Convergence of Reweighted ℓ_1 Minimization Algorithms and Unique Solution of Truncated ℓ_p Minimization

Xiaojun Chen* and Weijun Zhou†

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Abstract

Extensive numerical experiments have shown that the iteratively reweighted ℓ_1 minimization algorithm (IRL1) is a very efficient method for variable selection, signal reconstruction and image processing. However no convergence results have been given for the IRL1. In this paper, we first give a global convergence theorem of the IRL1 for the ℓ_2-ℓ_p $(0 < p < 1)$ minimization problem. We prove that any sequence generated by the IRL1 converges to a stationary point of the ℓ_2-ℓ_p minimization problem. Moreover, the stationary point is a global minimizer in certain domain and the convergence rate is approximately linear under certain conditions. We derive posteriori error bounds which can be used to construct practical stopping rules for the algorithm. Other contribution of this paper is to prove the uniqueness of solution of the truncated ℓ_p minimization problem under the truncated null space property which is weaker than the restricted isometry property.

Keywords. ℓ_p minimization, stationary points, nonsmooth and nonconvex optimization, pseudo convex, global convergence, truncated null space property.

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1 Introduction

Iteratively reweighted ℓ_1 minimization algorithms have been widely used for solving nonconvex optimization problems in variable selection, signal reconstruction and image processing [2, 3, 4, 6, 7, 16, 19]. Candès, Wakin, and Boyd [2] proposed the iteratively reweighted ℓ_1 minimization algorithm (IRL1) to solve the penalized likelihood signal restoration problems of the form [13]

$$\min_{x\in R^n} \|Ax-b\|_2^2 + \lambda\|x\|_p^p, \qquad 0 < p < 1, \tag{1.1}$$

where $A \in R^{m\times n}, b \in R^m$, λ is a positive penalty parameter and

$$\|x\|_p^p = \sum_{i=1}^{n} |x_i|^p.$$

*Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, China (maxjchen@polyu.edu.hk). This author's work was supported in part by a Hong Kong Research Grant Council.

†Department of Mathematics, Changsha University of Science and Technology, Changsha 410004, China (weijunzhou@126.com). This author's work was supported by the Hong Kong Polytechnic University Postdoctoral Fellowship Scheme and the NSF foundation (10901026 and 10771057) of China.

A version of the IRL1 for solving (1.1) is as follows:

$$x^{k+1} = \arg\min_{x\in R^n} f_k(x,\varepsilon) := \|Ax-b\|_2^2 + \lambda\|W^k x\|_1 \tag{1.2}$$

where the weight W^k =diag(w^k) is defined by the previous iterates and updated in each iteration as

$$w_i^k = \frac{p}{(|x_i^k|+\varepsilon)^{1-p}}, \qquad i=1,\dots,n.$$

Here ε is a positive parameter to ensure that the algorithm is well-defined.

At each iteration, the IRL1 (1.2) solves a convex ℓ_2-ℓ_1 minimization problem. Extensive numerical experiments have shown that the IRL1 (1.2) is a very efficient method for variable selection, signal reconstruction and image processing. However no convergence results have been given for (1.2).

In this paper, we first give a global convergence theorem of (1.2). We prove that any sequence generated by the IRL1 (1.2) converges to a stationary point x^* of the following ℓ_2-ℓ_p minimization problem.

$$\min_{x\in R^n} f(x,\varepsilon) := \|Ax-b\|_2^2 + \lambda\sum_{i=1}^n (|x_i|+\varepsilon)^p, \qquad 0<p<1. \tag{1.3}$$

Moreover, we show that the stationary point is a global minimizer of (1.3) in certain domain and the convergence rate is approximately linear under certain conditions. Moreover, we derive posteriori error bounds

$$\|x^k - x^*\|_2 \le \gamma\|x^{k+1}-x^k\|_2,$$

with a positive constant γ, which can be used to construct practical stopping rules for the algorithm.

The model (1.3) is a natural unconstrained version of the following constrained ℓ_p optimization problem

$$\min_{x\in R^n} \sum_{i=1}^n (|x_i|+\varepsilon)^p, \qquad \text{s.t.} \quad Ax=b, \tag{1.4}$$

which is an approximation of the ℓ_p minimization problem

$$\min_{x\in R^n} \|x\|_p^p, \qquad \text{s.t.} \quad Ax=b. \tag{1.5}$$

The models (1.1) and (1.3) are also called denoising models of (1.4) and (1.5).

Problems (1.4) and (1.5) have been widely used [2, 3, 4, 5, 6, 7, 14, 16] when the vector b contains little or no noise. Ge, Jiang and Ye [15] show that the ℓ_p minimization problem (1.5) is NP-hard. Following their proof in [15], it is not difficult to show that (1.4) is also NP-hard. An advantage of (1.4) is that its objective function is Lipschitz continuous. Moreover, the sparse signal can be exactly recovered by solving it when ε is sufficiently small. In fact, Foucart and Lai [14] proved the following result. Suppose that

$$\alpha_s\|x\|_2 \le \|Ax\|_2 \le \beta_s\|x\|_2, \qquad \forall \quad \|x\|_0 \le s, \tag{1.6}$$

where $\|x\|_0 = \#\{\, i \,|x_i \neq 0\}$. Set

$$\gamma_{2s} := \frac{\beta_{2s}^2}{\alpha_{2s}^2} \geq 1. \tag{1.7}$$

Lemma 1.1. [14] *Given $0 < p < 1$ and the original s-sparse vector x^*, if for some $t \geq s$,*

$$\gamma_{2t} - 1 < 4(\sqrt{2} - 1)(\frac{t}{s})^{\frac{1}{p} - \frac{1}{2}},$$

then there exists $\zeta > 0$ such that, for any nonnegative $\varepsilon \leq \zeta$, the vector x^* is exactly recovered by solving the problem (1.4). Here ζ depends only on n, p, x^*, γ_{2t}, and the ratio s/t.

Other contribution of this paper is to prove that any feasible solution $\bar{x}$ of the ℓ_p minimization problem (1.5) is a unique solution of a truncated ℓ_p minimization problem

$$\min_{x \in R^n} \|x_T\|_p^p, \qquad \text{s.t.} \quad Ax = b, \tag{1.8}$$

where $\|x_T\|_p^p = \sum_{i \in T} |x_i|^p$ and T is a subset of $\{1, \ldots, n\}$.

It was shown in [15] that the set of all basic feasible solutions of (1.5) is exactly the set of all of its local minimizers. However, checking if a local minimizer is a solution of (1.5) is still NP-hard. We present sufficient conditions for a local minimizer being a unique solution of a truncated ℓ_p minimization problem (1.8). The sufficient conditions extend the truncated null space property [20, 10] for ℓ_1 norm to ℓ_p norm. The truncated null space property is weaker than the restricted isometry property [1].

Our convergence analysis for IRL1 can be applied to the following truncated IRL1:

$$x^{k+1} = \arg\min_{x \in R^n} f_{T,k}(x, \varepsilon) := \|Ax - b\|_2^2 + \lambda \|(W^k x)_T\|_1 \tag{1.9}$$

for the ℓ_2-ℓ_p truncated minimization problem.

$$\min_{x \in R^n} f_T(x, \varepsilon) := \|Ax - b\|_2^2 + \lambda \sum_{i \in T} (|x_i| + \varepsilon)^p, \qquad 0 < p < 1. \tag{1.10}$$

We summary some notations and results in nonsmooth optimization [8], which will be used in this paper. It is known that a Lipschitz function $g : R^n \to R$ is almost everywhere differentiable and its subgradient is defined by

$$\partial g(y) = \text{co}\{ \lim_{\substack{y^k \to y \\ y^k \in D_g}} \nabla g(y^k)\},$$

where D_g is the set of points at which g is differentiable.

We say x^* is a stationary point of g if $0 \in \partial g(x^*)$. If g is a convex function, then x^* is a global minimizer of g in R^n if and only if x^* is a stationary point of g.

A function g is convex if and only if ∂g is a monotone operator, that is,

$$(y - x, \, \xi_y - \xi_x) \geq 0, \qquad \forall \, \xi_y \in \partial g(y), \quad \forall \, \xi_x \in \partial g(x).$$

We say a function $g : R^n \to R$ is strongly pseudoconvex at x on D if for every $\xi \in \partial g(x)$ and every $y \in D$,

$$\xi^T(y-x) \geq 0 \quad \Rightarrow \quad g(y) \geq g(x).$$

We say a function $g : R^n \to R$ is strongly pseudoconvex on D if g is strongly pseudoconvex at every point in D.

Throughout this paper, $\|\cdot\|$ denotes the ℓ_2 norm. The vector $e_i \in R^n$ is the ith column of the identity matrix. The vector $a_i \in R^m$ is the ith column of the matrix A. The cardinality of a subset $T \subset \{1, \ldots, n\}$ is denoted by $|T|$, and its complement set is denoted by T^C.

2 Convergence analysis

In this section, we give convergence analysis for the IRL1 (1.2). Note that both objective functions f and f_k are Lipschitz continuous for any fixed $\varepsilon > 0$. Hence we can define their subgradients in R^n. Moreover, both functions are nonnegative and satisfy

$$f(x,\varepsilon) \to \infty, \quad f_k(x,\varepsilon) \to \infty \quad \text{as} \quad \|x\| \to \infty. \tag{2.1}$$

Therefore, the solution sets of (1.2) and (1.3) are nonempty and bounded.

Lemma 2.1. *For any nonnegative constants α, β and $t \in (0,1)$, we have*

$$\alpha^{1-t}\beta^t \leq (1-t)\alpha + t\beta, \tag{2.2}$$

and equality holds if and only if $\alpha = \beta$.

Proof Young's inequality states that for any nonnegative constants μ and ν,

$$\mu\nu \leq \frac{1}{q}\mu^q + \frac{1}{r}\nu^r, \qquad (\,\frac{1}{q} + \frac{1}{r} = 1\,)$$

where equality holds if and only if $\mu^q = \nu^r$. Set $\frac{1}{q} = 1-t$, $\mu^q = \alpha$ and $\nu^r = \beta$ in this inequality. We obtain (2.2) and equality holds if and only if $\alpha = \beta$. □

Lemma 2.2. *Let $\{x^k\}$ be the sequence generated by the IRL1 (1.2). Then we have*

$$f(x^{k+1},\varepsilon) \leq f(x^k,\varepsilon) - \|A(x^{k+1}-x^k)\| - \delta(x^{k+1},x^k), \tag{2.3}$$

where $\delta(x^{k+1},x^k) \geq 0$ and equality holds if and only if $|x^{k+1}| = |x^k|$. If $p = \frac{1}{2}$, then

$$\delta(x^{k+1},x^k) = \lambda\frac{1}{2}\sum_{i=1}^{n}\frac{\left((|x_i^{k+1}|+\varepsilon)^{\frac{1}{2}} - (|x_i^k|+\varepsilon)^{\frac{1}{2}}\right)^2}{(|x_i^k|+\varepsilon)^{\frac{1}{2}}}.$$

Proof Since x^{k+1} is the solution of problem (1.2), by [8, Corollary 1, p39] we have

$$0 \in \partial f_k(x^{k+1},\varepsilon).$$

The function f_k is the sum of $n+1$ convex functions, namely, $\|Ax-b\|^2$ and $|x_i|$, $i=1,\ldots,n$. By the addition rule of subgradient for the sum of convex functions [8, Proposition 2.3.3], we have

$$\partial f_k(x,\varepsilon)=\lambda\sum_{i=1}^{n}\frac{\partial p|x_i|}{(|x_i^k|+\varepsilon)^{1-p}}e_i+2A^T(Ax-b).$$

Hence, we find

$$0\in\partial f_k(x^{k+1},\varepsilon)=\lambda\sum_{i=1}^{n}\frac{p}{(|x_i^k|+\epsilon)^{1-p}}\partial|x_i^{k+1}|e_i+2A^T(Ax^{k+1}-b), \tag{2.4}$$

which means that there exist $c_i\in\partial|x_i^{k+1}|, i=1,\cdots,n$ such that

$$\lambda\Big(\frac{pc_i}{(|x_i^k|+\epsilon)^{1-p}}\Big)_{1\le i\le n}+2A^T(Ax^{k+1}-b)=0. \tag{2.5}$$

By the definition of the subdifferential for $|x_i|$, we have

$$c_i=\begin{cases}1, & \text{if} \quad x_i^{k+1}>0,\\ -1, & \text{if} \quad x_i^{k+1}<0,\\ \alpha, & \text{if} \quad x_i^{k+1}=0, \quad \alpha\in[-1,1].\end{cases} \tag{2.6}$$

By (2.5), (2.6) and (2.2), we obtain

$$\begin{aligned}
& f(x^k,\epsilon)-f(x^{k+1},\epsilon)\\
= \;& \lambda\sum_{i=1}^{n}\Big((|x_i^k|+\epsilon)^p-(|x_i^{k+1}|+\epsilon)^p\Big)+\|Ax^{k+1}-Ax^k\|^2+2(Ax^k-Ax^{k+1})^T(Ax^{k+1}-b)\\
= \;& \|Ax^{k+1}-Ax^k\|^2+\lambda\sum_{i=1}^{n}\Big((|x_i^k|+\epsilon)^p-(|x_i^{k+1}|+\epsilon)^p+\frac{pc_i(x_i^{k+1}-x_i^k)}{(|x_i^k|+\epsilon)^{1-p}}\Big) \qquad (2.7)\\
\ge \;& \|Ax^{k+1}-Ax^k\|^2+\lambda\sum_{i=1}^{n}\Big((|x_i^k|+\epsilon)^p-(|x_i^{k+1}|+\epsilon)^p+\frac{p(|x_i^{k+1}|-|x_i^k|)}{(|x_i^k|+\epsilon)^{1-p}}\Big)\\
= \;& \|Ax^{k+1}-Ax^k\|^2+\lambda\sum_{i=1}^{n}\Big(\frac{(|x_i^k|+\epsilon)-(|x_i^k|+\epsilon)^{1-p}(|x_i^{k+1}|+\epsilon)^p+p(|x_i^{k+1}|-|x_i^k|)}{(|x_i^k|+\epsilon)^{1-p}}\Big)\\
= \;& \|Ax^{k+1}-Ax^k\|^2+\lambda\sum_{i=1}^{n}\Big(\frac{(1-p)(|x_i^k|+\epsilon)+p(|x_i^{k+1}|+\epsilon)-(|x_i^k|+\epsilon)^{1-p}(|x_i^{k+1}|+\epsilon)^p}{(|x_i^k|+\epsilon)^{1-p}}\Big)\\
= \;& \|Ax^{k+1}-Ax^k\|^2+\delta(x^{k+1},x^k)\\
\ge \;& \|Ax^{k+1}-Ax^k\|^2,
\end{aligned}$$

where the first inequality uses

$$c_ix_i^{k+1}=|x_i^{k+1}| \quad \text{and} \quad |c_i|\le 1$$

and the last inequality uses Lemma 2.1. □

Lemma 2.3. *Suppose that $g_1:R^n\to R$ and $-g_2:R^n\to R$ are convex on a closed convex set Ω, and $g_1(x)\ge 0$ and $g_2(x)>0$, for all $x\in\Omega$ then $h(x)=\dfrac{g_1(x)}{g_2(x)}$ is strongly pseudoconvex on Ω.*

Proof This lemma is a simple generalization of [17], which proved that the condition number of a symmetric positive definite matrix is pseudoconvex. For completeness, we give a proof of this lemma.

From the convexity assumption, for any $x, y \in \Omega$ and $\xi_1 \in \partial g_1(x), \xi_2 \in \partial g_2(x)$, we have

$$g_1(y) - g_1(x) \geq \xi_1^T(y - x),$$

and

$$-g_2(y) + g_2(x) \geq \xi_2^T(y - x).$$

Hence we obtain

$$\begin{aligned} g_1(y) - h(x)g_2(y) &= g_1(y) - g_1(x) + h(x)(-g_2(y) + g_2(x)) \\ &\geq \xi_1^T(y - x) + h(x)\xi_2^T(y - x) \\ &= g_2(x)\Big(\frac{\xi_1 g_2(x) - g_1(x)\xi_2}{g_2(x)^2}\Big)^T(y - x). \end{aligned}$$

By the quotient rule for the Clarke generalized gradient [8, Proposition 2.3.14], we find that $\frac{\xi_1 g_2(x) - g_1(x)\xi_2}{g_2(x)^2} \in \partial h(x)$, from that g_2 and g_1 are Clarke regular. Therefore we have $h(y) \geq h(x)$ if $\xi^T(y - x) \geq 0$ with $\xi \in \partial h(x)$. □

Lemma 2.4. *For constants $\alpha > 0, \varepsilon > 0$ and $p \in (0, 1)$, let*

$$\phi(t) = |t| + (\alpha t^2 + \beta t)(|t| + \varepsilon)^{1-p}.$$

Then ϕ is convex in $[0, \infty)$ and $(-\infty, 0]$ if

$$|\beta| \leq \frac{\alpha\varepsilon}{1 - p}. \tag{2.8}$$

Proof The function ϕ is differentiable in R except $t = 0$. To show the convexity of ϕ, we consider the second derivative of ϕ for $t \neq 0$.

First we consider $t > 0$. By simple calculation, we get

$$\phi^{''}(t) = (t + \varepsilon)^{-1-p}(c_1 t^2 + c_2 t + c_3),$$

where

$$c_1 = \alpha(2 + (4 - p)(1 - p)),$$

$$c_2 = (2 - p)((1 - p)\beta + 4\alpha\varepsilon),$$

$$c_3 = 2\varepsilon(\alpha\varepsilon + (1 - p)\beta).$$

Obviously, $c_i > 0, i = 1, 2$ and $c_3 \geq 0$. This implies that ϕ is convex for $t > 0$.

Now, we consider $t < 0$. In this case,

$$\phi(t) = -t + (\alpha t^2 + \beta t)(-t + \varepsilon)^{1-p}.$$

Similarly, we can find that for $t < 0$,

$$\phi^{''}(t) = (-t + \varepsilon)^{-1-p}(c_1 t^2 + c_4 t + c_5)$$

where

$$c_4 = (2-p)((1-p)\beta - 4\alpha\varepsilon),$$

$$c_5 = 2\varepsilon(\alpha\varepsilon - (1-p)\beta).$$

Obviously, $c_4 < 0$ and $c_5 \geq 0$. This implies that $\phi''(t) \geq 0$ and thus ϕ is convex for $t < 0$. By the continuity of ϕ and that for $t_1 t_2 > 0$

$$\phi(\mu t_1 + (1-\mu)t_2) \leq \mu\phi(t_1) + (1-\mu)\phi(t_2), \quad \text{for} \quad 0 \leq \mu \leq 1,$$

we can take $t_1 \to 0$ or $t_2 \to 0$, and claim that ϕ is convex in $[0, \infty)$ and $(-\infty, 0]$.

□

Theorem 2.1. *Let $\{x^k\}$ be a sequence generated by the IRL1 (1.2). Then the sequence $\{x^k\}$ converges to a stationary point x^* of (1.3). Moreover, the following statements hold.*

(1) If $\varepsilon \geq \left(\dfrac{\lambda(1-p)p}{2\|a_i\|^2}\right)^{\frac{1}{2-p}}$*, then*

$$f(x^*, \varepsilon) \leq f(x^* + te_i, \varepsilon), \quad \text{for} \quad t \in \begin{cases} [-x_i^*, \infty) & \text{if} \quad x_i^* \geq 0, \\ (-\infty, -x_i^*] & \text{if} \quad x_i^* \leq 0. \end{cases} \tag{2.9}$$

(2) If $|a_i^T(A_I x_I^* - b)| \leq \dfrac{\|a_i\|^2\varepsilon}{2(1-p)}$ *then (2.9) holds. Moreover, if* $a_i^T(A_I x_I^* - b) = 0$*, then* $x_i^* = 0$ *and*

$$f(x^*, \varepsilon) \leq f(x^* + te_i, \varepsilon), \quad \text{for} \quad t \in R, \tag{2.10}$$

where $A_I = [a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n]$ *and* $x_I^* = [x_1^*, \ldots, x_{i-1}^*, x_{i+1}^*, \ldots, x_n^*]^T$.

Proof By Lemma 2.2, the sequence $\{f(x^k, \varepsilon)\}$ is monotonically decreasing. Hence it converges. It is clear that the sequence $\{x^k\}$ is contained in the level set

$$\mathcal{L}(x^0) = \{\, x \,|\, f(x, \varepsilon) \leq f(x^0, \varepsilon)\}.$$

Obviously, $\mathcal{L}(x^0)$ is bounded from (2.1). Let $\{x^{n_k}\}$ be a subsequence of $\{x^k\}$ which converges to x^*.

By (2.3), we have $\delta(x^{k+1}, x^k) \to 0$, as $k \to \infty$. This implies that $\{|x^{n_k+1}|\}$ also converges to $|x^*|$. From (2.3) and (2.7), we have

$$\lim_{k\to\infty} f(x^k, \varepsilon) - f(x^{k+1}, \varepsilon) = \lim_{k\to\infty} \|A(x^{k+1} - x^k)\| = \lim_{k\to\infty} |x^k| - |x^{k+1}| = 0.$$

This, together with (2.7), implies

$$\lim_{k\to\infty} c_i(x_i^{k+1} - x_i^k) = 0, \qquad i = 1, \ldots, n.$$

Hence, from the definition of c_i and $\lim\limits_{n_k\to\infty} |x^{n_k}| - |x^{n_k+1}| = 0$, we obtain

$$\lim_{n_k\to\infty} (x_i^{n_k+1} - x_i^{n_k}) = 0, \qquad i = 1, \ldots, n.$$

Moreover, it implies that the whole sequence $\{x^k\}$ converges to x^*.

By the upper semi-continuous property of the subdifferential [8, Proposition 2.1.5], there exist $c_i^* \in \partial|x_i^*|$, $i = 1, \ldots, n$ such that

$$0 = \lambda\Big(\frac{pc_i^*}{(|x_i^*| + \epsilon)^{1-p}}\Big)_{1\leq i\leq n} + 2A^T(Ax^* - b) \in \partial f(x^*). \tag{2.11}$$

Hence x^* is a stationary point.

Now we prove (1) of this theorem. Let

$$\varphi(t) = \lambda\||x^* + te_i| + \varepsilon\|_p^p + \|A(x^* + te_i) - b\|^2. \tag{2.12}$$

The subdifferential of φ is

$$\partial\varphi(t) = \lambda\frac{p\,\mathrm{sign}(x_i^* + t)}{(|x_i^* + t| + \varepsilon)^{1-p}} + 2a_i^T(A(x^* + te_i) - b).$$

By (2.11), we have $0 \in \varphi(0)$, that is, 0 is a stationary point of φ. For t_1 and t_2 satisfying $(x_i^* + t_1)(x_i^* + t_2) > 0$, there is t_0 between t_1 and t_2 such that for any $\xi_1 \in \partial\varphi(t_1)$ and $\xi_2 \in \partial\varphi(t_2)$,

$$\xi_1 - \xi_2 = (-\frac{\lambda(1-p)p}{(|x_i^* + t_0| + \varepsilon)^{2-p}} + 2\|a_i\|^2)(t_1 - t_2) \geq (-\frac{\lambda(1-p)p}{\varepsilon^{2-p}} + 2\|a_i\|^2)(t_1 - t_2).$$

Hence if $\varepsilon \geq \left(\frac{\lambda(1-p)p}{2\|a_i\|^2}\right)^{\frac{1}{2-p}}$, then

$$(t_1 - t_2, \xi_1 - \xi_2) \geq 0.$$

Hence φ is convex, and 0 is the minimizer of φ in $(-x_i^*, \infty)$ if $x_i^* \geq 0$, and in $(-\infty, -x_i^*)$ if $x_i^* \leq 0$. This gives (2.9).

To prove the first part of (2) of this theorem, we show φ defined in (2.12) is pseudoconvex in $[-x_i^*, \infty)$ and $(-\infty, -x_i^*]$. The function φ can be rewritten as

$$\begin{aligned}\varphi(t) &= \lambda(|x_i^* + t| + \varepsilon)^p + \|a_i\|^2(x_i^* + t)^2 + 2a_i^T(A_Ix_I^* - b)(x_i^* + t) + c_0,\\ &= \lambda\Big(\frac{|x_i^* + t| + \varepsilon + \Big(\frac{\|a_i\|^2}{\lambda}(x_i^* + t)^2 + \frac{2a_i^T(A_Ix_I^* - b)}{\lambda}(x_i^* + t)\Big)(|x_i^* + t| + \varepsilon)^{1-p}}{(|x_i^* + t| + \varepsilon)^{1-p}}\Big) + c_0,\end{aligned}$$

where c_0 is a constant. Using Lemma 2.4, with

$$\alpha = \frac{\|a_i\|^2}{\lambda} \quad \text{and} \quad \beta = \frac{2a_i^T(A_Ix_I^* - b)}{\lambda},$$

we find that the function

$$|x_i^* + t| + \varepsilon + \Big(\frac{\|a_i\|^2}{\lambda}(x_i^* + t)^2 + \frac{2a_i^T(A_Ix_I^* - b)}{\lambda}(x_i^* + t)\Big)(|x_i^* + t| + \varepsilon)^{1-p}$$

is convex. Since $(|x_i^* + t| + \varepsilon)^{1-p}$ is concave, we find that φ is pseudoconvex by Lemma 2.3.

By the definition of the pseudo convexity and (2.11), we obtain (2.9).

If $a_i^T(A_I x_I^* - b) = 0$, then (2.11) implies that

$$0 \in \lambda\Big(\frac{pc_i^*}{(|x_i^*| + \epsilon)^{1-p}}\Big) + 2a_i^T a_i x_i^*.$$

Since $c_i^* = 1$ if $x_i^* > 0$ and $c_i^* = -1$ if $x_i^* < 0$, this encloser only holds at $x_i^* = 0$. Moreover, it is easy to see that in such case

$$\varphi(-x_i^*) = \varphi(0) \leq \varphi(t), \qquad \text{for} \quad t \in R,$$

that is,

$$f(x^* - x_i^* e_i, \varepsilon) = f(x^*, \varepsilon) \leq f(x^* + te_i), \quad \text{for} \quad t \in R.$$

We obtain the desired results. □

Remark 1. *Consider the constrained IRL1 [2]*

$$x^{k+1} = \arg\min_{x \in R^n} \|W^k x\|_1, \qquad \text{s.t.} \quad Ax = b \tag{2.13}$$

where the weight W^k =diag(w^k) is defined by

$$w_i^k = \frac{p}{(|x_i^k| + \varepsilon)^{1-p}}, \qquad i = 1, \ldots, n.$$

From the proof of Theorem 2.1, we can easily find that if the sequence $\{x^k\}$ generated by (2.13) converges to x^*, then x^* is a stationary point of (1.4), that is, there is $\mu \in R^m$ such that

$$\begin{aligned} 0 &= \Big(\frac{pc_i^*}{(|x_i^*| + \epsilon)^{1-p}}\Big)_{1 \leq i \leq n} + A^T\mu \in \partial_x L(x^*, \mu) \\ 0 &= Ax - b \end{aligned}$$

where $c_i \in \partial|x_i^*|$ and $L(x, \mu) = \sum_{i=1}^{n}(|x_i| + \varepsilon)^p + \mu(Ax - b)$ is the Lagrangian function.

In [7], it was shown that any local minimizer x^* of (1.1) satisfies

$$\textit{either} \quad |x_i^*| = 0 \quad \textit{or} \quad |x_i^*| \geq L, \qquad \forall\, i = 1, \cdots, n, \tag{2.14}$$

where

$$L := \Big(\frac{\lambda p(1-p)}{\max_{i \in \{i, \cdots, n\}} \|a_i\|^2}\Big)^{\frac{1}{2-p}}.$$

This lower bound for absolute value of nonzero elements of any local minimizer of (1.1) can be easily extended to the model (1.3). Also see Theorem 3.3 in [18]. We give the lower bound theory for (1.3) in the following theorem.

Theorem 2.2. *If $\epsilon < L$, then every local minimizer x^* of (1.3) satisfies*

$$\textit{either} \quad |x_i^*| = 0 \quad \textit{or} \quad |x_i^*| \geq L - \varepsilon, \qquad \forall\, i = 1, \cdots, n. \tag{2.15}$$

Now we use the lower bound L to derive the convergence rate of the IRL1 (1.2) and error bounds.

Theorem 2.3. *Assume that the sequence $\{x^k\}$ generated by (1.2) converges to a local minimizer x^* of (1.3). If*

$$\frac{\lambda p(1-p)}{L^{2-p}} < 2\lambda_{\min}(A_S^T A_S),$$

then for any positive constant $\varepsilon < L$, there exist positive constants $\gamma_i, i = 1,2,3$ and $c \in (0,1)$ such that for all sufficiently large k

$$\|x_S^k - x_S^*\| \le \gamma_1 \|x_S^k - x_S^{k+1}\| + \gamma_2 \|x_{S^C}^{k+1}\|,$$

and

$$\|x_S^{k+1} - x_S^*\| \le c\|x_S^k - x_S^*\| + \gamma_3 \|x_{S^C}^{k+1}\|.$$

Proof Denote

$$S = \{\, i \quad | \quad |x_i^*| \neq 0 \,\} \quad \text{and} \quad S_k = \{\, i \quad | \quad |x_i^k| \neq 0 \,\}.$$

Since $x^k \to x^*$, by (2.15) we have $S \subset S_k$ and there exists a small constant $\delta \in (0, \epsilon)$ such that for sufficiently large k, $|x_i^k| \ge L - \delta$, for $i \in S$.

Consider the function

$$g(z) = \sum_{i\in S} \lambda(|z_i| + \epsilon)^p + \|A_S z - b\|^2, \qquad z \in R^{|S|}.$$

Since $f(x^*) = g(x_S^*)$, it is easy to show that x_S^* is a local minimizer of $g(z)$. Therefore we have from the optimal condition for minimizing $g(z)$ that

$$\Big(\frac{\lambda p \,\mathrm{sign}(x_i^*)}{(|x_i^*| + \epsilon)^{1-p}}\Big)_{i\in S} + 2A_S^T(A_S x_S^* - b) = 0, \tag{2.16}$$

and the matrix

$$\mathrm{diag}\Big(\Big(\frac{\lambda p(p-1)}{(|x_i^*| + \epsilon)^{2-p}}\Big)_{i\in S}\Big) + 2A_S^T A_S$$

is semipositive definite, which implies that the matrix $A_S^T A_S$ is positive definite since $p - 1 < 0$.

Since x^{k+1} is a local minimizer of $f_k(x)$ and for sufficiently large k,

$$\mathrm{sign}(x_i^{k+1}) = \mathrm{sign}(x_i^k) = \mathrm{sign}(x_i^*), \qquad i \in S,$$

we have

$$\begin{pmatrix} \Big(\dfrac{\lambda p \,\mathrm{sign}(x_i^*)}{(|x_i^k| + \epsilon)^{1-p}}\Big)_{i\in S} \\ \Big(\dfrac{\lambda p \,\mathrm{sign}(x_i^{k+1})}{(|x_i^k| + \epsilon)^{1-p}}\Big)_{i\in S^C} \end{pmatrix} + 2\begin{pmatrix} A_S^T(Ax^{k+1} - b) \\ A_{S^C}(Ax^{k+1} - b) \end{pmatrix} = 0. \tag{2.17}$$

By (2.16) and (2.17), we have

$$B_S(x_S^k - x_S^*) = 2A_S^T A_S(x_S^k - x_S^{k+1}) - 2A_S^T A_{S^C} x_{S^C}^{k+1}, \tag{2.18}$$

and

$$x_S^{k+1} - x_S^* = -(2A_S^T A_S)^{-1} D_S (x_S^k - x_S^*) - (A_S^T A_S)^{-1} A_S^T A_{S^C} x_{S^C}^{k+1}, \tag{2.19}$$

where ζ_i is between x_i^* and x_i^k for any $i \in S$, and

$$D_S = \mathrm{diag}\Big(\Big(\frac{\lambda p(p-1)}{(|\zeta_i| + \epsilon)^{2-p}}\Big)_{i \in S}\Big), \qquad B_S = D_S + 2A_S^T A_S.$$

From $\mathrm{sign}(x_i^k) = \mathrm{sign}(x_i^*)$, we have $|\zeta_i| \geq L - \delta > 0$, for $i \in S$. Moreover, from the following inequalities

$$\frac{\lambda p(1-p)}{(|\zeta_i| + \epsilon)^{2-p}} \leq \frac{\lambda p(1-p)}{(L - \delta + \epsilon)^{2-p}} \leq \frac{\lambda p(1-p)}{(L)^{2-p}} < 2\lambda_{\min}(A_S^T A_S),$$

we obtain that B_S is nonsingular and we have from (2.18) and (2.19) that

$$\|x_S^k - x_S^*\| \leq 2\|B_S^{-1}\|\|A_S^T A_S\|\|x_S^k - x_S^{k+1}\| + 2\|B_S^{-1}\|\|A_S^T A_{S^C}\|\|x_{S^C}^{k+1}\|,$$

and

$$\|x_S^{k+1} - x_S^*\| \leq \|(2A_S^T A_S)^{-1} D_S\|\|x_S^k - x_S^*\| + \|(A_S^T A_S)^{-1} A_S^T A_{S^C}\|\|x_{S^C}^{k+1}\|.$$

Therefore, we complete the proof with $\gamma_1 = 2\|B_S^{-1}\|\|A_S^T A_S\|$, $\gamma_2 = 2\|B_S^{-1}\|\|A_S^T A_{S^C}\|$, $\gamma_3 = \|(A_S^T A_S)^{-1} A_S^T A_{S^C}\|$ and $c = \|(2A_S^T A_S)^{-1} D_S\|$. $\square$

If we know the index set S of nonzero elements x^* exactly, we can set $x_{S^C}^k = 0$ for all large k. Then from Theorem 2.3, we have

$$\|x^k - x^*\| = \|x_S^k - x_S^*\| \leq \gamma_1 \|x_S^k - x_S^{k+1}\| = \gamma_1 \|x^k - x^{k+1}\|$$

and

$$\|x^{k+1} - x^*\| = \|x_S^{k+1} - x_S^*\| \leq c\|x_S^k - x_S^*\| = c\|x^k - x^*\|.$$

3 Unique solution of truncated ℓ_p minimization

In the last section, we show that the IRL1 (1.2) converges to a stationary point of (1.3) which is a denoising problem of the ℓ_p minimization problem (1.5). Although (1.5) is an NP-hard problem, it is easy to find its local minimizers. In [15], Ge, Jiang and Ye showed that all basic feasible solutions of (1.5) are local minimizers of (1.5). In this section, we show that if x^* is a feasible solution of (1.5), then x^* is a unique global minimizer of a truncated ℓ_p minimization problem (1.8) under the the truncated null space property.

In [20], Wang and Yin proposed an iterative support detection method which solves a sequence of truncated ℓ_1 minimization problems

$$\min_{x \in R^n} \|x_T\|_1, \quad \text{s.t.} \quad Ax = b. \tag{3.1}$$

They introduced the truncated null space property of A in the ℓ_1 norm, an extension of the null space property studied in [9, 10, 11] which is more general than the widely used restricted isometry property [1].

The following definition is the truncated null space property of A in the ℓ_p norm.

Definition 3.1. *A matrix A satisfies the t-NSP of order K for $\gamma > 0$, $0 < t \le n$ if*

$$\|\eta_S\|_p \le \gamma \|\eta_{(T\cap S^C)}\|_p \tag{3.2}$$

holds for all sets $T \subset \{1, \cdots, n\}$ with $|T| = t$, all subsets $S \subset T$ with $|S| \le K$, and all $\eta \in N(A)$, the null space of A.

Following the notation in [20], we use t-NSP(t, K, γ) to denote the t-NSP of order K for γ and t, and use $\bar{\gamma}$ to replace γ and write t-NSP$(t, K, \bar{\gamma})$ if $\bar{\gamma}$ is the infimum to all the feasible γ satisfying (3.2).

The truncated null space property for $0 < p < 1$ is also a generalization of restricted isometry property. For $|T| = n$, we have the following result.

Lemma 3.1. *Given $0 < p < 1$, if for some $t_1 \ge K$,*

$$\gamma_{2t_1} - 1 < 4(\sqrt{2} - 1)\Big(\frac{t_1}{K}\Big)^{\frac{1}{p}-\frac{1}{2}},$$

where γ_{2t_1} is defined by (1.7), then the matrix A satisfies the t-NSP of order K for $\gamma < 1$ and $|T| = n$.

Proof It follows directly from the inequality (15) in Theorem 3.1 of [14]. □

The following three inequalities of the ℓ_p $(0 < p < 1)$ norm will be used in the proof of Theorem 3.1 and Theorem 3.2. For any vectors $u, v \in R^n$, we have

$$\|u\|_1 \le \|u\|_p, \quad \|u\|_p \le n^{\frac{1}{p}-\frac{1}{2}}\|u\|_2, \quad \|u+v\|_p^p \le \|u\|_p^p + \|v\|_p^p. \tag{3.3}$$

We denote the feasible solution set of (1.5) by

$$\mathcal{F} = \{\, x \mid Ax = b\}$$

and the index set of nonzero element of a given vector x by

$$S(x) = \{\, i \quad | \quad x_i \neq 0\,\}.$$

The following result is an extension of Theorem 3.1 in [20] from $p = 1$ to $0 < p < 1$, which provides a sufficient exact recovery condition for K-sparse vector.

Theorem 3.1. *Let $x^* \in \mathcal{F}$ and T be a subset of $\{1, \dots, n\}$. Let $S = T \cap S(x^*)$. If $S = \emptyset$, then x^* is a solution of (1.8). If $S \neq \emptyset$ and*

$$\|\eta_S\|_p \le \gamma \|\eta_{(T\cap S^C)}\|_p, \qquad \gamma < 1 \tag{3.4}$$

for all $\eta \in N(A)$, then x^* is the unique solution of (1.8).

Proof If $S = \emptyset$, then x_T^* with $x_i^* = 0, i \in T$ is a solution of (1.8). Suppose $S \neq \emptyset$. It is easy to see that the vector x^* is the unique solution of (1.8) if and only if for any $x \in N(A)$

$$\|x_T^* + x_T\|_p^p > \|x_T^*\|_p^p. \tag{3.5}$$

Since $\|x_S^*\|_p^p = \|x_T^*\|_p^p$, we have from the third inequality in (3.3) that

$$\begin{aligned} & \|x_T^* + x_T\|_p^p \\ = \ & \|x_S^* + x_S\|_p^p + \|0 + x_{T\cap S^C}\|_p^p \\ = \ & \|x_S^* + x_S\|_p^p - \|x_S^*\|_p^p + \|x_S\|_p^p + \|x_T^*\|_p^p + (\|x_{T\cap S^C}\|_p^p - \|x_S\|_p^p) \\ \geq \ & \|x_T^*\|_p^p + (\|x_{T\cap S^C}\|_p^p - \|x_S\|_p^p). \end{aligned}$$

By assumption (3.4) the above inequality shows that (3.5) holds. □

Corollary 3.1. *Let T be a subset of $\{1, \dots, n\}$. Assume that A satisfies t-NSP$(t, K, \bar{\gamma})$ for $t = |T|$ and $\bar{\gamma} < 1$. Then for any $x^* \in \mathcal{F}$, $\|x_T^*\|_0 \leq K$, $S(x^*) \cap T \neq \emptyset$, x^* is the unique minimizer of (1.8).*

In [20], Wang and Yin gave a class of matrices which satisfies the t-NSP property in ℓ_1 norm.

Lemma 3.2. [20] *Let $m < n$. Assume that $A \in R^{m\times n}$ is either a standard Gaussian matrix (i.e., one with i.i.d. standard normal entries) or a rank-m matrix with its m rows all orthogonal to an $(n-m)$-dimensional standard Gaussian linear subspace (i.e., existing a standard Gaussian matrix $B \in R^{n\times(n-m)}$ such that $AB = 0$). Given an index set T, with probability greater than $1 - e^{-c_0(n-m)}$, the matrix A satisfies t-NSP(t, K, γ) in ℓ_1 norm with*

$$\gamma = \frac{\sqrt{K}}{2\sqrt{k(d)} - \sqrt{K}}, \qquad k(d) := c\frac{m-d}{1 + \log(\frac{n-d}{m-d})}, \tag{3.6}$$

where $d = n - |T|$, and $c_0, c > 0$ are absolute constants independent of the dimensions m, n, and d.

The following theorem gives a class of matrices which satisfies the t-NSP property in ℓ_p norm.

Theorem 3.2. *Under the assumption of Lemma 3.2, given an index set T, with probability greater than $1 - e^{-c_0(n-m)}$, the matrix A satisfies t-NSP(t, K, γ) in ℓ_p norm with*

$$\gamma = \Big(\frac{K^{1-p/2}}{(4k(d))^{p/2} - K^{1-p/2}}\Big)^{1/p}, \quad k(d) := c\frac{m-d}{1 + \log(\frac{n-d}{m-d})}, \tag{3.7}$$

where $d = n - |T|$, and $c_0, c > 0$ are absolute constants independent of the dimensions m, n, and d.

Proof By Lemma 3.1 in [20], for all $S \subset T$ with $|S| \leq K$ we have

$$\sqrt{k(d)}\|v_T\| \leq \frac{1}{2}\|v_T\|_1, \quad \forall \quad v \in N(A), \quad v \neq 0.$$

By the second inequality and the first inequality in (3.3), we have

$$\|v_S\|_p \leq |S|^{1/p-1/2}\|v_S\| \leq K^{1/p-1/2}\|v_S\| \leq \frac{K^{1/p-1/2}}{2\sqrt{k(d)}}\|v_T\|_1 \leq \frac{K^{1/p-1/2}}{2\sqrt{k(d)}}\|v_T\|_p,$$

which together with the third inequality in (3.3) shows that

$$\|v_S\|_p^p \leq \Big(\frac{K^{1/p-1/2}}{2\sqrt{k(d)}}\Big)^p \|v_T\|_p^p \leq \Big(\frac{K^{1/p-1/2}}{2\sqrt{k(d)}}\Big)^p (\|v_S\|_p^p + \|v_{T\cap S^C}\|_p^p).$$

Therefore we have $\|v_S\|_p \leq \gamma \|v_{T\cap S^C}\|_p$, with γ defined by (3.7). $\square$

It is clear that Theorem 3.2 reduces to Lamma 3.2 in the case of $p = 1$. Moreover if $K < k(d)^{\frac{p}{2-p}}$, then $\gamma < 1$ by direct computation. Therefore by the above two theorems we have the following corollary.

Corollary 3.2. *Let $x^* \in \mathcal{F}$ and T be given such that $T \cap S(x^*) \neq \emptyset$. Let $m < n$. Assume that $A \in R^{m\times n}$ satisfies the assumption of Theorem 3.2. Then with probability greater than $1 - e^{-c_0(n-m)}$, the true sparse vector x^* is the unique solution of the problem (1.8) if $\|x_T^*\|_0 < k(d)^{\frac{p}{2-p}}$ where $k(d)$ is given by (3.7).*

The following two theorems extend Lemma 3.2 and Theorem 3.3 in [20] from $p = 1$ to $0 < p < 1$. Here we only present the results but omit the proof since they can be proved by very similar technique in [20].

Theorem 3.3. *Assume that A satisfies t-NSP$(t, K, \bar{\gamma})$ for a $t = |T|$ and $\bar{\gamma} < 1$. For $z, z' \in \mathcal{F}$, let $S \subset T$ be the set of indices corresponding to the largest K entries in z_T. We have*

$$\|(z - z')_{T\cap S^C}\|_p^p \leq \frac{1}{1-\bar{\gamma}^p}\Big(\|z'_T\|_p^p - \|z_T\|_p^p + 2(\sigma_K(z_T)_p)^p\Big), \tag{3.8}$$

where $\sigma_K(z)_p := \inf_{\|x\|_0 \leq K} \|z - x\|_p$.

Theorem 3.4. *Assume that A satisfies t-NSP$(t, K, \bar{\gamma})$ for $t = |T|$ and $\bar{\gamma} < 1$. Let x^* be the solution of the problem (1.8) and x be the true signal. Then we have $\|x_T^*\|_p \leq \|x_T\|_p$ and*

$$\|x^* - x\|_p \leq C_T \sigma_L(x_T)_p, \tag{3.9}$$

where

$$C_T = \Big(2\frac{1 + (1 + \max\{1, |T^C|/K\})\bar{\gamma}^p}{1-\bar{\gamma}^p}\Big)^{1/p}.$$

Remark 2. *Convergence analysis in the last section can be directly extended to the truncated IRL1 (1.9). For instance, we can claim that for a given index set T, any sequence generated by (1.9) is a stationary point of the ℓ_2-ℓ_p truncated minimization problem (1.10).*

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References

[1] E. Candès, J. Romberg and T. Tao, Stable signal recovery from incomplete and inaccurate information, Communications on Pure and Applied Mathematics, 59 (2005), 1207-1233.

[2] E. Candès, M. Wakin and S. Boyd, Enhancing sparsity by reweighted l_1 minimization, Journal of Fourier Analysis and Applications, 14 (2008), 877-905.

[3] R. Chartrand, Exact reconstruction of sparse signals via nonconvex minimization, IEEE Signal Processing Letters, 14 (2007), 707-710.

[4] R. Chartrand, Nonconvex regularization for shape preservation, IEEE International Conference on Image Processing (ICIP), 2007.

[5] R. Chartrand and V. Staneva, Restricted isometry properties and nonconvex compressive sensing, Inverse Problem, 24 (2008), 1-14.

[6] R. Chartrand and W. Yin, Iteratively reweighted algorithms for compressive sensing, In ICASSP 2008, pages 3869-3872, March 2008.

[7] X. Chen, F. Xu and Y. Ye, Lower bound theory of nonzero entries in solutions of l_2-l_p minimization, http://www.stanford.edu/ yyye/, submitted, 2009.

[8] F. H. Clarke, Optimization and Nonsmooth Analysis, John Wiley, New York, 1983.

[9] A. Cohen, W. Dahmen and R. Devore, Compressed sensing and best k-term approximation, Journal of the American Mathematical Society, 22 (2009), 211-231.

[10] I. Daubechies, R. DeVore, M. Fornasier and S. Gunturk, Iteratively re-weighted least squares minimization: Proof of faster than linear rate for sparse recovery, Information Sciences and Systems, 2008. CISS 2008. 42nd Annual Conference on, pages 26-29, March 2008.

[11] D. L. Donoho, Compressed sensing, IEEE Transactions on Information Theory, 52 (2006), 1289-1306.

[12] D. L. Donoho and Y. Tsaig, Fast solution of l_1-norm minimization problems when the solution may be sparse, IEEE Transactions on Information Theory, 54 (2008), 4789-4812.

[13] M. A. T. Figueiredo, J. M. Bioucas-Dias and R. D. Nowak, Majorization-minimization algorithms for wavelet-based image restoration, IEEE Trans. Image Processing, 16 (2007), 2980-2991.

[14] S. Foucart and M. J. Lai, Sparsest solutions of under-determined Linear Systems via l_q minimization for $0 < q \leq 1$, Applied and Computational Harmonic Analysis, 26 (2009), 395-407.

[15] D. Ge, X. Jiang Y. Ye, A note on complexity of L_p minimization, http://www.stanford.edu/ yyye/, 2010.

[16] M. Lai and Y. Wang, An unconstrained l_q minimization with $0 < q < 1$ for sparse solution of under-determined linear systems, http://www.math.uga.edu/ mjlai/, submitted, 2009.

[17] P. Marechal and J. Ye, Optimizing condition numbers, SIAM Journal on Optimization, 20 (2009), 935-947.

[18] M. Nikolova, Analysis of the recovery of edges in images and signals by minimizing nonconvex regularized least-squares, Multiscale Modeling and Simulation, 4 (2005), 960-991.

[19] M. Nikolova, M. K. Ng, S. Zhang and W. Ching, Efficient reconstruction of piecewise constant images using nonsmooth nonconvex minimization, SIAM Journal on Imaging Sciences, 1 (2008), 2-25.

[20] Y. Wang and W. Yin, Compressed sensing via iterative support detection, http://www.caam.rice.edu/ wy1/, submitted to SIAM Journal on Imaging Sciences, 2009.