Expected Residual Minimization Method for Stochastic Linear Complementarity Problems[1]

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January 13, 2004; Revised November 5, 2004

Abstract. This paper presents a new formulation for the stochastic linear complementarity problem (SLCP), which aims at minimizing an expected residual defined by an NCP function. We generate observations by the quasi-Monte Carlo methods and prove that every accumulation point of minimizers of discrete approximation problems is a minimum expected residual solution of the SLCP. We show that a sufficient condition for the existence of a solution to the expected residual minimization (ERM) problem and its discrete approximations is that there is an observation ω^i such that the coefficient matrix $M(\omega^i)$ is an R_0 matrix. Furthermore, we show that, for a class of problems with fixed coefficient matrices, the ERM problem becomes continuously differentiable and can be solved without using discrete approximation. Preliminary numerical results on a refinery production problem indicate that a solution of the new formulation is desirable.

Key words. Stochastic linear complementarity problem, NCP function, expected residual minimization, R_0 matrix.

AMS subject classifications. 90C33, 90C15

1 Introduction

The stochastic variational inequality problem is to find a vector $x \in R^n$ such that

$$x \in S, \quad F(x,\omega)^T(y-x) \geq 0 \quad \forall y \in S, \tag{1.1}$$

where $S \subseteq R^n$ is a nonempty closed convex set, $F : R^n \times \Omega \to R^n$ is a vector-valued function, and $(\Omega, \mathcal{F}, P)$ is a probability space with $\Omega \subseteq R^m$. When S is the nonnegative orthant $R^n_+ := \{x \in R^n \,|\, x \geq 0\}$, this problem is rewritten as the stochastic complementarity problem

$$F(x,\omega) \geq 0, \quad x \geq 0, \quad F(x,\omega)^T x = 0. \tag{1.2}$$

[1]This work was supported in part by a Grant-in-Aid for Scientific Research from Japan Society for the Promotion of Science.

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In general, there is no x satisfying (1.1) or (1.2) for *all* $\omega \in \Omega$. An existing approach, which may be called the *expected value method*, considers the following deterministic formulations of (1.1) and (1.2), respectively:

$$x \in S, \quad F_\infty(x)^T(y-x) \geq 0 \quad \forall y \in S, \tag{1.3}$$

and

$$F_\infty(x) \geq 0, \quad x \geq 0, \quad F_\infty(x)^T x = 0, \tag{1.4}$$

where $F_\infty(x) := \mathbf{E}[F(x,\omega)]$ is the expectation function of the random function $F(x,\omega)$ [12, 13]. Note that these problems are in general different from those which are obtained by simply replacing the random variable ω by its expected value $\mathbf{E}[\omega]$ in (1.1) or (1.2). Since the expectation function $F_\infty(x)$ is usually still difficult to evaluate exactly, one may construct a sequence of functions $\{F_k(x)\}$ that converges in a certain sense to $F_\infty(x)$, and solve a sequence of problems (1.3) or (1.4) in which $F_\infty(x)$ is replaced by $F_k(x)$. In practice, approximating functions $F_k(x)$ may be constructed by using discrete distributions $\{(\omega^i, p_i),\ i = 1, \ldots, k\}$ as

$$F_k(x) := \sum_{i=1}^{k} F(x, \omega^i) p_i,$$

where p_i is the probability of sample ω^i. Convergence properties of such approximation problems have been studied in [12, 13] by extending the earlier results [23] for stochastic optimization and deterministic variational inequality problems.

The deterministic complementarity problem has played an important role in studying equilibrium systems that arise in mathematical programming, operations research and game theory. There are numerous publications on complementarity problems. In particular, Cottle, Pang and Stone [6] and Facchinei and Pang [8] give comprehensive treatment of theory and methods in complementarity problems. Ferris and Pang [9] present a survey of applications in engineering and economics. On the other hand, in many practical applications, complenmentarity problems often involve uncertain data. However, references on stochastic complementarity problems are relatively scarce [1, 7, 12, 13, 14], compared with stochastic optimization problems for which abundant results are available in the literature; see [15, 18, 25] in particular for simulation-based approaches in stochastic optimization.

In this paper, we study a new deterministic formulation for the stochastic complementarity problem, which employs an *NCP function.* A function $\phi : R^2 \to R$ is called an NCP function if it has the property

$$\phi(a,b) = 0 \quad \Longleftrightarrow \quad a \geq 0,\ b \geq 0,\ ab = 0.$$

Two popular NCP functions are the "min" function

$$\phi(a,b) = \min(a,b)$$

and the Fischer-Burmeister (FB) function [10]

$$\phi(a,b) = a + b - \sqrt{a^2 + b^2}.$$

All NCP functions including the "min" function and FB function are equivalent in the sense that they can reformulate any complementarity problem as a system of nonlinear equations having the same solution set. Moreover, some NCP functions have the same growth rate. In particular, Tseng [27] showed that the "min" function and the FB function satisfy

$$\frac{2}{\sqrt{2}+2}|\min(a,b)| \leq |a+b-\sqrt{a^2+b^2}| \leq (\sqrt{2}+2)|\min(a,b)| \qquad \forall a,b \in R. \tag{1.5}$$

In the last decade, NCP functions have been used as a powerful tool for dealing with linear and nonlinear complementarity problems [2, 5, 11, 16, 17, 19, 22, 26].

In this paper, we propose the following deterministic formulation which is to find a vector $x \in R^n_+$ that minimizes an *expected residual* for the complementarity problem (1.2):

$$\min_{x \in R^n_+} \mathbf{E}[\|\Phi(x,\omega)\|^2], \tag{1.6}$$

where $\Phi : R^n \times \Omega \to R^n$ is defined by

$$\Phi(x,\omega) = \begin{pmatrix} \phi(F_1(x,\omega), x_1) \\ \vdots \\ \phi(F_n(x,\omega), x_n) \end{pmatrix}.$$

Our approach may be regarded as a natural extension of the least-squares method for a system of stochastic equations to the stochastic complementarity problem. Problem (1.6) will be referred to as the *expected residual minimization* (ERM) problem associated with the complementarity problem (1.2).

Throughout, we will focus on the stochastic linear complementarity problem (SLCP)

$$F(x,\omega) := M(\omega)x + q(\omega) \geq 0, \quad x \geq 0, \quad x^T F(x,\omega) = 0, \tag{1.7}$$

where $M(\omega) \in R^{n\times n}$ and $q(\omega) \in R^n$ are continuous random matrices and vectors. The norm $\|\cdot\|$ is the Euclidean norm $\|\cdot\|_2$.

We note that, if Ω has only one realization, then the ERM problem (1.6) associated with an SLCP reduces to the standard LCP and the solubility of (1.6) does not depend on the choice of NCP functions. However, the following example shows that we do not have such equivalence if Ω has more than one realization.

Example 1. Let $n = 1$, $m = 1$, $\Omega = \{\omega^1, \omega^2\} = \{0, 1\}$, $p_1 = p_2 = 1/2$, $M(\omega) = \omega(1-\omega)$ and $q(\omega) = 1 - 2\omega$. Then we have $M(\omega^1) = M(\omega^2) = 0$, $q(\omega^1) = 1$, $q(\omega^2) = -1$ and

$$\mathbf{E}[\|\Phi(x,\omega)\|^2] = \frac{1}{2}\sum_{i=1}^{2} \|\Phi(x,\omega^i)\|^2.$$

The objective function of the ERM problem (1.6) defined by the "min" function is

$$\frac{1}{2}[(\min(1,x))^2 + (\min(-1,x))^2] = \begin{cases} x^2 & x \leq -1 \\ \frac{1}{2}(x^2+1) & -1 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$$

and the problem has a unique solution $x^* = 0$. However, problem (1.6) defined by the FB function has no solution as the objective function

$$\frac{1}{2}[(1 + x - \sqrt{1 + x^2})^2 + (-1 + x - \sqrt{1 + x^2})^2]$$

is monotonically decreasing on $[0, \infty)$.

The remainder of the paper is organized as follows: In Section 2, we show that a sufficient condition for the existence of minimizers of the ERM problem and its discrete approximations is that there is an observation ω^i such that the coefficient matrix $M(\omega^i)$ is an R_0 matrix. Moreover, we prove that every accumulation point of minimizers of discrete approximation problems is a solution of the ERM problem. Especially, for a class of SLCPs with a fixed coefficient matrix $M(\omega) \equiv M$, we show that M being an R_0 matrix is a necessary and sufficient condition for the boundedness of the solution sets of the ERM problem and its discrete approximations with any $q(\omega)$. In Section 3, we show that a class of SLCPs with a fixed coefficient matrix, the ERM problem with the "min" function is smooth and can be solved without using discrete approximation. In Section 4, we present numerical results to compare the proposed ERM method with the expected value method on an example of SLCP. In Section 5, we make some remarks to conclude the paper.

2 Existence and convergence of solutions

Consider the following ERM problem:

$$\min_{x \geq 0} f(x) := \int_\Omega \|\Phi(x, \omega)\|^2 \rho(\omega) d\omega, \tag{2.1}$$

where $\rho : \Omega \to R_+$ is a continuous probability density function satisfying

$$\int_\Omega \rho(\omega) d\omega = 1 \quad \text{and} \quad \int_\Omega (\|M(\omega)\| + \|q(\omega)\|)^2 \rho(\omega) d\omega < \infty. \tag{2.2}$$

In order to find a solution of an ERM problem (1.6) numerically, it is necessary to study the objective function of (1.6) defined by an NCP function. There are a number of NCP functions [2, 17, 19, 22, 26]. In this paper, we focus on the "min" function and the FB function. We use $\Phi_1(x, \omega)$ and $\Phi_2(x, \omega)$ to distinguish the functions $\Phi(x, \omega)$ defined by the "min" function and the FB function, respectively. However, we retain the notation $\Phi(x, \omega)$ to represent both $\Phi_1(x, \omega)$ and $\Phi_2(x, \omega)$ when we discuss their common properties.

Note that, by the continuity of $\Phi_1(\cdot, \omega)$, the function

$$f_1(x) := \int_\Omega \|\Phi_1(x, \omega)\|^2 \rho(\omega) d\omega$$

is continuous. Moreover, by the continuous differentiability of $\|\Phi_2(\cdot, \omega)\|^2$ [16], the function

$$f_2(x) := \int_\Omega \|\Phi_2(x, \omega)\|^2 \rho(\omega) d\omega$$

is continuously differentiable.

Let the level sets of functions f and f_i, $i = 1, 2$, be denoted

$$D(\gamma) := \{\, x \,|\, f(x) \le \gamma\}$$

and

$$D_i(\gamma) := \{\, x \,|\, f_i(x) \le \gamma\}, \quad i = 1, 2,$$

respectively. From (1.5) and the definitions of $f_1(x)$ and $f_2(x)$, we have

$$\frac{2}{3+2\sqrt{2}} f_1(x) \le f_2(x) \le (6+4\sqrt{2}) f_1(x) \qquad \forall x \in R^n.$$

This implies

$$D_2(\gamma) \subseteq D_1\Big(\frac{3+2\sqrt{2}}{2}\gamma\Big) \tag{2.3}$$

and

$$D_1(\gamma) \subseteq D_2((6+4\sqrt{2})\gamma).$$

Recall that M is called an R_0 matrix if

$$x \ge 0,\ Mx \ge 0,\ x^T M x = 0 \quad \Longrightarrow \quad x = 0.$$

Lemma 2.1 *If $M(\bar\omega)$ is an R_0 matrix for some $\bar\omega \in \Omega$, then there is a closed sphere $B(\bar\omega, \delta) := \{\omega \,|\, \|\omega - \bar\omega\| \le \delta\}$ with $\delta > 0$ such that for every $\omega \in \bar B := B(\bar\omega, \delta) \cap \Omega$, $M(\omega)$ is an R_0 matrix.*

Proof: Assume that this lemma is not true. Then there is a sequence $\{\omega^k\} \subset \bar B$ such that

$$\lim_{k\to\infty} \omega^k = \bar\omega$$

and, for every $M(\omega^k)$, we can find $x^k \in R^n$ satisfying

$$x^k \ge 0, \quad x^k \ne 0, \quad M(\omega^k)x^k \ge 0, \quad (x^k)^T M(\omega^k) x^k = 0.$$

Put $v^k = \frac{x^k}{\|x^k\|}$. Then we have

$$v^k \ge 0, \quad \|v^k\| = 1, \quad M(\omega^k)v^k \ge 0, \quad (v^k)^T M(\omega^k) v^k = 0.$$

Letting $k \to \infty$, we obtain a vector $\bar v \in R^n$ satisfying

$$\bar v \ge 0, \quad \|\bar v\| = 1, \quad M(\bar\omega)\bar v \ge 0, \quad \bar v^T M(\bar\omega)\bar v = 0.$$

This contradicts the assumption that $M(\bar\omega)$ is an R_0 matrix. ∎

Lemma 2.2 *Assume that there exists an $\bar\omega \in \Omega$ such that $\rho(\bar\omega) > 0$ and $M(\bar\omega)$ is an R_0 matrix. Then, for any positive number γ, the level set $D(\gamma)$ is bounded.*

Proof: By the continuity of ρ and Lemma 2.1, there exist a closed sphere $B(\bar{\omega}, \delta)$ with $\delta > 0$ and a constant $\rho_0 > 0$ such that $M(\omega)$ is an R_0 matrix and $\rho(\omega) \geq \rho_0$ for all $\omega \in \bar{B} := B(\bar{\omega}, \delta) \cap \Omega$. Let us consider a sequence $\{x^k\} \subset R^n$. Then, by the continuity of $M(\cdot)$, $q(\cdot)$ and Φ, for each k, there exists an $\omega^k \in \bar{B}$ such that

$$\|\Phi(x^k, \omega^k)\| = \min_{\omega \in \bar{B}} \|\Phi(x^k, \omega)\|.$$

It then follows that

$$\begin{aligned} f(x^k) &\geq \int_{\bar{B}} \|\Phi(x^k, \omega)\|^2 \rho(\omega) d\omega \\ &\geq \|\Phi(x^k, \omega^k)\|^2 \rho_0 \int_{\bar{B}} d\omega \\ &\geq C\rho_0 \|\Phi(x^k, \omega^k)\|^2, \end{aligned}$$

where $C = \displaystyle\int_{\bar{B}} d\omega > 0$. To prove the lemma, it suffices show that $\|\Phi(x^k, \omega^k)\| \to +\infty$ whenever $\|x^k\| \to +\infty$.

Suppose $\|x^k\| \to +\infty$. It is not difficult to see that if $x_i^k \to -\infty$ or $(M(\omega^k)x^k + q(\omega^k))_i \to -\infty$ for some i, then we have $|\phi((M(\omega^k)x^k + q(\omega^k))_i, x_i^k)| \to +\infty$ and hence $\|\Phi(x^k, \omega^k)\| \to +\infty$. So we only need to consider the case where both $\{x_i^k\}$ and $\{(M(\omega^k)x^k + q(\omega^k))_i\}$ are bounded below for all i. Then, by dividing each element of these sequences by $\|x^k\|$ and passing to the limit, we obtain

$$(M(\hat{\omega})\hat{v})_i \geq 0, \quad \hat{v}_i \geq 0, \qquad i = 1, \ldots, n,$$

where $\hat{\omega}$ and $\hat{v}$ are accumulation points of $\{\omega^k\}$ and $\{\frac{x^k}{\|x^k\|}\}$, respectively. Note that $\hat{\omega} \in \bar{B}$ and $\|\hat{v}\| = 1$. Since $M(\hat{\omega})$ is an R_0 matrix and $\hat{v} \neq 0$, there must exist some i such that $(M(\hat{\omega})\hat{v})_i > 0$ and $\hat{v}_i > 0$. This implies $(M(\omega^k)x^k + q(\omega^k))_i \to +\infty$ and $x_i^k \to +\infty$, which in turn implies $|\phi((M(\omega^k)x^k + q(\omega^k))_i, x_i^k)| \to +\infty$. Hence we have $\|\Phi(x^k, \omega^k)\| \to +\infty$. This completes the proof. ∎

Now, we employ a quasi-Monte Carlo method for numerical integration [20]. In particular, we use a transformation function $\omega = u(\tilde{\omega})$ to go from an integral on Ω to the integral on the unit hypercube $[0,1]^m \subseteq R^m$ and generate observations $\{\tilde{\omega}^i,\ i = 1, \ldots, N\}$ in the unit hypercube. The function $f(x)$ can then be written as

$$\begin{aligned} f(x) &= \int_{\Omega} \|\Phi(x, \omega)\|^2 \rho(\omega) d\omega \\ &= \int_{[0,1]^m} \|\Phi(x, u(\tilde{\omega}))\|^2 \rho(u(\tilde{\omega})) u'(\tilde{\omega}) d\tilde{\omega} \\ &= \int_{[0,1]^m} \|\Phi(x, u(\tilde{\omega}))\|^2 \tilde{\rho}(\tilde{\omega}) d\tilde{\omega}, \end{aligned}$$

where $\tilde{\rho}(\tilde{\omega}) = \rho(u(\tilde{\omega}))u'(\tilde{\omega})$.

To simplify the notation, without confusion, we suppose $\Omega = [0,1]^m$ and let ω denote $\tilde{\omega}$ in the remainder of this section.

For each k, let

$$f^{(k)}(x) := \frac{1}{N_k} \sum_{\omega^i \in \Omega_k} \|\Phi(x, \omega^i)\|^2 \rho(\omega^i),$$

where $\Omega_k := \{\omega^i,\ i = 1, \dots, N_k\}$ is a set of observations generated by a quasi-Monte Carlo method such that $\Omega_k \subset \Omega$ and $N_k \to \infty$ as $k \to \infty$. In the remainder of this section, we will study the behavior of the following approximations to the ERM problem (2.1):

$$\min_{x \geq 0} f^{(k)}(x). \tag{2.4}$$

By the continuity of $\Phi_1(\cdot, \omega)$, the function

$$f_1^{(k)}(x) := \frac{1}{N_k} \sum_{\omega^i \in \Omega_k} \|\Phi_1(x, \omega^i)\|^2 \rho(\omega^i)$$

is continuous. Moreover, by the continuous differentiability of $\|\Phi_2(\cdot, \omega)\|^2$ [16], the function

$$f_2^{(k)}(x) := \frac{1}{N_k} \sum_{\omega^i \in \Omega_k} \|\Phi_2(x, \omega^i)\|^2 \rho(\omega^i)$$

is continuously differentiable.

Theorem 2.1 *For any fixed $x \in R^n_+$*

$$f(x) = \lim_{k \to \infty} f^{(k)}(x).$$

Proof: By the assumption of this theorem, for any fixed $x \in R^n_+$, we have

$$\begin{aligned} \|\Phi_1(x, \omega)\| &= \|\min(M(\omega)x + q(\omega), x)\| \\ &\leq \|M(\omega)x + q(\omega)\| \\ &\leq \|M(\omega)\|\|x\| + \|q(\omega)\| \\ &\leq (\|x\| + 1)(\|M(\omega)\| + \|q(\omega)\|), \end{aligned}$$

Hence the function $\|\Phi_1(x, \cdot)\|^2 \rho(\cdot)$ is continuous, nonnegative and bounded due to condition (2.2).

Since Φ_2 and Φ_1 have the same growth rate (1.5), $\|\Phi_2(x, \cdot)\|^2 \rho(\cdot)$ is also continuous, nonnegative and bounded. Therefore, we can claim that $\|\Phi(x, \cdot)\|^2 \rho(\cdot)$ is integrable, that is, $0 \leq f(x) < \infty$.

Finally, from the continuity of $\|\Phi(x, \cdot)\|^2 \rho(\cdot)$, and convergence analysis of distribution of sequences [20], we find that $f(x) = \lim_{k \to \infty} f^{(k)}(x)$ for each $x \in R^n_+$. ∎

Let us denote the set of optimal solutions to the ERM problem (2.1) by S and those of approximate ERM problems (2.4) by S_k.

Theorem 2.2 *Assume that there is an $\bar{\omega} \in \Omega$ such that $\rho(\bar{\omega}) > 0$ and $M(\bar{\omega})$ is an R_0 matrix. Then for all large k, S_k is nonempty and bounded. Let $x^{(k)} \in S_k$ for each k. Then every accumulation point of the sequence $\{x^{(k)}\}$ is contained in the set S.*

Proof: By Lemma 2.1, there are a $\bar{k} > 0$ and a closed sphere $B(\bar{\omega}, \delta) := \{\omega \,|\, \|\omega - \bar{\omega}\| \le \delta\}$ with $\delta > 0$ such that for all $k \ge \bar{k}$, $\Omega_k \cap B(\bar{\omega}, \delta)$ are nonempty and for every $\omega \in \Omega_k \cap B(\bar{\omega}, \delta)$, $M(\omega)$ is an R_0 matrix. Hence, we can show that for $k \ge \bar{k}$, S_k is nonempty and bounded in a similar manner to Lemma 2.2.

Let $\bar{x}$ be an accumulation point of $\{x^{(k)}\}$. For simplicity, we assume that $\{x^{(k)}\}$ itself converges to $\bar{x}$. Let $\gamma > f(\bar{x})$. Then from the continuity of f, we have

$$f(x^{(k)}) \le \gamma \quad \text{for all large} \quad k,$$

that is, $x^{(k)} \in D(\gamma)$ for all large k.

Now we show

$$|f^{(k)}(x^{(k)}) - f^{(k)}(\bar{x})| \to 0 \quad \text{as} \quad k \to \infty. \tag{2.5}$$

It is known that for any fixed ω, $\Phi(\cdot, \omega)$ is globally Lipschitzian, that is,

$$\|\Phi(x,\omega) - \Phi(y,\omega)\| \le L(\omega)\|x - y\| \qquad \text{for all } x, y \in R^n,$$

where $L(\omega)$ is a positive constant depending on ω. Moreover, we can show that

$$L(\omega) \le c_1(\|M(\omega)\| + \|q(\omega)\|)$$

for some positive constant c_1.

In a way similar to the proof of Theorem 2.1, for any $x \in D(\gamma)$, we obtain

$$\|\Phi(x,\omega)\| \le c_0(\|x\| + 1)(\|M(\omega)\| + \|q(\omega)\|)$$

for some constant $c_0 > 0$. Furthermore, since $D(\gamma)$ is closed and bounded by Lemma 2.2, we may define

$$c_2 := \max\{\|x\| \mid x \in D(\gamma)\}.$$

Therefore, for any $x, y \in D(\gamma)$, we obtain

$$\begin{aligned}
\left|\|\Phi(x,\omega)\|^2 - \|\Phi(y,\omega)\|^2\right| &= (\|\Phi(x,\omega)\| + \|\Phi(y,\omega)\|)\,|\|\Phi(x,\omega)\| - \|\Phi(y,\omega)\|| \\
&\le c_0c_1(2 + \|x\| + \|y\|)(\|M(\omega)\| + \|q(\omega)\|)^2\|x - y\| \\
&\le C(\|M(\omega)\| + \|q(\omega)\|)^2\|x - y\|,
\end{aligned}$$

where $C := 2c_0c_1(1 + c_2)$.

By the assumption (2.2) on the density function ρ, we obtain

$$\begin{aligned}
|f^{(k)}(x^{(k)}) - f^{(k)}(\bar{x})| &\le \frac{1}{N_k}\sum_{i=1}^{N_k}\left|\|\Phi(x^{(k)},\omega^i)\|^2 - \|\Phi(\bar{x},\omega^i)\|^2\right|\rho(\omega^i) \\
&\le \frac{1}{N_k}\sum_{i=1}^{N_k} C(\|M(\omega^i)\| + \|q(\omega^i)\|)^2\rho(\omega^i)\|x^{(k)} - \bar{x}\| \\
&\le K\|x^{(k)} - \bar{x}\|,
\end{aligned}$$

where K is a constant satisfying

$$K \geq \frac{C}{N_k} \sum_{i=1}^{N_k} (\|M(\omega^i)\| + \|q(\omega^i)\|)^2 \rho(\omega^i) \quad \text{for all large } k.$$

Hence (2.5) holds. Now by Theorem 2.1 and (2.5), we find

$$\begin{aligned} |f^{(k)}(x^{(k)}) - f(\bar{x})| &\leq |f^{(k)}(x^{(k)}) - f^{(k)}(\bar{x})| + |f^{(k)}(\bar{x}) - f(\bar{x})| \\ &\to 0 \qquad \text{as} \quad k \to \infty. \end{aligned}$$

However, by definition, we have

$$f^{(k)}(x^{(k)}) \leq f^{(k)}(x) \quad \text{for all } x \in R^n_+.$$

Therefore, combining the above results, we obtain

$$f(\bar{x}) = \lim_{k\to\infty} f^{(k)}(x^{(k)}) \leq \lim_{k\to\infty} f^{(k)}(x) = f(x) \quad \text{for all } x \in R^n_+.$$

This completes the proof. ∎

The following lemma shows that the converse of Lemma 2.2 is true, when $M(\omega) \equiv M$ and $q(\omega)$ is a linear function of ω.

Lemma 2.3 *Suppose that $M(\omega) \equiv M$. If M is not an R_0 matrix, and $q(\omega)$ is a linear function of ω, then there is a $\gamma > 0$ such that the level set $D(\gamma)$ is unbounded.*

Proof: Since M is not an R_0 matrix, there is an $x \neq 0$ such that

$$x \geq 0,\ Mx \geq 0,\ x^T Mx = 0,$$

which in particular implies that either $x_i = 0$ or $(Mx)_i = 0$ holds for each i. Hence we have

$$\min(x_i, (Mx + q(\omega))_i) = \begin{cases} 0 & x_i = 0,\ (Mx + q(\omega))_i \geq 0 \\ (Mx + q(\omega))_i & x_i = 0,\ (Mx + q(\omega))_i \leq 0 \\ x_i & (Mx)_i = 0,\ q_i(\omega) \geq x_i \\ q_i(\omega) & (Mx)_i = 0,\ q_i(\omega) \leq x_i. \end{cases} \tag{2.6}$$

Note that, since $Mx \geq 0$, we have $|(Mx+q(\omega))_i| \leq |q_i(\omega)|$ whenever $(Mx+q(\omega))_i \leq 0$. Thus it follows from (2.6) that

$$\frac{1}{\sqrt{2}+2} |\Phi_2(x,\omega)_i| \leq |\Phi_1(x,\omega)_i| = |\min(x_i, (Mx + q(\omega))_i)| \leq |q_i(\omega)|,$$

and hence we find

$$f(x) \leq (\sqrt{2}+2)^2 \int_\Omega \|q(\omega)\|^2 \rho(\omega) d\omega =: \gamma.$$

Since by assumption $q(\omega)$ is a linear function of ω, it follows from assumption (2.2) on $\rho(\omega)$ that we have $\gamma < \infty$.

Since the argument above holds for λx with any $\lambda > 0$, that is, $f(\lambda x) \leq \gamma$, we complete the proof. ∎

Theorem 2.3 *Suppose $M(\omega) \equiv M$ and $q(\omega)$ is a linear function of ω, i.e., $q(\omega) = \bar{q} + T\omega$, where $\bar{q} \in R^n$ and $T \in R^{n\times m}$.*

1. *If M is an R_0 matrix, then for any $\bar{q}$ and T, the sets S_k are nonempty and bounded. Let $x^{(k)} \in S_k$ for each k. Then any accumulation point of $\{x^{(k)}\}$ is a solution of the ERM problem (1.6).*

2. *If M is not an R_0 matrix, then there are $\bar{q}$ and T such that the sets S_k are unbounded for all k.*

Proof: Part 1 follows from Theorem 2.2 directly. To prove Part 2, let $x \neq 0$ satisfy

$$x \geq 0, \quad Mx \geq 0, \quad x^T Mx = 0$$

and choose $\bar{q}$ and T such that

$$\begin{aligned} x_i = 0 &\implies q_i(\omega) \geq 0 \quad \text{for all } \omega \in \Omega, \\ x_i > 0 &\implies q_i(\omega) \equiv 0 \quad \text{for all } \omega \in \Omega. \end{aligned}$$

Then we have

$$\|\Phi_2(\lambda x, \omega)\| \leq (\sqrt{2}+2)\|\Phi_1(\lambda x, \omega)\| = (\sqrt{2}+2)\|\min(\lambda x, M(\lambda x) + q(\omega))\| = 0$$

for all $\omega \in \Omega$ and $\lambda > 0$. Hence $f^{(k)}(\lambda x) = 0$, that is, $\lambda x \in S_k$ for all $\lambda > 0$. Since $x \neq 0$, S_k is unbounded. ∎

Remark 2.1. It is well known that for the LCP, the matrix M being a P matrix is a necessary and sufficient condition for the existence and uniqueness of the solution of the LCP with all q. Theorem 2.3 states that the matrix M being an R_0 matrix is a necessary and sufficient condition for the boundedness of the solution set of the ERM problem associated with the SLCP with a fixed matrix M and a random vector $q(\omega)$ that is a linear function of ω.

Remark 2.2 Recall that $G : R^n \to R^n$ is called an R_0 function in a domain $X \subseteq R^n$, if the Jacobian $\nabla G(x)$ is an R_0 matrix for any $x \in X$. (A slightly different definition of an R_0 function is also found in the literature. See [3, 27, 28].) Results for the SLCP with an R_0 matrix may be generalized to the stochastic nonlinear complementarity problem (SNCP) with an R_0 function. For instance, a nonlinear version of Lemma 2.1 may be stated as follows: Suppose that F is continuously differentiable with respect to x. If $F(\cdot, \bar{\omega})$ is an R_0 function for some $\bar{\omega} \in \Omega$ on a closed domain X, and $\nabla_x F$ is continuous on $X \times \Omega$, then there is a closed sphere $B(\bar{\omega}, \delta) = \{\omega \,|\, \|\omega - \bar{\omega}\| \leq \delta\}$ with $\delta > 0$ such that for every $\omega \in \bar{B} := B(\bar{x}, \delta) \cap \Omega$, $F(\cdot, \omega)$ is an R_0 function on X.

Remark 2.3. Assuming that $F(x, \cdot) = M(\cdot)x + q(\cdot)$ is continuous with respect to ω makes the condition for the existence of a solution very simple, that is, there is an $\bar{\omega} \in \Omega$ such that $M(\bar{\omega})$ is an R_0 matrix. Without the continuity in ω, we may establish the existence of a solution by assuming that F is a Carathéodory mapping,[4] i.e., $F(x,\cdot) = M(\cdot)x + q(\cdot)$ is measurable for each x, and there is a closed sphere B such that $M(\omega)$ is an R_0 matrix for every $\omega \in B \cap \Omega$ and

$$\int_{B\cap\Omega} dP(\omega) > 0,$$

where P is a probability distribution function of ω.

3 SLCP with fixed coefficient matrix

In this section, we consider a class of SLCPs, where

$$M(\omega) \equiv M \quad \text{and} \quad q(\omega) \equiv \bar{q} + T\omega,$$

where $M \in R^{n\times n}$, $\bar{q} \in R^n$ and $T \in R^{n\times m}$ are given constants. Moreover we will assume that matrix T has at least one nonzero element in each row.

Lemma 3.1 *Suppose that $\rho : R \to [0,\infty)$ satisfies (2.2). Then for any $c \neq 0$, the function*

$$\psi(a,b) := \int_{-\infty}^{\infty} (\min(a, b + c\omega))^2 \rho(\omega) d\omega$$

is a real-valued and continuously differentiable function.

Proof: It is easy to verify

$$\begin{aligned} \psi(a,b) &= \int_{-\infty}^{\infty} (a - \max(0, a - b - c\omega))^2 \rho(\omega) d\omega \\ &= a^2 - 2a \int_{-\infty}^{\infty} \max(0, a - b - c\omega) \rho(\omega) d\omega \\ &\quad + \int_{-\infty}^{\infty} (\max(0, a - b - c\omega))^2 \rho(\omega) d\omega. \end{aligned} \tag{3.1}$$

By conditions (2.2), for any a, b, we have $\psi(a,b) < \infty$. Next, we show ψ is continuously differentiable. Obviously the first term is continuously differentiable. If $c > 0$, then the second term is a class of smoothing function for the function $\max(0,\cdot)$. By the results in [4], it is continuously differentiable. If $c < 0$, we set $\tilde{\omega} = -\omega$ and $\tilde{\rho}(\tilde{\omega}) := \rho(\omega)$. Then $\tilde{\rho} : R \to [0,\infty)$ is continuous. Hence we can use the results in [4] to claim that

$$\begin{aligned} &\int_{-\infty}^{\infty} \max(0, a - b - c\omega) \rho(\omega) d\omega \\ &= \int_{-\infty}^{\infty} \max(0, a - b - |c|\tilde{\omega}) \tilde{\rho}(\tilde{\omega}) d\tilde{\omega} \end{aligned}$$

is also continuously differentiable with respect to a and b.

[4] F is called a Carathéodory mapping [24] if F is continuous in x for every ω and is measurable in ω for every x.

The continuously differentiability of the third term in (3.1) follows from the fact that $(\max(0, a-b-c\omega))^2 \rho(\omega)$ is continuously differentiable with respect to a and b. ∎

We illustrate Lemma 3.1 by the following uniform density function

$$\rho(\omega) = \begin{cases} 1 & \omega \in [0,\ 1] \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $c > 0$. We can deal with the case $c < 0$ in a similar manner. Calculating the integrals, we obtain

$$\int_{-\infty}^{\infty} \max(0, a-b-c\omega)\rho(\omega)d\omega$$
$$= \begin{cases} 0 & a-b \leq 0 \\ \frac{1}{2c}(a-b)^2 & 0 \leq a-b \leq c \\ a-b-\frac{c}{2} & a-b \geq c \end{cases}$$

and

$$\int_{-\infty}^{\infty} (\max(0, a-b-c\omega))^2 \rho(\omega)d\omega$$
$$= \begin{cases} 0 & a-b \leq 0 \\ \frac{1}{3c}(a-b)^3 & 0 \leq a-b \leq c \\ \frac{1}{3c}[(a-b)^3 - (a-b-c)^3] & a-b \geq c. \end{cases}$$

Summarizing these terms gives the function

$$\psi(a,b) = \begin{cases} a^2 & a-b \leq 0 \\ a^2 - \frac{a}{c}(a-b)^2 + \frac{1}{3c}(a-b)^3 & 0 \leq a-b \leq c \\ a^2 - 2a(a-b) + ac + \frac{1}{3c}[(a-b)^3 - (a-b-c)^3] & a-b \geq c, \end{cases}$$

which is continuously differentiable.

Theorem 3.1 *Let $\Omega = [\alpha_1, \beta_1] \times \ldots \times [\alpha_m, \beta_m]$ with $\alpha_i < \beta_i$, $j = 1, \ldots, m$. Suppose that $\omega_j, j = 1, \ldots, m$, are independent and the density function ρ satisfies (2.2). We also assume that matrix T has at least one nonzero element in each row. Then the function*

$$f_1(x) = \int_\Omega \| \min(x, Mx + \bar{q} + T\omega)\|^2 \rho(\omega)d\omega$$

is real-valued and continuously differentiable.

Proof: From (2.2), we have

$$f_1(x) < \infty.$$

For each j, let ρ_j denote the density function for ω_j. Due to the structure of the problem, f_1 can be written as

$$f_1(x) = \sum_{i=1}^{n} \int_{\alpha_m}^{\beta_m} \cdots \int_{\alpha_1}^{\beta_1} (\min(x_i, (Mx + \bar{q} + T\omega)_i)^2 \rho_1(\omega_1) \cdots \rho_m(\omega_m) d\omega_1 \cdots d\omega_m.$$

Recall that, for each i, there exists at least one j such that $T_{ij} \neq 0$. From Lemma 3.1, it then follows that

$$\begin{aligned}
&\eta_{ij}(x, \omega_1, \ldots, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_m) \\
&\quad := \int_{\alpha_j}^{\beta_j} (\min(x_i, (Mx + \bar{q} + T\omega)_i)^2 \rho_j(\omega_j) d\omega_j \\
&\quad = \int_{\alpha_j}^{\beta_j} (\min(x_i, (Mx + \bar{q})_i + \sum_{l \neq j} T_{il}\omega_l + T_{ij}\omega_j)^2 \rho_j(\omega_j) d\omega_j
\end{aligned}$$

is continuously differentiable in x, since $T_{ij} \neq 0$. Hence

$$\begin{aligned}
f_{1,i}(x) \quad &:= \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_m}^{\beta_m} (\min(x_i, (Mx + \bar{q} + T\omega)_i)^2 \rho_1(\omega_1) \cdots \rho_m(\omega_m) d\omega_1 \cdots d\omega_m \\
&= \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_{j-1}}^{\beta_{j-1}} \int_{\alpha_{j+1}}^{\beta_{j+1}} \cdots \int_{\alpha_m}^{\beta_m} \eta_{ij}(x, \omega_1, \ldots, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_m) \\
&\quad \rho_1(\omega_1) \cdots \rho_{j-1}(\omega_{j-1}) \rho_{j+1}(\omega_{j+1}) \cdots \rho_m(\omega_m) d\omega_1 \cdots d\omega_{j-1} d\omega_{j+1} \cdots d\omega_m
\end{aligned}$$

is a continuously differentiable function. Since each row of T has at least one nonzero element, we can claim that the function $f_1 = \sum_{i=1}^{n} f_{1,i}$ is continuously differentiable. ∎

This theorem suggests that it is possible to solve some special SLCPs without using discrete approximation. For example, we consider the following case: For each i, the ith row of matrix T has just one positive element t_i, and the density function ρ is defined by

$$\rho(\omega) = \begin{cases} 1 & \omega \in [0,1]^m \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

In this case, we can write f_1 explicitly as

$$f_1(x) = \sum_{i=1}^{n} f_{1,i}(x),$$

where

$$f_{1,i}(x) = \begin{cases} x_i^2 & x_i - y_i \leq 0 \\ x_i^2 - \frac{x_i}{t_i}(x_i - y_i)^2 + \frac{1}{3t_i}(x_i - y_i)^3 & 0 \leq x_i - y_i \leq t_i \\ x_i^2 - 2x_i(x_i - y_i) + x_i t_i + \frac{1}{3t_i}[(x_i - y_i)^3 & \\ \qquad -(x_i - y_i - t_i)^3] & x_i - y_i \geq t_i \end{cases}$$

and $y_i = (Mx + \bar{q})_i$. Moreover, it is notable, in this case, the ERM problem (1.6) defined by the "min" function is continuously differentiable. Therefore, we even do not need to use smoothing approximations.

In other cases, for example, ω_j are distributed normally or exponentially. In practice, we may always restrict ourselves to their 99% confidence interval $[\alpha_j, \beta_j]$. Hence Theorem 3.1 is particularly important for the purpose of applications. Moreover, we can use a transformation function to go from an integral on R^m to the integral of $[0, 1]^m$.

Now let us turn our attention to the FB function. The ERM problem (1.6) defined by the FB function is continuously differentiable. In the case where the ith row of matrix T has only one positive element t_i for each $i = 1, \ldots, n$, and the density function ρ is defined by (3.2), the function f_2 can be written as

$$\begin{aligned} f_2(x) &= \sum_{i=1}^{n} (x_i + y_i)^2 + \frac{t_i}{2} + \frac{1}{2t_i}\left(y_i\sqrt{x_i^2 + y_i^2} - (y_i + t_i)\sqrt{x_i^2 + (y_i + t_i)^2}\right) \\ &\quad + \frac{x_i^2}{2t_i} \log\left(\frac{\sqrt{x_i^2 + y_i^2} + y_i}{\sqrt{x_i^2 + (y_i + t_i)^2} + y_i + t_i}\right). \end{aligned} \tag{3.3}$$

Remark 3.1 Since f is not a convex function, it may be interesting to study stationary points of f and its approximations $f^{(k)}$. In the case where $\|\Phi(\cdot, \cdot)\|^2$ is continuously differentiable with respect to x, a stationary point of the ERM problem (2.1) is a solution of the equation

$$H(x) := \min(\nabla f(x), x) = \min\left(\int_\Omega \nabla_x \|\Phi(x, \omega)\|^2 \rho(\omega) d\omega, x\right) = 0, \tag{3.4}$$

while a stationary point of the approximate ERM problem (2.4) is a solution of the equation

$$H^{(k)}(x) := \min(\nabla f^{(k)}(x), x) = \min\left(\frac{1}{N_k} \sum_{\omega^i \in \Omega_k} \nabla_x \|\Phi(x, \omega^i)\|^2 \rho(\omega^i), x\right) = 0. \tag{3.5}$$

Solutions of (3.5) may be shown to converge to a solution of (3.4) under the assumption that $H(x)$ is real-valued for every x and $H(\cdot)$ is uniformly continuous. Moreover, we may construct a superlinearly convergent Newton-like method for solving the ERM problem via (3.4). However, the function $\|\Phi_1(\cdot, \cdot)\|^2$ is generally nondifferentiable, and computing the gradient $\nabla_x \|\Phi_2(x, \omega)\|^2$ is much more expensive than computing $\|\Phi_2(x, \omega)\|^2$. Therefore, an explicit expression of f like (3.3) will be useful theoretically and computationally, since it does not involve an integral any more.

4 Numerical example

To illustrate our model, we use a refinery production problem, which is based on an example in [15] and a market equilibrium model in [6].

A refinery has two products; gasoline and fuel oil. The production and the demand depend on the output of oil and the weather, respectively, which change every day with uncertainty.

On the supply side, the problem is to minimize the production cost with the technological constraint (4.1) and the demand requirement constraints (4.2)–(4.3):

$$\begin{array}{lrcll}
\min\limits_{u_1,u_2} & 2u_1 + 3u_2 & & & \\
\text{s.t.} & u_1 + u_2 & \leq & 100 & (4.1)\\
 & (2+\omega_1)u_1 + 6u_2 & \geq & \omega_3 y_1 + \omega_3 y_2 + 180 + \omega_3 & (4.2)\\
 & 3u_1 + (3.4-\omega_2)u_2 & \geq & \omega_4 y_1 - \omega_4 y_2 + 162 + \omega_4 & (4.3)\\
 & u_1 \geq 0, \quad u_2 \geq 0, & & &
\end{array}$$

where ω_3 and ω_4 are distributed normally, and ω_1 and ω_2 are distributed uniformly and exponentially, respectively, with the following parameters:

$$\begin{array}{lcl}
\text{distr } \omega_1 & \approx & \mathcal{U}[-0.8, 0.8]\\
\text{distr } \omega_2 & \approx & \mathcal{EXP}(\lambda = 2.5)\\
\text{distr } \omega_3 & \approx & \mathcal{N}(0, 12)\\
\text{distr } \omega_4 & \approx & \mathcal{N}(0, 9).
\end{array}$$

Let

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad c = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A = (-1, -1), \quad b = -100,$$

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix}, \quad B(\omega) = \begin{pmatrix} 2+\omega_1 & 6 \\ 3 & 3.4-\omega_2 \end{pmatrix}, \quad D(\omega) = \begin{pmatrix} \omega_3 & \omega_3 \\ \omega_4 & -\omega_4 \end{pmatrix}$$

and

$$d(y, \omega) = D(\omega)y + \begin{pmatrix} 180 + \omega_3 \\ 162 + \omega_4 \end{pmatrix}.$$

Then the problem on the supply side can be written as

$$\begin{array}{ll}
\min_u & c^T u\\
\text{s.t.} & Au \geq b\\
 & B(\omega)u \geq d(y, \omega), \ u \geq 0.
\end{array} \tag{4.4}$$

On the demand side, $d(y, \omega)$ is the market demand function with y representing the demand prices. Generally, $D(\omega)$ is not symmetric, that is, the demand function is not integrable.

The equilibrating condition is given by

$$y = \pi,$$

where π denotes the market supply prices corresponding to the constraints (4.2)–(4.3). Following the argument in [6], we may write the equilibrium conditions for each fixed ω as the linear complementarity problem

$$M(\omega)x + q(\omega) \geq 0, \quad x \geq 0, \quad (M(\omega)x + q(\omega))^T x = 0 \tag{4.5}$$

with

$$x := \begin{pmatrix} u \\ v \\ y \end{pmatrix}, \quad M(\omega) := \begin{pmatrix} 0 & -A^T & -B(\omega)^T \\ A & 0 & 0 \\ B(\omega) & 0 & -D(\omega) \end{pmatrix}, \quad q(\omega) := \begin{pmatrix} c \\ -b \\ -180 - \omega_3 \\ -162 - \omega_4 \end{pmatrix}.$$

A solution of problem (4.5) is dependent on the random variable ω. However, in practice, we do not know the value of the random variable before solving the problem. Now we apply the proposed ERM approach as well as the expected value method to (4.5). In the latter method, one solves the single deterministic LCP

$$\bar{M}x + \bar{q} \geq 0, \quad x \geq 0, \quad (\bar{M}x + \bar{q})^T x = 0, \tag{4.6}$$

where $\bar{M} = \mathbf{E}[M(\omega)]$ and $\bar{q} = \mathbf{E}[q(\omega)]$. In this example, these expected values are given as

$$\mathbf{E}[M(\omega)] = M(\mathbf{E}[\omega]) \quad \text{and} \quad \mathbf{E}[q(\omega)] = q(\mathbf{E}[\omega]).$$

Since the expected values of $\omega_1, \omega_2, \omega_3, \omega_4$ are $0, 0.4, 0, 0$, respectively, we have

$$\bar{M} = \begin{pmatrix} 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & -6 & -3 \\ -1 & -1 & 0 & 0 & 0 \\ 2 & 6 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} 2 \\ 3 \\ 100 \\ -180 \\ -162 \end{pmatrix}.$$

Note that the LCP (4.6) corresponds to the expected value formulation (1.4). An advantage of the expected value formulation (4.6) is that we may solve the problem fast, for example, by applying a Newton-type method [5] to the equivalent system of nonsmooth equations

$$\min(\bar{M}x + \bar{q}, x) = 0.$$

In fact, the solution to (4.6) is computed as

$$\bar{x} = (36, 18, 0.0, 0.25, 0.5)^T.$$

However, as shown below, the decision based on the expected value formulation (4.6) may not be reliable in terms of the feasibility for the constraints (4.2)–(4.3).

To compare the ERM approach with the expected value method, we will consider two cases. In the first case, we assume that only the demand vector d changes its values randomly, whereas the productivity matrix B is fixed with $\omega_1 \equiv 0$ and $\omega_2 \equiv 0.4$. In the second case, both of the vector d and matrix B change their values randomly according to the afore-mentioned distributions. Note that the expected value method yields the identical formulation (4.6) for both cases. In our implementation of the ERM method, we use the following method in [15] to approximate the continuous distributions by discrete ones.

- Generate samples $\omega_j^k,\ j=1,2,3,4,\ k=1,2,\ldots,K$, from their respective 99% confidence intervals (except for uniform distributions)

$$\begin{aligned}
\omega_1^k &\in \mathcal{I}_1 = [-0.8,\ 0.8] \\
\omega_2^k &\in \mathcal{I}_2 = [0.0,\ 1.84] \\
\omega_3^k &\in \mathcal{I}_3 = [-30.91,\ 30.91] \\
\omega_4^k &\in \mathcal{I}_4 = [-23.18,\ 23.18].
\end{aligned}$$

- For each j, divide $\mathcal{I}_j$ into m_j subintervals $\mathcal{I}_{j,i},\ i=1,2,\ldots,m_j$, with equal length.

- For each (j,i), calculate the mean $v_{j,i}$ of samples ω_j^k that belong to the subinterval $\mathcal{I}_{j,i}$.

- For each (j,i), estimate the probability of $v_{j,i}$ as $p_{j,i}=k_{j,i}/K$, where $k_{j,i}$ is the number of samples $\omega_j^k \in \mathcal{I}_{j,i}$.

- Let $N = m_1 \times m_2 \times m_3 \times m_4$ and set the joint distribution $\{(\omega^\ell, p_\ell),\ \ell = 1,2,\ldots,N\}$ as

$$\omega^\ell = \begin{pmatrix} v_{1,i_1} \\ v_{2,i_2} \\ v_{3,i_3} \\ v_{4,i_4} \end{pmatrix}, \qquad p_\ell = p_{1,i_1}p_{2,i_2}p_{3,i_3}p_{4,i_4}$$

  for $i_1=1,\ldots,m_1,\ i_2=1,\ldots,m_2,\ i_3=1,\ldots,m_3,\ i_4=1,\ldots,m_4$.

With these preparations, we obtain an (approximate) ERM problem

$$\min_{x\geq 0}\ f^K(x) := \sum_{\ell=1}^{N} p_\ell \|(\min(M(\omega^\ell)x + q(\omega^\ell), x)\|^2. \tag{4.7}$$

Note that the objective function f^K depends on K, the number of sample data used to construct the approximate ERM problem (4.7). In our numerical experiments, we solved problem (4.7) with various values of K by using *fmincon* in the Matlab (version 6.1) tool box for constrained optimization. We examined the following two cases:

$$\begin{aligned}
&\text{Case 1:}\quad \omega_1 \equiv 0, \quad \omega_2 \equiv 0.4, \quad m_3 = 15, \quad m_4 = 15 \\
&\text{Case 2:}\quad m_1 = 5, \quad m_2 = 9, \quad\ m_3 = 7, \quad\ m_4 = 11.
\end{aligned}$$

Table 1 and Table 2 show the solutions, denoted x^K, of problems (4.7) for $K = 10^i,\ i = 2,\ldots,6$, along with the corresponding objective values $f^K(x^K)$ and the empirical reliability $rel^K(x^K)$, that is, the probability for the solution x to be feasible for the constraints (4.2)–(4.3), which is defined by

$$rel^K(x) = \sum_{\ell=1}^{N} rel_\ell^K(x),$$

Table 1: Case 1: $\omega_1 \equiv 0, \omega_2 \equiv 0.4, m_3 = 15, m_4 = 15$

K	x^K	$f^K(x^K)$	$rel^K(x^K)$	$f^K(\bar{x})$	$rel^K(\bar{x})$
10^2	(46.5948, 40.6903, 0, 0.2773, 0.4289)	0.2824	0.99	214.8772	0.2710
10^3	(45.9339, 42.3753, 0, 0.2752, 0.4339)	0.2861	0.99	215.4773	0.2861
10^4	(46.5239, 41.0382, 0, 0.2751, 0.4340)	0.2861	0.99	214.5933	0.2388
10^5	(46.6777, 41.0575, 0, 0.2752, 0.4337)	0.2859	0.99	213.2355	0.2212
10^6	(46.6268, 41.0411, 0, 0.2753, 0.4337)	0.2859	0.99	212.9540	0.2700

Table 2: Case 2: $m_1 = 5, m_2 = 9, m_3 = 7, m_4 = 11$

K	x^K	$f^K(x^K)$	$rel^K(x^K)$	$f^K(\bar{x})$	$rel^K(\bar{x})$
10^2	(24.2152, 54.9518, 0, 0.2675, 0.4444)	0.3017	0.99	311.7898	0.2905
10^3	(25.2186, 53.9342, 0, 0.2701, 0.4399)	0.3016	0.99	313.8778	0.2996
10^4	(22.9901, 54.9858, 0, 0.2698, 0.4404)	0.3020	0.99	336.0607	0.2945
10^5	(21.9258, 56.2509, 0, 0.2699, 0.4404)	0.3019	0.99	339.6021	0.2972
10^6	(21.9272, 56.2320, 0, 0.2700, 0.4404)	0.3018	0.99	337.2366	0.2980

where

$$rel_\ell^K(x) = \begin{cases} p_\ell & \text{if } B(\omega^\ell)u \geq d(y, \omega^\ell) \\ 0 & \text{otherwise.} \end{cases}$$

Table 1 and Table 2 also show the values of $f^K(x)$ and $rel^K(x)$ for $K = 10^i$, $i = 2, \ldots, 6$, evaluated at the solution $\bar{x} = (36, 18, 0.0, 0.25, 0.5)^T$ of the LCP (4.6) in the expected value method. We may observe that $\bar{x}$ has a rather large residual value for each K and it satisfies the stochastic constraints (4.2)–(4.3) with probability no more than 0.3.

Our preliminary numerical results for the oil refinery problem indicate that the proposed ERM formulation yields a reasonable solution of the stochastic LCP (4.5). In particular, it has desirable properties with regard to the reliability for the random demand requirement constraints, as $B(\omega)u \geq d(y, \omega)$ holds with probability 0.99 for all cases. We note that the ERM problem has the nonnegativity constraints $x \geq 0$ only, and hence may be solved efficiently, although its objective function is nonlinear.

It is known that for a fixed $\omega \in \Omega$, the residual $\|\Phi(x, \omega)\|$ can be used to give some quantitative information about the distance between x and the solution set of the deterministic linear complementarity problem LCP$(M(\omega), q(\omega))$ [6, 21]. For example, assume that $M(\omega_\ell)$ is a P matrix. Then LCP$(M(\omega_\ell), q(\omega_\ell))$ has a unique solution x_ℓ. By Proposition 5.10.5 in [6], we have an absolute error bound for $x \in R^n$,

$$\|x_\ell - x\| \leq n \frac{1 + \|M(\omega_\ell)\|_\infty}{c(M(\omega_\ell))} \|\Phi_1(x, \omega_\ell)\|$$

where the quantity

$$c(M(\omega_\ell)) = \min_{\|z\|_\infty=1}\{\max_{1\le i\le n} z_i(M(\omega_\ell)z)_i\} > 0$$

is well defined. If $M(\omega_\ell)$ is a P matrix for every $\omega_\ell, \ell = 1, 2, \ldots, N$, then we obtain a total absolute error bound for $x \in R^n$,

$$\begin{aligned}
\sum_{\ell=1}^{N} p_\ell \|x_\ell - x\| &\le n\sum_{\ell=1}^{N} p_\ell \frac{1+\|M(\omega_\ell)\|_\infty}{c(M(\omega_\ell))}\|\Phi_1(x,\omega_\ell)\| \\
&\le n\left(\sum_{\ell=1}^{N} p_\ell \left(\frac{1+\|M(\omega_\ell)\|_\infty}{c(M(\omega_\ell))}\right)^2\right)^{\frac{1}{2}} \left(\sum_{\ell=1}^{N} p_\ell \|\Phi_1(x,\omega_\ell)\|^2\right)^{\frac{1}{2}} \\
&= L\sqrt{f^K(x)},
\end{aligned}$$

where the second inequality follows from Cauchy-Schwarz inequality and

$$L := n\left(\sum_{\ell=1}^{N} p_\ell \left(\frac{1+\|M(\omega_\ell)\|_\infty}{c(M(\omega_\ell))}\right)^2\right)^{\frac{1}{2}}.$$

This indicates that the value of $f^K(x)$ may quantify the total error of a given point x to the SLCP. From the numerical results shown in Tabel 1 and Table 2, we may observe that the solutions x^K of the ERM formulations are expected to have much smaller total absolute errors than the solution $\bar{x}$ obtained by the expected value method. By using Theorem 5.10.8 in [6], a similar observation may be made concerning the total relative error.

5 Concluding remarks

We have proposed the ERM formulation for the stochastic complementarity problem and studied some properties of the ERM problem for the SLCP. This may be considered an alternative to the expected value method, which is the only approach currently available for stochastic complementarity problems. Then a natural question arises: Which is superior, the ERM method or the expected value method? Unfortunately, it does not seem possible to give a definitive answer to this question. Each approach has the pros and cons. An obvious advantage of the expected value method is that it only needs to solve complementarity problems, while the ERM method is required to solve nonconvex optimization problems. Moreover, if the random variable ω has small variance, then the expected value method is expected to produce a reasonably good solution. On the other hand, if the variance of the random variable is not small, a solution obtained by the expected value may considerably violate the complementarity conditions for many realizations of the random variable. In such cases, the ERM method is expected to produce a solution that is more reliable in the sense that it satisfies the complementarity conditions more accurately on the whole. This is because the method in itself attempts to minimize the residual for the complementarity conditions, which seems to be a reasonable measure to quantify the goodness of a solution of the stochastic complementarity problem.

Acknowledgements. The authors are very grateful to Professor Jong-Shi Pang for his very constructive comments on this paper. The authors also thank the referees for their very helpful comments.

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