Global Error Bounds for the Extended Vertical LCP [1]

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Abstract. A new necessary and sufficient condition for the row $\mathcal{W}$-property is given. By using this new condition and a special row rearrangement, we provide two global error bounds for the extended vertical linear complementarity problem under the row $\mathcal{W}$-property, which extend the error bounds given in [2, 10] for the P-matrix linear complementarity problem, respectively. We show that one of the new error bounds is sharper than the other, and it can be computed easily for some special class of the row $\mathcal{W}$-property block matrix. Numerical examples are given to illustrate the error bounds.

Key words. Global error bound; row $\mathcal{W}$-property; extended vertical LCP.

1 Introduction

For a given matrix $M \in R^{n\times n}$ and a given vector $q \in R^n$, the standard linear complementarity problem [5], LCP(M, q) for short, is to find a vector $x \in R^n$ such that

$$x \geq 0,\ Mx + q \geq 0,\ x^T(Mx + q) = 0.$$

The LCP is equivalent to the following system of nonlinear equations

$$\min(x, Mx + q) = 0,$$

where the min operator denotes the componentwise minimum.

Consider a block matrix $\mathbf{M}$ and a block vector $\mathbf{q}$, where

$$\mathbf{M} = (M_0, M_1, \ldots, M_k),\ \mathbf{q} = (q_0, q_1, \ldots, q_k), \tag{1.1}$$

$M_j \in R^{n\times n}$ and $q_j \in R^n$ for $j = 0, 1, \ldots, k$. The extended vertical linear complementarity problem, EVLCP$(\mathbf{M}, \mathbf{q})$ for short, is to find a vector $x \in R^n$ such that

$$r(x) := \min(M_0x + q_0, M_1x + q_1, \cdots, M_kx + q_k) = 0. \tag{1.2}$$

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If $k = 1$, $M_0 = I$ and $q_0 = 0$, the EVLCP($\mathbf{M}, \mathbf{q}$) reduces to the LCP(M_1, q_1). Furthermore, if $M_0 = I$ and $q_0 = 0$, problem (1.2) is called the vertical LCP, which was introduced by Cottle and Dantzig [3]. The EVLCP has many applications in control theory [14], generalized bimatrix games [8], nonlinear networks [4], etc. The existence of solutions and algorithms for the EVLCP have been studied in many literatures e.g., see [5, 7, 11, 12, 15].

Gowda and Sznajder [7] showed that the EVLCP($\mathbf{M}, \mathbf{q}$) has a unique solution for any $\mathbf{q}$ if and only if $\mathbf{M}$ has the row $\mathcal{W}$-property, which indicates that the row $\mathcal{W}$-property is an extension of the P-matrix.

Error bounds for the LCP have been studied extensively [2, 5, 9, 10], which play important role in convergence analysis, sensitive analysis, and verification of computed solutions. For M being a P-matrix, a well-known global error bound for the LCP(M, q) is given by Mathias and Pang in [10]

$$\|x - x^*\|_\infty \le \frac{1 + \|M\|_\infty}{\alpha(M)} \|r(x)\|_\infty \quad \text{for any } x \in R^n, \tag{1.3}$$

where x^* is the unique solution of the LCP(M, q) and

$$\alpha(M) := \min_{\|x\|_\infty = 1} \{ \max_{1 \le i \le n} x_i (Mx)_i \}. \tag{1.4}$$

Recently, Chen and Xiang [2] present a new error bound for the P-matrix LCP(M, q) in $\|\cdot\|_p$ ($p \ge 1$, or $p = \infty$) norms,

$$\|x - x^*\|_p \le \max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_p \|r(x)\|_p \quad \text{for any } x \in R^n, \tag{1.5}$$

where D is a diagonal matrix whose diagonal elements are $d := (d_1, d_2, \ldots, d_n) \in [0, 1]^n$. It was shown in [2] that

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty \le \frac{1 + \|M\|_\infty}{\alpha(M)},$$

and

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_p \le \|\tilde{M}^{-1} \max(\Lambda, I)\|_p, \tag{1.6}$$

for M being an H-matrix with positive diagonals. Here $\tilde{M}$ is the comparison matrix of M and Λ is the diagonal part of M.

An $n \times n$ matrix M is called an M-matrix, if $M^{-1} \ge 0$ and $M_{ij} \le 0$ ($i \ne j$) for $i, j = 1, 2, \ldots, n$. M is called an H-matrix, if its comparison matrix $\tilde{M}$ is an M-matrix, where

$$\tilde{M}_{ii} = |M_{ii}|, \quad \tilde{M}_{ij} = -|M_{ij}| \quad \text{for } i \ne j.$$

An H-matrix with positive diagonals is a P-matrix [5].

In this paper, we extend the error bounds (1.3) and (1.5) for the P-matrix LCP and (1.6) for the H-matrix with positive diagonals to the EVLCP. In Section 2, we provide a new necessary and sufficient condition for the row $\mathcal{W}$-property. In Section 3, using the new condition, we extend the error bound (1.5) for the P-matrix LCP to the EVLCP under the row $\mathcal{W}$-property for $k = 1$. Moreover, we give computable error bounds for two classes of row $\mathcal{W}$-property block matrices, which include (1.6) as a special case. In [16], Xiu and Zhang extended the Mathias-Pang error bound (1.3) to the EVLCP with $k = 1$. We show that the error bound given in this paper is sharper than the Xiu-Zhang error bound. In Section 4, using a special row rearrangement, we extend the results of error bounds of the EVLCP in Section 3 and the Xiu-Zhang error bound for $k = 1$ to any natural number $k \geq 1$. In Section 5, we illustrate the error bounds by four numerical examples.

Throughout this paper, $A_{i\cdot}$ denotes the i-th row of a matrix A and a_i denotes the ith component of a vector a. The absolute matrix of A is denoted by $|A|$. The spectral radius of matrix A is denoted by $\rho(A)$. The min operator and max operator work component-wise, for both vectors and matrices. Let $N = \{1, 2, \ldots, n\}$ and $e = (1, 1, \ldots, 1)^T \in R^n$. Let $\|\cdot\|$ denote the p-norm for $p \geq 1$ or $p = \infty$. For any block $M_j \in R^{n\times n}$ in $\mathbf{M}$, Λ_j denotes the diagonal part of M_j and $B_j = \Lambda_j - M_j$. When $\mathbf{M}$ has the row $\mathcal{W}$-property, we use x^* to denote the unique solution of the EVLCP$(\mathbf{M}, \mathbf{q})$.

2 Row $\mathcal{W}$-Property

The row $\mathcal{W}$-property was introduced in [15]. Here we use one of its equivalent forms as the definition.

Definition 2.1 *[7] We say $\mathbf{M}$ has the row $\mathcal{W}$-property if*

$$\min(M_0x, M_1x, \ldots, M_kx) \leq 0 \leq \max(M_0x, M_1x, \ldots, M_kx) \;\Rightarrow\; x = 0. \tag{2.1}$$

Lemma 2.1 *[15] $\mathbf{M}$ has the row $\mathcal{W}$-property if and only if for arbitrary nonnegative diagonal matrices $X_0, X_1, \ldots, X_k \in R^{n\times n}$ with $\mathrm{diag}(X_0 + X_1 + \cdots + X_k) > 0$,*

$$\det(X_0M_0 + X_1M_1 + \cdots + X_kM_k) \neq 0.$$

To study error bounds for the EVLCP$(\mathbf{M}, \mathbf{q})$, we derive a necessary and sufficient condition for the row $\mathcal{W}$-property.

Lemma 2.2 *For $k = 1$, $\mathbf{M} = (M_0, M_1)$ has the row $\mathcal{W}$-property if and only if $(I - D)M_0 + DM_1$ is nonsingular for any $D = \mathrm{diag}(d)$ with $d \in [0, 1]^n$.*

Proof: If $\mathbf{M}$ has the row $\mathcal{W}$-property, then $(I - D)M_0 + DM_1$ is nonsingular for any $D = \mathrm{diag}(d)$ with $d \in [0, 1]^n$, since $\det((I - D)M_0 + DM_1) \neq 0$ by Lemma 2.1.

Conversely, assume that $(I-D)M_0 + DM_1$ is nonsingular for any $D = \mathrm{diag}(d)$ with $d \in [0,1]^n$. For arbitrary nonnegative diagonal matrices X_0, X_1 with $\mathrm{diag}(X_0 + X_1) > 0$, let the diagonal matrix $D = \mathrm{diag}(d) = (X_0 + X_1)^{-1}X_1$, which satisfies $d \in [0,1]^n$. Since

$$X_0M_0 + X_1M_1 = (X_0 + X_1)[(I-D)M_0 + DM_1]$$

is nonsingular, we have $\det(X_0M_0 + X_1M_1) \neq 0$. Hence $\mathbf{M}$ has the row $\mathcal{W}$-property by Lemma 2.1. ∎

Remark 2.1 *In [6] Gabriel and Moré proved that a matrix $A \in R^{n\times n}$ is a P-matrix if and only if $I - D + DA$ is nonsingular for any $D = \mathrm{diag}(d)$ with $d \in [0,1]^n$. Obviously, Lemma 2.2 is a generalization of their result.*

For $\mathbf{M}' = (M'_0, M'_1, \ldots, M'_k)$ and $\mathbf{q}' = (q'_0, q'_1, \ldots, q'_k)$, where $M'_j \in R^{n\times n}$ and $q'_j \in R^n$ for $j = 0, 1, \ldots, k$, we say that the pair $(\mathbf{M}', \mathbf{q}')$ is a row rearrangement of $(\mathbf{M}, \mathbf{q})$, if for each $i \in N$,

$$(M'_j)_{i.} = (M_{j_i})_{i.} \in \{(M_0)_{i.}, (M_1)_{i.}, \ldots, (M_k)_{i.}\} = \{(M'_0)_{i.}, (M'_1)_{i.}, \ldots, (M'_k)_{i.}\},$$

and

$$(q'_j)_i = (q_{j_i})_i \in \{(q_0)_i, (q_1)_i, \ldots, (q_k)_i\} = \{(q'_0)_i, (q'_1)_i, \ldots, (q'_k)_i\},$$

where $j_i \in \{0, 1, \ldots, k\}$. In this circumstance, we also say that $\mathbf{M}'$ is a row rearrangement of $\mathbf{M}$ and $\mathbf{q}'$ is a row rearrangement of $\mathbf{q}$, respectively. We use $\mathcal{R}(\mathbf{M})$ to denote the set of all row rearrangements of $\mathbf{M}$.

Proposition 2.1 *The block matrix $\mathbf{M} = (M_0, M_1, \ldots, M_k)$ has the row $\mathcal{W}$-property if and only if $(I-D)M'_j + DM'_l$ is nonsingular for any two blocks M'_j and M'_l of $\mathbf{M}' \in \mathcal{R}(\mathbf{M})$ and any $D = \mathrm{diag}(d)$ with $d \in [0,1]^n$.*

Proof: We first show that the implication (2.1) is equivalent to that for any two blocks M'_j and M'_l of $\mathbf{M}' \in \mathcal{R}(\mathbf{M})$,

$$\min(M'_jx, M'_lx) \le 0 \le \max(M'_jx, M'_lx) = 0 \;\Rightarrow\; x = 0. \tag{2.2}$$

(2.1) $\Rightarrow$ (2.2): Let M'_j and M'_l be any two blocks of $\mathbf{M}' \in \mathcal{R}(\mathbf{M})$ such that

$$\min(M'_jx, M'_lx) \le 0 \le \max(M'_jx, M'_lx),$$

for a vector $x \in R^n$. It is easy to see that

$$\begin{aligned}
\min(M_0x, M_1x, \ldots, M_kx) &\le \min(M'_jx, M'_lx) \\
&\le 0 \\
&\le \max(M'_jx, M'_lx) \\
&\le \max(M_0x, M_1x, \ldots, M_kx).
\end{aligned}$$

Hence we have $x = 0$ according to (2.1).

(2.2) $\Rightarrow$ (2.1): Suppose that

$$\min(M_0 x, M_1 x, \ldots, M_k x) \le 0 \le \max(M_0 x, M_1 x, \ldots, M_k x)$$

for a vector $x \in R^n$. Let us make the row rearrangement $\mathbf{M}' = (M_0', M_1', \ldots, M_k') \in \mathcal{R}(\mathbf{M})$ such that

$$(M_0' x)_i \le (M_1' x)_i \le \cdots \le (M_k' x)_i, \quad i \in N.$$

Thus we have

$$\min(M_0' x, M_k' x) \le 0 \le \max(M_0' x, M_k' x),$$

which implies $x = 0$ by employing (2.2).

The equivalence between the implications (2.1) and (2.2) implies that $\mathbf{M}$ has the row $\mathcal{W}$-property if and only if for any two blocks M_j' and M_l' of $\mathbf{M}' \in \mathcal{R}(\mathbf{M})$, the block matrix (M_j', M_l') has the row $\mathcal{W}$-property. Thus we have this proposition by Lemma 2.2. ∎

We can relate the row $\mathcal{W}$-property to a regular interval matrix. For two matrices $\underline{A}$, $\overline{A} \in R^{n\times n}$, we write $\underline{A} \le \overline{A}$ if $\underline{A}_{ij} \le \overline{A}_{ij}$ for any $i, j \in N$. Given $n \times n$ matrices $\Delta \ge 0$ and A_c, the square interval matrix [13] is defined by

$$A^I = [A_c - \Delta, A_c + \Delta] = \{A \ : \ A_c - \Delta \le A \le A_c + \Delta\}.$$

A^I is called regular if each $A \in A^I$ is nonsingular. If the spectral radius $\rho(|A_c^{-1}|\Delta) < 1$, then A^I is regular (Corollary 5.1 in [13]).

Proposition 2.2 *If the square interval matrix A^I is regular, then a block matrix $\mathbf{M}$ composed of any matrices $M_j \in A^I$ for $j = 0, 1, \ldots, k$, has the row $\mathcal{W}$-property.*

Proof: Suppose that a vector $\hat{x} \in R^n$ satisfies

$$\min(M_0\hat{x}, M_1\hat{x}, \ldots, M_k\hat{x}) \le 0 \le \max(M_0\hat{x}, M_1\hat{x}, \ldots, M_k\hat{x}).$$

Choose the row rearrangement $\mathbf{M}' = (M_0', M_1', \ldots, M_k') \in \mathcal{R}(\mathbf{M})$ such that

$$(M_0'\hat{x})_i \le 0 \le (M_k'\hat{x})_i, \quad i \in N.$$

Let $\hat{y} = M_0'\hat{x}$, then $\hat{x} = {M_0'}^{-1}\hat{y}$ since $M_0' \in A^I$ is nonsingular. Thus we obtain

$$\max_{1\le i\le n} \hat{y}_i(M_k' {M_0'}^{-1}\hat{y})_i = \max_{1\le i\le n} (M_0'\hat{x})_i(M_k'\hat{x})_i \le 0.$$

It is known (Theorem 1.2 [13]) that if A^I is regular, then $A_1 A_2^{-1}$ is a P-matrix for each $A_1, A_2 \in A^I$. Hence $M_k' {M_0'}^{-1}$ is a P-matrix, which implies $\hat{y} = 0$ by Theorem 3.3.4 [5]. We get $\hat{x} = 0$, and thus $\mathbf{M}$ has the row $\mathcal{W}$-property. ∎

3 Global error bounds for $k = 1$

In this section, we first extend the error bound (1.5) to the EVLCP under the row $\mathcal{W}$-property for the special case $k = 1$. For every $y, z, y^*, z^* \in R^n$, it is known [2] that

$$\min(y_i, z_i) - \min(y_i^*, z_i^*) = (1 - d_i)(y_i - y_i^*) + d_i(z_i - z_i^*), \quad i \in N \tag{3.1}$$

where $d_i \in [0, 1]$ is given by

$$d_i = \begin{cases} 0 & \text{if } z_i \geq y_i, z_i^* \geq y_i^* \\ 1 & \text{if } z_i \leq y_i, z_i^* \leq y_i^* \\ \dfrac{\min(y_i, z_i) - \min(y_i^*, z_i^*) + y_i^* - y_i}{z_i - z_i^* + y_i^* - y_i} & \text{otherwise.} \end{cases}$$

By putting $y = M_0 x + q_0$, $y^* = M_0 x^* + q_0$, and $z = M_1 x + q_1$, $z^* = M_1 x^* + q_1$ in (3.1), we get

$$r(x) = \min(M_0 x + q_0, M_1 x + q_1) = [(I - D)M_0 + DM_1](x - x^*), \tag{3.2}$$

where D is a diagonal matrix whose diagonal elements are $d = (d_1, d_2, \ldots, d_n) \in [0, 1]^n$.

By using (3.2) and Lemma 2.2, we obtain the global error bound for the EVLCP under the row $\mathcal{W}$-property when $k = 1$.

Theorem 3.1 *Suppose that the block matrix $\mathbf{M} = (M_0, M_1)$ has the row $\mathcal{W}$-property. Then for any $x \in R^n$,*

$$\|x - x^*\| \leq \max_{d \in [0,1]^n} \|[(I - D)M_0 + DM_1]^{-1}\| \|r(x)\|. \tag{3.3}$$

In what follows, we provide two sufficient conditions for $\mathbf{M}$ having the row $\mathcal{W}$-property and give some simple upper bounds for

$$\max_{d \in [0,1]^n} \|[(I - D)M_0 + DM_1]^{-1}\|.$$

Recall that $M_l = \Lambda_l - B_l$, where Λ_l is the diagonal part of M_l for $l = 0, 1$.

Theorem 3.2 *Suppose that the diagonals of M_0, M_1 are positive, and the spectral radius $\rho(\max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)) < 1$, then $\mathbf{M} = (M_0, M_1)$ has the row $\mathcal{W}$-property and*

$$\begin{aligned} &\max_{d \in [0,1]^n} \|[(I - D)M_0 + DM_1]^{-1}\| \\ &\qquad \leq \|[I - \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)]^{-1} \max(\Lambda_0^{-1}, \Lambda_1^{-1})\|. \end{aligned} \tag{3.4}$$

Proof: Denote $V = (I - D)\Lambda_0 + D\Lambda_1$ and $U = (I - D)B_0 + DB_1$. Since $(\Lambda_0)_{ii} > 0$ and $(\Lambda_1)_{ii} > 0$, the diagonal matrix V satisfies $V_{ii} > 0$ for $i = 1, \ldots, n$, and

$$(I - D)M_0 + DM_1 = V - U = V(I - V^{-1}U). \tag{3.5}$$

For the ith diagonal element of the diagonal matrix V^{-1}, we consider the function

$$\psi(t) = \frac{1}{(\Lambda_0)_{ii} + t(\Lambda_1 - \Lambda_0)_{ii}}, \quad \text{for} \quad t \in [0,1].$$

For $t \in [0,1]$, $\psi(t) > 0$. Moreover, if $(\Lambda_0)_{ii} - (\Lambda_1)_{ii} > 0$, we get $\psi'(t) > 0$; otherwise $\psi'(t) \le 0$. Thus

$$\max_{t\in[0,1]} \psi(t) = \begin{cases} 1/(\Lambda_1)_{ii} & \text{if } (\Lambda_0)_{ii} - (\Lambda_1)_{ii} > 0 \\ 1/(\Lambda_0)_{ii} & \text{otherwise.} \end{cases}$$

Hence we obtain

$$0 \le \min(\Lambda_0^{-1}, \Lambda_1^{-1}) \le V^{-1} \le \max(\Lambda_0^{-1}, \Lambda_1^{-1}), \quad \text{for} \quad d \in [0,1]^n. \tag{3.6}$$

We first prove (3.4) for M_0 and M_1 whose off-diagonal elements are non-positive. Thus $B_0 \ge 0$, $B_1 \ge 0$ and $U \ge 0$, and their diagonals are zero.

Now we consider $V^{-1}U$. For the element $(V^{-1}U)_{ij}$, let us consider the function

$$\phi(t) = \frac{(B_0)_{ij} + t(B_1 - B_0)_{ij}}{(\Lambda_0)_{ii} + t(\Lambda_1 - \Lambda_0)_{ii}}, \quad \text{for} \quad t \in [0,1].$$

For $t \in [0,1]$, $\phi(t) \ge 0$. Moreover, if $(B_1)_{ij}(\Lambda_0)_{ii} - (B_0)_{ij}(\Lambda_1)_{ij} > 0$, we have $\phi'(t) > 0$; otherwise $\phi'(t) \le 0$. Thus

$$\max_{t\in[0,1]} \phi(t) = \begin{cases} (B_1)_{ij}/(\Lambda_1)_{ii} & \text{if } (B_1)_{ij}(\Lambda_0)_{ii} - (B_0)_{ij}(\Lambda_1)_{ii} > 0 \\ (B_0)_{ij}/(\Lambda_0)_{ii} & \text{otherwise.} \end{cases}$$

Hence, we obtain

$$0 \le V^{-1}U \le \max(\Lambda_0^{-1}B_0, \Lambda_1^{-1}B_1). \tag{3.7}$$

By the assumption of this theorem, the spectral radius satisfies

$$\rho(V^{-1}U) \le \rho(\max(\Lambda_0^{-1}B_0, \Lambda_1^{-1}B_1)) < 1.$$

Thus, by using Lemma 2.1 , Chap. 6 in [1], we find

$$\begin{aligned} 0 \le (I - V^{-1}U)^{-1} &= I + V^{-1}U + \cdots + (V^{-1}U)^m + \cdots \\ &\le I + (\max(\Lambda_0^{-1}B_0, \Lambda_1^{-1}B_1)) + \cdots + (\max(\Lambda_0^{-1}B_0, \Lambda_1^{-1}B_1))^m + \cdots \\ &= [I - \max(\Lambda_0^{-1}B_0, \Lambda_1^{-1}B_1)]^{-1}. \end{aligned} \tag{3.8}$$

This, together with (3.5) and (3.6), gives

$$\begin{aligned} [(I-D)M_0 + DM_1]^{-1} &= (I - V^{-1}U)^{-1}V^{-1} \\ &\le [I - \max(\Lambda_0^{-1}B_0, \Lambda_1^{-1}B_1)]^{-1}\max(\Lambda_0^{-1}, \Lambda_1^{-1}). \end{aligned}$$

Therefore, (3.4) holds when $(M_0)_{ij} \le 0$ and $(M_1)_{ij} \le 0$ for $i \ne j$.

Now we show (3.4) under the assumptions of this theorem. Note that

$$\begin{aligned} V^{-1}U &\le V^{-1}|U| \\ &\le V^{-1}[(I-D)|B_0| + D|B_1|] \\ &\le \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|), \end{aligned}$$

where the last inequality comes from (3.7). We replace U by $(I-D)|B_0| + D|B_1|$.

Since for any $n \times n$ matrix C, $\rho(C) \le \rho(|C|)$, and for any nonnegative $n \times n$ matrices J and K, $J \le K$ implies $\rho(J) \le \rho(K)$, we have that for all $d \in [0,1]^n$,

$$\rho(V^{-1}U) \le \rho(|V^{-1}U|) \le \rho(V^{-1}|U|) \le \rho(\max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)) < 1.$$

Thus $I - V^{-1}U$ is nonsingular, and

$$\begin{aligned} |(I - V^{-1}U)^{-1}| &= |I + (V^{-1}U) + \cdots + (V^{-1}U)^m + \cdots| \\ &\le I + (V^{-1}|U|) + \cdots + (V^{-1}|U|)^m + \cdots \\ &\le I + (\max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)) + \cdots \\ &\qquad + (\max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|))^m + \cdots \\ &= [I - \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)]^{-1}. \end{aligned}$$

That is,

$$-[I - \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)]^{-1} \le (I - V^{-1}U)^{-1} \le [I - \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)]^{-1}.$$

This together with (3.6) gives

$$\begin{aligned} &-[I - \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)]^{-1} \max(\Lambda_0^{-1}, \Lambda_1^{-1}) \\ &\qquad \le -[I - \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)]^{-1} V^{-1} \\ &\qquad \le (I - V^{-1}U)^{-1} V^{-1} \\ &\qquad \le [I - \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)]^{-1} V^{-1} \\ &\qquad \le [I - \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)]^{-1} \max(\Lambda_0^{-1}, \Lambda_1^{-1}). \end{aligned}$$

This implies

$$|(I - V^{-1}U)^{-1} V^{-1}| \le [I - \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)]^{-1} \max(\Lambda_0^{-1}, \Lambda_1^{-1}). \tag{3.9}$$

Therefore by (3.5) and (3.9), we obtain that

$$[(I-D)M_0 + DM_1]^{-1} = (I - V^{-1}U)^{-1} V^{-1},$$

and

$$\begin{aligned} \|(I - V^{-1}U)^{-1} V^{-1}\| &\le \||(I - V^{-1}U)^{-1} V^{-1}|\| \\ &\le \|[I - \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)]^{-1} \max(\Lambda_0^{-1}, \Lambda_1^{-1})\|. \end{aligned} \tag{3.10}$$

By using Lemma 2.2, we know that $\mathbf{M} = (M_0, M_1)$ has the row $\mathcal{W}$-property. ∎

Remark 3.1 *If $M_0 = I$, and M_1 is an* H*-matrix with positive diagonals, then $B_0 = 0$ and $\rho(\max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)) = \rho(\Lambda_1^{-1}|B_1|) < 1$. By Theorem 3.2,*

$$\begin{aligned} \max_{d\in[0,1]^n} \|(I - D + DM_1)^{-1}\| &\leq \|[I - \Lambda_1^{-1}|B_1|]^{-1} \max(I, \Lambda_1^{-1})\| \\ &= \|\tilde{M}_1^{-1} \max(\Lambda_1, I)\|, \end{aligned} \tag{3.11}$$

which reduces to Theorem 2.1 in [2].

Corollary 3.1 *Suppose that the diagonals of M_0, M_1 are positive. If the matrix $\bar{M} = \min(\Lambda_0, \Lambda_1) - \max(|B_0|, |B_1|)$ is an M-matrix, then $\mathbf{M} = (M_0, M_1)$ has the row $\mathcal{W}$-property and (3.4) holds.*

Proof: Since $\bar{M}$ is an M-matrix, we have

$$\rho(\max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)) \leq \rho((\min(\Lambda_0, \Lambda_1))^{-1} \max(|B_0|, |B_1|)) < 1.$$

Hence this corollary follows from Theorem 3.2. ∎

Theorem 3.3 *Suppose that M_0, M_1 are strictly row diagonally dominant matrices, and $(M_0)_{ii}(M_1)_{ii} > 0$ for each $i \in N$, then $\mathbf{M} = (M_0, M_1)$ has the row $\mathcal{W}$-property and*

$$\max_{d\in[0,1]^n} \|[(I - D)M_0 + DM_1]^{-1}\|_\infty \leq \frac{1}{\min_{i\in N} \min((\tilde{M}_0 e)_i, (\tilde{M}_1 e)_i)}. \tag{3.12}$$

Proof: For any $d \in [0, 1]^n$, $G := (I - D)M_0 + DM_1$ is a strictly row diagonally dominant matrix, which is a P-matrix [5]. By Lemma 2.2, $\mathbf{M}$ has the row $\mathcal{W}$-property. Moreover, it is easy to verify that for any strictly row diagonally dominant matrix A,

$$\|A^{-1}\|_\infty \leq \frac{1}{\min_{i\in N} (\tilde{A}e)_i},$$

where $\tilde{A}$ is the comparison matrix of A.

Using the assumption of the theorem, we have for any $i \in N$, the comparison matrix of G satisfies

$$\begin{aligned} (\tilde{G}e)_i &= |(1 - d_i)(M_0)_{ii} + d_i(M_1)_{ii}| - \sum_{j=1,j\neq i}^{n} |(1 - d_i)(M_0)_{ij} + d_i(M_1)_{ij}| \\ &\geq (1 - d_i)|(M_0)_{ii}| + d_i|(M_1)_{ii}| - (1 - d_i) \sum_{j=1,j\neq i}^{n} |(M_0)_{ij}| - d_i \sum_{j=1,j\neq i}^{n} |(M_1)_{ij}| \\ &= (1 - d_i)(\tilde{M}_0 e)_i + d_i(\tilde{M}_1 e)_i \\ &\geq \min((\tilde{M}_0 e)_i, (\tilde{M}_1 e)_i) \\ &> 0. \end{aligned}$$

Thus for any $D = \mathrm{diag}(d)$ with $d \in [0,1]^n$,

$$\|[(I-D)M_0 + DM_1]^{-1}\|_\infty \;\leq\; \frac{1}{\min\limits_{i\in N}(\tilde{G}e)_i} \leq \frac{1}{\min\limits_{i\in N}\min((\tilde{M}_0 e)_i, (\tilde{M}_1 e)_i)}.$$

This completes the proof. ∎

Note that Theorem 3.3 includes the case that the diagonal elements $(M_0)_{ii}$ and $(M_1)_{ii}$ are both negative for some $i \in N$. Moreover, for the case that both M_0 and M_1 have positive diagonals, the two classes of the row $\mathcal{W}$-property block matrices discussed in Theorem 3.2 and Theorem 3.3 do not coincide; see the following simple example.

Example 3.1 *Consider the block matrix* $\mathbf{M} = (M_0, M_1)$, *where*

$$M_0 = \begin{pmatrix} 1 & 3/4 & 0 \\ 3/4 & 1 & 0 \\ 0 & 3/4 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 3/4 \\ 3/4 & 0 & 1 \end{pmatrix}.$$

It is easy to find that $\mathbf{M}$ *satisfies the condition in Theorem 3.3, and hence*

$$\max_{d\in[0,1]^n} \|[(I-D)M_0 + DM_1]^{-1}\|_\infty \leq \frac{1}{\min\limits_{i\in N}\min((\tilde{M}_0 e)_i, (\tilde{M}_1 e)_i)} = 4.$$

However, $\mathbf{M}$ *fails to satisfy the condition in Theorem 3.2, since*

$$\rho(\max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)) = 1.5 > 1.$$

Xiu and Zhang [16] extended the Mathias-Pang error bound (1.3) to the EVLCP$(\mathbf{M}, \mathbf{q})$ under the row $\mathcal{W}$-property for the special case of $k = 1$:

$$\|x - x^*\|_\infty \leq \frac{\|M_0 + M_1\|_\infty}{\alpha\{M_0, M_1\}}\|r(x)\|_\infty, \quad \text{for any } x \in R^n, \tag{3.13}$$

where

$$\alpha\{M_0, M_1\} := \min_{\|x\|_\infty = 1}\{\max_{1\leq i\leq n}(M_0 x)_i (M_1 x)_i\}. \tag{3.14}$$

At the end of this section, we show that the error bound (3.3) given in this paper is sharper than (3.13).

Theorem 3.4 *If* $\mathbf{M} = (M_0, M_1)$ *has the row* $\mathcal{W}$*-property, then for any* $x \in R^n$*, the following inequalities hold.*

$$\begin{aligned} &\max_{d\in[0,1]^n} \|[(I-D)M_0 + DM_1]^{-1}\|_\infty \|r(x)\|_\infty \\ &\leq \frac{\min\Big(\|M_0 + M_1\|_\infty, \max(\|M_0\|_\infty, \|M_1\|_\infty)\Big)}{\alpha\{M_0, M_1\}}\|r(x)\|_\infty \\ &\leq \frac{\|M_0 + M_1\|_\infty}{\alpha\{M_0, M_1\}}\|r(x)\|_\infty. \end{aligned} \tag{3.15}$$

Proof: For any diagonal matrix $D = \text{diag}(d)$ with $d \in [0,1]^n$, let $H = [(I-D)M_0 + DM_1]^{-1}$ and i_0 be the index such that $\sum_{j=1}^n |H_{i_0 j}| = \|[(I-D)M_0 + DM_1]^{-1}\|_\infty$.

Define $y = [(I-D)M_0 + DM_1]^{-1} z$, where $z = (\text{sgn}(H_{i_0 1}), \ldots, \text{sgn}(H_{i_0 n}))$. It is clear that $\|y\|_\infty = \|[(I-D)M_0 + DM_1]^{-1}\|_\infty$, and

$$z = (I-D)M_0 y + DM_1 y. \tag{3.16}$$

Furthermore, by the definition of $\alpha\{M_0, M_1\}$, we have

$$0 < \alpha\{M_0, M_1\}\|y\|_\infty^2 \leq \max_i (M_0 y)_i (M_1 y)_i.$$

Let j be the index such that $(M_0 y)_j (M_1 y)_j = \max_i (M_0 y)_i (M_1 y)_i$. Thus we know that $(M_0 y)_j$ and $(M_1 y)_j$ are of the same sign, since $(M_0 y)_j (M_1 y)_j > 0$. We claim that either $|(M_0 y)_j| \leq 1$ or $|(M_1 y)_j| \leq 1$. Because if this is not true, we have $(M_0 y)_j > 1$ and $(M_1 y)_j > 1$; or $(M_0 y)_j < -1$ and $(M_1 y)_j < -1$, which contradicts to (3.16) by noticing that $|z_j| \leq 1$. Then we have

$$0 < \alpha\{M_0, M_1\}\|y\|_\infty^2 \leq \max(|(M_0 y)_j|, |(M_1 y)_j|) \leq |((M_0 + M_1)y)_j|.$$

Hence $\alpha\{M_0, M_1\}\|y\|_\infty^2 \leq \max(\|M_0\|_\infty, \|M_1\|_\infty)\|y\|_\infty$ and

$$\|[(I-D)M_0 + DM_1]^{-1}\|_\infty \leq \frac{\max(\|M_0\|_\infty, \|M_1\|_\infty)}{\alpha\{M_0, M_1\}}.$$

Moreover, $\alpha\{M_0, M_1\}\|y\|_\infty^2 \leq \|M_0 + M_1\|_\infty \|y\|_\infty$ and

$$\|[(I-D)M_0 + DM_1]^{-1}\|_\infty \leq \frac{\|M_0 + M_1\|_\infty}{\alpha\{M_0, M_1\}}.$$

Therefore,

$$\|[(I-D)M_0 + DM_1]^{-1}\|_\infty \leq \frac{\min\Big(\|M_0 + M_1\|_\infty, \max(\|M_0\|_\infty, \|M_1\|_\infty)\Big)}{\alpha\{M_0, M_1\}}.$$

The second inequality in (3.15) is trivial. ∎

According to Theorem 3.2 and Theorem 3.3, we can easily compute an upper bound of the error bound (3.3) for two classes of the row $\mathcal{W}$-property block matrix. However, it is not easy to estimate an upper bound for the error bound (3.13). Hence, the new error bound (3.3) is not only sharp but also easy to compute in some special cases.

4 Global error bounds for $k \geq 1$

In this section, we construct a special row rearrangement of $(\mathbf{M}, \mathbf{q})$ and extend the error bounds (3.3) and (3.13) to the EVLCP for any natural number $k \geq 1$.

Theorem 4.1 *Suppose that the block matrix $\mathbf{M} = (M_0, M_1, \ldots, M_k)$ has the row $\mathcal{W}$-property. Then for any $x \in R^n$,*

$$\|x - x^*\| \leq \max_{\mathbf{M}' \in \mathcal{R}(\mathbf{M})} \max_{j<l \in \{0,1,\ldots,k\}} \max_{d \in [0,1]^n} \|[(I-D)M'_j + DM'_l]^{-1}\| \|r(x)\|, \tag{4.1}$$

where M'_j, $M'_l \in R^{n\times n}$ are any two blocks in $\mathbf{M}' \in \mathcal{R}(\mathbf{M})$.

Proof: For an arbitrary vector $x^0 \in R^n$, we construct a row rearrangement $(\mathbf{M}', \mathbf{q}')$ of $(\mathbf{M}, \mathbf{q})$, where $\mathbf{M}' = (M'_0, M'_1, \ldots, M'_k)$ and $\mathbf{q}' = (q'_0, q'_1, \ldots, q'_k)$ satisfy

$$\begin{cases} M'_0 x^* + q'_0 = 0, \\ M'_1 x^0 + q'_1 \leq M'_2 x^0 + q'_2 \leq \cdots \leq M'_k x^0 + q'_k. \end{cases}$$

Such $\mathbf{M}'$ and $\mathbf{q}'$ can be defined as follows. First we use x^* to determine the row rearrangement $(\widetilde{\mathbf{M}}, \widetilde{\mathbf{q}})$ where $\widetilde{\mathbf{M}} = (\widetilde{M}_0, \widetilde{M}_1, \ldots, \widetilde{M}_k)$ and $\widetilde{\mathbf{q}} = (\widetilde{q}_0, \widetilde{q}_1, \ldots, \widetilde{q}_k)$ such that for each $i \in N$,

$$(\widetilde{M}_0 x^* + \widetilde{q}_0)_i \leq (\widetilde{M}_1 x^* + \widetilde{q}_1)_i \leq \cdots \leq (\widetilde{M}_k x^* + \widetilde{q}_k)_i.$$

Let $M'_0 = \widetilde{M}_0$ and $q'_0 = \widetilde{q}_0$. Since x^* is the solution of the EVLCP$(\mathbf{M}, \mathbf{q})$, we have $M'_0 x^* + q'_0 = 0$. Then by using x^0, we make a new row rearrangement $(M'_1, M'_2, \ldots, M'_k)$ and $(q'_1, q'_2, \ldots, q'_k)$ of $(\widetilde{M}_1, \widetilde{M}_2, \ldots, \widetilde{M}_k)$ and $(\widetilde{q}_1, \widetilde{q}_2, \ldots, \widetilde{q}_k)$ such that for each $i \in N$,

$$(M'_1 x^0 + q'_1)_i \leq (M'_2 x^0 + q'_2)_i \leq \cdots \leq (M'_k x^0 + q'_k)_i,$$

that is,

$$M'_1 x^0 + q'_1 \leq M'_2 x^0 + q'_2 \leq \cdots \leq M'_k x^0 + q'_k.$$

The desired row rearrangement is obtained.

Since (M'_0, M'_1) has the row $\mathcal{W}$-property from the proof of Proposition 2.1, x^* is the unique solution of

$$\min(M'_0 x + q'_0, M'_1 x + q'_1) = 0. \tag{4.2}$$

Let $r'(x) := \min(M'_0 x + q'_0, M'_1 x + q'_1)$. Clearly, $r'(x^*) = 0$ and

$$\begin{aligned} r'(x^0) &= \min(M'_0 x^0 + q'_0, M'_1 x^0 + q'_1) \\ &= \min(M'_0 x^0 + q'_0, M'_1 x^0 + q'_1, M'_2 x^0 + q'_2, \ldots, M'_k x^0 + q'_k) \\ &= \min(M_0 x^0 + q_0, M_1 x^0 + q_1, \ldots, M_k x^0 + q_k) \\ &= r(x^0). \end{aligned} \tag{4.3}$$

Form (4.2), (4.3), and by using the error bound (3.3) for $k = 1$, we get that

$$\begin{aligned} \|x^0 - x^*\| &\leq \max_{d \in [0,1]^n} \|[(I-D)M'_0 + DM'_1]^{-1}\| \|r'(x^0)\| \\ &= \max_{d \in [0,1]^n} \|[(I-D)M'_0 + DM'_1]^{-1}\| \|r(x^0)\|. \end{aligned}$$

Since x^0 is arbitrarily chosen, we get the error bound (4.1). ∎

Remark 4.1 *It is not difficult to obtain a lower bound for the EVLCP under the row $\mathcal{W}$-property. For $k = 1$, by using (3.2), we can easily find that*

$$\|x - x^*\| \geq \frac{\|r(x)\|}{\max\limits_{d\in[0,1]^n} \|(I - D)M_0 + DM_1\|} \quad \text{for any } x \in R^n. \tag{4.4}$$

By employing (4.4) to (4.2), and noticing (4.3), we have that for any $x \in R^n$,

$$\|x - x^*\| \geq \frac{\|r(x)\|}{\max\limits_{\mathbf{M}'\in\mathcal{R}(\mathbf{M})} \max\limits_{j<l\in\{0,1,\ldots,k\}} \max\limits_{d\in[0,1]^n} \|(I - D)M'_j + DM'_l\|}.$$

By using the same row rearrangement, and the Xiu-Zhang error bound (3.13), we can extend the Mathias-Pang error bound (1.3) for the P-matrix LCP to the EVLCP under the row $\mathcal{W}$-property as follows.

Theorem 4.2 *Suppose that the block matrix $\mathbf{M} = (M_0, M_1, \ldots, M_k)$ has the row $\mathcal{W}$-property. Then for any $x \in R^n$,*

$$\|x - x^*\|_\infty \leq \max_{\mathbf{M}'\in\mathcal{R}(\mathbf{M})} \max_{j<l\in\{0,1,\ldots,k\}} \frac{\|M'_j + M'_l\|_\infty}{\alpha\{M'_j, M'_l\}} \|r(x)\|_\infty. \tag{4.5}$$

Following the proof of Theorem 3.4, we can show that the error bound (4.1) is sharper than (4.5).

The next two theorems for any natural number $k \geq 1$ are generalizations of Theorem 3.2 and Theorem 3.3.

Theorem 4.3 *Suppose that $\mathbf{M} = (M_0, M_1, \ldots, M_k)$, where M_j are matrices with positive diagonals for $j = 0, 1, \ldots, k$, and the spectral radius*

$$\rho(\max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|, \ldots, \Lambda_k^{-1}|B_k|)) < 1,$$

then $\mathbf{M}$ has the row $\mathcal{W}$-property and

$$\begin{aligned}&\max_{\mathbf{M}'\in\mathcal{R}(\mathbf{M})} \max_{j<l\in\{0,1,\ldots,k\}} \max_{d\in[0,1]^n} \|[(I - D)M'_j + DM'_l]^{-1}\| \\ &\quad \leq \|[I - \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|, \ldots, \Lambda_k^{-1}|B_k|)]^{-1} \max(\Lambda_0^{-1}, \Lambda_1^{-1}, \ldots, \Lambda_k^{-1})\|.\end{aligned} \tag{4.6}$$

Proof: For any $\mathbf{M}' \in \mathcal{R}(\mathbf{M})$ and any two blocks M'_j, $M'_l \in R^{n\times n}$ in $\mathbf{M}'$, let us denote

$$M'_j = \Lambda'_j - B'_j \quad \text{and} \quad M'_l = \Lambda'_l - B'_l,$$

where Λ'_j and Λ'_l are the diagonals of M'_j and M'_l, respectively. Then it is clear that

$$0 \leq \max((\Lambda'_j)^{-1}|B'_j|, (\Lambda'_l)^{-1}|B'_l|) \leq \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|, \ldots, \Lambda_k^{-1}|B_k|). \tag{4.7}$$

Hence,

$$\rho(\max((\Lambda'_j)^{-1}|B'_j|, (\Lambda'_l)^{-1}|B'_l|)) \leq \rho(\max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|, \ldots, \Lambda_k^{-1}|B_k|)) < 1,$$

which implies that

$$\begin{aligned}
0 &\le (I-\max((\Lambda_j')^{-1}|B_j'|,(\Lambda_l')^{-1}|B_l'|))^{-1}\\
&= I+(\max((\Lambda_j')^{-1}|B_j'|,(\Lambda_l')^{-1}|B_l'|))+\cdots+(\max((\Lambda_j')^{-1}|B_j'|,(\Lambda_l')^{-1}|B_l'|))^m+\cdots\\
&\le I+(\max(\Lambda_0^{-1}|B_0|,\ldots,\Lambda_k^{-1}|B_k|))+\cdots+(\max(\Lambda_0^{-1}|B_0|,\ldots,\Lambda_k^{-1}|B_k|))^m+\cdots\\
&= (I-\max(\Lambda_0^{-1}|B_0|,\cdots,\Lambda_k^{-1}|B_k|))^{-1}.
\end{aligned}$$

Moreover, by noticing that $0 \le \max((\Lambda_j')^{-1},(\Lambda_l')^{-1}) \le \max(\Lambda_0^{-1},\Lambda_1^{-1},\cdots,\Lambda_k^{-1})$, we obtain from Theorem 3.1 that

$$\begin{aligned}
&\max_{d\in[0,1]^n}\|[(I-D)M_j'+DM_l']^{-1}\|\\
&\quad\le \|[I-\max((\Lambda_j')^{-1}|B_j'|,(\Lambda_l')^{-1}|B_l'|)]^{-1}\max((\Lambda_j')^{-1},(\Lambda_l')^{-1})\|\\
&\quad\le \|[I-\max(\Lambda_0^{-1}|B_0|,\cdots,\Lambda_k^{-1}|B_k|)]^{-1}\max(\Lambda_0^{-1},\Lambda_1^{-1},\cdots,\Lambda_k^{-1})\|.
\end{aligned}$$

Therefore (4.6) holds. Furthermore, from Proposition 2.1, we find that $\mathbf{M}$ has the row $\mathcal{W}$-property. ∎

Theorem 4.4 *Suppose that each $M_j \in R^{n\times n}$ in $\mathbf{M}$ is strictly row diagonally dominant for $j = 0,1,\ldots,k$, and for each $i \in N$,*

$$(M_j)_{ii}(M_l)_{ii} > 0, \quad \textit{for any } j < l \in \{0,1,\ldots,k\},$$

then $\mathbf{M}$ has the row $\mathcal{W}$-property and

$$\begin{aligned}
&\max_{\mathbf{M}'\in\mathcal{R}(\mathbf{M})}\ \max_{j<l\in\{0,1,\ldots,k\}}\ \max_{d\in[0,1]^n}\|[(I-D)M_j'+DM_l']^{-1}\|_\infty\\
&\le \frac{1}{\min\limits_{i\in N}\min((\tilde{M}_0e)_i,(\tilde{M}_1e)_i,\ldots,(\tilde{M}_ke)_i)}.
\end{aligned}\tag{4.8}$$

We omit the proof since it is analogous to that of Theorem 3.3.

Theorem 4.3 and Theorem 4.4 provide two sufficient conditions for the row $\mathcal{W}$-property block matrix. Gowda and Sznajder [7] showed that $\mathbf{M}$ has the row $\mathcal{W}$-property, if each M_j is strictly diagonally dominant, with positive diagonal and nonpositive off-diagonal elements. The conditions in Theorem 4.4 are weaker than what they used.

5 Numerical examples

In this section, we provide four examples to illustrate the error bounds given in this paper.

Example 5.1 *Let $\mathbf{M} = (M_0, M_1)$ where*

$$M_0 = \begin{pmatrix} 2 & -2 \\ 0 & 4 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}.$$

It is easy to show that $\rho(\max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)) = \sqrt{1/2} < 1$. Hence $\mathbf{M}$ satisfies the conditions in Theorem 3.2, which implies that $\mathbf{M}$ has the row $\mathcal{W}$-property. By direct computation,

$$\max_{d\in[0,1]^2} \|[(I-D)M_0 + DM_1]^{-1}\|_\infty = \max_{d_1,d_2\in[0,1]} \frac{6-d_1}{(4-2d_1)(2+d_2)+4d_1} = \frac{3}{4},$$

and

$$\|[I - \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)]^{-1} \max(\Lambda_0^{-1}, \Lambda_1^{-1})\|_\infty = \frac{3}{2}.$$

On the other hand, let $\hat{x} = (1, 1/8)^T$, we find

$$\alpha\{M_0, M_1\} = \min_{\|x\|_\infty=1} \max_i (M_0x)_i(M_1x)_i \le \max_i (M_0\hat{x})_i(M_1\hat{x})_i = \frac{105}{32}.$$

Hence

$$\frac{\|M_0 + M_1\|_\infty}{\alpha\{M_0, M_1\}} \ge \frac{320}{105} > 3.$$

Example 5.2 *For $t \in [1, \infty)$, let $\mathbf{M} = (M_0, M_1)$ where*

$$M_0 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & t^2 \\ 0 & 1 \end{pmatrix}.$$

It is easy to check $\rho(\max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|)) = 0 < 1$. Hence $\mathbf{M}$ satisfies the conditions in Theorem 3.2, which implies that $\mathbf{M}$ has the row $\mathcal{W}$-property. By direct computation,

$$\begin{aligned}
\max_{d\in[0,1]^2} \|[(I-D)M_0 + DM_1]^{-1}\|_\infty &= \max_{d_1\in[0,1]} (1 + |-t + td_1 + t^2d_1|) \\
&= \|(I - \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|))^{-1} \max(\Lambda_0^{-1}, \Lambda_1^{-1})\|_\infty \\
&= 1 + t^2.
\end{aligned}$$

By choosing $\hat{x} = (-1, 1/t)^T$, we have

$$\alpha\{M_0, M_1\} = \min_{\|x\|_\infty=1} \max_i (M_0x)_i(M_1x)_i \le \max_i (M_0\hat{x})_i(M_1\hat{x})_i = \frac{1}{t^2}.$$

Thus

$$\frac{\|M_0 + M_1\|_\infty}{\alpha\{M_0, M_1\}} \ge (2 + t + t^2)t^2 = O(t^4).$$

Hence the error bound provided in this paper is much sharper than the Xiu-Zhang error bound [16], when $t \to \infty$.

We can also show that the new error bound is tight. For instance, let $t = 1$ and $\mathbf{q} = (q_0, q_1)$ where $q_0 = q_1 = (0, -1)^T$. Then $x^* = (-1, 1)^T$ is the unique solution of the EVLCP$(\mathbf{M}, \mathbf{q})$. The vector $\hat{x} = (1, 0)^T$ satisfies

$$\|\hat{x} - x^*\|_\infty = 2, \quad \max_{d\in[0,1]^2} \|[(I-D)M_0 + DM_1]^{-1}\|_\infty = 2, \quad \|r(\hat{x})\|_\infty = 1.$$

Hence (3.3) and (3.4) hold with the equality at $\hat{x}$.

Example 5.3 *Consider the block matrix* $\mathbf{M} = (M_0, M_1, M_2)$ *where*

$$M_0 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}.$$

It is easy to check that $\mathbf{M}$ *has the row* $\mathcal{W}$*-property. However the conditions in Theorem 4.3 fail, since the spectral radius* $\rho(\max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|, \Lambda_2^{-1}|B_2|)) = \sqrt{3} > 1$. *By direct computation,*

$$\mu_1 := \max_{d\in[0,1]^2} \|[(I-D)M_0 + DM_1]^{-1}\|_\infty = \max\Big(\max_{d_1\in[0,1]} \frac{2}{2+d_1}, 1\Big) = 1,$$

$$\mu_2 := \max_{d\in[0,1]^2} \|[(I-D)M_0 + DM_2]^{-1}\|_\infty = \max\Big(\max_{d_1\in[0,1]} \frac{2}{2+d_1}, 1\Big) = 1,$$

$$\mu_3 := \max_{d\in[0,1]^2} \|[(I-D)M_1 + DM_2]^{-1}\|_\infty = \max_{d_1\in[0,1]} \frac{3-d_1}{3+d_1} = 1.$$

Since $(M_0)_{2.} = (M_1)_{2.} = (M_2)_{2.}$, *we get the error bound coefficient in Theorem 4.1,*

$$\max_{\mathbf{M}'\in\mathcal{R}(\mathbf{M})} \max_{j<l\in\{0,1,2\}} \max_{d\in[0,1]^2} \|[(I-D)M_j' + DM_l']^{-1}\|_\infty = \max(\mu_1, \mu_2, \mu_3) = 1.$$

Example 5.4 *Let a positive diagonal matrix* $A_c \in R^{n\times n}$ *and a nonnegative matrix* $\Delta \in R^{n\times n}$ *be defined by*

$$A_c = \begin{pmatrix} b+\gamma\sin(\frac{1}{n}) & & & \\ & b+\gamma\sin(\frac{2}{n}) & & \\ & & \ddots & \\ & & & b+\gamma\sin(1) \end{pmatrix}, \quad \Delta = \begin{pmatrix} 0 & c & & \\ a & \ddots & \ddots & \\ & \ddots & \ddots & c \\ & & a & 0 \end{pmatrix},$$

where $b \geq a+c > 0$, $a \geq c \geq 0$, $\gamma > 0$. *It is easy to find that* $\rho(A_c^{-1}\Delta) < 1$, *and* $A^I = [A_c - \Delta, A_c + \Delta]$ *is a regular interval matrix. Choose*

$$M_j = A_c - \Delta + \frac{j}{k}2\Delta, \quad j = 0, 1, \ldots, k.$$

Thus for any natural number k, *we obtain the upper bound coefficient in (4.6)*

$$\begin{aligned} \tau : &= \|[I - \max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|, \ldots, \Lambda_k^{-1}|B_k|)]^{-1} \max(\Lambda_0^{-1}, \Lambda_1^{-1}, \ldots, \Lambda_k^{-1})\|_\infty \\ &= \|(I - A_c^{-1}\Delta)^{-1}A_c^{-1}\|_\infty \\ &= \|(A_c - \Delta)^{-1}\|_\infty. \end{aligned}$$

Moreover, since each M_j *is a strictly row diagonally dominant matrix, we can also apply (4.8) to get error bound coefficient*

$$\begin{aligned} \mu : &= \frac{1}{\min_{i\in N} \min((\tilde{M}_0 e)_i, (\tilde{M}_1 e)_i, \ldots, (\tilde{M}_k e)_i)} \\ &= \frac{1}{\min_{i\in N}((A_c - \Delta)e)_i}. \end{aligned}$$

Since $A_c - \Delta$ *is strictly row diagonally dominant, we have* $\tau \leq \mu$.

Table 5.1. Example 5.4, $n = 400$.

γ	a	b	c	τ	μ
n^{-2}	1.5	2	0.5	3.9416E2	3.2000E7
n^{-2}	1.5	2.2	0.5	5.0000E0	5.0000E0
1	1.5	3.0	1.5	2.4399E1	2.0000E2
1	1.5	3.3	1.5	3.0463E0	3.2787E0

References

[1] A. Berman and R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, SIAM Publisher: Philadelphia, 1994.

[2] X. Chen and S.H. Xiang, "Computation of error bounds for P-matrix linear complementarity problems," Math. Programming, vol. 106, pp. 513-525, 2006.

[3] R.W. Cottle and G.B. Dantzig, "A generalization of the linear complementarity problem," J. Comb. Theory, vol. 8, pp. 79-90, 1970.

[4] T. Fujisawa, and E.S. Kuh, "Piecewise-linear theory of nonlinear networks," SIAM J. Appl. Math., vol. 22, pp. 307-328, 1972.

[5] R.W. Cottle, J.-S. Pang and R.E. Stone, The Linear Complementarity Problem, Academic Press: New York, 1992.

[6] S.A. Gabriel, J.J. Moré, "Smoothing of mixed complementarity problem". In: M.C. Ferris and J.-S. Pang (eds.) Complementarity and Variational Problems: State of the Art, SIAM Publications: Philadelphia, pp. 105-116, 1997.

[7] M.S. Gowda and R. Sznajder, "The generalized order linear complementarity problem," SIAM J. Matrix Anal. Appl., vol. 15, pp. 779-795, 1994.

[8] M.S. Gowda and R. Sznajder, "A generalization of the Nash equilibrium theorem on bimatrix games," Inter. J. Game Theory, vol. 25, pp. 1-12, 1996.

[9] O.L. Mangasarian and J. Ren, "New improved error bounds for the linear complementarity problem," Math. Programming, vol. 66, pp. 241-257, 1994.

[10] R. Mathias and J.-S. Pang, "Error bounds for the linear complementarity problem with a **P**-Matrix," Linear Algebra Appl., vol. 132, pp. 123-136, 1990.

[11] J.M. Peng and Z.H. Lin, "A non-interior continuation method for generalized complementarity problems," Math. Programming, vol. 86, pp. 533-563, 1999.

[12] H.D. Qi and L.Z. Liao, "A smoothing Newton method for extended vertical linear complementarity problems," SIAM J. Matrix Anal. Appl., vol. 21, pp. 45-66, 1999.

[13] J. Rohn, "Systems of linear interval equations," Linear Algebra Appl., vol. 126, pp. 39-78, 1989.

[14] M. Sun, "Singular control problems in bounded intervals," Stochastics, vol. 21, pp. 303-344, 1987.

[15] R. Sznajder and M.S. Gowda, “Generalizations of P_0- and P-properties; extended vertical and horizontal linear complementarity problems,” Linear Algebra Appl., vol. 223/224, pp. 695-715, 1995.

[16] N. Xiu and J. Zhang, “A characteristic quantity of **P**-matrices,” Appl. Math. Lett., vol. 15, pp. 41-46, 2002.