

Convergence of the Reweighted ℓ_1 Minimization Algorithm

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Abstract

The iteratively reweighted ℓ_1 minimization algorithm (IRL1) has been widely used for variable selection, signal reconstruction and image processing. However the convergence of the IRL1 has not been proved. In this paper, we prove that any sequence generated by the IRL1 is bounded and any accumulation point is a stationary point of the ℓ_2 - ℓ_p minimization problem with $0 < p < 1$. Moreover, the stationary point is a global minimizer and the convergence rate is approximately linear under certain conditions. We derive posteriori error bounds which can be used to construct practical stopping rules for the algorithm.

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1 Introduction

The nonsmooth, non-Lipschitz ℓ_p ($0 < p < 1$) regularization has advantages over smooth, convex regularization for restoring image with near edges, sparse signal reconstruction and variable selection. Iteratively reweighted ℓ_1 minimization algorithms have been widely used for solving minimization problems with ℓ_p regularization

$$\min_{x \in R^n} \|Ax - b\|_2^2 + \lambda \|x\|_p^p, \quad 0 < p < 1, \quad (1.1)$$

where $A \in R^{m \times n}$, $b \in R^m$, λ is a positive penalty parameter and

$$\|x\|_p^p = \sum_{i=1}^n |x_i|^p.$$

See [1, 2, 4, 5, 13, 16, 18]. A version of the IRL1 for solving the ℓ_2 - ℓ_p minimization problem (1.1) is as follows:

$$x^{k+1} \in \arg \min_{x \in R^n} f_k(x, \varepsilon) := \|Ax - b\|_2^2 + \lambda \|W^k x\|_1 \quad (1.2)$$

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where the weight $W^k = \text{diag}(w^k)$ is defined by the previous iterates and updated in each iteration as

$$w_i^k = \frac{p}{(|x_i^k| + \varepsilon)^{1-p}}, \quad i = 1, \dots, n.$$

Here ε is a positive parameter to ensure that the algorithm is well-defined.

At each iteration, the IRL1 (1.2) solves a convex ℓ_2 - ℓ_1 minimization problem. Extensive numerical experiments have shown that the IRL1 (1.2) is an efficient method for variable selection, signal reconstruction and image processing. However no convergence results have been given for (1.2).

In this paper, we prove that any sequence generated by the IRL1 (1.2) is bounded and any accumulation point is a stationary point x^* of the following ℓ_2 - ℓ_p minimization problem.

$$\min_{x \in R^n} f(x, \varepsilon) := \|Ax - b\|_2^2 + \lambda \sum_{i=1}^n (|x_i| + \varepsilon)^p, \quad 0 < p < 1. \quad (1.3)$$

Moreover, we show that the stationary point is a global minimizer of (1.3) in certain domain and the convergence rate is approximately linear under certain conditions. Moreover, we derive posteriori error bounds

$$\|x^k - x^*\|_2 \leq \gamma \|x^{k+1} - x^k\|_2,$$

with a positive constant γ , which can be used to construct practical stopping rules for the algorithm.

Note that the problem (1.2) may have multiple solutions since the objective is not strictly convex. However, our convergence results hold for any choice x^{k+1} in the solution set $\arg \min_{x \in R^n} f_k(x, \varepsilon)$.

The model (1.3) can be considered as an approximation to the following constrained ℓ_p optimization problem

$$\min_{x \in R^n} \sum_{i=1}^n (|x_i| + \varepsilon)^p, \quad \text{s.t. } Ax = b, \quad (1.4)$$

which is an approximation of the ℓ_p minimization problem

$$\min_{x \in R^n} \|x\|_p^p, \quad \text{s.t. } Ax = b. \quad (1.5)$$

Problems (1.4) and (1.5) have been widely used [1, 2, 3, 4, 5, 10, 13] when the vector b contains little or no noise. The models (1.1) and (1.3) are also called denoising models of (1.4) and (1.5). Recently, it has been proved that these four problems are NP-hard in [6, 11]. An advantage of (1.4) and (1.3) is that their objective functions are Lipschitz continuous.

We summarize some notations and results in nonsmooth optimization [7], which will be used in this paper. It is known that a Lipschitz function $g : R^n \rightarrow R$ is almost everywhere differentiable and its subgradient is defined by

$$\partial g(y) = \text{co} \left\{ \lim_{\substack{y^k \rightarrow y \\ y^k \in D_g}} \nabla g(y^k) \right\},$$

where "co" denotes the convex hull and D_g is the set of points at which g is differentiable.

We say x^* is a stationary point of g if $0 \in \partial g(x^*)$. If g is a convex function, then x^* is a global minimizer of g in R^n if and only if x^* is a stationary point of g .

A function g is convex if and only if ∂g is a monotone operator, that is,

$$(y - x, \xi_y - \xi_x) \geq 0, \quad \forall \xi_y \in \partial g(y), \quad \forall \xi_x \in \partial g(x).$$

We say a function $g : R^n \rightarrow R$ is strongly pseudoconvex at x on D if for every $\xi \in \partial g(x)$ and every $y \in D$,

$$\xi^T(y - x) \geq 0 \quad \Rightarrow \quad g(y) \geq g(x).$$

We say a function $g : R^n \rightarrow R$ is strongly pseudoconvex on D if g is strongly pseudoconvex at every point in D .

Throughout this paper, $\|\cdot\|$ denotes the ℓ_2 norm. $|x|$ is the absolute value vector of x , that is, $|x| = (|x_1|, \dots, |x_n|)^T$. The vector $e_i \in R^n$ is the i th column of the identity matrix. The vector $a_i \in R^m$ is the i th column of the matrix A . The cardinality of a subset $T \subset \{1, \dots, n\}$ is denoted by $|T|$, and its complement set is denoted by T^C .

2 Convergence analysis

In this section, we give convergence analysis for the IRL1 (1.2). Note that both objective functions f and f_k are Lipschitz continuous for any fixed $\varepsilon > 0$. Hence we can define their subgradients in R^n . Moreover, both functions are nonnegative and satisfy

$$f(x, \varepsilon) \rightarrow \infty, \quad f_k(x, \varepsilon) \rightarrow \infty \quad \text{as} \quad \|x\| \rightarrow \infty. \quad (2.1)$$

Therefore, the solution sets of (1.2) and (1.3) are nonempty and bounded.

Lemma 2.1. *For any nonnegative constants α, β and $t \in (0, 1)$, we have*

$$\alpha^{1-t}\beta^t \leq (1-t)\alpha + t\beta, \quad (2.2)$$

and equality holds if and only if $\alpha = \beta$.

Proof. Young's inequality states that for any nonnegative constants μ and ν ,

$$\mu\nu \leq \frac{1}{q}\mu^q + \frac{1}{r}\nu^r, \quad \left(\frac{1}{q} + \frac{1}{r} = 1\right)$$

where equality holds if and only if $\mu^q = \nu^r$. Set $\frac{1}{q} = 1 - t$, $\mu^q = \alpha$ and $\nu^r = \beta$ in this inequality. We obtain (2.2) and equality holds if and only if $\alpha = \beta$. \square

Lemma 2.2. *Let $\{x^k\}$ be the sequence generated by the IRL1 (1.2). Then we have*

$$f(x^{k+1}, \varepsilon) \leq f(x^k, \varepsilon) - \|A(x^{k+1} - x^k)\| - \delta(x^{k+1}, x^k), \quad (2.3)$$

where $\delta(x^{k+1}, x^k) \geq 0$ and equality holds if and only if $|x^{k+1}| = |x^k|$. If $p = \frac{1}{2}$, then

$$\delta(x^{k+1}, x^k) = \lambda \frac{1}{2} \sum_{i=1}^n \frac{\left(\left(|x_i^{k+1}| + \varepsilon\right)^{\frac{1}{2}} - \left(|x_i^k| + \varepsilon\right)^{\frac{1}{2}}\right)^2}{\left(|x_i^k| + \varepsilon\right)^{\frac{1}{2}}}.$$

Proof. Since x^{k+1} is a solution of problem (1.2), the zero vector is contained in the generalized differential with respect to x , that is,

$$0 \in \partial f_k(x^{k+1}, \varepsilon).$$

See [7]. The function f_k is the sum of $n + 1$ convex functions, namely, $\|Ax - b\|^2$ and $|x_i|$, $i = 1, \dots, n$. By the addition rule of subgradient for the sum of convex functions [7, Corollary 3, p40], we have

$$\partial f_k(x, \varepsilon) = \lambda \sum_{i=1}^n \frac{p \partial |x_i|}{(|x_i^k| + \varepsilon)^{1-p}} e_i + 2A^T(Ax - b). \quad (2.4)$$

Hence, we find

$$0 \in \partial f_k(x^{k+1}, \varepsilon) = \lambda \sum_{i=1}^n \frac{p}{(|x_i^k| + \varepsilon)^{1-p}} \partial |x_i^{k+1}| e_i + 2A^T(Ax^{k+1} - b), \quad (2.5)$$

which means that there exist $c_i^{k+1} \in \partial |x_i^{k+1}|$, $i = 1, \dots, n$ such that

$$\lambda \left(\frac{p c_i^{k+1}}{(|x_i^k| + \varepsilon)^{1-p}} \right)_{1 \leq i \leq n} + 2A^T(Ax^{k+1} - b) = 0. \quad (2.6)$$

By the definition of the subdifferential for $|x_i|$, we have

$$c_i^{k+1} = \begin{cases} 1, & \text{if } x_i^{k+1} > 0, \\ -1, & \text{if } x_i^{k+1} < 0, \\ \alpha, & \text{if } x_i^{k+1} = 0, \quad \alpha \in [-1, 1]. \end{cases} \quad (2.7)$$

By (2.6), (2.7) and (2.2), we obtain

$$\begin{aligned} & f(x^k, \varepsilon) - f(x^{k+1}, \varepsilon) \\ &= \lambda \sum_{i=1}^n \left((|x_i^k| + \varepsilon)^p - (|x_i^{k+1}| + \varepsilon)^p \right) + \|Ax^{k+1} - Ax^k\|^2 + 2(Ax^k - Ax^{k+1})^T(Ax^{k+1} - b) \\ &= \|Ax^{k+1} - Ax^k\|^2 + \lambda \sum_{i=1}^n \left((|x_i^k| + \varepsilon)^p - (|x_i^{k+1}| + \varepsilon)^p + \frac{p c_i^{k+1} (x_i^{k+1} - x_i^k)}{(|x_i^k| + \varepsilon)^{1-p}} \right) \quad (2.8) \\ &\geq \|Ax^{k+1} - Ax^k\|^2 + \lambda \sum_{i=1}^n \left((|x_i^k| + \varepsilon)^p - (|x_i^{k+1}| + \varepsilon)^p + \frac{p(|x_i^{k+1}| - |x_i^k|)}{(|x_i^k| + \varepsilon)^{1-p}} \right) \\ &= \|Ax^{k+1} - Ax^k\|^2 + \lambda \sum_{i=1}^n \left(\frac{(|x_i^k| + \varepsilon) - (|x_i^k| + \varepsilon)^{1-p} (|x_i^{k+1}| + \varepsilon)^p + p(|x_i^{k+1}| - |x_i^k|)}{(|x_i^k| + \varepsilon)^{1-p}} \right) \\ &= \|Ax^{k+1} - Ax^k\|^2 + \lambda \sum_{i=1}^n \left(\frac{(1-p)(|x_i^k| + \varepsilon) + p(|x_i^{k+1}| + \varepsilon) - (|x_i^k| + \varepsilon)^{1-p} (|x_i^{k+1}| + \varepsilon)^p}{(|x_i^k| + \varepsilon)^{1-p}} \right) \\ &= \|Ax^{k+1} - Ax^k\|^2 + \delta(x^{k+1}, x^k) \\ &\geq \|Ax^{k+1} - Ax^k\|^2, \end{aligned}$$

where the first inequality uses

$$c_i^{k+1} x_i^{k+1} = |x_i^{k+1}| \quad \text{and} \quad |c_i^{k+1}| \leq 1$$

and the last inequality uses Lemma 2.1 to claim that

$$\delta_i(x^{k+1}, x^k) = \lambda \frac{(1-p)(|x_i^k| + \epsilon) + p(|x_i^{k+1}| + \epsilon) - (|x_i^k| + \epsilon)^{1-p}(|x_i^{k+1}| + \epsilon)^p}{(|x_i^k| + \epsilon)^{1-p}} \geq 0,$$

and $\delta(x^{k+1}, x^k) = \sum_{i=1}^n \delta_i(x^{k+1}, x^k) \geq 0$. \square

Lemma 2.3. *Suppose that $g_1 : R^n \rightarrow R$ and $-g_2 : R^n \rightarrow R$ are convex on a closed convex set Ω , and $g_1(x) \geq 0$ and $g_2(x) > 0$, for all $x \in \Omega$ then $h(x) = \frac{g_1(x)}{g_2(x)}$ is strongly pseudoconvex on Ω .*

Proof. This lemma is a simple generalization of [14, Proposition 5.2, p943], which proved that the condition number of a symmetric positive definite matrix is pseudoconvex. For completeness, we give a proof of this lemma.

From the convexity assumption, g_1 and g_2 are locally Lipschitz continuous and for any $x, y \in \Omega$ and $\xi_1 \in \partial g_1(x), \xi_2 \in \partial g_2(x)$, we have

$$g_1(y) - g_1(x) \geq \xi_1^T(y - x),$$

and

$$-g_2(y) + g_2(x) \geq -\xi_2^T(y - x).$$

Hence we obtain

$$\begin{aligned} g_1(y) - h(x)g_2(y) &= g_1(y) - g_1(x) + h(x)(-g_2(y) + g_2(x)) \\ &\geq \xi_1^T(y - x) - h(x)\xi_2^T(y - x) \\ &= (\xi_1 - h(x)\xi_2)^T(y - x) \\ &= g_2(x) \left(\frac{\xi_1 g_2(x) - g_1(x)\xi_2}{g_2(x)^2} \right)^T (y - x). \end{aligned}$$

By the quotient rule for the Clarke generalized gradient [7, Proposition 2.3.14], we find that $\frac{\xi_1 g_2(x) - g_1(x)\xi_2}{g_2(x)^2} \in \partial h(x)$, from that g_2 and g_1 are Clarke regular. Therefore we have $h(y) \geq h(x)$ if $\xi^T(y - x) \geq 0$ with $\xi \in \partial h(x)$. \square

Lemma 2.4. *For constants $\alpha > 0, \epsilon > 0$ and $p \in (0, 1)$, let*

$$\phi(t) = |t| + (\alpha t^2 + \beta t)(|t| + \epsilon)^{1-p}.$$

Then ϕ is convex in $[0, \infty)$ and $(-\infty, 0]$ if

$$|\beta| \leq \frac{\alpha \epsilon}{1-p}. \quad (2.9)$$

Proof. The function ϕ is differentiable in R except $t = 0$. To show the convexity of ϕ , we consider the second derivative of ϕ for $t \neq 0$.

First we consider $t > 0$. By simple calculation, we get

$$\phi''(t) = (t + \epsilon)^{-1-p}(c_1 t^2 + c_2 t + c_3),$$

where

$$\begin{aligned} c_1 &= \alpha(2 + (4 - p)(1 - p)), \\ c_2 &= (2 - p)((1 - p)\beta + 4\alpha\varepsilon), \\ c_3 &= 2\varepsilon(\alpha\varepsilon + (1 - p)\beta). \end{aligned}$$

Obviously, $c_i > 0, i = 1, 2$ and $c_3 \geq 0$. This implies that ϕ is convex for $t > 0$.

Now, we consider $t < 0$. In this case,

$$\phi(t) = -t + (\alpha t^2 + \beta t)(-t + \varepsilon)^{1-p}.$$

Similarly, we can find that for $t < 0$,

$$\phi''(t) = (-t + \varepsilon)^{-1-p}(c_1 t^2 + c_4 t + c_5)$$

where

$$\begin{aligned} c_4 &= (2 - p)((1 - p)\beta - 4\alpha\varepsilon), \\ c_5 &= 2\varepsilon(\alpha\varepsilon - (1 - p)\beta). \end{aligned}$$

Obviously, $c_4 < 0$ and $c_5 \geq 0$. This implies that $\phi''(t) \geq 0$ and thus ϕ is convex for $t < 0$. By the continuity of ϕ and that for $t_1 t_2 > 0$

$$\phi(\mu t_1 + (1 - \mu)t_2) \leq \mu\phi(t_1) + (1 - \mu)\phi(t_2), \quad \text{for } 0 \leq \mu \leq 1,$$

we can take $t_1 \rightarrow 0$ or $t_2 \rightarrow 0$, and claim that ϕ is convex in $[0, \infty)$ and $(-\infty, 0]$. \square

Theorem 2.1. *Let $\{x^k\}$ be a sequence generated by the IRL1 (1.2). Then the sequence $\{x^k\}$ is bounded and $\lim_{k \rightarrow \infty} (x^{k+1} - x^k) = 0$. Moreover, any accumulation point of $\{x^k\}$ is a stationary point x^* of (1.3).*

Proof. By Lemma 2.2, the sequence $\{f(x^k, \varepsilon)\}$ is monotonically decreasing and bounded below. Hence it converges. It is clear that the sequence $\{x^k\}$ is contained in the level set

$$\mathcal{L}(x^0) = \{x \mid f(x, \varepsilon) \leq f(x^0, \varepsilon)\}.$$

Obviously, $\mathcal{L}(x^0)$ is bounded from (2.1).

By (2.3), we have $\delta(x^{k+1}, x^k) \rightarrow 0$, as $k \rightarrow \infty$. From Lemma 2.1 and $\delta_i(x^{k+1}, x^k) \geq 0$, we have

$$\lim_{k \rightarrow \infty} f(x^k, \varepsilon) - f(x^{k+1}, \varepsilon) = \lim_{k \rightarrow \infty} \|A(x^{k+1} - x^k)\| = \lim_{k \rightarrow \infty} (|x^k| - |x^{k+1}|) = 0. \quad (2.10)$$

This, together with (2.8), implies

$$\lim_{k \rightarrow \infty} c_i^{k+1}(x_i^{k+1} - x_i^k) = 0, \quad i = 1, \dots, n, \quad (2.11)$$

where c_i^{k+1} is defined in (2.7). Note that $c_i^{k+1} = 0$ only if $x_i^{k+1} = 0$, and $|c_i^{k+1}| = 1$ if $x_i^{k+1} \neq 0$. For a fixed i , suppose that there is a subsequence $\{x_i^{k_j+1}\}$ such that $x_i^{k_j+1} = 0$,

then from (2.10) we have $\lim_{k \rightarrow \infty} x_i^k = 0$. Otherwise, we have $|c_i^{k+1}| = 1$ for sufficiently large k , which, together with (2.11), we have

$$\lim_{k \rightarrow \infty} (x_i^{k+1} - x_i^k) = 0, \quad i = 1, \dots, n. \quad (2.12)$$

Let $\{x^{n_k}\}$ be a subsequence of $\{x^k\}$ which converges to x^* . By (2.6) and (2.12), there exist $c_i^* \in \partial|x_i^*|$, $i = 1, \dots, n$ such that

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \lambda \left(\frac{p c_i^{n_k}}{(|x_i^{n_k} + x_i^{n_k-1} - x_i^{n_k}| + \varepsilon)^{1-p}} \right)_{1 \leq i \leq n} + 2A^T(Ax^{n_k} - b) \\ &= \lambda \left(\frac{p c_i^*}{(|x_i^*| + \varepsilon)^{1-p}} \right)_{1 \leq i \leq n} + 2A^T(Ax^* - b) \in \partial f(x^*, \varepsilon). \end{aligned} \quad (2.13)$$

Hence x^* is a stationary point of (1.3).

Let $A_I = [a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n]$ and $x_I^* = [x_1^*, \dots, x_{i-1}^*, x_{i+1}^*, \dots, x_n^*]^T$.

Theorem 2.2. *Let x^* be a stationary point of (1.3). The following statements hold.*

(1) *If for some i , $\varepsilon \geq \left(\frac{\lambda(1-p)p}{2\|a_i\|^2} \right)^{\frac{1}{2-p}}$ holds, then*

$$f(x^*, \varepsilon) \leq f(x^* + te_i, \varepsilon), \quad \text{for } t \in \begin{cases} [-x_i^*, \infty) & \text{if } x_i^* \geq 0, \\ (-\infty, -x_i^*] & \text{if } x_i^* \leq 0. \end{cases} \quad (2.14)$$

(2) *If for some i , $|a_i^T(A_I x_I^* - b)| \leq \frac{\|a_i\|^2 \varepsilon}{2(1-p)}$ holds, then (2.14) holds. Moreover, if for some i , $a_i^T(A_I x_I^* - b) = 0$ holds, then $x_i^* = 0$ and*

$$f(x^*, \varepsilon) \leq f(x^* + te_i, \varepsilon), \quad \text{for } t \in \mathbb{R}. \quad (2.15)$$

Proof. (1) Let

$$\varphi(t) = \lambda \| |x^* + te_i| + \varepsilon \|_p^p + \|A(x^* + te_i) - b\|^2. \quad (2.16)$$

The subdifferential of φ is

$$\partial\varphi(t) = \lambda \frac{p \operatorname{sign}(x_i^* + t)}{(|x_i^* + t| + \varepsilon)^{1-p}} + 2a_i^T(A(x^* + te_i) - b).$$

By (2.13), we have $0 \in \varphi(0)$, that is, 0 is a stationary point of φ . For t_1 and t_2 satisfying $(x_i^* + t_1)(x_i^* + t_2) > 0$, φ is continuously twice differentiable on $[t_1, t_2]$. Thus there is t_0 between t_1 and t_2 such that

$$\varphi'(t_1) - \varphi'(t_2) = \left(-\frac{\lambda(1-p)p}{(|x_i^* + t_0| + \varepsilon)^{2-p}} + 2\|a_i\|^2 \right) (t_1 - t_2).$$

Hence if $\varepsilon \geq \left(\frac{\lambda(1-p)p}{2\|a_i\|^2} \right)^{\frac{1}{2-p}}$, then

$$\begin{aligned} (t_1 - t_2)(\varphi'(t_1) - \varphi'(t_2)) &= \left(-\frac{\lambda(1-p)p}{(|x_i^* + t_0| + \varepsilon)^{2-p}} + 2\|a_i\|^2 \right) (t_1 - t_2)^2 \\ &\geq \left(-\frac{\lambda(1-p)p}{\varepsilon^{2-p}} + 2\|a_i\|^2 \right) (t_1 - t_2) \geq 0. \end{aligned}$$

Hence φ is convex in $[-x_i^*, \infty)$ if $x_i^* \geq 0$, and in $(-\infty, -x_i^*]$ if $x_i^* \leq 0$. This, together with $0 \in \partial\varphi(0)$ implies that 0 is the minimizer of φ in $(-x_i^*, \infty)$ if $x_i^* \geq 0$, and in $(-\infty, -x_i^*]$ if $x_i^* \leq 0$. This gives (2.14).

(2) To prove this part, we show φ defined in (2.16) is strongly pseudoconvex in $[-x_i^*, \infty)$ and $(-\infty, -x_i^*]$. The function φ can be rewritten as

$$\begin{aligned}\varphi(t) &= \lambda(|x_i^* + t| + \varepsilon)^p + \|a_i\|^2(x_i^* + t)^2 + 2a_i^T(A_I x_I^* - b)(x_i^* + t) + c_0, \\ &= \lambda\left(\frac{|x_i^* + t| + \varepsilon + \left(\frac{\|a_i\|^2}{\lambda}(x_i^* + t)^2 + \frac{2a_i^T(A_I x_I^* - b)}{\lambda}(x_i^* + t)\right)}{(|x_i^* + t| + \varepsilon)^{1-p}}\right) + c_0,\end{aligned}$$

where c_0 is a constant. Using Lemma 2.4, with

$$\alpha = \frac{\|a_i\|^2}{\lambda} \quad \text{and} \quad \beta = \frac{2a_i^T(A_I x_I^* - b)}{\lambda},$$

we find that the function

$$|x_i^* + t| + \varepsilon + \left(\frac{\|a_i\|^2}{\lambda}(x_i^* + t)^2 + \frac{2a_i^T(A_I x_I^* - b)}{\lambda}(x_i^* + t)\right)(|x_i^* + t| + \varepsilon)^{1-p}$$

is convex. Since $(|x_i^* + t| + \varepsilon)^{1-p}$ is concave, we find that φ is strongly pseudoconvex by Lemma 2.3.

By the definition of the strong pseudo convexity and (2.13), from $\varphi(0) = f(x^*, \varepsilon)$ and $\varphi(t) = f(x^* + te_i, \varepsilon)$, we obtain (2.14).

If $a_i^T(A_I x_I^* - b) = 0$, then (2.13) implies that

$$0 \in \lambda\left(\frac{pc_i^*}{(|x_i^*| + \varepsilon)^{1-p}}\right) + 2a_i^T a_i x_i^*. \quad (2.17)$$

Since $c_i^* = 1$ if $x_i^* > 0$ and $c_i^* = -1$ if $x_i^* < 0$, (2.17) only holds at $x_i^* = 0$. Moreover, it is easy to see that in such case with $x_i^* = 0$,

$$\varphi(-x_i^*) = \varphi(0) \leq \varphi(t), \quad \text{for } t \in R,$$

that is,

$$f(x^* - x_i^* e_i, \varepsilon) = f(x^*, \varepsilon) \leq f(x^* + te_i, \varepsilon), \quad \text{for } t \in R.$$

We obtain the desired results. \square

In [5], it was shown that any local minimizer x^* of (1.1) satisfies

$$\text{either } |x_i^*| = 0 \quad \text{or} \quad |x_i^*| \geq L_i, \quad \forall i = 1, \dots, n, \quad (2.18)$$

where

$$L_i := \left(\frac{\lambda p(1-p)}{2\|a_i\|^2}\right)^{\frac{1}{2-p}}.$$

This lower bound for absolute value of nonzero elements of any local minimizer of (1.1) can be easily extended to the model (1.3). We give the lower bound theory for (1.3) in the following theorem.

Theorem 2.3. *If $0 \leq \epsilon < L := \min_{1 \leq i \leq n} L_i$, then every local minimizer x^* of (1.3) satisfies*

$$\text{either } |x_i^*| = 0 \text{ or } |x_i^*| \geq L_i - \epsilon, \quad \forall i = 1, \dots, n. \quad (2.19)$$

Proof. This Theorem is a simple generalization of Theorem 2.1 in [5]. For completeness, we give a brief proof.

Let x^* be a local minimizer of (1.3) with $\|x^*\|_0 = k$, without loss of generality, we assume

$$x^* = (x_1^*, \dots, x_k^*, 0, \dots, 0)^T.$$

Let $z^* = (x_1^*, \dots, x_k^*)^T$ and $B \in R^{m \times k}$ be the submatrix of A , whose columns are the first k columns of A . For a fixed $\epsilon \geq 0$, define a function $g : R^k \rightarrow R$ by

$$g(z, \epsilon) = \|Bz - b\|^2 + \lambda \sum_{i=1}^k (|z_i| + \epsilon)^p + (n - k)\epsilon^p.$$

We have

$$f(x^*, \epsilon) = \|Ax^* - b\|^2 + \lambda \sum_{i=1}^n (|x_i^*| + \epsilon)^p = \|Bz^* - b\|^2 + \lambda \sum_{i=1}^k (|z_i^*| + \epsilon)^p + (n - k)\epsilon^p.$$

Since $|z_i^*| > 0, i = 1, \dots, k$, g is continuously differentiable at z^* . Moreover, in a neighbourhood of x^* ,

$$\begin{aligned} g(z^*, \epsilon) = f(x^*, \epsilon) &\leq \min\{f(x, \epsilon) | x_i = 0, i = k + 1, \dots, n\} \\ &= \min\{g(z, \epsilon) | z \in R^k\}, \end{aligned}$$

which implies that z^* is a local minimizer of the function g . Hence the second order necessary condition for

$$\min_{z \in R^k} g(z, \epsilon) \quad (2.20)$$

holds at z^* , which gives that the matrix

$$2B^T B + \lambda p(p - 1) \text{diag}((|z_i^*| + \epsilon)^{p-2})$$

is positive semi-definite. Therefore, we obtain

$$2e_i^T B^T B e_i + \lambda p(p - 1) \text{diag}((|z_i^*| + \epsilon)^{p-2}) \geq 0, \quad i = 1, \dots, k$$

where e_i is the i th column of the identity matrix of $R^{k \times k}$.

Note that $\|a_i\|^2 = e_i^T B^T B e_i$. We find that

$$(|z_i^*| + \epsilon)^{p-2} \leq \frac{2\|a_i\|^2}{\lambda p(1 - p)}, \quad i = 1, \dots, k$$

which implies that

$$|z_i^*| \geq \left(\frac{\lambda p(1 - p)}{2\|a_i\|^2} \right)^{\frac{1}{2-p}} - \epsilon = L_i - \epsilon, \quad i = 1, \dots, k.$$

Hence for any local minimizer x^* of (1.3) if $x_i^* \neq 0$, then $|x_i^*| \geq L_i - \epsilon$. \square

Now we derive the convergence rate of the IRL1 (1.2) and error bounds.

Theorem 2.4. Assume that the sequence $\{x^k\}$ generated by (1.2) converges to a local minimizer x^* of (1.3). Denote $S = \{i \mid |x_i^*| \neq 0\}$ and $\beta = \min_{i \in S} |x_i^*|$. If

$$\frac{\lambda p(1-p)}{\beta^{2-p}} \leq 2\lambda_{\min}(A_S^T A_S), \quad (2.21)$$

then there exist positive constants $\gamma_i, i = 1, 2, 3$ and $c \in (0, 1)$ such that for all sufficiently large k

$$\|x_S^k - x_S^*\| \leq \gamma_1 \|x_S^k - x_S^{k+1}\| + \gamma_2 \|x_{S^C}^{k+1}\|,$$

and

$$\|x_S^{k+1} - x_S^*\| \leq c \|x_S^k - x_S^*\| + \gamma_3 \|x_{S^C}^{k+1}\|.$$

Proof. Denote $S_k = \{i \mid |x_i^k| \neq 0\}$. Since $x^k \rightarrow x^*$, by (2.19) for sufficiently large k , we have $S \subset S_k$ and there exists a small constant $\delta \in (0, \epsilon)$ such that $|x_i^k| \geq \beta - \delta$, for $i \in S$.

Consider the function

$$g(z, \epsilon) = \sum_{i \in S} \lambda(|z_i| + \epsilon)^p + \|A_S z - b\|^2 + \lambda \sum_{i \in S^C} \epsilon^p, \quad z \in R^{|S|}.$$

From the proof of Theorem 2.3, we see that x_S^* is a local minimizer of $g(z, \epsilon)$. Therefore we have from the optimal condition for minimizing $g(z, \epsilon)$ that

$$\left(\frac{\lambda p \operatorname{sign}(x_i^*)}{(|x_i^*| + \epsilon)^{1-p}} \right)_{i \in S} + 2A_S^T (A_S x_S^* - b) = 0, \quad (2.22)$$

and the matrix

$$\operatorname{diag} \left(\left(\frac{\lambda p(p-1)}{(|x_i^*| + \epsilon)^{2-p}} \right)_{i \in S} \right) + 2A_S^T A_S$$

is positive semi-definite, which implies that the matrix $A_S^T A_S$ is positive definite since $p-1 < 0$.

Since x^{k+1} is a local minimizer of $f_k(x, \epsilon)$ and for sufficiently large k ,

$$\operatorname{sign}(x_i^{k+1}) = \operatorname{sign}(x_i^k) = \operatorname{sign}(x_i^*), \quad i \in S,$$

we have

$$\left(\begin{array}{c} \left(\frac{\lambda p \operatorname{sign}(x_i^*)}{(|x_i^k| + \epsilon)^{1-p}} \right)_{i \in S} \\ \left(\frac{\lambda p c_i^{k+1}}{(|x_i^k| + \epsilon)^{1-p}} \right)_{i \in S^C} \end{array} \right) + 2 \left(\begin{array}{c} A_S^T (A x^{k+1} - b) \\ A_{S^C} (A x^{k+1} - b) \end{array} \right) = 0, \quad (2.23)$$

where $c_i^{k+1} \in \partial|x_i^{k+1}|$. By (2.22) and (2.23), we have

$$B_S (x_S^k - x_S^*) = 2A_S^T A_S (x_S^k - x_S^{k+1}) - 2A_S^T A_{S^C} x_{S^C}^{k+1}, \quad (2.24)$$

and

$$x_S^{k+1} - x_S^* = -(2A_S^T A_S)^{-1} D_S (x_S^k - x_S^*) - (A_S^T A_S)^{-1} A_S^T A_{S^C} x_{S^C}^{k+1}, \quad (2.25)$$

where ζ_i is between x_i^* and x_i^k for any $i \in S$, and

$$D_S = \operatorname{diag} \left(\left(\frac{\lambda p(p-1)}{(|\zeta_i| + \epsilon)^{2-p}} \right)_{i \in S} \right), \quad B_S = D_S + 2A_S^T A_S.$$

From $\text{sign}(x_i^k) = \text{sign}(x_i^*)$, we have $|\zeta_i| \geq \beta - \delta > 0$, for $i \in S$. Moreover, from (2.21) and the following inequalities

$$\frac{\lambda p(1-p)}{(|\zeta_i| + \epsilon)^{2-p}} \leq \frac{\lambda p(1-p)}{(\beta - \delta + \epsilon)^{2-p}} < \frac{\lambda p(1-p)}{(\beta)^{2-p}} \leq 2\lambda_{\min}(A_S^T A_S), \quad (2.26)$$

we obtain that B_S is nonsingular and we have from (2.24) and (2.25) that

$$\|x_S^k - x_S^*\| \leq 2\|B_S^{-1}\| \|A_S^T A_S\| \|x_S^k - x_S^{k+1}\| + 2\|B_S^{-1}\| \|A_S^T A_{S^c}\| \|x_{S^c}^{k+1}\|,$$

and

$$\|x_S^{k+1} - x_S^*\| \leq \|(2A_S^T A_S)^{-1} D_S\| \|x_S^k - x_S^*\| + \|(A_S^T A_S)^{-1} A_S^T A_{S^c}\| \|x_{S^c}^{k+1}\|.$$

By (2.21) and (2.26), we have $\|(2A_S^T A_S)^{-1} D_S\| < 1$. Therefore, we complete the proof with $\gamma_1 = 2\|B_S^{-1}\| \|A_S^T A_S\|$, $\gamma_2 = 2\|B_S^{-1}\| \|A_S^T A_{S^c}\|$, $\gamma_3 = \|(A_S^T A_S)^{-1} A_S^T A_{S^c}\|$ and $c = \|(2A_S^T A_S)^{-1} D_S\|$. \square

Based on Theorem 2.3, for large k , entries x_i^k satisfying $|x_i^k| \ll L_i - \epsilon$ very likely converge to zero. If we can guess the index set S of nonzero elements x^* correctly and set $x_{S^c}^k = 0$ for all large k . Then from Theorem 2.4, we have

$$\|x^k - x^*\| = \|x_S^k - x_S^*\| \leq \gamma_1 \|x_S^k - x_S^{k+1}\| = \gamma_1 \|x^k - x^{k+1}\|$$

and

$$\|x^{k+1} - x^*\| = \|x_S^{k+1} - x_S^*\| \leq c \|x_S^k - x_S^*\| = c \|x^k - x^*\|.$$

3 Conclusion

Regularized minimization problems with ℓ_p regularization arise frequently in many fields such as finance, econometrics and signal processing. On the statistical side, the ℓ_p regularization is called the bridge penalty and minimizers of the minimization problem (1.1) with $\|x\|_p^p$ regularization are called bridge estimators [12]. Theoretical results show that the bridge estimators have various attractive features due to the concavity and non-Lipschitzian property of the regularization function $\|x\|_p^p$. However, the minimization problem (1.1) is nonconvex and non-Lipschitz. There are not many optimization theories and algorithms for solving (1.1). The reweighted ℓ_1 minimization algorithm (IRL1) is developed to solve (1.1). The IRL1 has been widely used for variable selection, signal reconstruction and image processing. Moreover, extensive numerical experiments showed that the IRL1 is efficient for many applications. However no convergence results have been given for IRL1. This paper proves that any sequence generated by the IRL1 is bounded and any accumulation point is a stationary point of the minimization problem (1.3). In general, a stationary point of the minimization problem (1.3) is not a minimizer of (1.1). Hence, this paper provides a negative certificate by showing that the IRL1 cannot find bridge estimators in most cases. On the positive side, Theorem 2.3 shows any local minimizer of (1.3) has certain sparsity. These results are important for developing algorithms for solving the nonconvex and non-Lipschitz minimization problem (1.1) and applications in variable selection, signal reconstruction and image processing.

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