Noname manuscript No.
(will be inserted by the editor)

Convergence of the reweighted ℓ_1 minimization algorithm for ℓ_2-ℓ_p minimization

Xiaojun Chen · Weijun Zhou

Received: 13 March 2013 / Accepted: date

Abstract The iteratively reweighted ℓ_1 minimization algorithm (IRL1) has been widely used for variable selection, signal reconstruction and image processing. In this paper, we show that any sequence generated by the IRL1 is bounded and any accumulation point is a stationary point of the ℓ_2-ℓ_p minimization problem with $0 < p < 1$. Moreover, the stationary point is a global minimizer and the convergence rate is approximately linear under certain conditions. We derive posteriori error bounds which can be used to construct practical stopping rules for the algorithm.

Keywords ℓ_p minimization · stationary points · nonsmooth and nonconvex optimization · pseudo convex · global convergence

Mathematics Subject Classification (2000) 90C26 · 90C46 · 90C90 · 65K10

1 Introduction

The nonsmooth, non-Lipschitz $\ell_p(0 < p < 1)$ regularization has advantages over smooth, convex regularization for restoring image with near edges, sparse

This paper is dedicated to Professor Masao Fukushima on the occasion of his 65th birthday. The first author's work was supported in part by the Hong Kong Research Grant Council, the second author's work was supported by the NSF foundation (10901026) of China and the Key Project of the Scientific Research Fund (12A004) of the Hunan Provincial Education Department.

Xiaojun Chen
Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, China
E-mail: maxjchen@polyu.edu.hk

Weijun Zhou
Department of Mathematics, Changsha University of Science and Technology, Changsha 410004, China
E-mail: weijunzhou@126.com

signal reconstruction and variable selection. Iteratively reweighted ℓ_1 minimization algorithms have been widely used for solving minimization problems with ℓ_p regularization

$$\min_{x\in R^n} \|Ax-b\|_2^2+\lambda\|x\|_p^p, \qquad 0<p<1, \tag{1}$$

where $A\in R^{m\times n}, b\in R^m$, λ is a positive penalty parameter and

$$\|x\|_p^p=\sum_{i=1}^n |x_i|^p.$$

See [3,4,6–8,12,20,23,26]. A version of the IRL1 for solving the ℓ_2-ℓ_p minimization problem (1) is as follows:

$$x^{k+1}\in \arg\min_{x\in R^n} f_k(x,\varepsilon):=\|Ax-b\|_2^2+\lambda\|W^k x\|_1 \tag{2}$$

where the weight W^k =diag(w^k) is defined by the previous iterates and updated in each iteration as

$$w_i^k=\frac{p}{(|x_i^k|+\varepsilon)^{1-p}}, \qquad i=1,\dots,n.$$

Here ε is a positive parameter to ensure that the algorithm is well-defined.

At each iteration, the IRL1 (2) solves a convex ℓ_2-ℓ_1 minimization problem. There are many efficient algorithms for solving the ℓ_2-ℓ_1 minimization problem. In particular, new algorithms for solving large scale ℓ_2-ℓ_1 minimization problems have been developed in recent few years. Nocedal et al. proposed second-order methods for convex ℓ_1 regularized optimization problems [2,24]. Fukushima [17] presented an SOR-type algorithm and a Jacobi-type algorithm that can effectively be applied to the ℓ_2-ℓ_1 problem by exploiting its special structure. The algorithms are globally convergent and can be implemented in a particularly simple manner. Moreover, the algorithms have close relations with coordinate minimization methods.

Extensive numerical experiments have shown that the IRL1 (2) is an efficient method for variable selection, signal reconstruction and image processing. In this paper, we prove that any sequence generated by the IRL1 (2) is bounded and any accumulation point is a stationary point x^* of the following ℓ_2-ℓ_p minimization problem.

$$\min_{x\in R^n} f(x,\varepsilon):=\|Ax-b\|_2^2+\lambda\sum_{i=1}^n(|x_i|+\varepsilon)^p, \qquad 0<p<1. \tag{3}$$

Moreover, the stationary point is a global minimizer of (3) in certain domain and the convergence rate is approximately linear under certain conditions. We derive posteriori error bounds

$$\|x^k-x^*\|_2\le\gamma\|x^{k+1}-x^k\|_2$$

with a positive constant γ, which can be used to construct practical stopping rules for the algorithm.

Note that the problem (2) may have multiple solutions since the objective is not strictly convex. However, convergence results in this paper hold for any choice x^{k+1} in the solution set $\arg\min_{x\in R^n} f_k(x,\varepsilon)$.

The model (3) can be considered as an approximation to the following constrained ℓ_p optimization problem

$$\min_{x\in R^n} \sum_{i=1}^{n}(|x_i|+\varepsilon)^p, \qquad \text{s.t.} \quad Ax=b, \tag{4}$$

which is an approximation of the ℓ_p minimization problem

$$\min_{x\in R^n} \|x\|_p^p, \qquad \text{s.t.} \quad Ax=b. \tag{5}$$

Problems (4) and (5) have been widely used [3–6, 8, 16, 20] when the vector b contains little or no noise. The models (1) and (3) are also called denoising models of (4) and (5). Recently, it has been proved that these four problems are NP-hard in [9, 18]. An advantage of (4) and (3) is that their objective functions are Lipschitz continuous. However, relaxing non-Lipschitzian continuity to Lipschitzian continuity will not change the hardness of the problems.

We summarize some notations and results in nonsmooth optimization [13], which will be used in this paper. It is known that a Lipschitz function $g: R^n \to R$ is almost everywhere differentiable and its subgradient is defined by

$$\partial g(y) = \text{co}\{\lim_{\substack{y^k\to y\\ y^k\in D_g}} \nabla g(y^k)\},$$

where "co" denotes the convex hull and D_g is the set of points at which g is differentiable.

We say x^* is a stationary point of g if $0\in\partial g(x^*)$. If g is a convex function, then x^* is a global minimizer of g in R^n if and only if x^* is a stationary point of g.

A function g is convex if and only if ∂g is a monotone operator, that is,

$$(y-x,\, \xi_y-\xi_x)\ge 0, \qquad \forall\,\xi_y\in\partial g(y), \quad \forall\,\xi_x\in\partial g(x).$$

We say a function $g: R^n\to R$ is strongly pseudoconvex at x on D if for every $\xi\in\partial g(x)$ and every $y\in D$,

$$\xi^T(y-x)\ge 0 \quad \Rightarrow \quad g(y)\ge g(x).$$

We say a function $g: R^n\to R$ is strongly pseudoconvex on D if g is strongly pseudoconvex at every point in D [21, Definition 5.1].

Throughout this paper, $\|\cdot\|$ denotes the ℓ_2 norm and $(x_i)_{1\le i\le n}$ stands for the vector $x\in R^n$. $|x|$ is the absolute value vector of x, that is, $|x| = (|x_1|,\cdots,|x_n|)^T$. The vector $e_i\in R^n$ is the ith column of the identity matrix. The vector $a_i\in R^m$ is the ith column of the matrix A. The cardinality of a subset $T\subset\{1,\ldots,n\}$ is denoted by $|T|$, and its complement set is denoted by T^C.

2 Convergence analysis

In this section, we give convergence analysis for the IRL1 (2). Note that both objective functions f and f_k are Lipschitz continuous for any fixed $\varepsilon > 0$. Hence we can define their subgradients in R^n. Moreover, both functions are nonnegative and satisfy

$$f(x,\varepsilon) \to \infty, \quad f_k(x,\varepsilon) \to \infty \quad \text{as} \quad \|x\| \to \infty. \tag{6}$$

Therefore, the solution sets of (2) and (3) are nonempty and bounded.

Lemma 1 *For any nonnegative constants α, β and $t \in (0,1)$, we have*

$$\alpha^{1-t}\beta^t \leq (1-t)\alpha + t\beta, \tag{7}$$

and equality holds if and only if $\alpha = \beta$.

Proof. Young's inequality states that for any nonnegative constants μ and ν,

$$\mu\nu \leq \frac{1}{q}\mu^q + \frac{1}{r}\nu^r, \qquad \left(\frac{1}{q} + \frac{1}{r} = 1\right)$$

where equality holds if and only if $\mu^q = \nu^r$. Set $\frac{1}{q} = 1 - t$, $\mu^q = \alpha$ and $\nu^r = \beta$ in this inequality. We obtain (7) and equality holds if and only if $\alpha = \beta$. □

Lemma 2 *Let $\{x^k\}$ be the sequence generated by the IRL1 (2). Then we have*

$$f(x^{k+1},\varepsilon) \leq f(x^k,\varepsilon) - \|A(x^{k+1} - x^k)\| - \delta(x^{k+1}, x^k), \tag{8}$$

where

$$\delta(x^{k+1}, x^k) = \lambda \sum_{i=1}^{n} \frac{(1-p)(|x_i^k| + \epsilon) + p(|x_i^{k+1}| + \epsilon) - (|x_i^k| + \epsilon)^{1-p}(|x_i^{k+1}| + \epsilon)^p}{(|x_i^k| + \epsilon)^{1-p}} \geq 0$$

and equality holds if and only if $|x^{k+1}| = |x^k|$.

Proof. Since x^{k+1} is a solution of problem (2), the zero vector is contained in the generalized differential with respect to x, that is,

$$0 \in \partial f_k(x^{k+1}, \varepsilon).$$

See [13]. The function f_k is the sum of $n+1$ convex functions, namely, $\|Ax-b\|^2$ and $|x_i|$, $i = 1, \ldots, n$. By the addition rule of subgradient for the sum of convex functions [13, Corollary 3, p40], we have

$$\partial f_k(x,\varepsilon) = \lambda \sum_{i=1}^{n} \frac{p\partial|x_i|}{(|x_i^k| + \varepsilon)^{1-p}} e_i + 2A^T(Ax - b). \tag{9}$$

Hence, we find

$$0 \in \partial f_k(x^{k+1},\varepsilon) = \lambda \sum_{i=1}^{n} \frac{p}{(|x_i^k| + \epsilon)^{1-p}} \partial|x_i^{k+1}| e_i + 2A^T(Ax^{k+1} - b), \tag{10}$$

which means that there exist $c_i^{k+1} \in \partial|x_i^{k+1}|, i = 1, \cdots, n$ such that

$$\lambda\Big(\frac{pc_i^{k+1}}{(|x_i^k|+\epsilon)^{1-p}}\Big)_{1\le i\le n} + 2A^T(Ax^{k+1}-b) = 0. \tag{11}$$

By the definition of the subdifferential for $|x_i|$, we have

$$c_i^{k+1} = \begin{cases} 1, & \text{if} \quad x_i^{k+1} > 0, \\ -1, & \text{if} \quad x_i^{k+1} < 0, \\ \alpha, & \text{if} \quad x_i^{k+1} = 0, \quad \alpha \in [-1,1]. \end{cases} \tag{12}$$

By (11), (12) and (7), we obtain

$$\begin{aligned}
&f(x^k,\epsilon) - f(x^{k+1},\epsilon) \\
&= \lambda\sum_{i=1}^n\Big((|x_i^k|+\epsilon)^p - (|x_i^{k+1}|+\epsilon)^p\Big) + \|Ax^{k+1}-Ax^k\|^2 + 2(Ax^k - Ax^{k+1})^T(Ax^{k+1}-b) \\
&= \|Ax^{k+1}-Ax^k\|^2 + \lambda\sum_{i=1}^n\Big((|x_i^k|+\epsilon)^p - (|x_i^{k+1}|+\epsilon)^p + \frac{pc_i^{k+1}(x_i^{k+1}-x_i^k)}{(|x_i^k|+\epsilon)^{1-p}}\Big) \qquad (13) \\
&\ge \|Ax^{k+1}-Ax^k\|^2 + \lambda\sum_{i=1}^n\Big((|x_i^k|+\epsilon)^p - (|x_i^{k+1}|+\epsilon)^p + \frac{p(|x_i^{k+1}|-|x_i^k|)}{(|x_i^k|+\epsilon)^{1-p}}\Big) \\
&= \|Ax^{k+1}-Ax^k\|^2 + \lambda\sum_{i=1}^n\Big(\frac{(|x_i^k|+\epsilon) - (|x_i^k|+\epsilon)^{1-p}(|x_i^{k+1}|+\epsilon)^p + p(|x_i^{k+1}|-|x_i^k|)}{(|x_i^k|+\epsilon)^{1-p}}\Big) \\
&= \|Ax^{k+1}-Ax^k\|^2 + \lambda\sum_{i=1}^n\Big(\frac{(1-p)(|x_i^k|+\epsilon) + p(|x_i^{k+1}|+\epsilon) - (|x_i^k|+\epsilon)^{1-p}(|x_i^{k+1}|+\epsilon)^p}{(|x_i^k|+\epsilon)^{1-p}}\Big) \\
&= \|Ax^{k+1}-Ax^k\|^2 + \delta(x^{k+1},x^k) \\
&\ge \|Ax^{k+1}-Ax^k\|^2,
\end{aligned}$$

where the first inequality uses

$$c_i^{k+1}x_i^{k+1} = |x_i^{k+1}| \quad \text{and} \quad |c_i^{k+1}| \le 1$$

and the last inequality uses Lemma 1 to claim that

$$\delta_i(x^{k+1},x^k) = \lambda\frac{(1-p)(|x_i^k|+\epsilon) + p(|x_i^{k+1}|+\epsilon) - (|x_i^k|+\epsilon)^{1-p}(|x_i^{k+1}|+\epsilon)^p}{(|x_i^k|+\epsilon)^{1-p}} \ge 0,$$

and $\delta(x^{k+1},x^k) = \sum_{i=1}^n \delta_i(x^{k+1},x^k) \ge 0$. □

Lemma 3 *Suppose that $g_1 : R^n \to R$ and $-g_2 : R^n \to R$ are convex on a closed convex set Ω, and $g_1(x) \ge 0$ and $g_2(x) > 0$, for all $x \in \Omega$ then $h(x) = \dfrac{g_1(x)}{g_2(x)}$ is strongly pseudoconvex on Ω.*

Proof. This lemma is a simple generalization of [21, Proposition 5.2, p943], which proved that the condition number of a symmetric positive definite matrix is pseudoconvex. For completeness, we give a proof of this lemma.

From the convexity assumption, g_1 and g_2 are locally Lipschitz continuous and for any $x, y \in \Omega$ and $\xi_1 \in \partial g_1(x), \xi_2 \in \partial g_2(x)$, we have

$$g_1(y) - g_1(x) \geq \xi_1^T (y - x),$$

and

$$-g_2(y) + g_2(x) \geq -\xi_2^T (y - x).$$

Hence we obtain

$$\begin{aligned} g_1(y) - h(x) g_2(y) &= g_1(y) - g_1(x) + h(x)(-g_2(y) + g_2(x)) \\ &\geq \xi_1^T (y - x) - h(x) \xi_2^T (y - x) \\ &= (\xi_1 - h(x)\xi_2)^T (y - x) \\ &= g_2(x) \Big(\frac{\xi_1 g_2(x) - g_1(x)\xi_2}{g_2(x)^2} \Big)^T (y - x). \end{aligned}$$

By the quotient rule for the Clarke generalized gradient [13, Proposition 2.3.14], we find that $\dfrac{\xi_1 g_2(x) - g_1(x)\xi_2}{g_2(x)^2} \in \partial h(x)$, from that g_2 and g_1 are Clarke regular. Therefore we have $h(y) \geq h(x)$ if $\xi^T (y - x) \geq 0$ with $\xi \in \partial h(x)$. $\square$

Lemma 4 *For constants $\alpha > 0, \varepsilon > 0$ and $p \in (0, 1)$, let*

$$\phi(t) = |t| + (\alpha t^2 + \beta t)(|t| + \varepsilon)^{1-p}.$$

Then ϕ is convex in $[0, \infty)$ and $(-\infty, 0]$ if

$$|\beta| \leq \frac{\alpha \varepsilon}{1 - p}. \tag{14}$$

Proof. The function ϕ is differentiable in R except $t = 0$. To show the convexity of ϕ, we consider the second derivative of ϕ for $t \neq 0$.

First we consider $t > 0$. By simple calculation, we get

$$\phi^{''}(t) = (t + \varepsilon)^{-1-p} (c_1 t^2 + c_2 t + c_3),$$

where

$$\begin{aligned} c_1 &= \alpha(2 + (4 - p)(1 - p)), \\ c_2 &= (2 - p)((1 - p)\beta + 4\alpha\varepsilon), \\ c_3 &= 2\varepsilon(\alpha\varepsilon + (1 - p)\beta). \end{aligned}$$

Obviously, $c_i > 0, i = 1, 2$ and $c_3 \geq 0$. This implies that ϕ is convex for $t > 0$.

Now, we consider $t < 0$. In this case,

$$\phi(t) = -t + (\alpha t^2 + \beta t)(-t + \varepsilon)^{1-p}.$$

Similarly, we can find that for $t < 0$,

$$\phi^{''}(t) = (-t+\varepsilon)^{-1-p}(c_1t^2 + c_4t + c_5)$$

where

$$c_4 = (2-p)((1-p)\beta - 4\alpha\varepsilon),$$
$$c_5 = 2\varepsilon(\alpha\varepsilon - (1-p)\beta).$$

Obviously, $c_4 < 0$ and $c_5 \geq 0$. This implies that $\phi''(t) \geq 0$ and thus ϕ is convex for $t < 0$. By the continuity of ϕ and that for $t_1t_2 > 0$

$$\phi(\mu t_1 + (1-\mu)t_2) \leq \mu\phi(t_1) + (1-\mu)\phi(t_2), \quad \text{for} \quad 0 \leq \mu \leq 1,$$

we can take $t_1 \to 0$ or $t_2 \to 0$, and claim that ϕ is convex in $[0,\infty)$ and $(-\infty, 0]$. $\square$

Theorem 1 *Let $\{x^k\}$ be a sequence generated by the IRL1 (2). Then the sequence $\{x^k\}$ is bounded and $\lim_{k\to\infty}(x^{k+1} - x^k) = 0$. Moreover, any accumulation point of $\{x^k\}$ is a stationary point x^* of (3).*

Proof. By Lemma 2, the sequence $\{f(x^k, \varepsilon)\}$ is monotonically decreasing and bounded below. Hence it converges. It is clear that the sequence $\{x^k\}$ is contained in the level set

$$\mathcal{L}(x^0) = \{\, x \mid f(x,\varepsilon) \leq f(x^0, \varepsilon)\}.$$

Obviously, $\mathcal{L}(x^0)$ is bounded from (6).

By (8), we have $\delta(x^{k+1}, x^k) \to 0$, as $k \to \infty$. From Lemma 2 and $\delta_i(x^{k+1}, x^k) \geq 0$, we have

$$\lim_{k\to\infty} f(x^k,\varepsilon) - f(x^{k+1},\varepsilon) = \lim_{k\to\infty} \|A(x^{k+1} - x^k)\| = \lim_{k\to\infty} (|x^k| - |x^{k+1}|) = 0. \tag{15}$$

This, together with (13), implies

$$\lim_{k\to\infty} c_i^{k+1}(x_i^{k+1} - x_i^k) = 0, \qquad i = 1,\dots,n, \tag{16}$$

where c_i^{k+1} is defined in (12). Note that $c_i^{k+1} = 0$ only if $x_i^{k+1} = 0$, and $|c_i^{k+1}| = 1$ if $x_i^{k+1} \neq 0$. For a fixed i, suppose that there is a subsequence $\{x_i^{k_j+1}\}$ such that $x_i^{k_j+1} = 0$, then from (15) we have $\lim_{k\to\infty} x_i^k = 0$. Otherwise, we have $|c_i^{k+1}| = 1$ for sufficiently large k, which, together with (16), we have

$$\lim_{k\to\infty} (x_i^{k+1} - x_i^k) = 0, \qquad i = 1,\dots,n. \tag{17}$$

Let $\{x^{n_k}\}$ be a subsequence of $\{x^k\}$ which converges to x^*. By (11) and (17), there exist $c_i^* \in \partial|x_i^*|$, $i = 1,\dots,n$ such that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \lambda\Big(\frac{pc_i^{n_k}}{(|x_i^{n_k} + x_i^{n_k-1} - x_i^{n_k}| + \varepsilon)^{1-p}}\Big)_{1\leq i\leq n} + 2A^T(Ax^{n_k} - b) \\ &= \lambda\Big(\frac{pc_i^*}{(|x_i^*| + \varepsilon)^{1-p}}\Big)_{1\leq i\leq n} + 2A^T(Ax^* - b) \in \partial f(x^*, \varepsilon). \end{aligned} \tag{18}$$

Hence x^* is a stationary point of (3). $\square$

Remark 1 Consider the constrained IRL1 [3]

$$x^{k+1} \in \arg\min_{x\in R^n} \|W^k x\|_1, \qquad \text{s.t.} \quad Ax = b \tag{19}$$

where the weight W^k =diag(w^k) is defined by

$$w_i^k = \frac{p}{(|x_i^k| + \varepsilon)^{1-p}}, \qquad i = 1, \ldots, n.$$

From the proof of Theorem 1, we can easily find that any accumulation point x^* of the sequence $\{x^k\}$ generated by (19) is a stationary point of (4), that is, there is $\mu \in R^m$ such that

$$\begin{aligned} 0 &= \Big(\frac{pc_i^*}{(|x_i^*| + \epsilon)^{1-p}}\Big)_{1\le i\le n} + A^T\mu \in \partial_x L(x^*, \mu) \\ 0 &= Ax - b \end{aligned}$$

where $c_i \in \partial|x_i^*|$ and $L(x, \mu) = \sum_{i=1}^{n}(|x_i| + \varepsilon)^p + \mu(Ax - b)$ is the Lagrangian function of problem (4).

Let $A_I = [a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n]$ and $x_I^* = [x_1^*, \ldots, x_{i-1}^*, x_{i+1}^*, \ldots, x_n^*]^T$.

Theorem 2 *Let x^* be a stationary point of (3). The following statements hold.*

(1) If for some i, $\varepsilon \ge \left(\frac{\lambda(1-p)p}{2\|a_i\|^2}\right)^{\frac{1}{2-p}}$ holds, then

$$f(x^*, \varepsilon) \le f(x^* + te_i, \varepsilon), \quad \text{for} \quad t \in \begin{cases} [-x_i^*, \infty) & \text{if} \quad x_i^* \ge 0, \\ (-\infty, -x_i^*] & \text{if} \quad x_i^* \le 0. \end{cases} \tag{20}$$

(2) If for some i, $\varepsilon \ge \frac{2(1-p)|a_i^T(A_I x_I^* - b)|}{\|a_i\|^2}$ holds, then (20) holds. Moreover, if for some i, $a_i^T(A_I x_I^* - b) = 0$ holds, then $x_i^* = 0$ and

$$f(x^*, \varepsilon) \le f(x^* + te_i, \varepsilon), \quad \text{for} \quad t \in R. \tag{21}$$

Proof. (1) Let

$$\varphi(t) = \lambda\||x^* + te_i| + \varepsilon\|_p^p + \|A(x^* + te_i) - b\|^2. \tag{22}$$

The subdifferential of φ is

$$\partial\varphi(t) = \lambda\frac{p\,\mathrm{sign}(x_i^* + t)}{(|x_i^* + t| + \varepsilon)^{1-p}} + 2a_i^T(A(x^* + te_i) - b).$$

By (18), we have $0 \in \partial\varphi(0)$, that is, 0 is a stationary point of φ. For t_1 and t_2 satisfying $(x_i^* + t_1)(x_i^* + t_2) > 0$, φ is continuously twice differentiable on $[t_1, t_2]$. Thus there is t_0 between t_1 and t_2 such that

$$\varphi'(t_1) - \varphi'(t_2) = (-\frac{\lambda(1-p)p}{(|x_i^* + t_0| + \varepsilon)^{2-p}} + 2\|a_i\|^2)(t_1 - t_2).$$

Hence if $\varepsilon \geq \left(\frac{\lambda(1-p)p}{2\|a_i\|^2}\right)^{\frac{1}{2-p}}$, then

$$(t_1 - t_2)(\varphi'(t_1) - \varphi'(t_2)) = (-\frac{\lambda(1-p)p}{(|x_i^* + t_0| + \varepsilon)^{2-p}} + 2\|a_i\|^2)(t_1 - t_2)^2$$
$$\geq (-\frac{\lambda(1-p)p}{\varepsilon^{2-p}} + 2\|a_i\|^2)(t_1 - t_2)^2 \geq 0.$$

Hence φ is convex in $[-x_i^*, \infty)$ if $x_i^* \geq 0$, and in $(-\infty, -x_i^*]$ if $x_i^* \leq 0$. This, together with $0 \in \partial\varphi(0)$ implies that 0 is the minimizer of φ in $(-x_i^*, \infty)$ if $x_i^* \geq 0$, and in $(-\infty, -x_i^*)$ if $x_i^* \leq 0$. This gives (20).

(2) To prove this part, we show φ defined in (22) is strongly pseudoconvex in $[-x_i^*, \infty)$ and $(-\infty, -x_i^*]$. The function φ can be rewritten as

$$\varphi(t) = \lambda(|x_i^* + t| + \varepsilon)^p + \|a_i\|^2(x_i^* + t)^2 + 2a_i^T(A_I x_I^* - b)(x_i^* + t) + c_0,$$
$$= \lambda\Big(\frac{|x_i^* + t| + \varepsilon + \Big(\frac{\|a_i\|^2}{\lambda}(x_i^* + t)^2 + \frac{2a_i^T(A_I x_I^* - b)}{\lambda}(x_i^* + t)\Big)(|x_i^* + t| + \varepsilon)^{1-p}}{(|x_i^* + t| + \varepsilon)^{1-p}}\Big) + c_0,$$

where c_0 is a constant. Using Lemma 4, with

$$\alpha = \frac{\|a_i\|^2}{\lambda} \quad \text{and} \quad \beta = \frac{2a_i^T(A_I x_I^* - b)}{\lambda},$$

we find that the function

$$|x_i^* + t| + \varepsilon + \Big(\frac{\|a_i\|^2}{\lambda}(x_i^* + t)^2 + \frac{2a_i^T(A_I x_I^* - b)}{\lambda}(x_i^* + t)\Big)(|x_i^* + t| + \varepsilon)^{1-p}$$

is convex. Since $(|x_i^* + t| + \varepsilon)^{1-p}$ is concave, we find that φ is strongly pseudoconvex by Lemma 3.

By the definition of the strong pseudo convexity and (18), from $\varphi(0) = f(x^*, \varepsilon)$ and $\varphi(t) = f(x^* + te_i, \varepsilon)$, we obtain (20).

If $a_i^T(A_I x_I^* - b) = 0$, then (18) implies that

$$0 = \lambda\Big(\frac{pc_i^*}{(|x_i^*| + \epsilon)^{1-p}}\Big) + 2a_i^T a_i x_i^*. \tag{23}$$

Since $c_i^* = 1$ if $x_i^* > 0$ and $c_i^* = -1$ if $x_i^* < 0$, (23) only holds at $x_i^* = 0$. Moreover, it is easy to see that in such case with $x_i^* = 0$,

$$\varphi(-x_i^*) = \varphi(0) \leq \varphi(t), \qquad \text{for} \quad t \in R,$$

that is,

$$f(x^* - x_i^* e_i, \varepsilon) = f(x^*, \varepsilon) \leq f(x^* + te_i, \varepsilon), \quad \text{for} \quad t \in R.$$

We obtain the desired results. □

In [8], it was shown that any local minimizer x^* of (1) satisfies

$$\textit{either} \quad |x_i^*| = 0 \quad \textit{or} \quad |x_i^*| \geq L_i, \qquad \forall\, i = 1, \cdots, n, \tag{24}$$

where

$$L_i := \Big(\frac{\lambda p(1-p)}{2\|a_i\|^2}\Big)^{\frac{1}{2-p}}.$$

This lower bound for absolute value of nonzero elements of any local minimizer of (1) can be easily extended to the model (3). We give the lower bound theory for (3) in the following theorem.

Theorem 3 *If $0 \le \epsilon < L := \min_{1\le i\le n} L_i$, then every local minimizer x^* of (3) satisfies*

$$\textit{either}\quad |x_i^*| = 0 \quad \textit{or} \quad |x_i^*| \ge L_i - \varepsilon, \qquad \forall\, i = 1, \cdots, n. \tag{25}$$

Proof. This Theorem is a simple generalization of Theorem 2.1 in [8]. For completeness, we give a brief proof.

Let x^* be a local minimizer of (3) with $\|x^*\|_0 = k$, without loss of generality, we assume

$$x^* = (x_1^*, \cdots, x_k^*, 0, \cdots, 0)^T.$$

Let $z^* = (x_1^*, \cdots, x_k^*)^T$ and $B \in R^{m\times k}$ be the submatrix of A, whose columns are the first k columns of A. For a fixed $\varepsilon \ge 0$, define a function $g : R^k \to R$ by

$$g(z,\varepsilon) = \|Bz - b\|^2 + \lambda \sum_{i=1}^{k} (|z_i| + \varepsilon)^p + (n-k)\varepsilon^p.$$

We have

$$f(x^*,\varepsilon) = \|Ax^* - b\|^2 + \lambda \sum_{i=1}^{n} (|x_i^*| + \varepsilon)^p = \|Bz^* - b\|^2 + \lambda \sum_{i=1}^{k} (|z_i^*| + \varepsilon)^p + (n-k)\varepsilon^p.$$

Since $|z_i^*| > 0, i = 1, \cdots, k$, g is continuously differentiable at z^*. Moreover, in a neighbourhood of x^*,

$$\begin{aligned} g(z^*,\varepsilon) = f(x^*,\varepsilon) &\le \min\{f(x,\varepsilon) | x_i = 0, i = k+1, \cdots, n\} \\ &= \min\{g(z,\varepsilon) | z \in R^k\}, \end{aligned}$$

which implies that z^* is a local minimizer of the function g. Hence the second order necessary condition for

$$\min_{z\in R^k} g(z,\varepsilon) \tag{26}$$

holds at z^*, which gives that the matrix

$$2B^TB + \lambda p(p-1)\mathrm{diag}((|z^*| + \varepsilon)^{p-2})$$

is positive semi-definite. Therefore, we obtain

$$2e_i^T B^T B e_i + \lambda p(p-1)\mathrm{diag}((|z_i^*| + \varepsilon)^{p-2}) \ge 0, \quad i = 1, \cdots, k$$

where e_i is the ith column of the identity matrix of $R^{k\times k}$.

Note that $\|a_i\|^2 = e_i^T B^T B e_i$. We find that

$$(|z_i^*| + \varepsilon)^{p-2} \leq \frac{2\|a_i\|^2}{\lambda p(1-p)}, \quad i = 1, \cdots, k$$

which implies that

$$|z_i^*| \geq \Big(\frac{\lambda p(1-p)}{2\|a_i\|^2}\Big)^{\frac{1}{2-p}} - \varepsilon = L_i - \varepsilon, \quad i = 1, \cdots, k.$$

Hence for any local minimizer x^* of (3) if $x_i^* \neq 0$, then $|x_i^*| \geq L_i - \varepsilon$. □

In the proof of Theorem 3, we use the second order necessary optimality condition of a subproblem to derive the lower bound (25). Similarly, we can use the first order necessary optimality condition of the subproblem to derive other lower bound as Theorem 3.1 in [8]. Moreover, the lower bound theory of nonzero entries in local minimizers can be extended to vectors satisfying the first order and second order necessary optimality conditions.

For $x \in R^n$, let X=diag(x) and $S = \{i | x_i \neq 0\}$. For problem (1), the necessary optimality conditions are as follows [8].

First order: $2XA^T(Ax - b) + \lambda p|x|^p = 0$

Second order: $2XA^TAX + \lambda p(p-1)\text{diag}(|x|^p)$ is positive semi-definite.

These two conditions can be equivalently written as

First order: $(2A^T(Ax-b))_S + \lambda p|x_S|^{p-1} = 0, \quad x_{S^C} = 0$

Second order: $2A_S^TA_S + \lambda p(p-1)\text{diag}(|x_S|^{p-2})$ is positive semi-definite, $x_{S^C} = 0$.

For problem (3), the necessary optimality conditions are as follows.

First order: $(2A^T(Ax-b) + \lambda p(|x| + \varepsilon)^{p-1})_S = 0, \quad x_{S^C} = 0$

Second order: $2A_S^TA_S + \lambda p(p-1)\text{diag}((|x_S| + \varepsilon)^{p-2})$ is positive semi-definite, $x_{S^C} = 0$.

It is easy to see that if x satisfies the second order necessary optimality condition of (1), then x satisfies the second order necessary optimality condition of (3), from the following inequality

$$\lambda p(p-1)|x_i|^{p-2} < \lambda p(p-1)(|x_i| + \varepsilon)^{p-2}, \quad \text{for} \quad i \in S.$$

Moreover, we have the following proposition for problems (1) and (3).

Proposition 1 *For any sequence $\{\varepsilon_k\}$, which satisfies $\varepsilon_k \to 0^+$, the following statements hold.*

(i) Let $\{x_{\varepsilon_k}\}$ be a sequence of vectors satisfying the first order necessary optimality condition of (3) with $\varepsilon = \varepsilon_k$, then any accumulation point of $\{x_{\varepsilon_k}\}$ satisfies the first order necessary optimality condition of (1).

(ii) Let $\{x_{\varepsilon_k}\}$ be a sequence of vectors satisfying the second order necessary optimality condition of (3) with $\varepsilon = \varepsilon_k$, then any accumulation point of $\{x_{\varepsilon_k}\}$ satisfies the second order necessary optimality condition of (1).

(iii) Let $\{x_{\varepsilon_k}\}$ be a sequence of global minimizers of (3) with $\varepsilon = \varepsilon_k$, then any accumulation point of $\{x_{\varepsilon_k}\}$ is a global minimizer of (1).

Proof. The proof for parts (i) and (ii) can be easily derived by the definition. (See [11].) To prove part (iii), let x^* be a global minimizer of (1) and x_{ε_k} be a global minimizer of problem (3) with $\varepsilon = \varepsilon_k$, and $\bar{x}$ be an accumulation point of $\{x_{\varepsilon_k}\}$. Without loss of generality, we assume that $x_{\varepsilon_k} \to \bar{x}$ as $\varepsilon_k \to 0^+$. Then from

$$0 \le \|Ax_{\varepsilon_k} - b\|_2^2 + \lambda \|x_{\varepsilon_k}\|_p^p \le f(x_{\varepsilon_k}, \varepsilon_k) \le f(x^*, \varepsilon_k),$$

we have $f(\bar{x}) \le f(x^*)$ when $\varepsilon_k \to 0^+$. Hence $\bar{x}$ is a global minimizer of (1). □

Now we derive the convergence rate of the IRL1 (2) and error bounds.

Theorem 4 *Assume that the sequence $\{x^k\}$ generated by (2) converges to a local minimizer x^* of (3). Denote $S = \{i|\ x_i^* \neq 0\}$ and $\beta = \min_{i\in S} |x_i^*|$. If*

$$\frac{\lambda p(1-p)}{\beta^{2-p}} \le 2\lambda_{\min}(A_S^T A_S), \tag{27}$$

then there exist positive constants $\gamma_i, i = 1, 2, 3$ and $c \in (0, 1)$ such that for all sufficiently large k

$$\|x_S^k - x_S^*\| \le \gamma_1 \|x_S^k - x_S^{k+1}\| + \gamma_2 \|x_{S^C}^{k+1}\|,$$

and

$$\|x_S^{k+1} - x_S^*\| \le c\|x_S^k - x_S^*\| + \gamma_3 \|x_{S^C}^{k+1}\|.$$

Proof. Denote $S_k = \{\, i|\ |x_i^k| \neq 0\,\}$. Since $x^k \to x^*$, by (25) for sufficiently large k, we have $S \subset S_k$ and there exists a small constant $\delta \in (0, \epsilon)$ such that $|x_i^k| \ge \beta - \delta$, for $i \in S$.

Consider the function

$$g(z, \varepsilon) = \sum_{i\in S} \lambda(|z_i| + \epsilon)^p + \|A_S z - b\|^2 + \lambda \sum_{i\in S^C} \varepsilon^p, \qquad z \in R^{|S|}.$$

From the proof of Theorem 3, we see that x_S^* is a local minimizer of $g(z, \varepsilon)$. Therefore we have from the optimal condition for minimizing $g(z, \varepsilon)$ that

$$\Big(\frac{\lambda p \,\mathrm{sign}(x_i^*)}{(|x_i^*| + \epsilon)^{1-p}}\Big)_{i\in S} + 2A_S^T(A_S x_S^* - b) = 0, \tag{28}$$

and the matrix

$$\mathrm{diag}\Big(\Big(\frac{\lambda p(p-1)}{(|x_i^*| + \epsilon)^{2-p}}\Big)_{i\in S}\Big) + 2A_S^T A_S$$

is positive semi-definite, which implies that the matrix $A_S^T A_S$ is positive definite since $p - 1 < 0$.

Since x^{k+1} is a global minimizer of $f_k(x, \varepsilon)$ and for sufficiently large k,

$$\mathrm{sign}(x_i^{k+1}) = \mathrm{sign}(x_i^k) = \mathrm{sign}(x_i^*), \qquad i \in S,$$

we have

$$\begin{pmatrix} \left(\frac{\lambda p\,\mathrm{sign}(x_i^*)}{(|x_i^k|+\epsilon)^{1-p}}\right)_{i\in S} \\ \left(\frac{\lambda p\, c_i^{k+1}}{(|x_i^k|+\epsilon)^{1-p}}\right)_{i\in S^C} \end{pmatrix} + 2\begin{pmatrix} A_S^T(Ax^{k+1}-b) \\ A_{S^C}(Ax^{k+1}-b) \end{pmatrix} = 0, \tag{29}$$

where $c_i^{k+1} \in \partial|x_i^{k+1}|$. By (28) and (29), we have

$$B_S(x_S^k - x_S^*) = 2A_S^T A_S(x_S^k - x_S^{k+1}) - 2A_S^T A_{S^C} x_{S^C}^{k+1}, \tag{30}$$

and

$$x_S^{k+1} - x_S^* = -(2A_S^T A_S)^{-1} D_S(x_S^k - x_S^*) - (A_S^T A_S)^{-1} A_S^T A_{S^C} x_{S^C}^{k+1}, \tag{31}$$

where ζ_i is between x_i^* and x_i^k for any $i \in S$, and

$$D_S = \mathrm{diag}\Big(\Big(\frac{\lambda p(p-1)}{(|\zeta_i|+\epsilon)^{2-p}}\Big)_{i\in S}\Big), \qquad B_S = D_S + 2A_S^T A_S.$$

From $\mathrm{sign}(x_i^k) = \mathrm{sign}(x_i^*)$, we have $|\zeta_i| \geq \beta - \delta > 0,$ for $i \in S$. Moreover, from (27) and the following inequalities

$$\frac{\lambda p(1-p)}{(|\zeta_i|+\epsilon)^{2-p}} \leq \frac{\lambda p(1-p)}{(\beta-\delta+\epsilon)^{2-p}} < \frac{\lambda p(1-p)}{(\beta)^{2-p}} \leq 2\lambda_{\min}(A_S^T A_S), \tag{32}$$

we obtain that B_S is nonsingular and we have from (30) and (31) that

$$\|x_S^k - x_S^*\| \leq 2\|B_S^{-1}\|\|A_S^T A_S\|\|x_S^k - x_S^{k+1}\| + 2\|B_S^{-1}\|\|A_S^T A_{S^C}\|\|x_{S^C}^{k+1}\|,$$

and

$$\|x_S^{k+1} - x_S^*\| \leq \|(2A_S^T A_S)^{-1} D_S\|\|x_S^k - x_S^*\| + \|(A_S^T A_S)^{-1} A_S^T A_{S^C}\|\|x_{S^C}^{k+1}\|.$$

By (27) and (32), we have $\|(2A_S^T A_S)^{-1} D_S\| < 1$. Therefore, we complete the proof with $\gamma_1 = 2\|B_S^{-1}\|\|A_S^T A_S\|$, $\gamma_2 = 2\|B_S^{-1}\|\|A_S^T A_{S^C}\|$, $\gamma_3 = \|(A_S^T A_S)^{-1} A_S^T A_{S^C}\|$ and $c = \|(2A_S^T A_S)^{-1} D_S\|$. □

Based on Theorem 3, for large k, entries x_i^k satisfying $|x_i^k| << L_i - \varepsilon$ very likely converge to zero. If we can guess the index set S of nonzero elements x^* correctly and set $x_{S^C}^k = 0$ for all large k, then from Theorem 4, we have

$$\|x^k - x^*\| = \|x_S^k - x_S^*\| \leq \gamma_1\|x_S^k - x_S^{k+1}\| = \gamma_1\|x^k - x^{k+1}\|$$

and

$$\|x^{k+1} - x^*\| = \|x_S^{k+1} - x_S^*\| \leq c\|x_S^k - x_S^*\| = c\|x^k - x^*\|.$$

3 Conclusion

Regularized minimization problems with ℓ_p regularization arise frequently in many fields such as finance, econometrics and signal processing. On the statistical side, the ℓ_p regularization is called the bridge penalty and minimizers of the minimization problem (1) with $\|x\|_p^p$ regularization are called bridge estimators [19]. Theoretical results show that the bridge estimators have various attractive features due to the concavity and non-Lipschitzian property of the regularization function $\|x\|_p^p$. However, the minimization problem (1) is nonconvex and non-Lipschitz. It is shown in [9] that (1) and its smoothed version (3) are strongly NP-hard. The reweighted ℓ_1 minimization algorithm (IRL1) is developed to solve (1). The IRL1 has been widely used for variable selection, signal reconstruction and image processing. Moreover, extensive numerical experiments showed that the IRL1 is efficient for many applications. We prove that any sequence generated by the IRL1 is bounded and any accumulation point is a stationary point of the minimization problem (3). In general, a stationary point of the minimization problem (3) is not a minimizer of (1). However, on the positive side, Theorem 3 shows any local minimizer of (3) has certain sparsity. These results are important for developing algorithms for solving the nonconvex and non-Lipschitz minimization problem (1) and applications in variable selection, signal reconstruction and image processing.

Acknowledgements We would like to thank the two referees for their very helpful comments and suggestions.

References

1. W. Bian and X. Chen, Smoothing neural network for constrained non-Lipschitz optimization with applications, IEEE Transanction on Neural Networks, 23 (2012), 399-411.
2. R.H. Byrd, G.M.Chin, J. Nocedal and Y. Wu, Sample size selection in optimization methods for machine learning, Mathematical Programming, 134 (2012), 127-155.
3. E. Candès, M. Wakin and S. Boyd, Enhancing sparsity by reweighted l_1 minimization, Journal of Fourier Analysis and Applications, 14 (2008), 877-905.
4. R. Chartrand, Exact reconstruction of sparse signals via nonconvex minimization, IEEE Signal Processing Letters, 14 (2007), 707-710.
5. R. Chartrand and V. Staneva, Restricted isometry properties and nonconvex compressive sensing, Inverse Problem, 24 (2008), 1-14.
6. R. Chartrand and W. Yin, Iteratively reweighted algorithms for compressive sensing, In ICASSP 2008, pages 3869-3872, March 2008.
7. X. Chen, Smoothing methods for nonsmooth, nonconvex minimization, Mathematical Programming, 134 (2012), 71-99.
8. X. Chen, F. Xu and Y. Ye, Lower bound theory of nonzero entries in solutions of l_2-l_p minimization, SIAM Journal on Scientific Computing, 32 (2010), 2832-2852.
9. X. Chen, D. Ge, Z. Wang and Y. Ye, Complexity of unconstrained L_2-L_p minimization, to appear in Mathematical Programming.
10. X. Chen, M. Ng and C. Zhang, Nonconvex l_p regularization and box constrained model for image restoration, IEEE Transaction on Image Processing, 21 (2012), 4709-4721.
11. X. Chen, L. Niu and Y. Yuan, Optimality conditions and smoothing trust region Newton method for non-Lipschitz optimization, 2012, Department of Applied Mathematics, The Hong Kong Polytechnic University.

12. X. Chen and W. Zhou, Convergence of reweighted l_1 minimization algorithms and unique solution of truncated l_p minimization, 2010, Department of Applied Mathematics, The Hong Kong Polytechnic University.
13. F. H. Clarke, Optimization and Nonsmooth Analysis, John Wiley, New York, 1983.
14. D. L. Donoho and Y. Tsaig, Fast solution of l_1-norm minimization problems when the solution may be sparse, IEEE Transactions on Information Theory, 54 (2008), 4789-4812.
15. M. A. T. Figueiredo, J. M. Bioucas-Dias and R. D. Nowak, Majorization-minimization algorithms for wavelet-based image restoration, IEEE Transactions on Image Processing, 16 (2007), 2980-2991.
16. S. Foucart and M. J. Lai, Sparsest solutions of under-determined linear systems via l_q minimization for $0 < q \leq 1$, Applied and Computational Harmonic Analysis, 26 (2009), 395-407.
17. M. Fukushima, SOR- and Jacobi-type iterative methods for solving ℓ_1-ℓ_2 problems by way of Fenchel duality, Optimization Letters 6 (2012), 679-686.
18. D. Ge, X. Jiang and Y. Ye, A note on complexity of L_p minimization, Mathematical Programming, 129 (2011), 285-299.
19. J. Huang, J. L. Horowitz and S. Ma, Asymptotic properties of bridge estimators in sparse high-dimensional regression models, The Annals of Statistics, 36 (2008), 587-613.
20. M. Lai and Y. Wang, An unconstrained l_q minimization with $0 < q < 1$ for sparse solution of under-determined linear systems, SIAM Journal on Optimization, 21 (2011), 82-101.
21. P. Marechal and J. Ye, Optimizing condition numbers, SIAM Journal on Optimization, 20 (2009), 935-947.
22. M. Nikolova, Analysis of the recovery of edges in images and signals by minimizing nonconvex regularized least-squares, Multiscale Modeling and Simulation, 4 (2005), 960-991.
23. M. Nikolova, M. K. Ng, S. Zhang and W. Ching, Efficient reconstruction of piecewise constant images using nonsmooth nonconvex minimization, SIAM Journal on Imaging Sciences, 1 (2008), 2-25.
24. J. Nocedal, Second-order methods for stochastic, semi-smooth and nonlinear programming, Plenary lecture, International Symposium on Mathematical Programming, Berlin 2012.
25. Y. Wang and W. Yin, Sparse signal reconstruction via iterative support detection, SIAM Journal on Imaging Sciences, 3 (2010), 462–491.
26. Z. Xu, H. Zhang, Y. Wang and X. Chang, $L_{\frac{1}{2}}$ regularization, Science China Information Science, 53 (2010), 1159-1169.