. *Ann. Inst. Henri Poincaré (B) Probabilité et statistiques* **37** 503–522.

[6] KOMLOS, J., MAJOR, P. AND TUSNÁDY, G. (1975). An approximation of partial sums of independent RVs, and the sample DF, part I. *Z. W. Verw. Geb.* **32** 111–131.

[7] LI, W.V. AND SHAO, Q.-M. (2001). Gaussian processes: inequalities, small ball probabilities and applications. *Handbook of Statistics, Vol. 19, Stochastic processes: Theory and methods*, Edited by C.R. Rao and D. Shanbhag, 533–598.

[8] MAUMY. M. (2004). Strong Approximations for the Compound Empirical Process. *Annales de l'ISUP.* **48** 69–83.

[9] REVUZ, D. AND YOR, M. (1998). *Continuous Martingales and Brownian Motion.* Grund. Math. Wiss. 293, Springer-Verlag.

[10] SAKHANENKO, A. I. (1985). Convergence rate in the invariance principle for non-identically distributed variables with exponential moments. In: Advances in Probability Theory: Limit Theorems for Sums of Random Variables (A. A. Borovkov, ed.) 2–73. Springer, New York.

[11] TALAGRAND, M. (1996). New concentration inequalities in product spaces. *Invent. Math.* **26** 505–563.

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```
#Table 1: R-code

# return tt and delta
curstatgen=function(n,lam,tau,tt,delta)
{

  x=rexp(n+1,lam)
  tt[2:(n+1)] <- runif(n,0,tau)
  delta[2:(n+1)] <- rep(0,n)
  delta[x<=tt[1:(n+1)]] <- 1
  delta[1] <- 0

  obs_tt <- tt[2:(n+1)]
  obs_delta <- delta[2:(n+1)]

  obs <- data.frame(cbind(obs_tt,obs_delta))
  obs <- obs[order(obs$obs_tt),]

  tt[2:(n+1)] <- obs$obs_tt
  delta[2:(n+1)] <- obs$obs_delta

  return(data.frame(tt,delta))
}

# retrun cs

cumsum <- function(m,cs,delta)
{
  cs[2] <- delta[2]
  if(m>=2)
    for(i in 2:m)
      cs[i+1] = cs[i-1+1] + delta[i+1];
    return(cs)
}


# return y

convexmin <- function(n,cumw,cs,y)
{
  y[2] = cs[2]/cumw[2];

  if(n>=2)
    for(i in 2:n)
    {
      y[i+1] = (cs[i+1]-cs[i-1+1])/(cumw[i+1]-cumw[i-1+1]);
      if(y[i-1+1]>y[i+1])
      {
        j = i;
        while(y[j-1+1] > y[i+1] && j>1)
        {
          j <- j - 1;
          if(j>1)
            y[i+1] = (cs[i+1]-cs[j-1+1])/(cumw[i+1]-cumw[j-1+1])
          else
            y[i+1] = cs[i+1]/cumw[i+1]
          if((i-1)>=j)
            for(m in j:(i-1))    y[m+1] = y[i+1]
        }
      }
    }
  return(y)
}

F.b=function(n,m,b, t_0, data0,F)
{
  t.b=t_0+b*log(n)/n^(1/3)
  MLE.b = 0
  if(t.b<data0[2])
      MLE.b <- 0

    for(j in 1:(m-1))
    {
      if(data0[j+1]<= t.b && t.b<data0[j+1+1])
        MLE.b <- F[j+1]
    }
  if(data0[m+1]<= t.b)
      MLE.b <- F[m+1]

  MLE.b
}

#  NumIt <- 1000
```

```r
Power=function(t_0,n,lam,tau,M)
{
  F0=1-exp(-(t_0/2))
  bn=n^{1/30.1}
  hn=bn/n^(1/3.1)
  delta <- rep(0,n+1)
  delta2 <- rep(0,n+1)
  data <- rep(0,n+1)
  data0 <- rep(0,n+1)
  data2 <- rep(0,n+1)

  F= rep(0,n+1)
  cumw= rep(0,n+1)
  cs= rep(0,n+1)
  cumw1= rep(0,n+1)
  cs1= rep(0,n+1)
  y= rep(0,n+1)
  y1= rep(0,n+1)
  y2= rep(0,n+1)

  lowbound <- 0
  upbound <- 0


  F[1] <-0
  cumw[1] <- 0
  cs[1]<- 0
  cumw1[1] <- 0
  cs1[1] <- 0
  cumw[2:(n+1)] <- 1:n
  cumw1[2:(n+1)] <- cumw[2:(n+1)]

  y[1] <- 0
  y1[1] <- 0
  y2[1] <- 0

  s=0
  s.LR=0

  for (r in 1:M)
  {
    aa <- curstatgen(n,lam,tau,data,delta)
    data <- aa$tt
    delta <- aa$delta
    cs <- cumsum(n,cs,delta)
    y <- convexmin(n,cumw,cs,y)
    T=data[2:(n+1)]
    g=(1/(n*hn))*0.5*sum(as.numeric(abs((t_0-T)/hn)<=1))



    j <- 0;

    for(i in 1:n)
    {
      if (y[i+1]>y[i-1+1])
      {
        j <- j + 1
        F[j+1] <- y[i+1]
        data0[j+1] <- data[i+1]
      }
    }

    m <- j

    MLE.0=F.b(n,m,0,t_0, data0,F)
    MLE.b=F.b(n,m,bn,t_0, data0,F)
    F.0=MLE.0
    f=n^(1/3)/(bn*log(n))*(MLE.b-MLE.0)
    z=0.9982*(4*F.0*(1-F.0)*f/g)^(1/3)/n^(1/3);
    s=s+as.numeric(abs(F.0-F0)>z)


    L<-0

    for(i in 1:n)
    {
      if(delta[i+1]==1)
        L <- L + log(y[i+1])
      else
        L <- L + log(1-y[i+1])
    }

    {
      j <- 1
```

```
      while (data[j+1]<t_0 && j<=n)
        j <- j + 1
      m1 <- j-1

      a <- y[m1+1];

      if((n-m1)>=1)
        for(i in 1:(n-m1))
          delta2[i+1]=1-delta[n-i+1+1];

      if((n-m1)>=1)
        for(i in 1:(n-m1))
          cs1[i+1]=cs1[i-1+1]+delta2[i+1]

      y1 <- convexmin(m1,cumw,cs,y1)
      y2 <- convexmin(n-m1,cumw1,cs1,y2)

      b <- 0
      j <- 0

      while (b<2.26916 & a+j*0.0001<=0.9999)
      {
        L0=0;

        j <- j + 1

        if(m1>=1)
          for(i in 1:m1)
          {
            if(delta[i+1]==1)
              L0 <- L0 + log(min(a+j*0.0001,y1[i+1]))
            else
              L0 <- L0 + log(1-min(a+j*0.0001,y1[i+1]))
          }

        if((n-m1)>=1)
          for(i in 1:(n-m1))
          {
            if(delta2[i+1]==1)
              L0 <- L0 + log(min(1-a-j*0.0001,y2[i+1]))
            else
              L0 <- L0 + log(1-min(1-a-j*0.0001,y2[i+1]))
          }

        b <- 2*(L-L0)
      }

      upbound=a+j*0.0001;

      j <- 0
      b <- 0

      while(b<2.26916 && a-j*0.0001>=0.0001)
      {
        L0 <- 0

        j <- j + 1

        if(m1>=1)
          for(i in 1:m1)
          {
            if (delta[i+1]==1)
              L0 <- L0 + log(min(a-j*0.0001,y1[i+1]))
            else
              L0 <- L0 + log(1-min(a-j*0.0001,y1[i+1]))
          }

        if((n-m1)>=1)
          for(i in 1:(n-m1))
          {
            if(delta2[i+1]==1)
              L0 <- L0 + log(min(1-a+j*0.0001,y2[i+1]))
            else
              L0 <- L0 + log(1-min(1-a+j*0.0001,y2[i+1]))
          }

        b <- 2*(L-L0)

      }

      lowbound <- a-j*0.0001

      if(F0<lowbound || F0>upbound)
        s.LR <- s.LR + 1
}
```

```
  }

  p=rep(0, 2)
    p[1]=s/M
    p[2] <- s.LR/M

  p
}


#t_0=1

p0=c(Power(1,100,  0.5, 2, 1000),Power(1,200, 0.5, 2, 1000), Power(1,300, 0.5, 2, 1000))

p1=c(Power(1,100,  0.7, 3, 1000),Power(1,200, 0.7, 3, 1000), Power(1,300, 0.7, 3, 1000))

p2=c(Power(1,100,  0.9, 3, 1000),Power(1,200, 0.9, 3, 1000), Power(1,300, 0.9, 3, 1000))

p3=c(Power(1,100,  0.3, 3, 1000),Power(1,200, 0.3, 3, 1000), Power(1,300, 0.3, 3, 1000))

p4=c(Power(1,100,  0.1, 3, 1000),Power(1,200, 0.1, 3, 1000), Power(1,300, 0.1, 3, 1000))

p5=c(Power(1,100,  0.7, 5, 1000),Power(1,200, 0.7, 5, 1000), Power(1,300, 0.7, 5, 1000))

p6=c(Power(1,100,  0.9, 5, 1000),Power(1,200, 0.9, 5, 1000), Power(1,300, 0.9, 5, 1000))

p7=c(Power(1,100,  0.3, 5, 1000),Power(1,200, 0.3, 5, 1000), Power(1,300, 0.3, 5, 1000))

p8=c(Power(1,100,  0.1, 5, 1000),Power(1,200, 0.1, 5, 1000), Power(1,300, 0.1, 5, 1000))

table1=rbind.data.frame(p0,p1,p2,p3,p4,p5,p6,p7,p8)

Table1=write.table(table1,file="Table1.txt",row.names=FALSE, col.names=FALSE)
```

```r
#Table 2: R-code

# return tt and delta
curstatgen=function(n,alp,lam,tau,tt,delta)
{
  b=1/lam
  x=rweibull(n+1,alp,b)
  tt[2:(n+1)] <- runif(n,0,tau)
  delta[2:(n+1)] <- rep(0,n)
  delta[x<=tt[1:(n+1)]] <- 1
  delta[1] <- 0

  obs_tt <- tt[2:(n+1)]
  obs_delta <- delta[2:(n+1)]

  obs <- data.frame(cbind(obs_tt,obs_delta))
  obs <- obs[order(obs$obs_tt),]

  tt[2:(n+1)] <- obs$obs_tt
  delta[2:(n+1)] <- obs$obs_delta

  return(data.frame(tt,delta))
}

# retrun cs

cumsum <- function(m,cs,delta)
{
  cs[2] <- delta[2]
  if(m>=2)
    for(i in 2:m)
      cs[i+1] = cs[i-1+1] + delta[i+1];
    return(cs)
}


# return y

convexmin <- function(n,cumw,cs,y)
{
  y[2] = cs[2]/cumw[2];

  if(n>=2)
    for(i in 2:n)
    {
      y[i+1] = (cs[i+1]-cs[i-1+1])/(cumw[i+1]-cumw[i-1+1]);
      if(y[i-1+1]>y[i+1])
      {
        j = i;
        while(y[j-1+1] > y[i+1] && j>1)
        {
          j <- j - 1;
          if(j>1)
            y[i+1] = (cs[i+1]-cs[j-1+1])/(cumw[i+1]-cumw[j-1+1])
          else
            y[i+1] = cs[i+1]/cumw[i+1]
          if((i-1)>=j)
            for(m in j:(i-1))    y[m+1] = y[i+1]
        }
      }
    }
  return(y)
}

F.b=function(n,m,b, t_0, data0,F)
{
  t.b=t_0+b*log(n)/n^(1/3)
  MLE.b = 0
  if(t.b<data0[2])
      MLE.b <- 0

    for(j in 1:(m-1))
    {
      if(data0[j+1]<= t.b && t.b<data0[j+1+1])
        MLE.b <- F[j+1]
    }
  if(data0[m+1]<= t.b)
      MLE.b <- F[m+1]

  MLE.b
}

#  NumIt <- 1000

Power=function(t_0,n,alp,lam,tau,M)
```

```
{
  F0=1-exp(-(t_0/2)^(1/2))
  bn=n^{1/30.1}
  hn=bn/n^(1/3.1)
  delta <- rep(0,n+1)
  delta2 <- rep(0,n+1)
  data <- rep(0,n+1)
  data0 <- rep(0,n+1)
  data2 <- rep(0,n+1)

  F= rep(0,n+1)
  cumw= rep(0,n+1)
  cs= rep(0,n+1)
  cumw1= rep(0,n+1)
  cs1= rep(0,n+1)
  y= rep(0,n+1)
  y1= rep(0,n+1)
  y2= rep(0,n+1)

  lowbound <- 0
  upbound <- 0


  F[1] <-0
  cumw[1] <- 0
  cs[1]<- 0
  cumw1[1] <- 0
  cs1[1] <- 0
  cumw[2:(n+1)] <- 1:n
  cumw1[2:(n+1)] <- cumw[2:(n+1)]

  y[1] <- 0
  y1[1] <- 0
  y2[1] <- 0

  s=0
  s.LR=0

  for (r in 1:M)
  {
    aa <- curstatgen(n,alp,lam,tau,data,delta)
    data <- aa$tt
    delta <- aa$delta
    cs <- cumsum(n,cs,delta)
    y <- convexmin(n,cumw,cs,y)
    T=data[2:(n+1)]
    g=(1/(n*hn))*0.5*sum(as.numeric(abs((t_0-T)/hn)<=1))



    j <- 0;

    for(i in 1:n)
    {
      if (y[i+1]>y[i-1+1])
      {
        j <- j + 1
        F[j+1] <- y[i+1]
        data0[j+1] <- data[i+1]
      }
    }

    m <- j

    MLE.0=F.b(n,m,0,t_0, data0,F)
    MLE.b=F.b(n,m,bn,t_0, data0,F)
    F.0=MLE.0
    f=n^(1/3)/(bn*log(n))*(MLE.b-MLE.0)
    z=0.9982*(4*F.0*(1-F.0)*f/g)^(1/3)/n^(1/3);
    s=s+as.numeric(abs(F.0-F0)>z)


    L<-0

    for(i in 1:n)
    {
      if(delta[i+1]==1)
        L <- L + log(y[i+1])
      else
        L <- L + log(1-y[i+1])
    }

    {
      j <- 1
      while (data[j+1]<t_0 && j<=n)
```

```
      j <- j + 1
    m1 <- j-1

    a <- y[m1+1];

    if((n-m1)>=1)
      for(i in 1:(n-m1))
        delta2[i+1]=1-delta[n-i+1+1];

    if((n-m1)>=1)
      for(i in 1:(n-m1))
        cs1[i+1]=cs1[i-1+1]+delta2[i+1]

    y1 <- convexmin(m1,cumw,cs,y1)
    y2 <- convexmin(n-m1,cumw1,cs1,y2)

    b <- 0
    j <- 0

    while (b<2.26916 & a+j*0.0001<=0.9999)
    {
      L0=0;

      j <- j + 1

      if(m1>=1)
        for(i in 1:m1)
        {
          if(delta[i+1]==1)
            L0 <- L0 + log(min(a+j*0.0001,y1[i+1]))
          else
            L0 <- L0 + log(1-min(a+j*0.0001,y1[i+1]))
        }

      if((n-m1)>=1)
        for(i in 1:(n-m1))
        {
          if(delta2[i+1]==1)
            L0 <- L0 + log(min(1-a-j*0.0001,y2[i+1]))
          else
            L0 <- L0 + log(1-min(1-a-j*0.0001,y2[i+1]))
        }

      b <- 2*(L-L0)
    }

    upbound=a+j*0.0001;

    j <- 0
    b <- 0

    while(b<2.26916 && a-j*0.0001>=0.0001)
    {
      L0 <- 0

      j <- j + 1

      if(m1>=1)
        for(i in 1:m1)
        {
          if (delta[i+1]==1)
            L0 <- L0 + log(min(a-j*0.0001,y1[i+1]))
          else
            L0 <- L0 + log(1-min(a-j*0.0001,y1[i+1]))
        }

      if((n-m1)>=1)
        for(i in 1:(n-m1))
        {
          if(delta2[i+1]==1)
            L0 <- L0 + log(min(1-a+j*0.0001,y2[i+1]))
          else
            L0 <- L0 + log(1-min(1-a+j*0.0001,y2[i+1]))
        }

      b <- 2*(L-L0)

    }

    lowbound <- a-j*0.0001

    if(F0<lowbound || F0>upbound)
      s.LR <- s.LR + 1
}
```

```
  }

  p=rep(0, 2)
    p[1]=s/M
    p[2] <- s.LR/M

  p
}


#t_0=1

p0=c(Power(1,100, 0.5, 0.5, 2, 1000),Power(1,200, 0.5, 0.5, 2, 1000), Power(1,300, 0.5, 0.5, 2, 1000))

p1=c(Power(1,100, 0.5, 0.7, 3, 1000),Power(1,200, 0.5, 0.7, 3, 1000), Power(1,300, 0.5, 0.7, 3, 1000))

p2=c(Power(1,100, 0.5, 0.9, 3, 1000),Power(1,200, 0.5, 0.9, 3, 1000), Power(1,300, 0.5, 0.9, 3, 1000))

p3=c(Power(1,100, 0.5, 0.3, 3, 1000),Power(1,200, 0.5, 0.3, 3, 1000), Power(1,300, 0.5, 0.3, 3, 1000))

p4=c(Power(1,100, 0.5, 0.1, 3, 1000),Power(1,200, 0.5, 0.1, 3, 1000), Power(1,300, 0.5, 0.1, 3, 1000))

p5=c(Power(1,100, 0.5, 0.7, 5, 1000),Power(1,200, 0.5, 0.7, 3, 1000), Power(1,300, 0.5, 0.7, 5, 1000))

p6=c(Power(1,100, 0.5, 0.9, 5, 1000),Power(1,200, 0.5, 0.9, 3, 1000), Power(1,300, 0.5, 0.9, 5, 1000))

p7=c(Power(1,100, 0.5, 0.3, 5, 1000),Power(1,200, 0.5, 0.3, 3, 1000), Power(1,300, 0.5, 0.3, 5, 1000))

p8=c(Power(1,100, 0.5, 0.1, 5, 1000),Power(1,200, 0.5, 0.1, 3, 1000), Power(1,300, 0.5, 0.1, 5, 1000))
table2=rbind.data.frame(p0,p1,p2,p3,p4,p5,p6,p7,p8)
Table12=write.table(table2,file="Table2.txt",row.names=FALSE, col.names=FALSE)
```

```
#Table 3: R-code

eta <- 0.9

F_0 <- function(t)
{
  if(t < 0) 0
  else 1-exp(-t/2)
}

F_0_inv <- function(u)
{
  -2*log(1-u)
}

f_0 <- function(t)
{
  if(t <= 0) 0
  else (1/2)*exp(-t/2)
}

B <- function(z,t_0,c)
{
  if(-c <= z & z <= c)
    eta * f_0(t_0) * (c-abs(z))
  else
    0
}

F_n <- function(t,t_0,c,n)
{
  F_0(t) + n^(-1/3) * B(n^(1/3)*(t-t_0),t_0,c)
}

G_inv <- function(u,t_0,c,n)
{
  if(u <= F_0(t_0 - c*n^(-1/3)))
    F_0_inv(u)
  else if(u <= F_n(t_0,t_0,c,n))
    t_0 - c*n^(-1/3) + c*n^(-1/3)*(u-F_0(t_0-c*n^(-1/3)))/(F_n(t_0,t_0,c,n)-F_0(t_0-c*n^(-1/3)))
  else if(u <= F_0(t_0 + c*n^(-1/3)))
    t_0 + c*n^(-1/3)*(u-F_n(t_0,t_0,c,n))/(F_0(t_0+c*n^(-1/3))-F_n(t_0,t_0,c,n))
  else
    F_0_inv(u)
}
curstatgen <- function(t_0,c,n,tau,tt,delta)
{
  u=runif(n+1)
  x=u
  for (i in 1:(n+1)) {x[i]=G_inv(u[i],t_0,c,n)}
  tt[2:(n+1)] <- runif(n,0,tau)
  delta[2:(n+1)] <- rep(0,n)
  delta[x<=tt[1:(n+1)]] <- 1
  delta[1] <- 0

  obs_tt <- tt[2:(n+1)]
  obs_delta <- delta[2:(n+1)]

  obs <- data.frame(cbind(obs_tt,obs_delta))
  obs <- obs[order(obs$obs_tt),]

  tt[2:(n+1)] <- obs$obs_tt
  delta[2:(n+1)] <- obs$obs_delta

  return(data.frame(tt,delta))
}

# retrun cs

cumsum <- function(m,cs,delta)
{
  cs[2] <- delta[2]
  if(m>=2)
    for(i in 2:m)
      cs[i+1] = cs[i-1+1] + delta[i+1];
    return(cs)
}


# return y

convexmin <- function(n,cumw,cs,y)
{
  y[2] = cs[2]/cumw[2];
```

```
   if(n>=2)
     for(i in 2:n)
       {
         y[i+1] = (cs[i+1]-cs[i-1+1])/(cumw[i+1]-cumw[i-1+1]);
         if(y[i-1+1]>y[i+1])
         {
            j = i;
            while(y[j-1+1] > y[i+1] && j>1)
            {
              j <- j - 1;
              if(j>1)
                y[i+1] = (cs[i+1]-cs[j-1+1])/(cumw[i+1]-cumw[j-1+1])
              else
                y[i+1] = cs[i+1]/cumw[i+1]
              if((i-1)>=j)
                for(m in j:(i-1))    y[m+1] = y[i+1]
            }
         }
       }
   return(y)
}

F.b=function(n,m,b, t_0, data0,F)
{
   t.b=t_0+b*log(n)/n^(1/3)
   MLE.b = 0
   if(t.b<data0[2])
       MLE.b <- 0

     for(j in 1:(m-1))
     {
        if(data0[j+1]<= t.b && t.b<data0[j+1+1])
          MLE.b <- F[j+1]
     }
   if(data0[m+1]<= t.b)
       MLE.b <- F[m+1]

   MLE.b
}

#   NumIt <- 1000
Power=function(t_0,c,n,tau,M)
{

   F0=F_0(t_0)

   bn=n^{1/30.1}
   hn=bn/n^(1/3.1)
   delta <- rep(0,n+1)
   delta2 <- rep(0,n+1)
   data <- rep(0,n+1)
   data0 <- rep(0,n+1)
   data2 <- rep(0,n+1)

   F= rep(0,n+1)
   cumw= rep(0,n+1)
   cs= rep(0,n+1)
   cumw1= rep(0,n+1)
   cs1= rep(0,n+1)
   y= rep(0,n+1)
   y1= rep(0,n+1)
   y2= rep(0,n+1)

   lowbound <- 0
   upbound <- 0


   F[1] <-0
   cumw[1] <- 0
   cs[1]<- 0
   cumw1[1] <- 0
   cs1[1] <- 0
   cumw[2:(n+1)] <- 1:n
   cumw1[2:(n+1)] <- cumw[2:(n+1)]

   y[1] <- 0
   y1[1] <- 0
   y2[1] <- 0

   s=0
   s.LR=0

   for (r in 1:M)
   {
      aa <- curstatgen(t_0,c,n,tau,data,delta)
```

```
data <- aa$tt
delta <- aa$delta
cs <- cumsum(n,cs,delta)
y <- convexmin(n,cumw,cs,y)

T=data[2:(n+1)]

g=(1/(n*hn))*0.5*sum(as.numeric(abs((t_0-T)/hn)<=1))




j <- 0;

for(i in 1:n)
{
  if (y[i+1]>y[i-1+1])
  {
    j <- j + 1
    F[j+1] <- y[i+1]
    data0[j+1] <- data[i+1]
  }
}

m <- j

MLE.0=F.b(n,m,0,t_0, data0,F)
MLE.b=F.b(n,m,bn,t_0, data0,F)
F.0=MLE.0
f=n^(1/3)/(bn*log(n))*(MLE.b-MLE.0)
z=0.9982*(4*F.0*(1-F.0)*f/g)^(1/3)/n^(1/3);
s=s+as.numeric(abs(F.0-F0)>z)

L<-0

for(i in 1:n)
{
  if(delta[i+1]==1)
    L <- L + log(y[i+1])
  else
    L <- L + log(1-y[i+1])
}

{
  j <- 1
  while (data[j+1]<t_0 && j<=n)
    j <- j + 1
  m1 <- j-1

  a <- y[m1+1];

  if((n-m1)>=1)
    for(i in 1:(n-m1))
      delta2[i+1]=1-delta[n-i+1+1];

  if((n-m1)>=1)
    for(i in 1:(n-m1))
      cs1[i+1]=cs1[i-1+1]+delta2[i+1]

  y1 <- convexmin(m1,cumw,cs,y1)
  y2 <- convexmin(n-m1,cumw1,cs1,y2)

  b <- 0
  j <- 0

  while (b<2.26916 & a+j*0.0001<=0.9999)
  {
    L0=0;

    j <- j + 1

    if(m1>=1)
      for(i in 1:m1)
      {
        if(delta[i+1]==1)
          L0 <- L0 + log(min(a+j*0.0001,y1[i+1]))
        else
          L0 <- L0 + log(1-min(a+j*0.0001,y1[i+1]))
      }

    if((n-m1)>=1)
      for(i in 1:(n-m1))
      {
        if(delta2[i+1]==1)
          L0 <- L0 + log(min(1-a-j*0.0001,y2[i+1]))
        else
```

```
                     L0 <- L0 + log(1-min(1-a-j*0.0001,y2[i+1]))
                }

            b <- 2*(L-L0)
        }

    upbound=a+j*0.0001;

    j <- 0
    b <- 0

    while(b<2.26916 && a-j*0.0001>=0.0001)
    {
      L0 <- 0

      j <- j + 1

      if(m1>=1)
        for(i in 1:m1)
        {
          if (delta[i+1]==1)
            L0 <- L0 + log(min(a-j*0.0001,y1[i+1]))
          else
            L0 <- L0 + log(1-min(a-j*0.0001,y1[i+1]))
        }

      if((n-m1)>=1)
        for(i in 1:(n-m1))
        {
          if(delta2[i+1]==1)
            L0 <- L0 + log(min(1-a+j*0.0001,y2[i+1]))
          else
            L0 <- L0 + log(1-min(1-a+j*0.0001,y2[i+1]))
        }

      b <- 2*(L-L0)

    }

    lowbound <- a-j*0.0001

    if(F0<lowbound || F0>upbound)
      s.LR <- s.LR + 1
  }

  }

  p=rep(0, 2)
  p[1]=s/M
  p[2] <- s.LR/M

  p
}

p11=c(1,Power(0.5,1,100,3,1000),Power(0.5,1,200,3,1000),Power(0.5,1,300,3,1000))
p12=c(2,Power(0.5,2,100,3,1000),Power(0.5,2,200,3,1000),Power(0.5,2,300,3,1000))
p13=c(3,Power(0.5,3,100,3,1000),Power(0.5,3,200,3,1000),Power(0.5,3,300,3,1000))
p14=c(4,Power(0.5,4,100,3,1000),Power(0.5,4,200,3,1000),Power(0.5,4,300,3,1000))
p15=c(5,Power(0.5,5,100,3,1000),Power(0.5,5,200,3,1000),Power(0.5,5,300,3,1000))

p21=c(1,Power(1.0,1,100,3,1000),Power(1.0,1,200,3,1000),Power(1.0,1,300,3,1000))
p22=c(2,Power(1.0,2,100,3,1000),Power(1.0,2,200,3,1000),Power(1.0,2,300,3,1000))
p23=c(3,Power(1.0,3,100,3,1000),Power(1.0,3,200,3,1000),Power(1.0,3,300,3,1000))
p24=c(4,Power(1.0,4,100,3,1000),Power(1.0,4,200,3,1000),Power(1.0,4,300,3,1000))
p25=c(5,Power(1.0,5,100,3,1000),Power(1.0,5,200,3,1000),Power(1.0,5,300,3,1000))

p31=c(1,Power(1.5,1,100,3,1000),Power(1.5,1,200,3,1000),Power(1.5,1,300,3,1000))
p32=c(2,Power(1.5,2,100,3,1000),Power(1.5,2,200,3,1000),Power(1.5,2,300,3,1000))
p33=c(3,Power(1.5,3,100,3,1000),Power(1.5,3,200,3,1000),Power(1.5,3,300,3,1000))
p34=c(4,Power(1.5,4,100,3,1000),Power(1.5,4,200,3,1000),Power(1.5,4,300,3,1000))
p35=c(5,Power(1.5,5,100,3,1000),Power(1.5,5,200,3,1000),Power(1.5,5,300,3,1000))

p41=c(1,Power(2.5,1,100,3,1000),Power(2.5,1,200,3,1000),Power(2.5,1,300,3,1000))
p42=c(2,Power(2.5,2,100,3,1000),Power(2.5,2,200,3,1000),Power(2.5,2,300,3,1000))
p43=c(3,Power(2.5,3,100,3,1000),Power(2.5,3,200,3,1000),Power(2.5,3,300,3,1000))
p44=c(4,Power(2.5,4,100,3,1000),Power(2.5,4,200,3,1000),Power(2.5,4,300,3,1000))
p45=c(5,Power(2.5,5,100,3,1000),Power(2.5,5,200,3,1000),Power(2.5,5,300,3,1000))

table3=rbind.data.frame(p11,p12,p13,p14,p15, p21,p22,p23,p24,p25,p31,p32,p33,p34,p35,p41,p42,p43,p44,p45)


Table3=write.table(table3,file="Table3.txt",row.names=FALSE, col.names=FALSE)
```

```
#Table 4: R-code

eta <- 0.9

F_0 <- function(t)
{
  if(t < 0) 0
  else 1-exp(-(t/2)^(1/2))
}

F_0_inv <- function(u)
{
  2*(-log(1-u))^(2)
}

f_0 <- function(t)
{
  if(t <= 0) 0
  else (1/4)*(t/2)^(-1/2)*exp(-(t/2)^(1/2))
}

B <- function(z,t_0,c)
{
  if(-c <= z & z <= c)
    eta * f_0(t_0) * (c-abs(z))
  else
    0
}

F_n <- function(t,t_0,c,n)
{
  F_0(t) + n^(-1/3) * B(n^(1/3)*(t-t_0),t_0,c)
}

G_inv <- function(u,t_0,c,n)
{
  if(u <= F_0(t_0 - c*n^(-1/3)))
    F_0_inv(u)
  else if(u <= F_n(t_0,t_0,c,n))
    t_0 - c*n^(-1/3) + c*n^(-1/3)*(u-F_0(t_0-c*n^(-1/3)))/(F_n(t_0,t_0,c,n)-F_0(t_0-c*n^(-1/3)))
  else if(u <= F_0(t_0 + c*n^(-1/3)))
    t_0 + c*n^(-1/3)*(u-F_n(t_0,t_0,c,n))/(F_0(t_0+c*n^(-1/3))-F_n(t_0,t_0,c,n))
  else
    F_0_inv(u)
}
curstatgen <- function(t_0,c,n,tau,tt,delta)
{
  u=runif(n+1)
  x=u
  for (i in 1:(n+1)) {x[i]=G_inv(u[i],t_0,c,n)}
  tt[2:(n+1)] <- runif(n,0,tau)
  delta[2:(n+1)] <- rep(0,n)
  delta[x<=tt[1:(n+1)]] <- 1
  delta[1] <- 0

  obs_tt <- tt[2:(n+1)]
  obs_delta <- delta[2:(n+1)]

  obs <- data.frame(cbind(obs_tt,obs_delta))
  obs <- obs[order(obs$obs_tt),]

  tt[2:(n+1)] <- obs$obs_tt
  delta[2:(n+1)] <- obs$obs_delta

  return(data.frame(tt,delta))
}

# retrun cs

cumsum <- function(m,cs,delta)
{
  cs[2] <- delta[2]
  if(m>=2)
    for(i in 2:m)
      cs[i+1] = cs[i-1+1] + delta[i+1];
    return(cs)
}


# return y

convexmin <- function(n,cumw,cs,y)
{
  y[2] = cs[2]/cumw[2];
```

```
   if(n>=2)
     for(i in 2:n)
     {
       y[i+1] = (cs[i+1]-cs[i-1+1])/(cumw[i+1]-cumw[i-1+1]);
       if(y[i-1+1]>y[i+1])
       {
         j = i;
         while(y[j-1+1] > y[i+1] && j>1)
         {
           j <- j - 1;
           if(j>1)
             y[i+1] = (cs[i+1]-cs[j-1+1])/(cumw[i+1]-cumw[j-1+1])
           else
             y[i+1] = cs[i+1]/cumw[i+1]
           if((i-1)>=j)
             for(m in j:(i-1))    y[m+1] = y[i+1]
         }
       }
     }
   return(y)
}

F.b=function(n,m,b, t_0, data0,F)
{
   t.b=t_0+b*log(n)/n^(1/3)
   MLE.b = 0
   if(t.b<data0[2])
       MLE.b <- 0

     for(j in 1:(m-1))
     {
       if(data0[j+1]<= t.b && t.b<data0[j+1+1])
         MLE.b <- F[j+1]
     }
   if(data0[m+1]<= t.b)
       MLE.b <- F[m+1]

   MLE.b
}

#  NumIt <- 1000
Power=function(t_0,c,n,tau,M)
{

   F0=F_0(t_0)

   bn=n^{1/30.1}
   hn=bn/n^(1/3.1)
   delta <- rep(0,n+1)
   delta2 <- rep(0,n+1)
   data <- rep(0,n+1)
   data0 <- rep(0,n+1)
   data2 <- rep(0,n+1)

   F= rep(0,n+1)
   cumw= rep(0,n+1)
   cs= rep(0,n+1)
   cumw1= rep(0,n+1)
   cs1= rep(0,n+1)
   y= rep(0,n+1)
   y1= rep(0,n+1)
   y2= rep(0,n+1)

   lowbound <- 0
   upbound <- 0


   F[1] <-0
   cumw[1] <- 0
   cs[1]<- 0
   cumw1[1] <- 0
   cs1[1] <- 0
   cumw[2:(n+1)] <- 1:n
   cumw1[2:(n+1)] <- cumw[2:(n+1)]

   y[1] <- 0
   y1[1] <- 0
   y2[1] <- 0

   s=0
   s.LR=0

   for (r in 1:M)
   {
     aa <- curstatgen(t_0,c,n,tau,data,delta)
```

```
data <- aa$tt
delta <- aa$delta
cs <- cumsum(n,cs,delta)
y <- convexmin(n,cumw,cs,y)

T=data[2:(n+1)]
g=(1/(n*hn))*0.5*sum(as.numeric(abs((t_0-T)/hn)<=1))


j <- 0;

for(i in 1:n)
{
  if (y[i+1]>y[i-1+1])
  {
    j <- j + 1
    F[j+1] <- y[i+1]
    data0[j+1] <- data[i+1]
  }
}

m <- j

MLE.0=F.b(n,m,0,t_0, data0,F)
MLE.b=F.b(n,m,bn,t_0, data0,F)
F.0=MLE.0
f=n^(1/3)/(bn*log(n))*(MLE.b-MLE.0)
z=0.9982*(4*F.0*(1-F.0)*f/g)^(1/3)/n^(1/3);
s=s+as.numeric(abs(F.0-F0)>z)

L<-0

for(i in 1:n)
{
  if(delta[i+1]==1)
    L <- L + log(y[i+1])
  else
    L <- L + log(1-y[i+1])
}

{
  j <- 1
  while (data[j+1]<t_0 && j<=n)
    j <- j + 1
  m1 <- j-1

  a <- y[m1+1];

  if((n-m1)>=1)
    for(i in 1:(n-m1))
      delta2[i+1]=1-delta[n-i+1+1];

  if((n-m1)>=1)
    for(i in 1:(n-m1))
      cs1[i+1]=cs1[i-1+1]+delta2[i+1]

  y1 <- convexmin(m1,cumw,cs,y1)
  y2 <- convexmin(n-m1,cumw1,cs1,y2)

  b <- 0
  j <- 0

  while (b<2.26916 & a+j*0.0001<=0.9999)
  {
    L0=0;

    j <- j + 1

    if(m1>=1)
      for(i in 1:m1)
      {
        if(delta[i+1]==1)
          L0 <- L0 + log(min(a+j*0.0001,y1[i+1]))
        else
          L0 <- L0 + log(1-min(a+j*0.0001,y1[i+1]))
      }

    if((n-m1)>=1)
      for(i in 1:(n-m1))
      {
        if(delta2[i+1]==1)
          L0 <- L0 + log(min(1-a-j*0.0001,y2[i+1]))
        else
          L0 <- L0 + log(1-min(1-a-j*0.0001,y2[i+1]))
      }

    j <- j + 1
```

```
        b <- 2*(L-L0)
      }

    upbound=a+j*0.0001;

    j <- 0
    b <- 0

    while(b<2.26916 && a-j*0.0001>=0.0001)
    {
      L0 <- 0

      j <- j + 1

      if(m1>=1)
        for(i in 1:m1)
        {
          if (delta[i+1]==1)
            L0 <- L0 + log(min(a-j*0.0001,y1[i+1]))
          else
            L0 <- L0 + log(1-min(a-j*0.0001,y1[i+1]))
        }

      if((n-m1)>=1)
        for(i in 1:(n-m1))
        {
          if(delta2[i+1]==1)
            L0 <- L0 + log(min(1-a+j*0.0001,y2[i+1]))
          else
            L0 <- L0 + log(1-min(1-a+j*0.0001,y2[i+1]))
        }

      b <- 2*(L-L0)

    }

    lowbound <- a-j*0.0001

    if(F0<lowbound || F0>upbound)
      s.LR <- s.LR + 1
  }

  }

  p=rep(0, 2)
  p[1]=s/M
  p[2] <- s.LR/M

  p
}


p11=c(1,Power(0.5,1,100,3,1000),Power(0.5,1,200,3,1000),Power(0.5,1,300,3,1000))
p12=c(2,Power(0.5,2,100,3,1000),Power(0.5,2,200,3,1000),Power(0.5,2,300,3,1000))
p13=c(3,Power(0.5,3,100,3,1000),Power(0.5,3,200,3,1000),Power(0.5,3,300,3,1000))
p14=c(4,Power(0.5,4,100,3,1000),Power(0.5,4,200,3,1000),Power(0.5,4,300,3,1000))
p15=c(5,Power(0.5,5,100,3,1000),Power(0.5,5,200,3,1000),Power(0.5,5,300,3,1000))

p21=c(1,Power(1.0,1,100,3,1000),Power(1.0,1,200,3,1000),Power(1.0,1,300,3,1000))
p22=c(2,Power(1.0,2,100,3,1000),Power(1.0,2,200,3,1000),Power(1.0,2,300,3,1000))
p23=c(3,Power(1.0,3,100,3,1000),Power(1.0,3,200,3,1000),Power(1.0,3,300,3,1000))
p24=c(4,Power(1.0,4,100,3,1000),Power(1.0,4,200,3,1000),Power(1.0,4,300,3,1000))
p25=c(5,Power(1.0,5,100,3,1000),Power(1.0,5,200,3,1000),Power(1.0,5,300,3,1000))

p31=c(1,Power(1.5,1,100,3,1000),Power(1.5,1,200,3,1000),Power(1.5,1,300,3,1000))
p32=c(2,Power(1.5,2,100,3,1000),Power(1.5,2,200,3,1000),Power(1.5,2,300,3,1000))
p33=c(3,Power(1.5,3,100,3,1000),Power(1.5,3,200,3,1000),Power(1.5,3,300,3,1000))
p34=c(4,Power(1.5,4,100,3,1000),Power(1.5,4,200,3,1000),Power(1.5,4,300,3,1000))
p35=c(5,Power(1.5,5,100,3,1000),Power(1.5,5,200,3,1000),Power(1.5,5,300,3,1000))

p41=c(1,Power(2.5,1,100,3,1000),Power(2.5,1,200,3,1000),Power(2.5,1,300,3,1000))
p42=c(2,Power(2.5,2,100,3,1000),Power(2.5,2,200,3,1000),Power(2.5,2,300,3,1000))
p43=c(3,Power(2.5,3,100,3,1000),Power(2.5,3,200,3,1000),Power(2.5,3,300,3,1000))
p44=c(4,Power(2.5,4,100,3,1000),Power(2.5,4,200,3,1000),Power(2.5,4,300,3,1000))
p45=c(5,Power(2.5,5,100,3,1000),Power(2.5,5,200,3,1000),Power(2.5,5,300,3,1000))

table4=rbind.data.frame(p11,p12,p13,p14,p15, p21,p22,p23,p24,p25,p31,p32,p33,p34,p35,p41,p42,p43,p44,p45)

Table4=write.table(table4,file="Table4.txt",row.names=FALSE, col.names=FALSE)
```