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Robust Wardrop's user equilibrium assignment under stochastic demand and supply: Expected residual minimization approach

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A B S T R A C T

Various models of traffic assignment under stochastic environment have been proposed recently, mainly by assuming different travelers' behavior against uncertainties. This paper focuses on the expected residual minimization (ERM) model to provide a robust traffic assignment with an emphasis on the planner's perspective. The model is further extended to obtain a stochastic prediction of the traffic volumes by the technique of path choice approach. We show theoretically the existence and the robustness of the ERM solution. In addition, we employ an improved solution algorithm for solving the ERM model. Numerical experiments are carried out to illustrate the characteristics of the proposed model, by comparing with other existing models.

1. Introduction

The main role of traffic or transportation model is to provide a forecast of future traffic state. The output from the model is often used in highway and public transport project design and evaluation. The current state of the art of traffic modeling involves various modeling paradigms ranging from the traditional four-step model, activity based model, to dynamic traffic assignment. An underlying structure of these modeling paradigms is the interaction between the demand and supply sides of the traffic system. The travel demand is normally defined as an origin–destination (OD) matrix or captured by a demand function. On the supply side, the performance of a road or highway is represented by a speed-flow function (in either static or dynamic framework). The forecast provided by the traffic model is then based on the equilibrium state between the demand and supply of travel which are deterministic inputs of the models. However, such long term forecasts often involve a high degree of uncertainty of the inputs (e.g. future travel demand in the next 10 years). Thus, the validity of the project evaluation or any infrastructure design may also be subject to this uncertainty (Ashley, 1980; Mahmassani, 1984).

Several other works addressed the issue of uncertainty in demand modeling and system evaluation (Ashley, 1980, see; Mahmassani, 1984; Zhao and Kockelman, 2002). Ashley (1980) proposed a modification of the conventional four-step model by incorporating uncertainties of modeling parameters (e.g. percentage growth, behavioral parameters, road speeds, etc.). At each step of the four-step model, a statistical simulation is performed to draw samples of these parameters as inputs to the calculation in that modeling step. The random outputs from each step is then propagated through the subsequent modeling steps. Zhao and Kockelman (2002) adopted a similar framework to illustrate the application of the approach with the case study of the Dallas–Forth Worth network. Essentially, this approach overcomes the deficiency in modeling calibration and

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prediction which rely only on a point estimate. The parameters involved in the models are treated as random variables resulting in stochastic vehicle/passenger flows.

This paper focuses on the last stage of traffic model which is the traffic assignment. A concept which is widely adopted to define an equilibrium point of the traffic assignment is Wardrop's user equilibrium (Wardrop, 1952). Under Wardrop's equilibrium, travelers will only travel on the cheapest route in terms of his/her generalized nonadditive travel cost which may include travel time, out of pocket expense, etc.. Thus, at the equilibrium point no traveler can change his/her route unilaterally to reduce his/her own travel cost. The traditional approach of traffic assignment requires deterministic inputs of travel demand and supply (e.g. OD matrix and speed-flow relationship) in which the model will then provide a deterministic prediction of the future traffic condition (e.g. congestion level on each link in the next 10 years).

Recently, several transport network modeling approaches have been proposed to consider uncertainties from both demand and supply sides of the system. In particular, the concept of stochastic network (see e.g. Watling, 2002; Clark and Watling, 2005; Sumalee et al., 2009) is developed to include the stochastic demand and supply characteristics into the traffic assignment model. The stochastic network framework takes the inputs of OD demand and/or road capacity as random variables. Watling (2002) proposed a framework of stochastic network model considering the stochastic travel demand which follows a stationary Poisson process and a probabilistic route choice model. Shao et al. (2006) and Sumalee and Xu (in press) proposed a similar model but used a normal distribution to represent the stochastic demand. Zhou and Chen (2008) on the other hand, adopted the log-normal stochastic demand in their model formulation. On the supply side, Lo and Tung (2003) introduced the stochastic link capacity, which is assumed to follow a uniform distribution, into the stochastic network model. Sumalee et al. (2009) introduced both demand and supply uncertainties which are assumed to follow a log-normal distribution. Chen and Zhou (2010) recently proposed an α-reliability model to represent equilibrium route choice assignment under stochastic demand and supply. In all cases, a key operational feature of the stochastic network model is the stochastic prediction of the equilibrium flows based on the stochastic inputs of OD demand and road capacity, i.e. equilibrium path and link flows will follow some statistical distributions. This, in some way, can be viewed as an attempt to consider uncertainties in the forecast of future traffic condition based on uncertain inputs.

However, all of the proposed stochastic network models mainly focus on the short-term uncertainty of the travel demand and supply. The definition of stochastic demand and supply, in fact, stems from the day-to-day variability of demand and supply in the network. For instance, it is evident that the number of travelers between each OD pair varies from day to day due to the intrinsic stochastic nature of travel demand and human behavior. Similarly, the road capacity may also change from day to day due to incidents or weather effect (Lam et al., 2008). The framework of stochastic network model emphasizes on capturing the effect of day-to-day uncertainties of travel condition on travelers in which a prediction of potential traffic states is made based upon these uncertainties (Bell and Cassir, 2002). This model can be viewed as a user-oriented model. Under this framework, attentions have been paid on developing a risk-averse based traffic equilibrium model to include uncertainty into traveler's decision (see e.g. Bell and Cassir, 2002; Connors and Sumalee, 2009).

On the other hand, from the planner's perspective the prediction of the future traffic condition should be ***robust*** against possible uncertainties of the future demand and supply. A forecast of the traffic condition (in terms of traffic volumes and minimum travel costs) is considered robust if this forecast deviates as little as possible from the set of equilibrium states resulted from the traffic assignment with different potential OD matrices and capacities. Let S_1, S_2 and S_3 be the solution set of Wardrop's user equilibrium assignment corresponding to the three different scenarios of the future demand, respectively. A forecasted traffic equilibrium $\hat{\mathbf{x}}$ is considered robust if the expected distance of $\hat{\mathbf{x}}$ to S_1, S_2 and S_3 is minimum. Thus, if $\hat{\mathbf{x}}$ is a robust prediction against the uncertain demand, any project evaluation and design based on $\hat{\mathbf{x}}$ should also be robust against anticipated uncertainty. This is probably one of the main concerns of the transport planners in using traffic model to evaluate or design a transport project. This paper, thus, proposes the robust Wardrop's equilibrium assignment model which can take the stochastic OD demand generated from the procedures suggested in Ashley (1980) or Zhao and Kockelman (2002), and provides the "robust" prediction of future traffic flows for the planning and evaluation purposes. The robust prediction proposed in this paper is different from the concepts previously proposed in the literature (which thoroughly reviewed above), and provides a more rigorous analysis of the prediction and project evaluation robustness as to be described later in the paper (see e.g. Theorem 2 and Remark 2 in Section 2.3).

Note that the notion of *robust Wardrop equilibrium* was used previously by Ordóñez and Stier-Moses (2007). However, their model focuses on the traveler's perspective in which the flow prediction is based on the Wardrop's user equilibrium principle considering the worst-case of uncertain link travel times.

Failure to include uncertainty properly in a traffic assignment model may lead to a very expensive, or even fatal decision if the anticipated random variable is not realized. This paper focuses on the robust traffic assignment model with an emphasis on the planner's perspective in obtaining a robust prediction against the uncertain future demand and capacity. The model is based on the *expected residual minimization* (ERM) introduced by Chen and Fukushima (2005) for general stochastic nonlinear complementarity problem (SNCP). The ERM model minimizes the expected value of loss at all possible scenarios due to failures in equilibrium, which also gives a small expected distance from its solution to the solution set S_ω of the NCP corresponding to each possible scenario under some conditions (see e.g. Chen et al., 2009). Note that obtaining one solution of the NCP for a given scenario ω might be difficult due to the nonadditive travel cost. In addition, the whole solution set S_ω may also in general be nonsingleton and even nonconvex.

Zhang and Chen (2008) applied the ERM model to the traffic equilibrium under uncertainty. Compared to that paper, this paper has the following main distinctive contribution. (i) The motivation of this paper is to present the ERM model for assessing the robustness of a traffic assignment to the transportation field. The aim of Zhang and Chen (2008) is to provide sufficient conditions that ensure the boundedness of the solution set of the ERM model. Traffic assignment is used as an example to show these sufficient conditions hold in some applications. (ii) The robustness of the solution of the ERM model for traffic assignment is provided theoretically in this paper, which has not been addressed before and hence constitutes an original contribution. (iii) The SPG method adopted here is promising to solve realistic larger network as illustrated in the numerical results. In Zhang and Chen (2008), no algorithm with convergence result has been proposed.

Given a ERM solution, we also propose an approach to generate the statistical distribution of the traffic state following the method adopted in Sumalee and Xu (in press). The paper compares the prediction results of the proposed model with those from other stochastic network assignment models.

This paper is organized as follows. In the following section, we begin with the basic settings for traffic network under stochastic environment. We then focus on presenting the proposed expected residual minimization (ERM) model, together with the path choice proportion approach, and briefly review two other existing models for robust assignment in the stochastic network. We also analyze theoretically some properties including the existence and robustness of the solution. Section 3 describes how to apply the smoothing projected gradient (SPG) method proposed by Zhang and Chen (2009) for solving the ERM model. In Section 4, numerical examples on two small-size networks as well as a moderate-size network are provided to demonstrate that the ERM model and the SPG method are promising in providing a robust traffic assignment under uncertainties. We then conclude the paper in the final section.

2. Model formulation

We will use the following notations in the paper. $z_+ = \max(z,0)$ for any given vector z, $|S|$ denotes the cardinality of a given finite set S, and $|\cdot|$ refers to the Euclidean norm. Given a set $\Omega \subseteq R^m$ of random vectors ω, let $\mathscr{P}\{\widehat{\Omega}\} = \mathscr{P}\{\omega \in \Omega : \omega \in \widehat{\Omega}\}$ be the probability of $\omega \in \Omega$ that belongs to the subset $\widehat{\Omega}$ of Ω. Let suppΩ be the support set of Ω. It is known that supp$\Omega = \Omega$ when $\Omega = \{\omega^1,\ldots,\omega^i,\ldots\}$ consisting of countable discrete points with $\mathscr{P}\{\omega^i\} > 0$ for all i; and supp Ω is the closure of the set $S = \{\omega \in \Omega\!: \rho(\omega) > 0\}$ when $\omega \in \Omega$ is a continuous random vector with density function $\rho(\omega)$.

2.1. Stochastic network framework

We consider a *strongly connected* network $[\mathscr{N},\mathscr{A}]$, where $\mathscr{N}$ is the set of nodes and $\mathscr{A}$ is the set of links. We denote by K the set of all possible paths with cardinality $|K|$, and W the origin–destination (OD) movements with cardinality $|W|$.

Let K^r be a set of paths connecting the rth OD, and $\Omega \subseteq R^m$ be the set of uncertain factors such as weather, accidents, etc. Let $\mathbf{Q}(\omega)$ be a demand vector with entries $Q^r(\omega)$ representing the stochastic travel demand on the rth OD, and $\mathbf{C}(\omega)$ be a capacity vector with entries $C_a(\omega)$ denoting the stochastic capacity on link a, for uncertain factor $\omega \in \Omega$. The probability distributions of random vectors $\mathbf{Q}(\omega)$ and $\mathbf{C}(\omega)$ are known.

For a realization of random vectors $\mathbf{Q}(\omega)$ and $\mathbf{C}(\omega)$, $\omega \in \Omega$, an assignment of flows to all paths is denoted by the vector $\mathbf{F}(\omega)$, whose component $F_k^r(\omega)$ denotes the flow on the kth path connecting the rth OD, while an assignment of flows to all links is represented by the vector $\mathbf{V}(\omega)$ whose component $V_a(\omega)$ denotes the stochastic flow on link a. The relation between $\mathbf{F}(\omega)$ and $\mathbf{V}(\omega)$ is presented by

$$\mathbf{V}(\omega) = \Delta\mathbf{F}(\omega),$$

where $\Delta = (\delta_{a,k})$ is the link–path incidence matrix with entries $\delta_{a,k} = 1$ if link a is on path k, and $\delta_{a,k} = 0$ otherwise. A random unknown vector $\mathbf{U}(\omega)$ with components $U_r(\omega)$ represents the stochastic minimum travel cost for the rth OD.

Let $\Gamma = (\gamma_{r,k})$ denote the OD-path incidence matrix with entries $\gamma_{r,k} = 1$ if path k connects the rth OD, and $\gamma_{r,k} = 0$ otherwise. Thus each row of Γ is a nonzero vector since the network is strongly connected, and Γ has full row rank since one path connects only one OD movement.

Given the path flow vector f, we know that the link flow vector $V = \Delta f$. The link travel time function $T(V,\omega)$ is a stochastic vector, and each of its entries $T_a(V,\omega)$ is assumed to follow a generalized Bureau of Public Roads (GBPR) function,

$$T_a(V,\omega) = t_a^0\left(1 + b_a\left(\frac{V_a}{C_a(\omega)}\right)^{n_a}\right), \tag{1}$$

where $t_a^0,\ b_a$ and n_a are given parameters. We employ the nonadditive path travel cost function $\Phi(f,\omega)$ extended from Gabriel and Bernstein (1997) by

$$\Phi(f,\omega) = \eta_1\Delta^T T(\Delta f,\omega) + \Psi(\Delta^T T(\Delta f,\omega)) + \Lambda(f,\omega). \tag{2}$$

Here $\eta_1 > 0$ is the time-based operating costs factor, Ψ is the function converting time T to money, and Λ is the perturbed financial cost function. Various factors may cause the nonadditivity of the cost function such as the route-specific toll schemes and the nonlinear value of time.

Later we always assume that the following assumption holds.

Assumption 1. The travel cost function $\Phi(f,\omega)$ for any given path flow f, and the uncertain demand $\mathbf{Q}(\omega)$, are bounded for $\omega \in \Omega$ almost everywhere (a.e.).

Let us denote

$$x = \begin{pmatrix} f \\ u \end{pmatrix}, \quad G(x,\omega) = \begin{pmatrix} \Phi(f,\omega) - \Gamma^T u \\ \Gamma f - \mathbf{Q}(\omega) \end{pmatrix}, \tag{3}$$

where $x \in R^n$ is a deterministic vector with $n = |K| + |W|$. Here $f \in R^{|K|}$ is a path flow pattern and $u \in R^{|W|}$ is a travel cost vector corresponding to f.

For a fixed $\check{\omega} \in \Omega$, the NCP formulation for Wardrop's user equilibrium, denoted by NCP $(G(x,\check{\omega}))$, seeks $x \in R^n$ such that

$$x \geqslant 0, \quad G(x,\check{\omega}) \geqslant 0, \quad x^T G(x,\check{\omega}) = 0. \tag{4}$$

At any solution, it is known that u coincides with the vector of minimum OD travel costs corresponding to the equilibrium flow pattern f. Both path flows and the minimum OD travel costs are considered as decision variables, constituting useful information of Wardrop's user equilibrium for the planner. Moreover, the NCP$(G(x,\check{\omega}))$ is equivalent to the system of nonlinear equations

$$\min(x, G(x,\check{\omega})) = 0.$$

In the case that the travel demands and road capacities are endogenously considered to be random variables, we may not expect to find a traffic path flow pattern f and a related travel cost u such that they constitute a Wardrop's user equilibrium under all realized demand and supply. That is, in general, there is no solution x satisfying the following SNCP

$$\min(x, G(x,\omega)) = 0 \quad \text{for all } \omega \in \Omega.$$

Hence it is meaningful to extend Wardrop's user equilibrium to a *user equilibrium under uncertainty* for the planner in the following two ways.

First, it should provide a deterministic equilibrium pattern for the planner, which deviates as little as possible from the set of equilibrium states under any possible scenario. Secondly, the planners can estimate the distribution of the random flow pattern, based on the vector of path choice proportions from the deterministic equilibrium pattern. From the above information, planner can then make a robust decision.

2.2. ERM model

We now explain the ERM model, which can meet the above two requirements for the traffic assignment under uncertainty mentioned earlier. The ERM model seeks a robust traffic assignment under stochastic environment by solving

$$\min_{x \in R^n_+} g(x) := E[\| \min(x, G(x,\omega))\|^2], \tag{5}$$

where $E[\cdot]$ refers to the expectation operator.

For any $\omega \in \Omega$, let S_ω denote the solution set of NCP $(G(x,\omega))$ and dist(x,S_ω) represent the Euclidean distance function from the vector x to the solution set S_ω. We define the residual function $r_\omega(\cdot)$: $R^n \to R_+$ of NCP $(G(x,\omega))$ by

$$r_\omega(x) := \| \min(x, G(x,\omega))\|.$$

In the field of NCP, $r_\omega(\cdot)$: $R^n \to R_+$ is called a residual of NCP$(G(x,\omega))$, if $r_\omega(x) = 0$ if and only if $x \in S_\omega$, i.e., dist$(x,S_\omega) = 0$. The objective function g plays the role of penalizing the vector $x \in R^n_+$ that is far away from solution set S_ω, $\omega \in \Omega$. In the next section, we will prove that by minimizing (5) the expected distance $E[\text{dist}(x_{ERM},S_\omega)]$ tends to be small. Let $x_{ERM} = (f^T_{ERM}, u^T_{ERM})^T$ be the solution of (5) and S_{ERM} be the solution set of the ERM model. The planner can choose f_{ERM} as a robust path flow pattern and u_{ERM} as a robust minimum travel cost corresponding to f_{ERM} under uncertainties.

One may question if we can also get a robust solution by solving:

$$\min_{x \in R^n_+} E[\text{dist}(x, S_\omega)]$$

instead of the ERM. We argue that although the above minimization problem meets the robust requirement, it is impractical to compute dist(x,S_ω) since S_ω in general is not a singleton nor convex if nonadditive travel cost is adopted. Another possible alternative is to obtain a solution $x_\omega \in S_\omega$ for each scenario ω and solve

$$\min_{x \in R^n_+} E[\|x - x_\omega\|^2].$$

However there exists numerous scenarios ω when a continuous distribution is used. To compute one solution for each scenario is computationally prohibitive. Furthermore, an arbitrary chosen $x_\omega \in S_\omega$ might lead to a bias prediction with respect to x_ω in the case that S_ω contains more than one element.

A stochastic traffic flow $\widetilde{\mathbf{F}}_{ERM}(\omega)$ which can be derived from f_{ERM} has not been studied by Zhang and Chen (2008). Here we outline the proportion technique to obtain a stochastic traffic flow $\widetilde{\mathbf{F}}(\omega)$ from a deterministic traffic assignment f.

We introduce the concept of vector of path choice proportions $\mathbf{p} = (p_k^r) \in R^{|K|}$ for a deterministic path flow pattern f, where K is the set of possible paths. The entry

$$p_k^r = \frac{f_k}{\sum_{j \in K^r} f_j} \tag{6}$$

is the proportion of flow on path $k \in K$ between the rth OD. It is clear that

$$\sum\nolimits_{k \in K^r} p_k^r = 1, \quad \text{for any } r\text{th OD movement}.$$

In a static setting of traffic network, if f lies in the set of Wardrop's user equilibria, the path choice proportions $p_k^r,\ k \in K^r$, determine the allocation of the given demand Q^r on the set of possible paths connecting the rth OD. The random path flow can then be expressed as

$$\widetilde{F}_k^r(\omega) = p_k^r Q^r(\omega), \tag{7}$$

and hence the demand conservation $\sum_{k \in K^r} \widetilde{F}_k^r(\omega) = Q^r(\omega)$ holds for each fixed ω. The random path flow $\widetilde{F}_k^r(\omega)$ follows the same type of distribution of $Q^r(\omega)$. We have

$$E\left[\widetilde{F}_k^r(\omega)\right] = E\left[p_k^r Q^r(\omega)\right] = p_k^r E[Q^r(\omega)],$$

and

$$Var\left[\widetilde{F}_k^r(\omega)\right] = Var\left[p_k^r Q^r(\omega)\right] = \left(p_k^r\right)^2 Var[Q^r(\omega)].$$

Note that variance of the demand flow is not conserved here since

$$\sum\nolimits_{k \in K^r} Var\left[\widetilde{F}_k^r(\omega)\right] = \sum\nolimits_{k \in K^r} \left(p_k^r\right)^2 Var[Q^r(\omega)] \neq Var[Q^r(\omega)].$$

The implication of this non-conservation is a lower path flow variance compared to the OD demand variance. This will also imply the underestimation of the variance of the link flow. Nevertheless, one can introduce the covariance term of the path flows, $cov(F_i, F_j)$, to ensure the variance conservation following Lam et al. (2008). However, the non-conservation property of variance should not affect the main result on the robustness of the ERM solution, since the model aims to find a deterministic robust prediction of flows. The variance of path/link flow is only generated through the path choice proportion concept which will be carried out after the ERM model is solved.

We can write $\widetilde{\mathbf{F}}(\omega)$ in the form of the multiplication of matrices as

$$\widetilde{\mathbf{F}}(\omega) = \mathrm{diag}(\mathbf{p}) \Gamma^T \mathbf{Q}(\omega),$$

where $\mathrm{diag}(\mathbf{p})$ is the diagonal matrix satisfying $(\mathrm{diag}(\mathbf{p}))_{ii} = \mathbf{p}_i$ for each i. Moreover,

$$E[\widetilde{\mathbf{F}}(\omega)] = \mathrm{diag}(\mathbf{p}) \Gamma^T E[\mathbf{Q}(\omega)].$$

At follows, we briefly review two relevant models on traffic assignment under stochastic environment, which are also extended from Wardrop's user equilibrium, but in different manners compared with the ERM model.

- Expected value (EV) model

The EV model assumes that to contend with random demand and supply, the travelers select paths to minimize their expected travel cost $\overline{\Phi}(f) = E[\Phi(f, \omega)]$, and the expected demand $\overline{\mathbf{Q}} = E[\mathbf{Q}(\omega)]$ is considered for the network.

Let us denote

$$\overline{G}(x) = E[G(x, \omega)] = \begin{pmatrix} \overline{\Phi}(f) - \Gamma^T u \\ \Gamma f - \overline{\mathbf{Q}} \end{pmatrix}.$$

Thus the EV model solves Wardrop's user equilibrium NCP $(\overline{G}(x))$,

$$\min(x, \overline{G}(x)) = 0. \tag{8}$$

Let us signify the solution of the EV model by $x_{EV} = \left(f_{EV}^T, u_{EV}^T\right)^T$. It is worth mentioning that f_{EV} in general is not unique, while the link flow vector v_{EV} is unique if $\overline{\Phi}$ is strictly monotone with respect to f.[1]

[1] We may also write Wardrop's user equilibrium in the variational inequality (VI) form. The link flow vector $v_{EV} = \Delta f_{EV}$ is Wardrop's user equilibrium if f_{EV} satisfies

$$\overline{\Phi}(f_{EV})^T (f - f_{EV}) \geqslant 0, \quad \forall f \in \mathscr{F} = \{f : \Gamma f = E[Q(\omega)],\ f \geqslant 0\}.$$

This formulation ignores any travel time variability and presupposes that travelers consider only the deterministic mean path costs.

- Best worst-case (BW) model

The so-called BW model borrows the idea from robust optimization of stochastic programming by assuming that each user selects path to minimize for each i the worst-case cost $\widehat{\Phi}_i(f) = \max_{\omega\in\text{supp}\Omega}\Phi_i(f,\omega)$, and the worst-case demand $\widehat{\mathbf{Q}}_i = \max_{\omega\in\text{supp}\Omega}\mathbf{Q}_i(\omega)$ will be assigned to the network. Note that the boundedness of $\Phi(f,\omega)$ and $\mathbf{Q}(\omega)$ over the domain Ω a.e. in Assumption 1 guarantees that $\widehat{\Phi}(f)$ and $\widehat{\mathbf{Q}}$ are well-defined and finite. Denote

$$\widehat{G}(x) = \begin{pmatrix} \widehat{\Phi}(f) - \Gamma^T u \\ \Gamma f - \widehat{\mathbf{Q}} \end{pmatrix}.$$

The BW model coincides to a Wardrop's user equilibrium NCP $(\widehat{G}(x))$,

$$\min(x, \widehat{G}(x)) = 0. \tag{9}$$

A special case of the BW model called a *robust Wardrop equilibrium* is proposed by Ordóñez and Stier-Moses (2007), where the path travel cost $\Phi(f,\omega) = \Delta^T T(V,\omega)$ is additive and only the link travel time

$$T_a(V,\omega) = l_a(V_a) + \omega_a\gamma_a$$

explicitly incorporates uncertainty. Here γ_a is an upper bound of possible deviation from the load-dependent nominal value $l_a(V_a)$, and the random vector $\omega = (\omega_a)$ belongs to

$$\Omega = \{\omega : \omega_a \in [0,1],\ (\Delta^T\omega)_k \leqslant \delta \text{ for any link } a \text{ and path } k\},$$

corresponding to a given uncertainty budget δ.

The aforementioned three models coincide to the same NCP for Wardrop's user equilibrium if Ω is a singleton. Otherwise, the EV/BW model is equivalent to a certain NCP for Wardrop's user equilibrium, where EV takes the average and BW adopts the worst-case of stochastic demand and travel cost to deal with uncertainties. The ERM model, on the other hand, is a non-smooth nonconvex optimization problem on R^n_+, which is not equivalent to a NCP.

From the computational point of view, the EV model, NCP $(\overline{G}(x))$, is a standard NCP with a smooth function $\overline{G}(x)$ in ordinary setting. The EV model can be solved efficiently if $\overline{G}(x)$ is monotone. However, the monotonicity of $\overline{G}(x)$ is in general not guaranteed due to the nonadditive cost. The BW model leads to computationally intensive method, because the nonsmoothness of $\widehat{\Phi}(x)$ caused by the maximum operator that functions on numerous/continuous random vector $\omega \in \Omega$. We will illustrate the SPG method for solving the ERM model in Section 3.

2.3. Existence and robustness of ERM

In this section, we establish the existence and robustness of the solution provided by the ERM model for the traffic assignment under stochastic environment. We assume the following condition on the travel cost function, which is easy to meet in practice.

Assumption 2. For fixed $\omega \in \Omega$ a.e., the travel cost function $\Phi_k(f,\omega)$ on each path k is a nonnegative nondecreasing smooth function of flow f.

In transportation field, much attention has been paid on whether the function $G(x,\check{\omega})$ in NCP$(G(x,\check{\omega}))$ for Wardrop's user equilibrium is monotone, where $\check{\omega}$ refers to a static scenario. This is often not the case when the nonadditive travel cost is adopted. However, we find that the special structure of $G(x,\check{\omega})$ allows it to be an R_0 function, and moreover, $G(x,\omega)$ to be a stochastic R_0 function where ω refers to random vector in Ω, regardless of the nonadditive travel cost function which is adopted. The existence of ERM solution relates closely to the stochastic R_0 function $G(x,\omega)$.

Let us review the concepts of R_0 function as well as stochastic R_0 function.

Definition 1 Chen, 2001. A function J: $R^n \to R^n$ is called an R_0 function on R^n_+ if for every infinite sequence $\{x^k\} \subseteq R^n_+$ satisfying

$$\lim_{k\to\infty}\|x^k\| = \infty, \quad \limsup_{k\to\infty}\|(-J(x^k))_+\| < \infty,$$

there exists $i \in \{1,2,\ldots,n\}$ such that $\limsup_{k\to\infty} \min\left(x_i^k, J_i(x^k)\right) = \infty$.

It is clear that J is an R_0 function if and only if $\|\min(x,J(x))\|^2$ is coercive, i.e., $\|\min(x,J(x))\|^2 \to \infty$ as $\|x\| \to \infty$. The coercivity implies the boundedness of level set

$$L_\tau = \{x : \|\min(x,J(x))\|^2 \leqslant \tau\} \quad \text{for any } \tau \geqslant 0.$$

Choosing the special $\tau = 0$, we obtain the boundedness of solution set L_0 of NCP$(J(x))$.

Definition 2 Zhang and Chen, 2008. A function $\mathscr{G} : R^n \times \Omega \to R^n$ is called a stochastic R_0 function on R^n_+ if for every infinite sequence $\{x^k\} \subseteq R^n_+$ satisfying

$$\lim_{k\to\infty}\|x^k\| = \infty, \quad \limsup_{k\to\infty}\|(-\mathscr{G}(x^k,\omega))_+\| < \infty \quad \text{for } \omega\in\Omega \text{ a.e.,} \tag{10}$$

there exists $i \in \{1,2,\ldots,n\}$ such that $\mathscr{P}\{\omega\in\Omega : \limsup_{k\to\infty}\min(x_i^k,\mathscr{G}_i(x^k,\omega)) = \infty\} > 0$.

Let us explain the above definition in detail. The stochastic R_0 function $\mathscr{G}(x,\omega)$, where the random vector $\omega\in\Omega$, indeed requires that for any $\{x^k\}\subseteq R^n_+$ satisfying (10), there must exist a subset $\widehat{\Omega}\subseteq\Omega$ with probability $\mathscr{P}\{\widehat{\Omega}\} > 0$ and a subsequence $\{x^{k_j}\}\subseteq\{x^k\}$ such that

$$\min\left(x_i^{k_j},\mathscr{G}_i(x^{k_j},\omega)\right)\to\infty \quad \text{for any } \omega\in\widehat{\Omega}.$$

We do not restrict the type of random vector ω. We only require the positive probability of $\widehat{\Omega}$, where for discrete random vector ω,

$$\mathscr{P}\{\widehat{\Omega}\} = \sum_{\omega^j\in\widehat{\Omega}}\mathscr{P}\{\omega^j\},$$

and for continuous random vector ω with density function $\rho(\omega)$,

$$\mathscr{P}\{\widehat{\Omega}\} = \int_{\omega\in\widehat{\Omega}}\rho(\omega)d\omega > 0.$$

Zhang and Chen (2008) investigated the special structure of the particular stochastic function

$$G(x,\omega) = \begin{pmatrix}\Phi(f,\omega)-\Gamma^T u\\ \Gamma f - \mathbf{Q}(\omega)\end{pmatrix}$$

for Wardrop's user equilibrium given in (3). It is shown in Proposition 3.1 (Zhang and Chen, 2008) that $G(x,\omega)$ is a stochastic R_0 function under Assumptions 1 and 2, where $\omega\in\Omega$ is random vector. If $\Omega=\{\breve{\omega}\}$ is a singleton, the stochastic R_0 function $G(x,\breve{\omega})$ reduces to the R_0 function, which indicates that $G(x,\breve{\omega})$ is an R_0 function for each fixed $\breve{\omega}\in\Omega$ a.e..

Roughly speaking, $G(x,\omega)$ being a stochastic R_0 function lies in that the two parts of x and $G(x,\omega)$ are closely correlated. For $\{x^k = ({f^k}^T,{u^k}^T)^T\}\subseteq R^n_+$ such that (10) holds, we can prove that there exist a subsequence $\{x^{k_j}\}\subseteq\{x^k\}$, a subset $\widehat{\Omega}\subseteq\Omega$ with positive probability, and a certain index l or i, such that for $\omega\in\widehat{\Omega}$, either

$$\min\left(f_l^{k_j},(\Phi(f^{k_j},\omega)-\Gamma^T u^{k_j})_l\right)\to\infty,$$

or

$$\min\left(u_i^{k_j},(\Gamma f^{k_j}-\mathbf{Q}(\omega))_i\right)\to\infty.$$

Together with Remark 3.1 (Zhang and Chen, 2008), we immediately have the following existence results.

Theorem 1. *Suppose that Assumptions 1 and 2 hold. Then the solution set S_{ERM} of the ERM model (5) is nonempty and bounded.*

Lemma 1. *Suppose that Assumptions 1 and 2 hold. Then for fixed $\omega\in\Omega$ a.e., the solution set S_ω is nonempty and bounded.*

Proof. For fixed $\omega\in\Omega$ a.e., S_ω is nonempty by Theorem 5.3 in Aashtiani and Magnanti (1981), since the network is strongly connected, the demand $\mathbf{Q}(\omega)$ is bounded, and $\Phi_k(f,\omega)$ on each path k is a nonnegative continuous function of flow f by Assumptions 1 and 2. Furthermore, $G(x,\omega)$ is an R_0 function indicates that the solution set S_ω is bounded. □

Our main motivation to propose the ERM model for the traffic assignment under stochastic environment lies in that it might provide the planner a robust solution under any possible realization of random variables. Below, we show theoretically the robustness of the solution.

Assumption 3. For fixed $\omega\in\Omega$ a.e., the residual $r_\omega(x) = \|\min(x,G(x,\omega))\|$ is a local error bound for NCP$(G(x,\omega))$ in R^n_+.

Recall that $r_\omega(x)$ is a local error bound for NCP$(G(x,\omega))$ if there exists some constant $\eta_\omega > 0$ and $\epsilon_\omega > 0$ such that, for each $x\in R^n_+$, with $r_\omega(x)\leqslant\epsilon_\omega$,

$$\mathrm{dist}(x,S_\omega)\leqslant\eta_\omega r_\omega(x).$$

Assumption 3 holds if for fixed $\omega\in\Omega$ a.e., $H_\omega(x) = \min(x,G(x,\omega))$ is BD-regular at all solutions in S_ω (Chen, 2001, Theorem 3.1).

The locally Lipschitzian function $H_\omega(x)$ is said to be BD-regular at x if all elements in

$$\partial_B H_\omega(x) = \{\lim\nabla H_\omega(x^k) : x^k\to x, x^k\in D_{H_\omega}\}$$

are nonsingular, where D_{H_ω} is the set of points where H_ω is F-differentiable. We give a sufficient condition to guarantee the BD-regularity of $H_\omega(x)$.

Proposition 1. *$H_\omega(x)$ is BD-regular at $x^* = (f^{*T}, u^{*T})^T \in S_\omega$, if the principal submatrix $(\nabla\Phi(f^*,\omega))_{\gamma'\gamma'}$ is nonsingular for any index subset γ' of $\mathscr{K} = \{1,2,\ldots,|K|\}$ satisfying $\gamma \subseteq \gamma' \subseteq \gamma \cup \beta$, where*

$$\beta = \{i \in \mathscr{K} : f_i^* = 0 = G_i(x^*,\omega)\},$$
$$\gamma = \{i \in \mathscr{K} : f_i^* > 0 = G_i(x^*,\omega)\}.$$

Proof. Any element $M \in \partial_B H_\omega(x^*)$ can be expressed by

$$M = \begin{pmatrix} (\nabla\Phi(f^*,\omega))_{\gamma'\mathscr{K}} & (-\Gamma^T)_{\gamma'\mathscr{W}} \\ I_{\bar{\gamma}'\mathscr{K}} & O_{\bar{\gamma}'\mathscr{W}} \\ \Gamma_{\mathscr{W}\mathscr{K}} & O_{\mathscr{W}\mathscr{W}} \end{pmatrix},$$

for some subset γ'. Here $\mathscr{W} = \{1,2,\ldots,|W|\}, \bar{\gamma}' = \mathscr{K} \setminus \gamma'$, and O refers to the zero matrix. Clearly we have

$$\det(M) = \det\left(\begin{pmatrix} (\nabla\Phi(f^*,\omega))_{\gamma'\gamma'} & (-\Gamma^T)_{\gamma'\mathscr{W}} \\ \Gamma_{\mathscr{W}\gamma'} & O_{\mathscr{W}\mathscr{W}} \end{pmatrix}\right).$$

Note that $(\nabla\Phi(f^*,\omega))_{\gamma'\gamma'}$ is invertible from the assumption, and $\Gamma_{\mathscr{W}\gamma'}$ is of full row rank, since there is at least one path $k \in \gamma'$ connecting each OD movement. Hence det $M \neq 0$ and M is nonsingular, which indicates that $H_\omega(x)$ is BD-regular at x^*. □

The BD-regular condition required in Proposition 1 may be difficult to verify in a general network case. Nevertheless, we can verify this condition under a more restricted setting.

Proposition 2. *Consider a network in which there does not exist any unused path with minimum path travel time (i.e. $\beta = \emptyset$). The BD-regular condition can be ensured if: the nonadditive path travel costs of the used path ($k \in \gamma$) is strongly dominated by the path flow on that path itself, i.e.,*

$$\frac{\partial\Phi_k(f^*,\omega)}{\partial f_k} > \sum\nolimits_{k \neq k' \in \gamma}\left|\frac{\partial\Phi_k(f^*,\omega)}{\partial f_{k'}}\right|, \quad \text{for any } k \in \gamma.$$

Proof. This condition indeed guarantees that for any $\gamma' \subseteq \gamma$, and any solution $x^* = (f^{*T}, u^{*T})^T \in S_\omega$, the submatrix $(\nabla\Phi(f^*,\omega))_{\gamma'\gamma'}$ is a strictly row diagonally dominant matrix, which is known to be nonsingular. By Proposition 1, $H_\omega(x)$ is BD-regular at any $x^* \in S_\omega$. □

Lemma 2. *Suppose that Assumptions 1–3 hold. Then for fixed $\omega \in \Omega$ a.e., and any compact set $X \subset R^n_+$, there exists a positive constant η_ω such that*

$$\text{dist}(x, S_\omega) \leqslant \eta_\omega r_\omega(x), \quad \forall x \in X.$$

Proof. Assume on the contrary that the lemma is false. Then for fixed $\omega \in \Omega$ a.e. satisfying Assumption 3, and for each integer k, there exists an $x^k \in X$ such that

$$\text{dist}(x^k, S_\omega) > k r_\omega(x^k).$$

Note that for fixed $\omega \in \Omega$ a.e., $r_\omega(x)$ is a local error bound for NCP($G(x,\omega)$). Hence for such ω, there exist an integer $\bar{k} > 0$ and a scalar $\epsilon > 0$ such that $r_\omega(x^k) > \epsilon$ for all $k > \bar{k}$. This indicates that

$$\text{dist}(x^k, S_\omega) \to \infty,$$

which is impossible since x^k contains in a compact set X, and S_ω is bounded. This completes the proof. □

Theorem 2. *Suppose that $\Omega = \{\omega^1, \omega^2, \ldots, \omega^N\}$ is a finite set with the probability of each element to be positive, and Assumptions 1–3 hold. Then for any compact set $X \subset R^n_+$, there exists a positive constant η such that*

$$E[\text{dist}(x, S_\omega)] \leqslant \eta\sqrt{g(x)}.$$

Proof. According to Lemma 2, we know that for $\omega^j \in \Omega$ and any compact set $X \subset R^n_+$, there exists a positive constant η_{ω^j} such that

$$\text{dist}(x, S_{\omega^j}) \leqslant \eta_{\omega^j} r_{\omega^j}(x), \quad \forall x \in X.$$

Let $\eta = \max_j \eta_{\omega^j}$, and we get immediately

$$E[\mathrm{dist}(x, S_\omega)] \leqslant E[\eta \| \min(x, G(x, \omega))\|] \leqslant \eta \sqrt{g(x)}. \qquad \square$$

Remark 1. From the above theorem, we have

$$E[\mathrm{dist}(x_{ERM}, S_\omega)] \leqslant \eta \sqrt{g(x_{ERM})} = \eta \sqrt{\min_{x \in R^n_+} g(x)},$$

which implies that the $E[\mathrm{dist}(x_{ERM}, S_\omega)]$ is likely to be small, and hence x_{ERM} of the ERM model can be considered as a robust solution, no matter what realization occurs.

Remark 2. The predicted traffic volume is often used in evaluating the cost and benefit of a transport project. As discussed earlier, the uncertainty of future demand may affect the reliability or robustness of the project evaluation. As explained in Theorem 2 and Remark 1, the proposed robust Wardrop's user equilibrium assignment model provides a robust prediction of future traffic volume. This robust traffic volume can then be used by transport planners to evaluate the project in a robust way. For instance, let $\zeta(V)$ denote the evaluation function of a transport project (e.g. total travel time), where V is the link flow. Suppose that the link flow V_ω of Wardrop's user equilibrium is unique under each realization $\omega \in \Omega = \{\omega^1, \omega^2, \ldots, \omega^N\}$, and ζ is a globally Lipschitz continuous function on $R_+^{|\mathscr{A}|}$, i.e., there exists a constant $L > 0$ such that

$$\|\zeta(V) - \zeta(V')\| \leqslant L\|V - V'\| \quad \text{for all} \quad V, \ V' \in R_+^{|\mathscr{A}|}.$$

Denote $\hat{x}_\omega = (\hat{f}_\omega, \hat{u}_\omega)^T$ to be the solution in S_ω satisfying

$$\mathrm{dist}(x_{ERM}, S_\omega) = \|x_{ERM} - \hat{x}_\omega\|.$$

We can easily derive that

$$\begin{aligned} E[\|\zeta(V_{ERM}) - \zeta(V_\omega)\|] &\leqslant E[L\|V_{ERM} - V_\omega\|] = E[L\|\Delta f_{ERM} - \Delta\hat{f}_\omega\|] \leqslant L\|\Delta\|E[\|f_{ERM} - \hat{f}_\omega\|] \leqslant L\|\Delta\|E[\|x_{ERM} - \hat{x}_\omega\|] \\ &= L\|\Delta\|E[\mathrm{dist}(x_{ERM}, S_\omega)] \leqslant L\|\Delta\|\eta\sqrt{g(x_{ERM})} \leqslant L\|\Delta\|\eta\sqrt{\min_{x \in R^n_+} g(x)}. \end{aligned}$$

This indicates that the evaluation of project adopting the outputs from the ERM model will also inherit the "robust" property.

Remark 3. In this section, only the existence and robustness properties of the ERM model are presented. We do not have the similar equivalence property as the NCP and VI for static traffic equilibrium, since the optimal objective value of the ERM model is in general nonzero. Moreover, the uniqueness property is difficult to provide due to the nonconvexity of the objective function.

3. Solution algorithm of the ERM model

The ERM model is in general a nonsmooth nonconvex minimization problem on R^n_+, with the objective function

$$g(x) = E[\|\min(x, G(x, \omega))\|^2] = E[H_\omega(x)^T H_\omega(x)].$$

The ERM model can be solved by the smoothing projected gradient (SPG) method (Zhang and Chen, 2009), which is extended from the classical projected gradient (PG) method.

Let $\tilde{g} : R^n \times R_+$ be a smoothing function of g, that is, $\tilde{g}(\cdot, \mu)$ is continuously differentiable in R^n for any $\mu > 0$, and for any $x \in R^n$,

$$\lim_{z \to x, \mu \downarrow 0} \tilde{g}(z, \mu) = g(x), \tag{11}$$

and $\{\lim_{z \to x, \mu \downarrow 0} \nabla_x \tilde{g}(z, \mu)\}$ is nonempty and bounded. Here $\nabla_x \tilde{g}(z, \mu)$ is the gradient of $\tilde{g}(\cdot, \mu)$ at point z. The SPG method is described in detail as follows.

Algorithm 1 (*SPG algorithm by Zhang and Chen (2009)*). Let $\hat{\varrho}, \varrho_1$ and ϱ_3 be positive constants, where $\varrho_1 \ll \varrho_3$. Let ϱ_2, σ, σ_1 and σ_2 be constants in (0,1), where $\sigma_1 \leqslant \sigma_2$. Choose $x^0 \in R^n_+$ and $\mu_0 > 0$. For $k \geqslant 0$:

1. If $\|[x^k - \nabla_x \tilde{g}(x^k, \mu_k)]_+ - x^k\| = 0$, let $x^{k+1} = x^k$ and go to step 3. Otherwise, go to step 2.
2. (PG method) Let $y^{0,k} = x^k$. For $j \geqslant 0$:

$$y^{j,k}(\alpha) = [y^{j,k} - \alpha \nabla_x \tilde{g}(y^{j,k}, \mu_k)]_+ \tag{12}$$

and $y^{j+1,k} = y^{j,k}(\alpha_{j,k})$ where $\alpha_{j,k}$ is chosen so that,

$$\tilde{g}(y^{j+1,k},\mu_k) \leqslant \tilde{g}(y^{j,k},\mu_k) + \sigma_1(\nabla_x\tilde{g}(y^{j,k},\mu_k), y^{j+1,k} - y^{j,k})$$

and

$$\varrho_3 \geqslant \alpha_{j,k} \geqslant \varrho_1, \quad \text{or} \quad \alpha_{j,k} \geqslant \varrho_2\bar{\alpha}_{j,k} > 0,$$

such that $\bar{y}^{j+1,k} = y^{j,k}(\bar{\alpha}_{j,k})$ satisfies

$$\tilde{g}(\bar{y}^{j+1,k},\mu_k) > \tilde{g}(y^{j,k},\mu_k) + \sigma_2(\nabla_x\tilde{g}(y^{j,k},\mu_k), \bar{y}^{j+1,k} - y^{j,k}). \tag{13}$$

If $\frac{\|y^{j+1,k}-y^{j,k}\|}{\alpha_{j,k}} < \hat{\varrho}\mu_k$, set $x^{k+1} = y^{j+1,k}$ and go to step 3.

3. Choose $\mu_{k+1} \leqslant \sigma\mu_k$.

Each iteration in the SPG algorithm only requires the calculation of the function value of the smoothing function $\tilde{g}$ and its gradient, without any computationally intensive operations. Note that the nonsmoothness of g comes essentially from the "min" operator. Hence its smoothing function $\tilde{g}$ can be constructed easily (see e.g. Chen and Ye (1999) for reference). We provide a concrete smoothing function $\tilde{g}$ along with its gradient as an example.

Let $\rho(s)$ be the uniform density function

$$\rho(s) = \begin{cases} 1 & \text{if } -\frac{1}{2} \leqslant s \leqslant \frac{1}{2}, \\ 0 & \text{otherwise}. \end{cases}$$

The Chen–Mangasarian family of smoothing approximation for the "min" operator

$$\min(a,b) = a - \max(0, a-b),$$

can be computed by

$$\begin{aligned} \phi(a,b,\mu) &= a - \int_{-\infty}^{\infty} \max(0, a-b-\mu s)\rho(s)ds \\ &= \begin{cases} b & \text{if } a-b \geqslant \frac{\mu}{2}, \\ a - \frac{1}{2\mu}\left(a-b+\frac{\mu}{2}\right)^2 & \text{if } -\frac{\mu}{2} < a-b < \frac{\mu}{2}, \\ a & \text{if } a-b \leqslant -\frac{\mu}{2}. \end{cases} \end{aligned}$$

The smoothing function $\tilde{g}$ can then be defined by

$$\tilde{g}(x,\mu) = E[\widetilde{H}_\omega(x,\mu)^T\widetilde{H}_\omega(x,\mu)], \tag{14}$$

where $\widetilde{H}_\omega : R^n \times R_{++} \to R^n$ is given by

$$\widetilde{H}_\omega(x,\mu) = \begin{pmatrix} \phi(x_1, G_1(x,\omega), \mu) \\ \vdots \\ \phi(x_n, G_n(x,\omega), \mu) \end{pmatrix}.$$

The gradient of $\tilde{g}$ can be computed by

$$\nabla_x\tilde{g}(x,\mu) = 2E[\nabla_x\widetilde{H}_\omega(x,\mu)\widetilde{H}_\omega(x,\mu)],$$

where for each $i = 1,2,\ldots,n$, the ith row of $\nabla_x\widetilde{H}_\omega(x,\mu) \in R^{n\times n}$ is defined by

$$\begin{aligned} (\nabla_x\widetilde{H}_\omega(x,\mu)^T)_{i.} &= I_{i.} - (I_{i.} - \nabla_x G(x,\omega))_{i.}\int_{-\infty}^{\frac{x_i - G_i(x,\omega)}{\mu}} \rho(s)ds, \\ &= \begin{cases} (\nabla_x G(x,\omega))_{i.} & \text{if } x_i - G_i(x,\omega) \geqslant \frac{\mu}{2}, \\ I_{i.} - (I_{i.} - (\nabla_x G(x,\omega))_{i.})\left(\frac{x_i - G_i(x)}{\mu} + \frac{1}{2}\right) & \text{if } -\frac{\mu}{2} < x_i - G_i(x,\omega) < \frac{\mu}{2}, \\ I_{i.} & \text{if } x_i - G_i(x,\omega) \leqslant -\frac{\mu}{2}. \end{cases} \end{aligned}$$

The SPG method is very easy to implement and attractive for large-scale problems. It is shown by Zhang and Chen (2009) that the SPG method is well-defined and globally convergent to a Clarke stationary point associated with $\tilde{g}$ under mild assumptions. It is worth pointing out that we may not obtain a global optimal point, or even a local minimizer. Nevertheless, if we choose the initial point of the SPG method good enough, e.g., to be a solution of the EV model, we may get a more robust solution than that of the EV model.

4. Numerical results

We present our computational results in this section. The purpose of the numerical experiments is to illustrate the characteristics of the ERM model for the user equilibrium assignment under uncertainties in both demand and supply sides, compared with the Wardrop's user equilibrium model, the BW model, and the EV model.

When the path travel cost is additive, the method of successive averages (MSA) is often applied for solving the NCPs corresponding to Wardrop's user equilibrium. For the case of nonadditive path travel cost, the semismooth Newton method (e.g. Luca et al., 1996) is further adopted for solving the NCPs in order to get a solution x_{BW} of the BW model, x_{EV} of the EV model, and a solution x_ω of Wardrop's user equilibrium under each scenario.

The convergence of the semismooth Newton method for a fixed NCP, e.g. NCP($G(x,\hat{\omega})$), can be evaluated by the residual

$$r_{2,\hat{\omega}}(x) = 0.5\left(\sum_{i=1}^{n}\sqrt{x_i^2 + G_i(x,\hat{\omega})^2} - x_i - G_i(x,\hat{\omega})\right)^2,$$

which should be close to zero to indicate convergence. Of course, other residuals such as $r_{\hat{\omega}}(x)$ can also play the role of indicator for convergence. The semismooth Newton method uses $r_{2,\hat{\omega}}(x)$ as the objective function to minimize due to its smoothness. Thus $r_{2,\hat{\omega}}(x)$ is computed already in each iteration and hence adopting it as indicator is convenient. Using x_{EV} as an initial point, we employ the SPG method in Algorithm 1 to obtain a solution x_{ERM} of the ERM model, with parameters

$$\mu_0 = 1, \quad \varrho_1 = \frac{1}{2}, \quad \varrho_2 = \frac{1}{4}, \quad \varrho_3 = 10^3, \quad \sigma = \frac{1}{2}, \quad \sigma_1 = \sigma_2 = 10^{-3} \quad \text{or} \quad 10^{-2}.$$

The parameter $\hat{\rho}_3$ varies with different scales of the networks. We use $\hat{\rho}_3 = 10^3$ for Examples 1 and 2 of small networks, and $\hat{\rho}_3 = 10^8$ for Example 3 where a larger network is considered. We stop the SPG algorithm and set $x_{ERM} = x^k$ if $\|x^k - x^{k-1}\| \leqslant 10^{-12}$ or the total number of iterations exceeds a given maximum iteration which varies for different problems.

4.1. Example 1: a simple 5-link network

To demonstrate the properties of the ERM model, we first use a small tractable 5-link network shown in Fig. 1, which is subject to three discrete one-dimensional random demand vectors.

There are two two-way roads: a mountain road (L_1,R_1), a sea-side road (L_2,R_2), and one one-way ordinary road L_3 connecting the two cities West and East. The links L_1, L_2 and L_3 direct from West to East, and the links R_1 and R_2 are the returns. Let $\Omega = \{\omega^1, \omega^2, \omega^3\}$ with $\omega^1 = 0$, $\omega^2 = 1$, $\omega^3 = 2$ represent the set of different future scenarios, with probabilities $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{4}$, $p_3 = \frac{1}{4}$, respectively.

The uncertainties in demand and supply sides are mainly due to different demand growth and supply change. The demand $\mathbf{Q}(\omega^1) = (260, 170)^T$ and $\mathbf{Q}(\omega^2) = \mathbf{Q}(\omega^3) = (160, 70)^T$, with the 1st and 2nd components for the demand connecting the OD pair – West to East, and the return, respectively. In practice, the simulation approach adopted in Ashley (1980) or Zhao and Kockelman (2002) for trip generation, trip distribution, and modal split can be utilized to generate the distribution of possible future demand. Although the stochastic demand generated by this approach follows a continuous statistical distribution, one can always define a number of discrete scenarios of demand pattern from this continuous distribution. It is noteworthy that our proposed robust traffic assignment can handle both discrete and continuous stochastic demand cases.

This example employs the GBPR function (1) with the parameters $n_a \equiv 1$ and $b_a = \frac{1}{2t_a^0}$, where t_a^0 and $C_a(\omega)^{-1}$ are listed in Table 1. We adopt the asymmetric path travel cost function (2) with parameter $\eta = 1$, $\Psi \equiv \mathbf{0}$, and

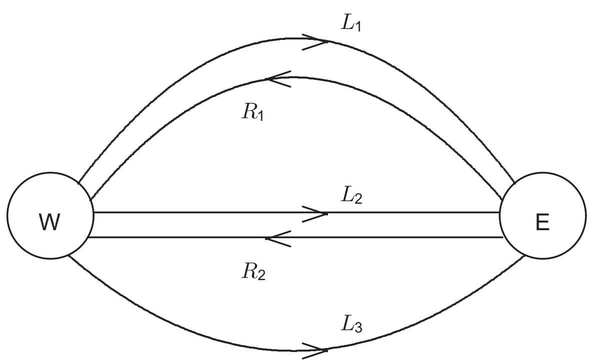


Fig. 1. 5-link network.

Table 1
Input network data.

Parameter	Link no.				
	1 (L_1)	2 (L_2)	3 (L_3)	4 (R_1)	5 (R_2)
t_a^0	1000	950	1500	1000	1300
$C_a(\omega)^{-1}$	$10 + 40\alpha(\omega)$	$15 + 60\beta(\omega)$	10	$20 + 80\alpha(\omega)$	$25 + 100\beta(\omega)$

$$\Lambda(f,\omega) = \begin{pmatrix} 0 & 0 & 0 & 20\alpha(\omega) & 0 \\ 0 & 0 & 0 & 0 & 20\beta(\omega) \\ 0 & 0 & 0 & 0 & 0 \\ 8\alpha(\omega) & 0 & 0 & 0 & 0 \\ 0 & 4\beta(\omega) & 0 & 0 & 0 \end{pmatrix} f,$$

where $\alpha(\omega) = \frac{1}{2}\omega(\omega-1)$ and $\beta(\omega) = \omega(2-\omega)$. Here the asymmetric term Λ comes from the interaction of the correlated two-way roads.

Various deterministic assignment patterns from Wardrop's user equilibrium, the BW, the EV, and the ERM models are listed in Table 2. The vector of path choice proportions $\mathbf{p} = (p_k^r)$ is important in the traffic assignment under uncertainty for the planner, which reflects the preference of choosing the possible paths, and plays essential role of generating stochastic traffic flow pattern $\widetilde{\mathbf{F}}(\omega)$. We present vectors of path choice proportions in Table 3. We list in Table 4 the values $g(x)$, $E[\|x - x_\omega\|]$, $E[\|V - V_\omega\|]$, $E[\|u - u_\omega\|]$, and $E[\|\widetilde{\mathbf{V}}(\omega) - V_\omega\|]$, which indicate the distance of a traffic assignment pattern under uncertainty to the Wardrop's user equilibrium under each realization. Here the notations $x_\omega = (f_\omega^T, u_\omega^T)^T$, V refers to the link flow, and $\widetilde{\mathbf{V}}(\omega) = \Delta\widetilde{\mathbf{F}}(\omega)$ represents the stochastic link flow. In this example, it is easy to see that $V_\omega = f_\omega$ and $\widetilde{\mathbf{V}}(\omega) = \widetilde{\mathbf{F}}(\omega)$.

From Tables 2 and 3, we can see that the traffic assignment patterns from the BW, the EV, and the ERM models are quite different. The major difference occurs on link L_3, which is the link with the highest free-flow travel time but without capacity variation. Among the three models, the BW model allocates the demand to L_3 most, and predicts the highest OD travel cost whereas the ERM model allocates the least flow to L_3 and predicts the lowest OD travel cost.

It is easy to check that the link travel time function $T(f,\omega)$ is strictly monotone for each scenario, which implies that $x_\omega = (f_\omega^T, u_\omega^T)^T = \left(V_\omega^T, u_\omega^T\right)^T$ is unique for each scenario, according to Theorem 6.2 of Aashtiani and Magnanti (1981). That is, $S_{\omega^j} = \{x_{\omega^j}\}$ is a singleton for $j = 1,2,3$ and

$$E[\|x - x_\omega\|] = E[\mathrm{dist}(x, S_\omega)].$$

Moreover, $\nabla\Phi(f,\omega)$ is strictly row diagonally dominant for each ω and hence $H_\omega(x)$ is BD-regular. Thus Assumption 3 holds, and there must exist a positive scalar η such that

$$E[\mathrm{dist}(x, S_\omega)] \leqslant \eta\sqrt{\min_{x\in R_+^n} g(x)}.$$

We find from Table 4 that the SPG method reduces greatly the function value $g(x)$ at x_{ERM} from the initial point x_{EV}. Table 4 shows that x_{ERM} has much smaller expected distance to Wardrop's user equilibrium assignments for all possible scenarios than x_{EV} and x_{BW}. We also notice that from this example the stochastic link flow $\widetilde{\mathbf{V}}(\omega)$ obtained from the deterministic flow pattern x_{ERM} is closer to V_ω under realizations than that from x_{EV} as well as x_{BW}.

4.2. Example 2: the Nguyen and Dupuis network

We also illustrate and compare the models by the Nguyen and Dupuis network shown in Fig. 2, which contains 13 nodes, 19 directed links, and 4 OD movements $1 \to 2$, $1 \to 3$, $4 \to 2$, and $4 \to 3$. The free-flow travel time t_a^0, and the mean of link capacity $E[C_a(\omega)]$ of the network are the same as those used by Yin et al. (2009).

Table 2
Various deterministic traffic assignment patterns.

Path flow (link sequence)	Traffic assignment patterns					
Travel cost (OD)	x_{ω^1}	x_{ω^2}	x_{ω^3}	x_{BW}	x_{EV}	x_{ERM}
f_1 (L_1)	132.5	122.4	15.1	0.0	61.9	107.5
f_2 (L_2)	95.0	15.2	102.0	9.4	52.5	78.7
f_3 (L_3)	32.5	22.4	42.9	250.6	95.6	8.8
f_4 (R_1)	107.8	65.3	16.9	97.4	71.7	73.2
f_5 (R_2)	62.2	4.7	53.1	72.6	48.3	34.7
u_1 ($W \to E$)	1662.5	1612.1	1714.7	2753.1	1978.1	1540.9
u_2 ($E \to W$)	2077.8	1653.2	1964.2	5872.2	2558.8	1733.3

Table 3
Various vectors of path choice proportions.

Path flow / Travel cost	Traffic assignment patterns					
	x_{ω^1}	x_{ω^2}	x_{ω^3}	x_{BW}	x_{EV}	x_{ERM}
p_1^1	0.5096	0.7651	0.0943	0.0000	0.2949	0.5515
p_2^1	0.3654	0.0947	0.6373	0.0361	0.2498	0.4035
p_3^1	0.1250	0.1401	0.2684	0.9639	0.4553	0.0450
p_4^2	0.6340	0.9331	0.2410	0.5732	0.5979	0.6786
p_5^2	0.3660	0.0669	0.7590	0.4268	0.4021	0.3214

Table 4
Robust criteria of various traffic assignment patterns.

Various criteria	Traffic assignment patterns					
	x_{ω^1}	x_{ω^2}	x_{ω^3}	x_{BW}	x_{EV}	x_{ERM}
g	2.05e4	3.06e5	7.94e5	2.60e7	1.50e6	1.15e4
$E[\Vert x - x_\omega \Vert]$	158.94	311.13	188.00	4086.10	703.27	295.43
$E[\Vert V - V_\omega \Vert]$	64.36	92.93	113.51	262.83	108.72	76.89
$E[\Vert u - u_\omega \Vert]$	138.15	295.65	144.39	4077.61	694.42	281.69
$E[\Vert \widetilde{\mathbf{V}}(\omega) - V_\omega \Vert]$	39.48	99.18	120.98	218.56	99.39	55.42

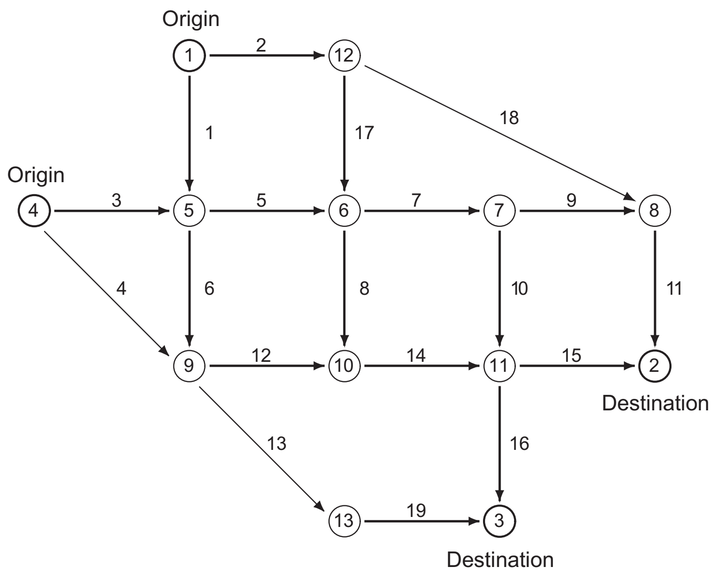


Fig. 2. Nguyen and Dupuis network.

Suppose the planner would like to forecast the robust traffic equilibrium pattern in the next ten years. According to the prediction of economic tendency, the demand vector (with the components following the order of OD movements $1 \to 2$, $1 \to 3$, $4 \to 2$, and $4 \to 3$) have three possible scenarios

$$\mathbf{Q}^1 = [800\ 800\ 1200\ 1200]^T, \quad \mathbf{Q}^2 = [400\ 1600\ 600\ 400]^T, \quad \mathbf{Q}^3 = [200\ 400\ 300\ 100]^T,$$

with probabilities $p_1 = \frac{1}{4},\ p_2 = \frac{1}{4}$, and $p_3 = \frac{1}{2}$. Here $\mathbf{Q}^1$ and $\mathbf{Q}^2$ correspond to the optimistic predictions that the economy will flourish and a new port will be built at either destination 2 or 3, respectively. The demand $\mathbf{Q}^3$ corresponds to the pessimistic estimation of future economy.

The link capacity $C_a(\omega)$ follows a log-normal distribution $C_a(\omega) \sim LN(\mu_{c,a}, \sigma_{c,a})$. The probability density function of the log-normal distribution is

$$\Pr(C_a(\omega)|\mu_{c,a}, \sigma_{c,a}) = \frac{1}{C_a(\omega)\sigma_{c,a}\sqrt{2\pi}} \exp\left(\frac{-(\ln C_a(\omega) - \mu_{c,a})^2}{2\sigma_{c,a}^2}\right).$$

From the mean $E[C_a(\omega)]$ and the coefficient of variation $CV[C_a(\omega)]$ in Table 5, we can obtain the parameters $\mu_{c,a}$ and $\sigma_{c,a}$ by

Table 5
Link cost parameters.

Link	Free-flow travel time t_a^0	Link capacity, C_a			
		Mean	Coefficient of variation		
			Case 1	Case 2	Case 3
1	7.0	800	0.10	0.30	0.30
2	9.0	400	0.10	0.10	0.10
3	9.0	200	0.10	0.10	0.10
4	12.0	800	0.10	0.30	0.30
5	3.0	350	0.10	0.10	0.10
6	9.0	400	0.10	0.10	0.10
7	5.0	800	0.10	0.10	0.10
8	13.0	250	0.10	0.10	0.10
9	5.0	250	0.10	0.10	0.10
10	9.0	300	0.10	0.10	0.10
11	9.0	550	0.10	0.30	0.30
12	10.0	550	0.10	0.30	0.30
13	9.0	600	0.10	0.10	0.10
14	6.0	700	0.10	0.30	0.30
15	9.0	500	0.10	0.30	0.30
16	8.0	300	0.10	0.10	0.10
17	7.0	200	0.10	0.10	0.10
18	14.0	400	0.10	0.10	0.10
19	11.0	600	0.10	0.10	0.10

$$\mu_{c,a} = \ln(E[C_a(\omega)]) - \frac{1}{2}\ln(1 + (CV[C_a(\omega)])^2) \quad \text{and} \quad \sigma_{c,a} = \sqrt{\ln(1 + (CV[C_a(\omega)])^2)}.$$

We choose for this example the GBPR link travel time function

$$T_a(V,\omega) = t_a^0\left(1 + 0.15\left(\frac{V_a}{C_a(\omega)}\right)^4\right),$$

and the nonadditive path travel cost function

$$\Phi_k(f,\omega) = \sum \delta_{a,k} T_a(\Delta f,\omega) + \left(\sum \delta_{a,k} T_a(\Delta f,\omega)\right)^2 + \Lambda_k(f,\omega).$$

We consider three cases of stochastic environment for the Nguyen and Dupuis network. The coefficients of variation for $C_a(\omega)$ are listed in Table 5 for the three cases. In Cases 1 and 2, the path-specific cost $\Lambda_k(f,\omega) = 0$ for all paths, while $\Lambda_k(f,\omega) = 200$ for paths $k = 1, 9, 14, 20$ and zero for other paths in Case 3.

For the expectation operators appeared in EV and ERM, the Monte-Carlo method is employed to randomly generate $N = 1000$ samples of $(\mathbf{Q}(\omega^i), \mathbf{C}(\omega^i))$ for $i = 1,2,\ldots,N$, where $\mathbf{Q}(\omega^i)$ are chosen from $\mathbf{Q}^1$, $\mathbf{Q}^2$, $\mathbf{Q}^3$ with the given probability, and each entry of $\mathbf{C}(\omega^i)$ follows the respective log-normal distribution independently. $G^N(x)$ and $g^N(x)$ are used to approximate $E[G(x,\omega)]$ and $g(x)$ in the EV and ERM models respectively by

$$G^N(x) := \frac{1}{N}\sum_{i=1}^{N} G(x,\omega^i), \quad g^N(x) := \frac{1}{N}\sum_{i=1}^{N} \|\min(x, G(x,\omega^i))\|^2.$$

We record the computational results for traffic assignment patterns x_{EV} and x_{ERM} in Table 6, and link flow patterns $v_{EV} = \Delta f_{EV}$ and $v_{ERM} = \Delta f_{ERM}$ in Table 7. Furthermore, we list some robust indicators in Table 8. We do not provide x_{BW} since it is in general difficult to solve, as we mentioned at the end of Section 2.2.

From Table 6 both traffic flow patterns f_{EV} and f_{ERM} depend heavily on paths $k = 1, 2, 9, 10, 14, 15, 20$. The difference lies in that in all of the three cases f_{ERM} tends to employ more paths than f_{EV}, which alleviates the burden on the above heavily used paths. Moreover, the tendency is strengthened as the variation of link capacity increases in Case 2 and the path-specific cost is involved in Case 3. The ERM model suggests lower travel cost than the EV model. In Table 7, we find that the link flow pattern V_{EV} never uses links 8 and 17 in the three cases, which may due to the relatively high free-flow travel time and limited mean link capacity of the two links. In contrast, the link flow pattern V_{ERM} pays more attention on the two links as the variation of capacity increases and the path-specific cost is added in Case 2 and Case 3. We try to explain the above phenomenon as follows.

The EV model only considers the expected travel cost in deciding which path (eventually which link) to be used. The ERM model considers the weighted distance from the solution sets, that is, ERM considers both probability and magnitude (of flow and minimum cost). Under the ERM, the realization with a low probability may be highly influential on the solution due to the high value of demand and/or minimum travel cost. When the capacity variation is high (i.e. Cases 2 and 3), the realization with a very low capacity and low probability may have the same level of impact on the solution of ERM compared to the realization with an average capacity but higher chance. This is due to the fact that ERM considers both the probability of

Table 6
Traffic assignment patterns of EV and ERM.

Path flow (link sequence)	x_{EV}			x_{ERM}		
Travel cost (OD)	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
$f_1(2-18-11)$	365.9	382.1	364.5	276.1	212.9	203.3
$f_2(1-5-7-9-11)$	34.1	17.9	35.5	34.1	29.0	57.3
$f_3(1-5-7-10-15)$	0	0	0	0	0	31.8
$f_4(1-5-8-14-15)$	0	0	0	0	0	21.1
$f_5(1-6-12-14-15)$	0	0	0	0	0	0
$f_6(2-17-7-9-11)$	0	0	0	30.2	42.0	23.7
$f_7(2-17-7-10-15)$	0	0	0	5.5	10.5	9.0
$f_8(2-17-8-14-15)$	0	0	0	0	2.4	0
$f_9(4-12-14-15)$	550.2	531.3	529.6	483.2	366.0	214.1
$f_{10}(3-5-7-9-11)$	249.8	268.7	270.4	236.5	216.4	139.1
$f_{11}(3-5-7-10-15)$	0	0	0	0	22.8	128.4
$f_{12}(3-5-8-14-15)$	0	0	0	0	14.5	117.7
$f_{13}(3-6-12-14-15)$	0	0	0	0	0	65.3
$f_{14}(1-6-13-19)$	310.5	311.4	289.9	287.3	273.8	316.9
$f_{15}(1-5-7-10-16)$	289.5	288.6	310.1	262.0	226.2	85.0
$f_{16}(1-5-8-14-16)$	0	0	0	0	0.3	56.1
$f_{17}(1-6-12-14-16)$	0	0	0	0	0	3.7
$f_{18}(2-17-7-10-16)$	0	0	0	14.5	32.7	52.3
$f_{19}(2-17-8-14-16)$	0	0	0	7.7	24.7	33.2
$f_{20}(4-13-19)$	200.0	200.0	200.0	162.9	145.3	114.9
$f_{21}(4-12-14-16)$	0	0	0	0	0	0
$f_{22}(3-6-13-19)$	0	0	0	0	0	0
$f_{23}(3-5-7-10-16)$	0	0	0	0	0	41.6
$f_{24}(3-5-8-14-16)$	0	0	0	0	0	30.9
$f_{25}(3-6-12-14-16)$	0	0	0	0	0	0
$u_1(\text{OD1} \to 2)$	1449.6	1892.3	2053.6	1249.1	1169.4	1347.4
$u_2(\text{OD1} \to 3)$	1870.6	2435.8	2620.8	1656.3	1593.4	1821.5
$u_3(\text{OD4} \to 2)$	1540.6	1574.5	1750.0	1451.4	1433.9	1737.8
$u_4(\text{OD4} \to 3)$	1279.2	1430.6	1580.6	1182.6	1140.8	1322.3

Table 7
Link flow patterns of EV and ERM.

Link	v_{EV}			v_{ERM}		
	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
1	634.1	617.9	635.5	583.4	529.3	571.8
2	365.9	382.1	364.5	334.1	325.1	321.4
3	249.8	268.7	270.4	236.5	253.6	523.0
4	750.2	731.3	729.6	646.2	511.3	329.0
5	573.4	575.2	616.1	532.6	509.1	709.0
6	310.5	311.4	289.9	287.3	273.8	385.9
7	573.4	575.2	616.1	582.9	579.5	568.1
8	0	0	0	7.7	41.8	259.0
9	284.0	286.6	305.9	300.8	287.4	220.1
10	289.5	288.6	310.1	282.1	292.1	348.0
11	649.8	668.7	670.4	576.9	500.2	423.4
12	550.2	531.3	529.6	483.2	366.0	283.1
13	510.5	511.4	489.9	450.2	419.1	431.8
14	550.2	531.3	529.6	490.9	407.9	542.1
15	550.2	531.3	529.6	488.7	416.1	587.4
16	289.5	288.6	310.1	284.3	283.9	302.7
17	0	0	0	57.9	112.3	118.2
18	365.9	382.1	364.5	276.1	212.9	203.3
19	510.5	511.4	489.9	450.2	419.1	431.8

the realization and the magnitude of the violation of the equilibrium condition of the ERM solution. In this case, the low capacity scenario may yield a high travel cost in which the violation of the UE condition under this realization may have a significant influence on the objective function of the ERM (despite its low probability). Thus, it is natural to observe the result in which the ERM spread flows on a larger set of paths.

Now we turn our attention to the data shown in Table 8. It is easy to see that the SPG method succeeds in decreasing the objective function $g(x)$ of the ERM model at x_{ERM} from the initial point x_{EV}. Before analyzing the expected distance, we obtain by direct computation that the Jacobian matrix of the path cost function $\Phi(f,\omega)$ is

Table 8
Robust indicators of various traffic assignment patterns for Example 2.

Various criteria	x_{EV}			x_{ERM}		
	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
g	5.39e5	1.54e6	1.73e6	5.04e5	5.75e5	5.87e5
$E[\|V - V_\omega\|]$	1.19e3	1.19e3	1.19e3	1.18e3	1.17e3	1.31e3
$E[\|u - u_\omega\|]$	5.04e3	1.14e4	1.14e4	5.00e3	1.13e4	1.13e4
$E[\|\widetilde{\mathbf{V}}(\omega) - V_\omega\|]$	468.0	489.6	555.5	468.4	515.3	906.3

$$\nabla \Phi(f,\omega) = (I + 2\text{diag}(\Delta^T T(V,\omega)))\Delta^T \nabla T(V,\omega)\Delta,$$

where $\text{diag}(\Delta^T T(V,\omega))$ is the diagonal matrix with ith diagonal element of $(\Delta^T T(V,\omega))_i$ for $i = 1,2,\ldots,|K|$, and $\nabla T(V,\omega)$ is the Jacobian matrix of the link cost function $T(V,\omega)$ with respect to the link flow vector V. Clearly $\nabla\Phi(f,\omega)$ is not a positive semi-definite matrix and hence $\Phi(f,\omega)$ is not monotone. Without monotonicity, S_ω may not be a singleton, and the algorithm for solving NCP($G(x,\omega)$) may become inefficient for some realizations ω. The inefficiency of the semismooth Newton method is also verified in our computational experience.

Fortunately, the link flow patterns V_ω and the minimum travel cost u_ω are unique in the first two cases, since NCP($G(x,\omega)$) is equivalent to a monotone NCP with additive path travel cost function by Theorem 3.1 of Agdeppa et al. (2007). On the other hand, the uniqueness of V_ω in Case 3 cannot be guaranteed because of the nonzero path-specific cost $\Lambda(f,\omega)$. The expected distances $E[\|V - V_\omega\|]$ and $E[\|u - u_\omega\|]$ in the first two cases indicate that (V_{ERM}, u_{ERM}) is closer to (V_ω, u_ω) under different scenarios compared to (V_{EV}, u_{EV}). The ERM model thus provides more robust traffic assignment patterns under stochastic environment in these two cases.

4.3. Example 3: the Sioux Falls network

We demonstrate that our ERM model and SPG algorithm are promising for planning applications in practice by testing on a larger network in this subsection. The well-known Sioux Falls network as shown in Fig. 3 is adopted, which consists of 24 nodes, 76 links, 528 OD movements. The total of 1179 paths are pre-generated as possible travel routes between different OD pairs. For this example, we adopt the GBPR link travel cost function

$$T_a(V,\omega) = t_a^0\left(1 + 0.15\left(\frac{V_a}{C_a}\right)^2\right),$$

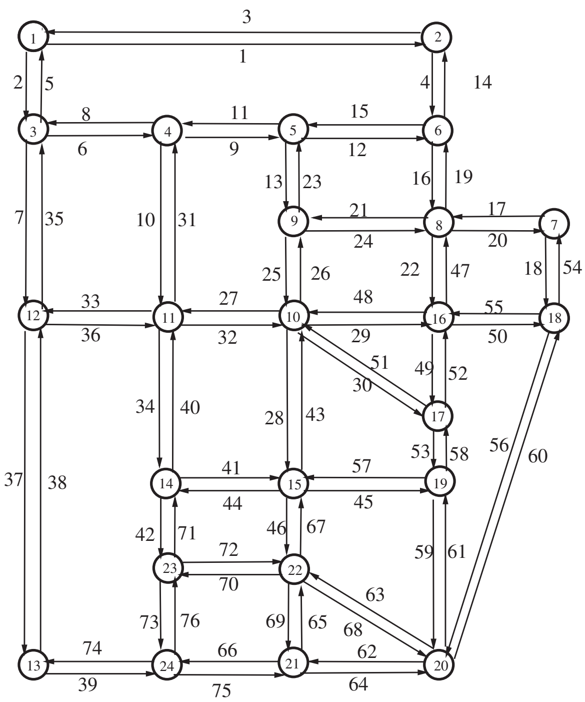


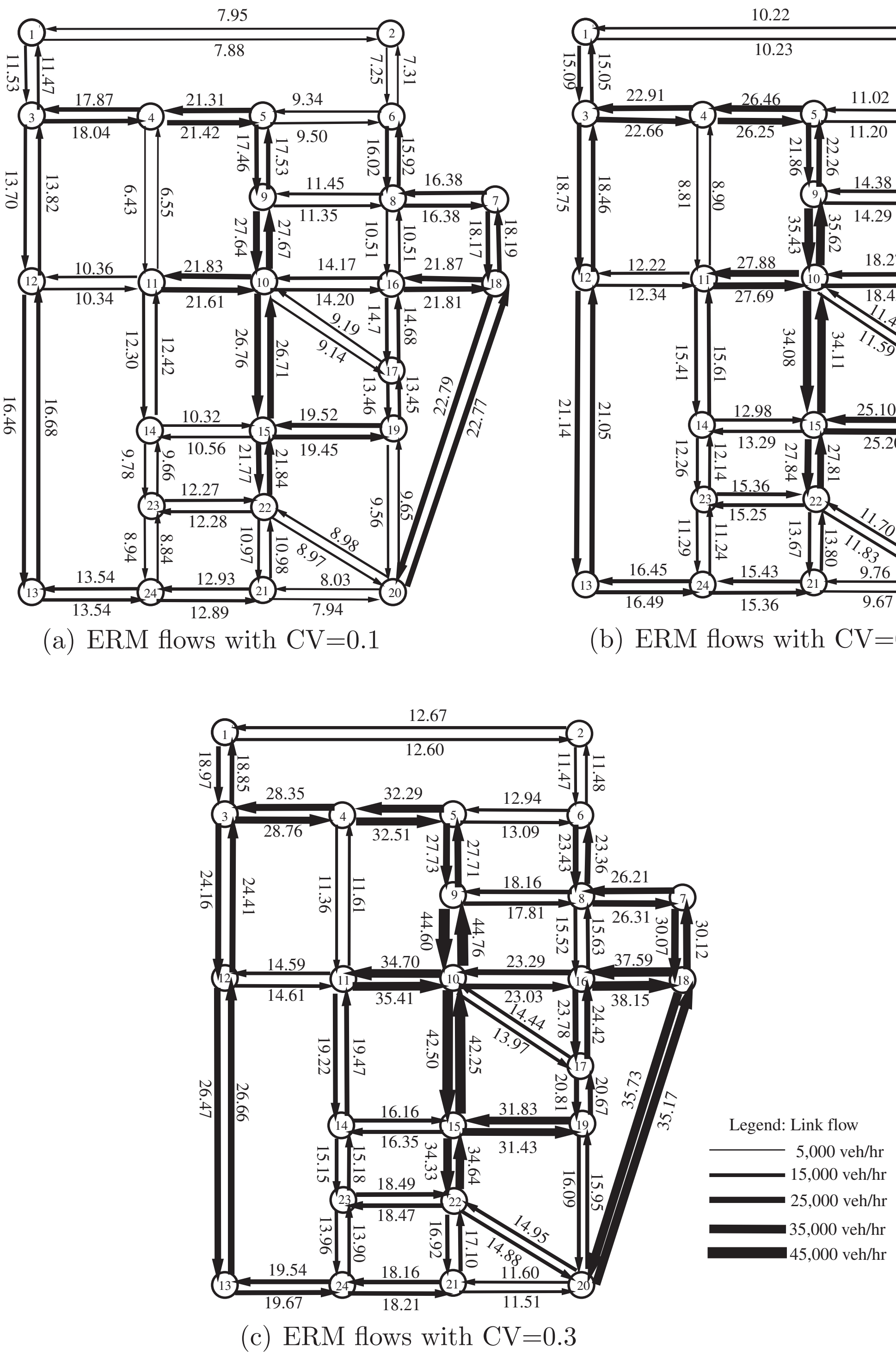
Fig. 3. Sioux Falls network.

The parameters of the GBPR functions follow those adopted in Suwansirikul et al. (1987). For simplicity, we choose the additive path travel cost function

$$\Phi(f,\omega) = \Delta^T T(\Delta f,\omega),$$

and assume that each OD travel demand is an independent random variable, following a log-normal distribution, with the mean OD demand as those used in Suwansirikul et al. (1987). We consider three cases of stochastic settings for the stochastic OD demand. The coefficients of variation for each $Q^r(\omega)$ are supposed to be 0.1, 0.2 and 0.3 in three cases, respectively.

The objective function of the ERM model involves multi-dimensional expectation operator. We use the sample average approximation (SAA) method by Monte-Carlo technique for approximating the objective function. A large number of



(a) ERM flows with CV=0.1

(b) ERM flows with CV=0.2

(c) ERM flows with CV=0.3

Fig. 4. ERM flows for three tests (CV = 0.1, 0.2, 0.3) with unit 1.0 e3.

Table 9
Comparison of x_{ERM} in three cases for Example 3.

Various criteria	x_{ERM}		
	Case 1 (CV = 0.1)	Case 2 (CV = 0.2)	Case 3 (CV = 0.3)
$g(x_{EV})$	8.68e8	8.83e8	9.08e8
$g(x_{ERM})$	1.47e5	3.58e5	7.77e5
mean(V_{ERM}./V_{EV})	1.24	1.58	1.96

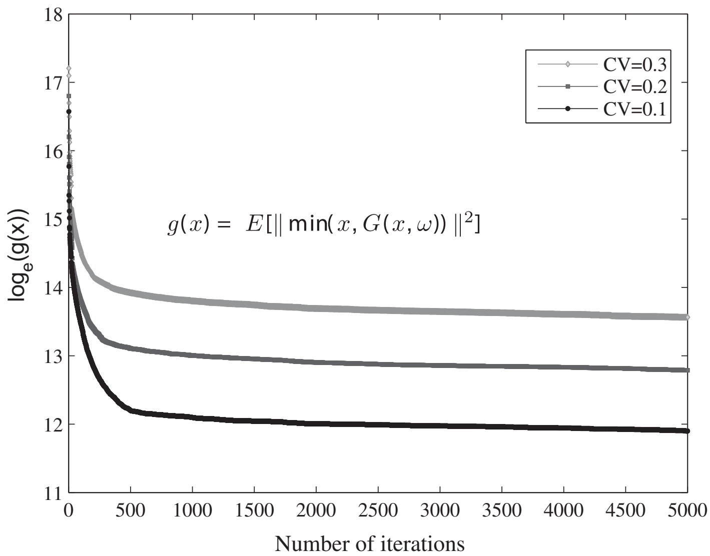


Fig. 5. Convergence of the proposed SPG algorithm for the Sioux Falls network.

sampling can guarantee the given accuracy requirement of approximation. The sufficient number of sampling relates to the dimension and the distribution of the random vector. For this example, we generate $N = 10{,}000$ samples of $\mathbf{Q}(\omega^i)$ for $i = 1, 2, \ldots, N$, where each entry follows the respective log-normal distribution independently.

The MSA method is used to solve the EV model (to obtain x_{EV}) where the path travel cost is additive. The link flow patterns of the ERM model for the three different cases are displayed in Fig. 4a–c. Here the link flow is displayed on each link with the unit 1.0 e3, and the width of each link is proportional to the link flow.

In this test, the OD demands are stochastic elements whereas the link capacities are considered deterministic. We list some useful aggregated information of the ERM solutions under the three cases below.

Here the mean ratio of link flow V_{ERM} to V_{EV} is defined by

$$\text{mean}(V_{ERM}./V_{EV}) = \frac{1}{76}\sum_{i=1}^{76} V_{ERM}(i)/V_{EV}(i).$$

From Table 9, we find that the SPG algorithm largely reduces the objective value of the ERM model at the computed solution x_{ERM} from the initial point x_{EV} in each case. It is reasonable that the objective value $g(x_{ERM})$ increases as the coefficient of variation increases from Case 1 to Case 3, because it is harder to find a solution that takes into account all the equilibrium conditions when larger variation of random OD demand is considered. Furthermore, from the planners' perspective, it is safer to expect heavier link flow when larger variation might occur.

Fig. 5 depicts the convergence behavior of the SPG algorithm for finding a minimizer of the ERM model tested on the Sioux Falls network with CV of 0.1, 0.2, and 0.3, respectively. Note that we generate a sample $\{\omega^1, \omega^2, \ldots, \omega^N\}$ of $N = 10{,}000$ by Monte-Carlo sampling techniques for each case. Hence most computation time is used to obtain the values of the objective function and its gradient at each iteration. The steepest descent direction adopted for line search in step 2 of the SPG algorithm ensures the convergence, but the line search step also uses considerable computation time. The algorithm terminated after reaching the 5000 maximum iterations. The figure shows that under each stochastic setting the trend of the objective value of the ERM model which decreased with the number of the iterations of the algorithm. Specifically, the value is reduced substantially in the first 500 iterations. When the function values are close to the minimum one, the reduction became slow. The test was carried out on a Dell PC (3.00 GHz, 2.00GB of RAM) with the use of Red Flag Linux Desktop 6.0 and Matlab R2009a (Version 7.8.0.347). The CPU times for the Sioux Falls network are 1.23 e3, 1.21 e3, and 1.19 e3 minutes under the three different settings of CV = 0.1, 0.2, and 0.3, respectively.

5. Conclusions and further studies

In this paper, we consider Wardrop's user equilibrium assignment under stochastic environment. We focus on the ERM model, which is flexible to accommodate nonadditive path travel cost and endogenous uncertainties in both the demand and supply sides. By using the ERM model, a deterministic traffic assignment pattern is provided, as well as a stochastic traffic flow pattern by further employing the technique of path choice proportion. We show theoretically the existence and robustness of the solution obtained by the ERM model under some conditions.

Compared with the EV and BW models for traffic assignment under uncertainty, the robustness of x_{ERM} is provided theoretically for the first time in the sense that the expected distance $E[\mathrm{dist}(x_{ERM}, S_\omega)]$ tends to be small. We apply the SPG method for solving the ERM model. Numerical experiments on the two small-size examples and a moderate-size example show that the SPG method is effective. Moreover, we found that the traffic assignment patterns from the EV, BW and ERM models are quite different, and the pattern from the ERM model is more robust than that from the EV and the BW models.

There exist some future extensions worth pursuing based on this paper. Firstly, mild assumptions to guarantee Assumption 3 are needed to make the robustness of the ERM model more applicable. Secondly, it is an interesting task to modify the formulation of the objective function of the model to reduce the effect of the magnitude of violation from a realization with low probability on the final solution. Thirdly, it is also worthwhile to develop a more efficient SPG algorithm for real-sized network by taking into account the special traffic network structure. Fourthly, how to generate appropriate probabilities and scenarios for the ERM model in realistic network deserves deep investigation.

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