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Computational error bounds for a differential linear variational inequality

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The differential linear variational inequality consists of a system of n ordinary differential equations (ODEs) and a parametric linear variational inequality as the constraint. The right-hand side function in the ODEs is not differentiable and cannot be evaluated exactly. Existing numerical methods provide only approximate solutions. In this paper we present a reliable error bound for an approximate solution $x^h(t)$ delivered by the time-stepping method, which takes all discretization and roundoff errors into account. In particular, we compute two trajectories $x_j^h(t) \pm \epsilon_j^h(t)$ to determine the existence region of the exact solution $x_j(t)$, i.e., $x_j^h(t) - \epsilon_j^h(t) \leqslant x_j(t) \leqslant x_j^h(t) + \epsilon_j^h(t)$ for each $j \in \{1, \ldots, n\}$. Moreover, we have $\epsilon_j^h(t) = \mathcal{O}(h)$. Numerical examples of bridge collapse, earthquake-induced structural pounding and circuit simulation are given to illustrate the efficiency of the error bound.

Keywords: ordinary differential equations; linear variational inequalities; time-stepping method; error bounds.

1. Introduction

Given a nonempty, closed and convex set $K \subseteq R^m$ and a function $F: K \to R^m$, the variational inequality problem is to find a vector $y^* \in K$ such that

$$(y - y^*)^{\mathrm{T}} F(y^*) \geqslant 0 \quad \forall\, y \in K.$$

Here we restrict our study to the case where the subset K is a box and the mapping F is affine, that is,

$$K = \{y \in R^m \,|\, l \leqslant y \leqslant u\},$$

where $l \in \{R \cup -\infty\}^m$ and $u \in \{R \cup \infty\}^m$ with $l < u$, and

$$F(y) = My + q,$$

where $M \in R^{m\times m}$ and $q \in R^m$. Such a problem is called a box-constrained linear variational inequality problem or mixed linear complementarity problem (see Billups & Ferris, 1997; Ferris & Pang, 1997;

Chen *et al.*, 1998; Chen & Ye, 1999). We denote this variational inequality by $\mathrm{VI}(l, u, q, M)$ and its solution set by $\mathrm{SOL}(l, u, q, M)$.

Consider the following autonomous differential linear variational inequality problem (DLVI):

$$\begin{cases} \dot{x}(t) = Ax(t) + By(t), \\ y(t) \in \mathrm{SOL}(l, u, Qx(t), M), \\ x(0) = x^0 \in R^n, \quad t \in [0, T], \end{cases} \tag{1.1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $Q \in R^{m\times n}$ and $M \in R^{m\times m}$. When $l_i = 0$ and $u_i = \infty$, for $i = 1, \ldots, m$, (1.1) reduces to the differential linear complementarity system (see Han *et al.*, 2008):

$$\begin{cases} \dot{x}(t) = Ax(t) + By(t), \\ 0 \leqslant y(t) \perp My(t) + Qx(t) \geqslant 0, \\ x(0) = x^0 \in R^n, \quad t \in [0, T], \end{cases} \tag{1.2}$$

where $\perp$ means orthogonal.

The DLVI provides a new and powerful modelling paradigm for many applications in engineering and economics (see Chen & Mahmoud, 2008; Heemels *et al.*, 2000; Schumacher, 2004). It is also closely related to some existing mathematical models. For instance, it can be rewritten as an integral equation

$$x(t) = x(0) + \int_0^t [Ax(\tau) + By(\tau)]\,\mathrm{d}\tau$$

with the variational inequality constraint

$$y(t) \in \mathrm{SOL}(l, u, Qx(t), M).$$

For an integral equation with complementarity constraints we refer to Gauthier *et al.* (2007). Another instance is that the differential linear complementarity system (1.2) can be reformulated into the convolution complementarity problem (CCP): given $k(\cdot)$ and $q(\cdot)$, find $u(\cdot)$ satisfying

$$0 \leqslant u(t) \perp (k * u)(t) + q(t) \geqslant 0,$$

where

$$(k * u)(t) = \int_0^t k(\tau)u(t - \tau)\,\mathrm{d}\tau$$

is the convolution of k and u. For a comprehensive treatment of the P-matrix CCP we refer to Stewart (2006).

In this paper we study the time-stepping method (see Pang & Stewart, 2008) for solving the DLVI, which begins with the division of the time interval $[0, T]$ into N_h subintervals

$$0 = t_{h,0} < t_{h,1} < \cdots < t_{h,N_h} = T,$$

where $t_{h,i+1} - t_{h,i} = h = T/N_h$, $i = 0, \ldots, N_h - 1$. Starting from a given vector $x^{h,0} = x^0 \in R^n$ we compute $y^{h,0} \in \mathrm{SOL}(l, u, Qx^{h,0}, M)$ and two finite families of vectors

$$\{x^{h,1}, x^{h,2}, \ldots, x^{h,N_h}\} \subset R^n \quad \text{and} \quad \{y^{h,1}, y^{h,2}, \ldots, y^{h,N_h}\} \subset R^m$$

by the recursion: for $i = 0, 1, \ldots, N_h - 1$,

$$\begin{aligned} x^{h,i+1} &= x^{h,i} + h\left\{A[\theta x^{h,i} + (1-\theta)x^{h,i+1}] + By^{h,i+1}\right\}, \\ y^{h,i+1} &\in \mathrm{SOL}(l, u, Qx^{h,i+1}, M), \end{aligned} \tag{1.3}$$

where $\theta \in [0, 1]$ is a scalar. The time-stepping method has been studied extensively (see Pang & Stewart, 2008). There are also some numerical methods for integral equations, for example the collocation methods (see Brunner, 2004). However, the question of how to tailor the methods for the complementarity or variational inequality constrained case still remains open.

We are aware that an approximate solution of the ordinary differential equations (ODEs) is always polluted by discretization and roundoff errors. For the DLVI (1.1) we have additional errors induced by the numerical solution of variational inequalities. For an approximate solution to be of practical use, it is imperative for us to have error bounds for it. For existing numerical methods no error bounds have yet been given.

In this paper we suppose that M is a P-matrix, i.e., all principal minors of M are positive. The P-matrix assumption is also imposed in Stewart (2006), but the paper was not concerned with error bounds. Note that every positive definite matrix and every M-matrix belong to the class of P-matrices. If M is a P-matrix, then the DLVI (1.1) has a unique solution $x \in C^1[0, T]$ with $y \in C^0[0, T]$, and the solution set $\mathrm{SOL}(l, u, Qx(t), M)$ contains a unique vector $y(t)$ for every $t \in [0, T]$. Moreover, the assumption guarantees that the solution set $\mathrm{SOL}(l, u, Qx^{h,i}, M)$ contains a unique vector $y^{h,i}$ for every $h > 0$ and every $i = 0, \ldots, N_h$.

It is worth noting that the results presented in this paper can be extended to other classes of matrices and the uniqueness of the solution in $\mathrm{SOL}(l, u, Qx(t), M)$ is not essential. For instance, we can extend the results to the class of Z-matrices, i.e., all off-diagonal elements of M are nonpositive. It is known that the least-element solution in $\mathrm{SOL}(l, u, q, M)$ is unique (see Chen & Xiang, 2010; Wang & Yuan, 2011) and it is Lipschitz continuous (see Mangasarian & Shiau, 1987) with respect to q. Hence, the results in this paper can be used to establish error bounds for the following least-element Z-matrix DLVI.

Least-element DLVI

$$\begin{aligned} \dot{x}(t) &= Ax(t) + By(t), \\ y(t) &= \operatorname{argmin}\ e^{\mathrm{T}}v, \\ &\qquad \text{subject to } v \in \mathrm{SOL}(l, u, Qx(t), M), \\ x(0) &= x_0, \quad t \in [0, T], \end{aligned} \tag{1.4}$$

where e is the vector in R^m all of whose elements are 1. The time-stepping method for (1.4) can be tailored as follows:

$$\begin{aligned} x^{h,i+1} &= x^{h,i} + h\left\{A[\theta x^{h,i} + (1-\theta)x^{h,i+1}] + By^{h,i+1}\right\}, \\ y^{h,i+1} &= \operatorname{argmin} e^{\mathrm{T}}v, \\ &\qquad \text{subject to } v \in \mathrm{SOL}(l, u, Qx^{h,i+1}, M), \end{aligned}$$

where $y^{h,i+1}$ can be computed by solving a linear programming problem (see Mangasarian & Shiau, 1987; Cottle *et al.*, 1992; Chen & Xiang, 2010; Wang & Yuan, 2011).

We define a piecewise linear function by the interpolant

$$\begin{aligned} x^h(t) &:= \frac{t_{h,i+1} - t}{h} x^{h,i} + \frac{t - t_{h,i}}{h} x^{h,i+1}, \quad t \in [t_{h,i}, t_{h,i+1}], \\ y^h(t) &\in \mathrm{SOL}(l, u, Qx^h(t), M), \end{aligned} \tag{1.5}$$

where $i = 0, \ldots, N_h - 1$.

The aim of this paper is to present computable and sharp error bounds $\epsilon_x^{h,i}$ and $\epsilon_y^{h,i}$ for the approximate solution defined by the linear interpolant (1.5) with the time-stepping method (1.3), such that

$$\|x(t) - x^h(t)\|_\infty \leqslant \epsilon_x^{h,i} \tag{1.6}$$

and

$$\|y(t) - y^h(t)\|_\infty \leqslant \epsilon_y^{h,i} \tag{1.7}$$

for all $t \in [t_{h,i}, t_{h,i+1}]$ and $i = 0, \ldots, N_h - 1$.

Our numerical code was written in MATLAB 7 with the use of INTLAB, a toolbox for reliable computation, which takes all roundoff errors into account (see Rump, 1999). Numerical examples of bridge collapse, earthquake-induced structural pounding and circuit simulation are used to show the efficiency of the error bounds.

Throughout this paper, $\|\cdot\|$ denotes the norm $\|\cdot\|_\infty$.

2. Estimation of Lipschitz constants

In this section we use the Euler method to estimate Lipschitz constants of the solution $x(t)$, $y(t)$ of DLVI (1.1) over $[t_{h,i}, t_{h,i+1}]$. The Lipschitz constants will be used to compute the error bounds of approximate solutions obtained by the time-stepping method.

The Euler method is the simplest numerical method for the initial value problem of ODEs. Starting from a given vector $x^{h,0} = x^0 \in R^n$ and $y^{h,0} \in \text{SOL}(l, u, Qx^{h,0}, M)$, the Euler method computes two families of vectors

$$\{x^{h,1}, x^{h,2}, \ldots, x^{h,N_h}\} \subset R^n \quad \text{and} \quad \{y^{h,1}, y^{h,2}, \ldots, y^{h,N_h}\} \subset R^m$$

by the recursion: for $i = 0, 1, \ldots, N_h - 1$,

$$\begin{aligned} x^{h,i+1} &= x^{h,i} + h(Ax^{h,i} + By^{h,i}), \\ y^{h,i+1} &\in \text{SOL}(l, u, Qx^{h,i+1}, M). \end{aligned} \tag{2.1}$$

The piecewise linear function $x^h(t)$ defined by (1.5) with $\{x^{h,i}\}$ and $\{y^{h,i}\}$ generated by the Euler method (2.1) satisfies

$$x^h(t) = x^{h,i} + \int_{t_{h,i}}^{t} (Ax^{h,i} + By^{h,i})\,\mathrm{d}s. \tag{2.2}$$

This equality is easily verified from the following calculation:

$$\begin{aligned} x^h(t) &= \frac{1}{h}\left[(t_{h,i+1} - t)x^{h,i} + (t - t_{h,i})x^{h,i+1}\right] \\ &= \frac{1}{h}\left[(t_{h,i+1} - t_{h,i})x^{h,i} + (t - t_{h,i})(x^{h,i+1} - x^{h,i})\right] \\ &= \frac{1}{h}\left[hx^{h,i} + h(t - t_{h,i})(Ax^{h,i} + By^{h,i})\right] \\ &= x^{h,i} + (t - t_{h,i})(Ax^{h,i} + By^{h,i}) \\ &= x^{h,i} + \int_{t_{h,i}}^{t} (Ax^{h,i} + By^{h,i})\,\mathrm{d}s. \end{aligned}$$

It is known that for a fixed P-matrix M, the solution function $z(q)$ of the P-matrix linear complementarity problem

$$0 \leqslant z \perp Mz + q \geqslant 0$$

is piecewise linear (see Luo *et al.*, 1996). Using a similar argument we can show that the solution function $z(q)$ of the P-matrix linear variational inequality problem

$$(w - z(q))^{\mathrm{T}}(Mz(q) + q) \geqslant 0 \quad \text{for } l \leqslant w \leqslant u,$$

is piecewise linear. Hence, we deduce that $y^h(t)$ defined by (1.5) is piecewise linear since $x^h(t)$ is piecewise linear.

In our error analysis we need the following lemma on the perturbation bound for the variational inequality problem $\mathrm{VI}(l, u, q, M)$.

LEMMA 2.1 Suppose that $M \in R^{m\times m}$ is a P-matrix. Then for any $q \in R^m$ the $\mathrm{VI}(l, u, q, M)$ has a unique solution. Let $v \in \mathrm{SOL}(l, u, q, M)$ and $v' \in \mathrm{SOL}(l, u, q', M)$ for $q, q' \in R^m$. Then we have

$$\|v - v'\| \leqslant \beta_M \|q - q'\|, \tag{2.3}$$

where

$$\beta_M = \max_{d\in[0,1]^n} \|(I - D + DM)^{-1} D\|, \tag{2.4}$$

and $D = \mathrm{diag}(d)$ with $d = (d_i) \in R^m$, $0 \leqslant d_i \leqslant 1$, $i = 1, \ldots, m$.

Proof. It is known that if M is a P-matrix, then for any $q \in R^m$ there is a unique vector $v \in \mathrm{SOL}(l, u, q, M)$ (see, e.g., Facchinei & Pang, 2003). It is easy to verify that $v \in \mathrm{SOL}(l, u, q, M)$ if and only if v is a solution of the system of nonsmooth equations

$$H_q(v) := \mathrm{mid}(v - l, v - u, Mv + q) = 0,$$

where 'mid' is the componentwise median operator. For $v \in \mathrm{SOL}(l, u, q, M)$ and $v' \in \mathrm{SOL}(l, u, q', M)$ let $z = Mv + q$ and $z' = Mv' + q'$. Note that $v, v' \in [l, u]$ and

$$v - l > v - u \quad \text{and} \quad v' - l > v' - u.$$

It is easy to find that

$$0 = (H_q(v) - H_{q'}(v'))_i = (1 - d_i)(v_i - v_i') + d_i(z_i - z_i'),$$

where

$$d_i = \begin{cases} 0 & \text{if } v_i - u_i \geqslant z_i,\ v_i' - u_i \geqslant z_i', \\ 0 & \text{if } v_i - l_i \leqslant z_i,\ v_i' - l_i \leqslant z_i', \\ 1 & \text{if } v_i - u_i < z_i < v_i - l_i,\ v_i' - u_i < z_i' < v_i' - l_i, \\ \frac{v_i - v_i'}{v_i - z_i - (v_i' - z_i')} & \text{otherwise}, \end{cases}$$

$$= \begin{cases} 0 & \text{if } v_i - z_i, v_i' - z_i' \in [u_i, \infty), \\ 0 & \text{if } v_i - z_i, v_i' - z_i' \in (-\infty, l_i], \\ 1 & \text{if } v_i - z_i, v' - z_i' \in (l_i, u_i), \\ \frac{v_i - v_i'}{v_i - z_i - (v_i' - z_i')} & \text{otherwise}. \end{cases}$$

We can show that $d_i \in [0, 1]$ by considering $v_i - z_i$ and $v_i' - z_i'$ in the three intervals $[u_i, \infty)$, $(-\infty, l_i]$ and (l_i, u_i). For example, consider

$$v_i - z_i \in [u_i, \infty) \quad \text{and} \quad v_i' - z_i' \in (-\infty, l_i].$$

Then

$$v_i - z_i - (v_i' - z_i') \geqslant u_i - l_i$$

and

$$v_i - u_i \geqslant z_i \quad \text{and} \quad v_i' - l_i \leqslant z_i'.$$

Hence, from $\text{mid}(v - l, v - u, z) = \text{mid}(v' - l, v' - u, z') = 0$, we have $v_i = u_i$ and $v_i' = l_i$. This implies that

$$0 < u_i - l_i = v_i - v_i' \leqslant v_i - z_i - (v_i' - z_i'),$$

and thus $d_i \in [0, 1]$.

Let $D = \text{diag}(d)$. We obtain

$$0 = (I - D)(v - v') + D[M(v - v') + q - q'].$$

Rearranging the terms in the above equality we achieve

$$(I - D + DM)(v - v') = -D(q - q').$$

Since M is a P-matrix then $I - D + DM$ is also a P-matrix and so is invertible (see Gabriel & Moré, 1997). Hence, we have

$$v - v' = -(I - D + DM)^{-1} D(q - q'),$$

which yields the error estimate (2.3). □

Chen & Xiang (2007) showed that the constant β_M can be estimated by the norm of M^{-1} for some special matrices. In particular, we have

- $\beta_M \leqslant \|\tilde{M}^{-1}\|$ if M is an H-matrix with positive diagonals, where $\tilde{M}$ is the comparison matrix of M;
- $\beta_M = \|M^{-1}\|$ if M is an M-matrix and
- $\beta_M \leqslant \sqrt{m}\|M^{-1}\|_2$ if M is a symmetric positive definite matrix.

The following theorem shows that the system (1.1) has a unique solution and provides an *a posteriori* error estimate with a computable Lipschitz constant L for any step size $h < 1/L$. This result is useful for practical applications and reliable computation.

THEOREM 2.2 Suppose that M is a P-matrix. Then for any $T > 0$, (1.1) has a unique solution (x, y) in $[0, T]$, where x is continuously differentiable and y is Lipschitz continuous. Moreover, for any positive number $h < 1/L$, $L = \|A\| + \beta_M \|B\| \|Q\|$, the sequence $(x_{k,i}(t), y_{k,i}(t))$ defined by setting $x_{0,i}(t) = x(t_{h,i})$ and $y_{0,i}(t) \in \text{SOL}(l, u, Qx_{0,i}(t), M)$ for $t \in [t_{h,i}, t_{h,i+1}]$ and by the recursion,

$$\begin{aligned} x_{k+1,i}(t) &= x(t_{h,i}) + \int_{t_{h,i}}^{t} (Ax_{k,i}(s) + By_{k,i}(s))\,\mathrm{d}s, \\ y_{k+1,i}(t) &\in \text{SOL}(l, u, Qx_{k+1,i}(t), M), \end{aligned} \tag{2.5}$$

converges uniformly on $[t_{h,i}, t_{h,i+1}]$ to the unique solution (x, y) of (1.1) as $k \to \infty$. Moreover, we have the following *a posteriori* error estimate for $t \in [t_{h,i}, t_{h,i+1}]$

$$\|x(t) - x_{k+1,i}(t)\| \leqslant \frac{Lh}{1 - Lh} \|x_{k+1,i}(t) - x_{k,i}(t)\|. \tag{2.6}$$

Proof. The assumption that M is a P-matrix ensures that there is a unique $y(t) \in \mathrm{SOL}(l, u, Qx(t), M)$ for any $x(t) \in R^n$. Hence, we can write y as an implicit function of x,

$$y(t) = y(x(t)).$$

From Lemma 2.1 we find for any $\tilde{x}(t) \in R^n$,

$$\|y(x(t)) - y(\tilde{x}(t))\| \leqslant \beta_M \|Q\| \|x(t) - \tilde{x}(t)\|.$$

Let

$$G(x, t) = Ax(t) + By(t).$$

Then we have

$$\|G(x, t) - G(\tilde{x}, t)\| \leqslant (\|A\| + \beta_M \|B\| \|Q\|) \|x(t) - \tilde{x}(t)\| = L\|x(t) - \tilde{x}(t)\|.$$

By the Picard–Lindelöf theorem we know that (1.1) has a unique solution (x, y) in $[-h, h]$ and x is continuously differentiable and y is Lipschitz continuous.

Note that h is determined by the Lipschitz constant L, which is independent of the initial point $(x(0), 0)$ and T. Hence, we can repeat the argument on each interval $[t_{h,i}, t_{h,i+1}]$ and claim that for any $T > 0$ (1.1) has a unique solution (x, y) in $[0, T]$, where x is continuously differentiable and y is Lipschitz continuous.

Now we show the convergence of $\{x_{k,i}\}$ and the *a posteriori* error estimate (2.6). Since G is Lipschitz continuous we can choose a bounded and closed domain D and a positive number Γ such that $(x(t_{h,i}), t_{h,i}) \in D$ and

$$\|G(x, t)\| \leqslant \Gamma, \quad (x, t) \in D.$$

Define an operator $\mathcal{T}: X \to X$

$$\mathcal{T}(x)(t) = x(t_{h,i}) + \int_{t_{h,i}}^{t} (Ax(s) + By(s))\, \mathrm{d}s,$$

where X is the closed subset

$$X := \{x \in C[t_{h,i-1}, t_{h,i+1}]: \|x - x(t_{h,i})\| \leqslant \Gamma h\}$$

of the Banach space $C[t_{h,i-1}, t_{h,i+1}]$. It is clear that $\mathcal{T}$ maps X into itself since $\mathcal{T}$ is continuous and

$$\|\mathcal{T}x(t) - x(t_{h,i})\| \leqslant \Gamma h.$$

Moreover, the following shows that the mapping $\mathcal{T}$ is contractive on X,

$$\|(\mathcal{T}(x) - \mathcal{T}(\tilde{x}))(t)\| = \left\| \int_{t_{h,i}}^{t} (A[x(s) - \tilde{x}(s)] + B[y(s) - \tilde{y}(s)])\, \mathrm{d}s \right\|$$

$$
\begin{aligned}
&\leqslant (t - t_{h,i})\left[\|A\| \max_{s\in[0,t]} \|x(s) - \tilde{x}(s)\| + \|B\| \max_{s\in[0,t]} \|y(s) - \tilde{y}(s)\|\right]\\
&\leqslant h(\|A\| + \beta_M\|B\|\|Q\|) \max_{s\in[0,t]} \|x(s) - \tilde{x}(s)\|\\
&\leqslant Lh\|x - \tilde{x}\|.
\end{aligned}
$$

The convergence and the *a posteriori* error estimate (2.6) follow from the Banach fixed point theorem (see Kress, 1998; Pang & Stewart, 2009). □

REMARK 2.3 In Kanat *et al.* (2006, Proposition 2.1) the existence of the solution of the differential linear complementarity system (1.2) is guaranteed by a more moderate assumption: $\{By\colon y \in \mathrm{SOL}(l, u, Qx, M)\}$ is a singleton for any $x \in R^n$. This is fulfilled in the setting that M is a P-matrix. Here we use the assumption that M is a P-matrix to obtain a computable Lipschitz constant L in the estimate (2.6). The constant L is necessary for deriving reliable error bounds.

Now we provide computable error bounds for $x^h(t)$ and $y^h(t)$ defined by the linear interpolation (1.5) with the Euler method (2.1).

THEOREM 2.4 Suppose that M is a P-matrix. Let $L = \|A\| + \beta_M\|B\|\|Q\|$, and $h < 1/L$. Assume that

$$\|x(t_{h,i}) - x^{h,i}\| \leqslant b_x^{h,i}.$$

Let

$$b_x^{h,i+1} = \frac{1}{1-Lh} b_x^{h,i} + \frac{Lh^2}{1-Lh}\|Ax^{h,i} + By^{h,i}\|.$$

Then we have for $t \in [t_{h,i}, t_{h,i+1}]$,

$$\|x(t) - x^h(t)\| \leqslant b_x^{h,i+1} \tag{2.7}$$

and

$$\|y(t) - y^h(t)\| \leqslant \beta_M\|Q\|b_x^{h,i+1}. \tag{2.8}$$

Proof. From the error estimate (2.6) of Theorem 2.2 we have that for any $t \in [t_{h,i}, t_{h,i+1}]$,

$$
\begin{aligned}
\|x(t) - x_{1,i}(t)\| &\leqslant \frac{Lh}{1-Lh}\|x_{1,i}(t) - x_{0,i}(t)\|\\
&\leqslant \frac{Lh}{1-Lh}\left\|\int_{t_{h,i}}^{t} [Ax_{0,i}(s) + By_{0,i}(s)]\,\mathrm{d}s\right\|\\
&\leqslant \frac{Lh^2}{1-Lh}\|Ax(t_{h,i}) + By(t_{h,i})\|\\
&\leqslant \frac{Lh^2}{1-Lh}\|A[x(t_{h,i}) - x^{h,i}] + B[y(t_{h,i}) - y^{h,i}] + Ax^{h,i} + By^{h,i}\|\\
&\leqslant \frac{Lh^2}{1-Lh}(Lb_x^{h,i} + \|Ax^{h,i} + By^{h,i}\|),
\end{aligned}
$$

where the last inequality uses

$$\|A[x(t_{h,i}) - x^{h,i}] + B[y(t_{h,i}) - y^{h,i}]\| \leqslant (\|A\| + \beta_M\|B\|\|Q\|)b_x^{h,i}.$$

From (2.5) and (2.2) we have

$$\begin{aligned}\|x_{1,i}(t) - x^h(t)\| &\leqslant \left\| x(t_{h,i}) + \int_{t_{h,i}}^{t} [Ax_{0,i}(s) + By_{0,i}(s)]\,\mathrm{d}s - x^{h,i} - \int_{t_{h,i}}^{t} (Ax^{h,i} + By^{h,i})\,\mathrm{d}s \right\| \\ &\leqslant \|x(t_{h,i}) - x^{h,i}\| + \left\| \int_{t_{h,i}}^{t} [A(x(t_{h,i}) - x^{h,i}) + B(y(t_{h,i}) - y^{h,i})]\,\mathrm{d}s \right\| \\ &\leqslant b_x^{h,i} + hLb_x^{h,i}.\end{aligned}$$

We achieve the error bound (2.7) by adding the above two inequalities and by using

$$\|x(t) - x^h(t)\| \ \leqslant\ \|x(t) - x_{1,i}(t)\| + \|x_{1,i}(t) - x^h(t)\|.$$

The error bound (2.8) can be obtained by Lemma 2.1 with the relations

$$y(t) \in \mathrm{SOL}(l, u, Qx(t), M) \quad \text{and} \quad y^h(t) \in \mathrm{SOL}(l, u, Qx^h(t), M)$$

for $t \in [t_{h,i}, t_{h,i+1}]$. This completes the proof. □

As the Euler method starts from $x^{h,0} = x(t_{h,0})$, the initial error $b_x^{h,0} = 0$ is available. Hence, the error bounds (2.7) and (2.8) can be computed practically.

We end this section with an estimate of the Lipschitz constants of the solution $x(t)$ and $y(t)$ over $[t_{h,i}, t_{h,i+1}]$. The Lipschitz constants will be used in the next section to derive the computable error bounds for the time-stepping method.

THEOREM 2.5 In the setting of Theorem 2.4 the solution $(x(t), y(t))$ of (1.1) satisfies

$$\|x(s) - x(t)\| \leqslant L_x^{h,i}|s - t|$$

and

$$\|y(s) - y(t)\| \leqslant L_y^{h,i}|s - t|$$

for $t \in [t_{h,i}, t_{h,i+1}]$, where the Lipschitz constants $L_x^{h,i}$ and $L_y^{h,i}$ are defined as follows:

$$L_x^{h,i} = Lb_x^{h,i+1} + (1 + Lh)\|Ax^{h,i} + By^{h,i}\|$$

and

$$L_y^{h,i} = \beta_M\|Q\|L_x^{h,i}.$$

Moreover, the Lipschitz constant $L_x^{h,i}$ satisfies

$$L_x^{h,i} - \|Ax^{h,i} + By^{h,i}\| = \mathcal{O}(h). \tag{2.9}$$

Proof. First we estimate the bounds of $\|Ax^h(t) + By^h(t)\|$ over $[t_{h,i}, t_{h,i+1}]$. It is clear that

$$\|x^h(t) - x^{h,i}\| \ = \frac{t - t_{h,i}}{h}\|x^{h,i+1} - x^{h,i}\| \ \leqslant\ \|x^{h,i+1} - x^{h,i}\| \ =\ h\|Ax^{h,i} + By^{h,i}\|$$

and

$$\|y^h(t) - y^{h,i}\| \leqslant \beta_M \|Q\| \|x^h(t) - x^{h,i}\|.$$

So we have

$$\begin{aligned}\|Ax^h(t) + By^h(t)\| &= \|A(x^h(t) - x^{h,i}) + B(y^h(t) - y^{h,i}) + (Ax^{h,i} + By^{h,i})\| \\ &\leqslant (\|A\| + \beta_M \|B\| \|Q\|)h\|Ax^{h,i} + By^{h,i}\| + \|Ax^{h,i} + By^{h,i}\| \\ &= (1 + Lh)\|Ax^{h,i} + By^{h,i}\|.\end{aligned} \tag{2.10}$$

From (2.7), (2.8) and (2.10) we deduce

$$\begin{aligned}\|\dot{x}(t)\| &= \|Ax(t) + By(t)\| \\ &= \|A(x(t) - x^h(t)) + B(y(t) - y^h(t)) + Ax^h(t) + By^h(t)\| \\ &\leqslant Lb_x^{h,i+1} + \|Ax^h(t) + By^h(t)\| \\ &\leqslant Lb_x^{h,i+1} + (1 + Lh)\|Ax^{h,i} + By^{h,i}\| \\ &=: L_x^{h,i}.\end{aligned}$$

Using Lemma 2.1 with the relations $y(t) \in \mathrm{SOL}(l, u, Qx(t), M)$ and $y(s) \in \mathrm{SOL}(l, u, Qx(s), M)$, for $t, s \in [0, T]$, we obtain

$$\|y(t) - y(s)\| \leqslant \beta_M \|Q\| \|x(t) - x(s)\| \leqslant \beta_M \|Q\| L_x^{h,i} |t - s|.$$

Now we show (2.9). By the Lipschitz continuity of $Ax(t) + By(t)$ and the convergence of the Euler method, there are positive scalars c and $\bar{h}$ such that for all $h \in (0, \bar{h}]$ and all $i = 0, 1, \ldots, N_h$,

$$\|Ax^{h,i} + By^{h,i}\| \leqslant c.$$

Let us set

$$\alpha_0 = \frac{1}{1 - Lh} \quad \text{and} \quad \alpha_1 = \frac{Lh^2}{1 - Lh}.$$

Then from the definition of $b_x^{h,i+1}$ and $b_x^{h,0} = 0$ we have

$$\begin{aligned}b_x^{h,i} &\leqslant \alpha_0 b_x^{h,i-1} + \alpha_1 c \\ &\leqslant \alpha_0^i b_x^{h,0} + (1 + \alpha_0 + \cdots + \alpha_0^{i-1})\alpha_1 c \\ &= \frac{(\alpha_0^i - 1)\alpha_1 c}{\alpha_0 - 1} \\ &= h(\alpha_0^i - 1)c.\end{aligned}$$

It is easy to see that

$$\alpha_0^i = (\alpha_0 - 1 + 1)^i \leqslant e^{\mathrm{i}(\alpha_0 - 1)} = e^{\frac{\mathrm{i}Lh}{1 - Lh}} \leqslant e^{\frac{TL}{1 - Lh}}.$$

Hence, $b_x^{h,i} = \mathcal{O}(h)$. From the definition of $L_x^{h,i}$ we obtain (2.9). □

3. Error bounds for time-stepping method

In this section we present computable error bounds for approximate solutions generated by the time-stepping method (1.3) with a given parameter $\theta \in [0, 1]$.

Let $L = \|A\| + \beta_M \|B\| \|Q\|$, and let $\bar{h}$ be a positive constant satisfying

$$\bar{h} < \frac{1}{L - \theta\|A\|} = \frac{1}{(1-\theta)\|A\| + \beta_M \|B\| \|Q\|}.$$

Let $\{x^{h,1}, \ldots, x^{h,N_h}\}$ and $\{y^{h,1}, \ldots, y^{h,N_h}\}$ be generated by the time-stepping method (1.3) with the step size $h = T/N_h \in (0, \bar{h}]$.

Similarly to (2.2) we can show that the piecewise linear function $x^h(t)$ defined by the linear interpolation (1.5) with the time-stepping method (1.3) satisfies

$$x^h(t) = x^{h,i} + \int_{t_{h,i}}^{t} \left\{A[\theta x^{h,i} + (1-\theta)x^{h,i+1}] + By^{h,i+1}\right\} \mathrm{d}s. \tag{3.1}$$

The following theorem presents computable error bounds for $x^h(t)$ and $y^h(t)$.

THEOREM 3.1 Suppose that M is a P-matrix. Assume that

$$\|x(t_{h,i}) - x^{h,i}\| \leqslant \epsilon_x^{h,i}.$$

Let

$$\epsilon_x^{h,i+1} = \frac{1 + h\theta\|A\|}{1 + h\theta\|A\| - Lh}\epsilon_x^{h,i} + \frac{1}{2}\frac{LL_x^{h,i}}{1 + h\theta\|A\| - Lh}h^2,$$

where $L_x^{h,i}$ is the Lipschitz constant of $x(t)$ over $[t_{h,i}, t_{h,i+1}]$. Then we have for $t \in [t_{h,i}, t_{h,i+1}]$

$$\|x(t) - x^h(t)\| \leqslant \epsilon_x^{h,i+1} \tag{3.2}$$

and

$$\|y(t) - y^h(t)\| \leqslant \beta_M \|Q\| \epsilon_x^{h,i+1}. \tag{3.3}$$

Moreover, we have

$$\epsilon_x^{h,0} \leqslant \cdots \leqslant \epsilon_x^{h,i} \leqslant \epsilon_x^{h,i+1} \leqslant \cdots \leqslant \epsilon_x^{h,N_h} = \mathcal{O}(h). \tag{3.4}$$

Proof. Subtracting (3.1) from

$$\begin{aligned} x(t) &= x(t_{h,i}) + \int_{t_{h,i}}^{t} [Ax(s) + By(s)]\,\mathrm{d}s, \\ &= x(t_{h,i}) + \int_{t_{h,i}}^{t} [\theta Ax(s) + (1-\theta)Ax(s) + By(s)]\,\mathrm{d}s, \end{aligned}$$

we have

$$\begin{aligned} x(t) - x^h(t) = x(t_{h,i}) - x^{h,i} &+ \int_{t_{h,i}}^{t} \theta A[x(s) - x^{h,i}]\,\mathrm{d}s + \int_{t_{h,i}}^{t} (1-\theta)A[x(s) - x^{h,i+1}]\,\mathrm{d}s \\ &+ \int_{t_{h,i}}^{t} B[y(s) - y^{h,i+1}]\,\mathrm{d}s, \end{aligned}$$

$$= x(t_{h,i}) - x^{h,i} + \int_{t_{h,i}}^{t} \theta A[x(s) - x(t_{h,i}) + x(t_{h,i}) - x^{h,i}]\,\mathrm{d}s$$
$$+ \int_{t_{h,i}}^{t} (1-\theta)A[x(s) - x(t_{h,i+1}) + x(t_{h,i+1}) - x^{h,i+1}]\,\mathrm{d}s$$
$$+ \int_{t_{h,i}}^{t} B[y(s) - y(t_{h,i+1}) + y(t_{h,i+1}) - y^{h,i+1})]\,\mathrm{d}s.$$

Using the Lipschitz continuity of $x(t)$ and $y(t)$ with the Lipschitz constants $L_x^{h,i}$ and $L_y^{h,i} = \beta_M \|Q\| L_x^{h,i}$, we obtain

$$\|x(t) - x^h(t)\| \leqslant \|x(t_{h,i}) - x^{h,i}\| + \int_{t_{h,i}}^{t} \theta\|A\|[L_x^{h,i}(s - t_{h,i}) + \|x(t_{h,i}) - x^{h,i}\|]\,\mathrm{d}s$$
$$+ \int_{t_{h,i}}^{t} (1-\theta)\|A\|[L_x^{h,i}(t_{h,i+1} - s) + \|x(t_{h,i+1}) - x^{h,i+1}\|]\,\mathrm{d}s$$
$$+ \int_{t_{h,i}}^{t} \|B\|\beta_M\|Q\|[L_x^{h,i}(t_{h,i+1} - s) + \|x(t_{h,i+1}) - x^{h,i+1}\|]\,\mathrm{d}s,$$

for $t \in [t_{h,i}, t_{h,i+1}]$, where we use

$$\|y(t_{h,i+1}) - y^{h,i+1}\| \leqslant \beta_M\|Q\|\|x(t_{h,i+1}) - x^{h,i+1}\|.$$

Taking the maximum of the two sides of the above inequality and noting that

$$\|x(t_{h,i+1}) - x^{h,i+1}\| \leqslant \max_{t\in[t_{h,i},t_{h,i+1}]} \|x(t) - x^h(t)\|,$$

we get

$$\max_{t\in[t_{h,i},t_{h,i+1}]} \|x(t) - x^h(t)\| \leqslant h[(1-\theta)\|A\| + \beta_M\|B\|\|Q\|] \max_{t\in[t_{h,i},t_{h,i+1}]} \|x(t) - x^h(t)\|$$
$$+ (1 + h\theta\|A\|)\epsilon_x^{h,i} + \frac{1}{2}h^2 L L_x^{h,i}.$$

Arranging the terms, we obtain

$$\max_{t\in[t_{h,i},t_{h,i+1}]} \|x(t) - x^h(t)\| \leqslant \frac{1 + h\theta\|A\|}{1 + h\theta\|A\| - Lh}\epsilon_x^{h,i} + \frac{1}{2}\frac{L L_x^{h,i}}{1 + h\theta\|A\| - Lh}h^2 = \epsilon_x^{h,i+1}.$$

This delivers the estimate (3.2). The estimate (3.3) follows from Lemma 2.1.

Now we show (3.4). It is obvious that $\epsilon_x^{h,i} \leqslant \epsilon_x^{h,i+1}$ for $i = 0, \ldots, N_h$. To show $\epsilon_x^{h,N_h} = \mathcal{O}(h)$ we note from (2.9) and the Lipschitz continuity of (x, y), and from the convergence of the Euler method, that there is a positive constant L_x, which is independent of h, such that

$$\max_{1\leqslant i\leqslant N_h - 1} L_x^{h,i} \leqslant L_x.$$

Denote

$$\alpha_0 = \frac{1 + h\theta\|A\|}{1 + h\theta\|A\| - Lh}$$

and

$$\alpha_1 = \frac{1}{2}\frac{L L_x}{1 + h\theta\|A\| - Lh}.$$

Then we can write $\epsilon_x^{h,i+1} \leqslant \alpha_0 \epsilon_x^{h,i} + \alpha_1 h^2$. Using this inequality repeatedly and noting that

$$\epsilon_x^{h,0} = \|x(t_{h,0}) - x^{h,0}\| = 0,$$

we obtain the estimate

$$\epsilon_x^{h,i} \leqslant \alpha_0^{i-1}\epsilon_x^{h,0} + \alpha_1 h^2(1 + \alpha_0 + \cdots + \alpha_0^i) = \frac{\alpha_0^i - 1}{\alpha_0 - 1}\alpha_1 h^2 = \frac{1}{2}L_x(\alpha_0^i - 1)h.$$

It is easy to see

$$\alpha_0 - 1 = \frac{Lh}{1 + h\theta\|A\| - Lh} \quad \text{and} \quad \frac{\alpha_1}{\alpha_0 - 1} = \frac{1}{2}\frac{(L_x)}{h}.$$

In a similar way to the proof of Theorem 2.5, we obtain (3.4). □

We conclude this section by showing that the error bound

$$\epsilon_x^{h,i+1} = \frac{1 + h\theta\|A\|}{1 + h\theta\|A\| - Lh}\epsilon_x^{h,i} + \frac{1}{2}\frac{LL_x^{h,i}}{1 + h\theta\|A\| - Lh}h^2$$

given in Theorem 3.1 (for the time-stepping method) is tighter than the error bound

$$b_x^{h,i+1} = \frac{1}{1 - Lh}b_x^{h,i} + \frac{Lh^2}{1 - Lh}\|Ax^{h,i} + By^{h,i}\|$$

given in Theorem 2.4 (for the Euler method). This is the novelty of our verification method. We first use the Euler method to define $b_x^{h,i+1}$ and use it to get a Lipschitz constant

$$L_x^{h,i} = Lb_x^{h,i+1} + (1 + Lh)\|Ax^{h,i} + By^{h,i}\|.$$

Next we present a sharp error bound $\epsilon_x^{h,i+1}$ by using the time-stepping method and the Lipschitz constant $L_x^{h,i}$.

THEOREM 3.2 For a given small positive number c, let $\mathcal{T}_c = \{t \mid \|Ax(t) + By(t)\| \geqslant c,\ t \in [0, T]\}$. Then there is an $\hat{h} \leqslant \bar{h}$ such that for all $h \in (0, \hat{h}]$, we have

$$\epsilon_x^{h,i+1} < b_x^{h,i+1},$$

if $\epsilon_x^{h,i} \leqslant b_x^{h,i}$, and $[ih, (i+1)h] \subseteq \mathcal{T}_c$.

Proof. By (2.9), there is an $\hat{h} < \bar{h}$ such that for all $h \in (0, \hat{h}]$, we have

$$L_x^{h,i} < 2\|Ax^{h,i} + By^{h,i}\| \quad \text{if } [ih, (i+1)h] \subseteq \mathcal{T}_c.$$

This, together with $\epsilon_x^{h,i} \leqslant b_x^{h,i}$, implies that

$$\begin{aligned}
\epsilon_x^{h,i+1} &= \frac{1 + h\theta\|A\|}{1 + h\theta\|A\| - Lh}\epsilon_x^{h,i} + \frac{1}{2}\frac{LL_x^{h,i}}{1 + h\theta\|A\| - Lh}h^2 \\
&\leqslant \frac{1}{1 - Lh}\epsilon_x^{h,i} + \frac{1}{2}\frac{LL_x^{h,i}}{1 - Lh}h^2 \\
&\leqslant \frac{1}{1 - Lh}b_x^{h,i} + \frac{1}{2}\frac{LL_x^{h,i}}{1 - Lh}h^2 \\
&< \frac{1}{1 - Lh}b_x^{h,i} + \frac{Lh^2}{1 - Lh}\|Ax^{h,i} + By^{h,i}\| = b_x^{h,i+1}.
\end{aligned}$$

□

REMARK 3.3 Note that $\epsilon_x^{h,0} = b_x^{h,0} = 0$. If at the initial point $Ax(0) + By(0) = 0$, then it is easy to find that the unique solution $x(t)$ of (1.1) satisfies $Ax(t) + By(t) = 0$, for $t \in [0, T]$, and $b_x^{h,i} \equiv 0$, $Ax^{h,i} + By^{h,i} \equiv 0$, $L_x^{h,i} \equiv 0$, $\epsilon_x^{h,i} \equiv 0$. Suppose that $Ax(0) + By(0) \neq 0$ then from Theorem 3.2, there are $t_c \leqslant T$ and $h_c > 0$, such that $\epsilon_x^{h,i} < b_x^{h,i}$ for all small $ih \in (0, t_c)$ and $h \in (0, h_c)$.

REMARK 3.4 The error bounds increase with respect to t (see Fig. 1). This is normal for the validated solution of the ODE and could be tightened by using a Taylor model of high order (see Berz & Makino, 2004), and in this case a piecewise integration technique must be adopted (see Wang & Wu, 2009).

4. Validated solution for DLVI

In this section we use the error bounds given in the last two sections to present an algorithm for delivering the validated solution for the nonautonomous system

$$\begin{cases} \dot{x}(t) = Ax(t) + By(t) + f(t), \\ y(t) \in \mathrm{SOL}(l, u, Qx(t) + g(t), M), \\ x(0) = x^0 \in R^n, \quad t \in [0, T], \end{cases} \tag{4.1}$$

where $f\colon [0, T] \to R^n$ and $g\colon [0, T] \to R^m$.

The Euler method for the nonautonomous system (4.1) is defined by

$$x^{h,i+1} = x^{h,i} + h[Ax^{h,i} + By^{h,i} + f(t_{h,i})],$$
$$y^{h,i+1} \in \mathrm{SOL}(l, u, Qx^{h,i+1} + g(t_{h,i+1}), M).$$

The time-stepping method (1.3) for the nonautonomous system (4.1) is defined by

$$\begin{aligned} x^{h,i+1} &= x^{h,i} + h\{A[\theta x^{h,i} + (1-\theta)x^{h,i+1}] + By^{h,i+1} + f(t_{h,i+1})\}, \\ y^{h,i+1} &\in \mathrm{SOL}(l, u, Qx^{h,i+1} + g(t_{h,i+1}), M). \end{aligned} \tag{4.2}$$

It is easy to extend error bounds for the Euler method, the Lipschitz constants $L_x^{h,i}$ and error bounds for the time-stepping method for the autonomous system (1.1) to the nonautonomous system (4.1). We summarize these results in the following theorem.

THEOREM 4.1 Suppose that M is a P-matrix. Let $L = \|A\| + \beta_M \|B\| \|Q\|$, let $\theta \in [0, 1]$ be a given parameter and let $\bar{h}$ be given such that

$$\bar{h} < \frac{1}{L - \theta\|A\|}.$$

Let $\{x^{h,1}, \ldots, x^{h,N_h}\}$ and $\{y^{h,1}, \ldots, y^{h,N_h}\}$ be generated by the time-stepping method (4.2) with the step size $h \in (0, \bar{h}]$ and the parameter θ. Assume that

$$\|x(t_{h,i}) - x^{h,i}\| \leqslant \epsilon_x^{h,i}.$$

Let

$$b_x^{h,i+1} = \frac{1}{1 - Lh}\epsilon_x^{h,i} + \frac{Lh^2}{1 - Lh}\|Ax^{h,i} + By^{h,i} + f(t_{h,i})\|,$$
$$L_x^{h,i} = Lb_x^{h,i+1} + (1 + Lh)\|Ax^{h,i} + By^{h,i} + f(t_{h,i})\|,$$

and

$$\epsilon_x^{h,i+1} = \frac{1 + h\theta\|A\|}{1 + h\theta\|A\| - Lh}\epsilon_x^{h,i} + \frac{1}{2}\frac{LL_x^{h,i} + L_f^{h,i}}{1 + h\theta\|A\| - Lh}h^2,$$

where $L_f^{h,i}$ is the Lipschitz constant of f over $[t_{h,i}, t_{h,i+1}]$. Then we have for $t \in [t_{h,i}, t_{h,i+1}]$,

$$\|x(t) - x^h(t)\| \leqslant \epsilon_x^{h,i+1} \tag{4.3}$$

and

$$\|y(t) - y^h(t)\| \leqslant \beta_M\|Q\|\epsilon_x^{h,i+1}, \tag{4.4}$$

where $(x(t), y(t))$ is the exact solution of (1.1) and $(x^h(t), y^h(t))$ is defined by (1.5).

Theorem 4.1 provides a method to compute the error bound for $(x^h(t), y^h(t))$ delivered by (4.2). This method can be implemented in two steps.

(i) Make use of the error bounds (2.7) based on the Euler method to estimate the Lipschitz constant $L_x^{h,i}$ of x over the interval $[t_{h,i}, t_{h,i+1}]$:

$$L_x^{h,i} = Lb_x^{h,i+1} + (1 + Lh)\|Ax^{h,i} + By^{h,i} + f(t_{h,i})\| + L_f^{h,i},$$

where $L_f^{h,i}$ is the Lipschitz constant of f over $[t_{h,i}, t_{h,i+1}]$.

(ii) Compute the error bounds (4.3) and (4.4) by using these Lipschitz constants $L_x^{h,i}$ and $L_f^{h,i}$.

An algorithm for numerical implementation can be given as follows.

ALGORITHM 4.2 (Computing validated solution via time-stepping method). Compute $\|A\|$, $\|B\|$, $\|Q\|$ and β_M, where β_M is defined by (2.4). Choose a step size $0 < h < 1/L$, where $L = \|A\| + \beta_M\|B\|\|Q\|$. Set $x^{h,0} = x^0$, $y^{h,0} \in \mathrm{SOL}(l, u, Qx^{h,0}, M)$ and $b_x^{h,0} = \epsilon_x^{h,0} = \epsilon_y^{h,0} = 0$.

We compute the approximate solutions $x^{h,i+1}$ and $y^{h,i+1}$ and their error bounds $\epsilon_x^{h,i+1}$ and $\epsilon_y^{h,i+1}$ by the following recursion for $i = 0, \ldots, N_h - 1$:

$$\begin{aligned}
b_x^{h,i+1} &:= \frac{1}{1 - Lh}\epsilon_x^{h,i} + \frac{Lh^2}{1 - Lh}\|Ax^{h,i} + By^{h,i} + f(t_{h,i})\|,\\
L_x^{h,i} &:= Lb_x^{h,i+1} + (1 + Lh)\|Ax^{h,i} + By^{h,i} + f(t_{h,i})\|,\\
x^{h,i+1} &:= x^{h,i} + h\left\{A[\theta x^{h,i} + (1-\theta)x^{h,i+1}] + By^{h,i+1} + f(t_{h,i+1})\right\},\\
y^{h,i+1} &\in \mathrm{SOL}(l, u, Qx^{h,i+1} + g(t_{h,i+1}), M),\\
\epsilon_x^{h,i+1} &:= \frac{1 + h\theta\|A\|}{1 + h\theta\|A\| - Lh}\epsilon_x^{h,i} + \frac{1}{2}\frac{LL_x^{h,i} + L_f^{h,i}}{1 + h\theta\|A\| - Lh}h^2,\\
\epsilon_y^{h,i+1} &:= \beta_M\|Q\|\epsilon_x^{h,i+1}.
\end{aligned}$$

The output of Algorithm 4.2 consists of four piecewise linear functions

$$x^h(t) - \epsilon^h(t),\ y^h(t) - \epsilon_y^h(t) \quad \text{and} \quad x^h(t) + \epsilon^h(t),\ y^h(t) + \epsilon_y^h(t),$$

which bound the exact solution $(x(t), y(t))$ of (4.1) over the interval $[0, T]$ as

$$x^h(t) - \epsilon^h(t) \leqslant x(t) \leqslant x^h(t) + \epsilon^h(t)$$

and

$$y^h(t) - \epsilon_y^h(t) \leqslant y(t) \leqslant y^h(t) + \epsilon_y^h(t).$$

Here $\epsilon^h(t)$ and $\epsilon_y^h(t)$ are the piecewise linear functions defined, with $e = (1, \dots, 1)^{\mathrm{T}} \in R^n$ and $e_y = (1, \dots, 1)^{\mathrm{T}} \in R^m$, by

$$\epsilon_j^h(t) = \frac{1}{h}\big((t_{h,i+1} - t)\epsilon_x^{h,i+1} + (t - t_{h,i})\epsilon_x^{h,i+2}\big)e$$

and

$$\epsilon_y^h(t) = \frac{1}{h}\big((t_{h,i+1} - t)\epsilon_y^{h,i+1} + (t - t_{h,i})\epsilon_y^{h,i+2}\big)e_y$$

for $t \in [t_{h,i}, t_{h,i+1})$, respectively. Here we use

$$\|x(t) - x^h(t)\| \leqslant \epsilon_x^{h,i+1} \leqslant \epsilon_x^{h,i+2}$$

to get

$$|x_j(t) - x_j^h(t)| \leqslant \epsilon_x^{h,i+1} \leqslant \epsilon_j^h(t)$$

for $t \in [t_{h,i}, t_{h,i+1})$ and $j \in \{1, \dots, n\}$.

REMARK 4.3 An important issue that we have to mention is the solution's dependence on the initial value x^0. Under the P-matrix assumption the solution $(x(t), y(t))$ of the DLVI (1.1) is continuously dependent on the initial value x^0 and so are the error bounds proposed here. However, the error bounds could be much tighter for some special initial value. Let $y^0 \in \mathrm{SOL}(l, u, Qx^0, M)$ satisfy the so-called strict complementarity condition: for any $i \in \{1, \dots, m\}$

$$(y^0 - l)_i \neq (Qx^0 + My^0 + g(0))_i \quad \text{and} \quad (y^0 - u)_i \neq (Qx^0 + My^0 + g(0))_i.$$

Denote

$$\begin{aligned} \tau &= \{i\colon (y^0 - l)_i < (Qx^0 + My^0 + g(0))_i\}, \\ \sigma &= \{i\colon (y^0 - u)_i < (Qx^0 + My^0 + g(0))_i < (y^0 - l)_i\}, \\ \varsigma &= \{i\colon (y^0 - u)_i > (Qx^0 + My^0 + g(0))_i\}. \end{aligned}$$

Then there must be a $\bar{t} > 0$ such that $(x(t), y(t)) \in \mathcal{D}$ for any $t \in [0, \bar{t}]$, where

$$\mathcal{D} = \left\{ (x, y)\colon \begin{array}{ll} (y - l)_i < (Qx + My + g(t))_i, & i \in \tau \\ (y - u)_i < (Qx + My + g(t))_i < (y - l)_i, & i \in \sigma \\ (y - u)_i > (Qx + My + g(t))_i, & i \in \varsigma \end{array} \right\}.$$

In this case our error bounds can be sharpened by replacing the constant β_M by $\|K^{-1}\|$, which could be much smaller, where

$$K = \begin{pmatrix} I_{|\tau|} & 0 & 0 \\ M_{\sigma\tau} & M_{\sigma\sigma} & M_{\sigma\varsigma} \\ 0 & 0 & I_{|\varsigma|} \end{pmatrix},$$

and $|\tau|$ and $|\varsigma|$ denote the cardinality of the index sets τ and ς, respectively. For a time point $t \in (t_{h,i}, t_{h,i+1}]$, from Theorem 2.5, we know

$$\|y(t) - y(0)\| \leqslant \sum_{j=0}^{i} L_y^{h,j} t.$$

Therefore, by this means, it can be known how many steps $\|K^{-1}\|$ can be used.

As an example consider the DLVI (4.1), where $l = 0$, $u = \infty$, $f(t) \equiv -2$, $g(t) \equiv 0$ and

$$A = -1, \quad B = (2, -1), \quad Q = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}.$$

For the choice of the initial value $x(0) = 1$, the DLVI has the solution $(x(t), y(t)) \in \mathcal{D}$ for $t \in [0, \log 2)$, where $x(t) = y_1(t) = 2 - e^t$, $y_2(t) = 0$ and

$$\mathcal{D} = \{(x, y)\colon y_1 > (Qx + My)_1,\ y_2 < (Qx + My)_2\}.$$

It is easy to compute that $\beta_M = 1 + |\mu|$ and $\|K^{-1}\| = 1$. And obviously $\beta_M >> \|K^{-1}\|$ when $|\mu|$ is large. By using $\|K^{-1}\|$ instead of β_M we can obtain much tighter error bounds over the time interval $[0, \log 2)$. However, for the initial value $x(0) = 0$, the strict complementarity condition is not fulfilled, and in this case we have the solution $(x(t), y(t)) \in \mathcal{D}$, where

$$\mathcal{D} = \{(x, y)\colon y_1 < (Qx + My)_1,\ y_2 > (Qx + My)_2\}.$$

We can compute $\|K^{-1}\| = \beta_M = 1 + |\mu|$ and therefore have to adopt the worst error estimate:

$$\epsilon_x^{h,i+1} = \frac{1 + h\theta\|A\|}{1 + h\theta\|A\| - Lh}\epsilon_x^{h,i} + \frac{1}{2}\frac{LL_x^{h,i} + L_f^{h,i}}{1 + h\theta\|A\| - Lh}h^2,$$
$$\epsilon_y^{h,i+1} = \beta_M\|Q\|\epsilon_x^{h,i+1}$$

by using β_M (where $L = \|A\| + \beta_M\|B\|\|Q\|$), which gives a safe bound for any initial value.

5. Numerical examples

In this section we apply Algorithm 4.2 to three examples to illustrate the efficiency of our error bounds. We use the semismooth-Newton method (see Kanzow & Fukushima, 1998) to solve variational inequalities. The algorithm is coded in MATLAB 7. The roundoff errors are taken into account by using INTLAB, a MATLAB toolbox for self-validating algorithms (see Rump, 1999; Alefeld & Mayer, 2000).

EXAMPLE 5.1 The collapse of the Tacoma Narrows suspension bridge in 1940 has attracted considerable attention from engineers and mathematicians. Lazer & McKenna (1990) contended that nonlinear effects were the main factors leading to the large oscillations of the bridge. We consider a simple version of their model (see Zill, 2001):

$$m\ddot{x} + q(x) = g(t),$$

where

$$q(x) = \begin{cases} \alpha x & \text{if } x \geqslant 0, \\ \beta x & \text{if } x < 0. \end{cases}$$

Here m is the mass of the section of the roadway, g is the applied force, q is an upward restoring force when $u \geqslant 0$ and a downward restoring force when $u < 0$ and α and β are the Hooke's constants for the tension and compression, respectively. Note that if $\alpha \geqslant \beta$, then we can write

$$q(x) = \alpha x + \max\{0, (\beta - \alpha)x\}.$$

Let $y = \max\{0, (\beta - \alpha)x\}$. It is easy to see that

$$0 \leqslant y \perp y + (\alpha - \beta)x \geqslant 0.$$

Introducing $x_1 = x$ and $x_2 = \dot{x}_1$, we obtain the following DLVI:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -\dfrac{\alpha}{m}x_1(t) - \dfrac{1}{m}y(t) + \dfrac{1}{m}g(t), \\ 0 \leqslant y(t) \perp y + (\alpha - \beta)x_1 \geqslant 0, \\ x_1(0) = 0, \ x_2(0) = \gamma, \qquad t \in [0, T]. \end{cases}$$

In our numerical experiments we adopt the choices from Zill (2001)

$$\alpha = 4, \quad \beta = 1, \quad g(t) = \sin(4t).$$

We report in Table 1 the error bounds $\epsilon_x^{h,N+1}$ at the endpoint of the time interval for different choices of the parameter γ, θ and the step size h. The numerical results indicate that the error bounds increase with respect to θ. Moreover, Algorithm 4.2 delivers tight error bounds for a suitable step size. To illustrate this we plot in the first row of Fig. 1 the approximate solutions $x_1^h(t)$ and $x_2^h(t)$, accompanied by the trajectories $x_1^h(t) \pm \epsilon_1^h(t)$ and $x_2^h(t) \pm \epsilon_2^h(t)$, which bound from above and from below the exact solutions $x_1(t)$ and $x_2(t)$, respectively. We plot in second row the error bounds at the endpoint with reference to the parameter θ and to the step size h, respectively. We set $h = 0.002$ when θ is varying, and set $\theta = 1$ when h is varying. The results for y are omitted as they are completely dependent of those for x.

EXAMPLE 5.2 (Maison & Kasai, 1992). We consider the linear spring model of seismic pounding between two adjacent structures. For $i = 1, 2$, let m_i be the masses, r_i be the viscous damping coefficients and k_i be the initial stiffness for the two structures, respectively. The coupling equation of motion for the two adjacent structures subjected to horizontal ground motion $\ddot{u}_g$ has the form:

$$\begin{cases} m_1\ddot{x}_1(t) + r_1\dot{x}_1 + k_1x_1 + f(x_1, x_2) = -m_1\ddot{u}_g, \\ m_2\ddot{x}_2(t) + r_2\dot{x}_2 + k_2x_2 - f(x_1, x_2) = -m_2\ddot{u}_g, \\ x_1(0) = x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0, \qquad t \in [0, T], \end{cases} \tag{5.1}$$

where for $i = 1, 2$, $\ddot{x}_i$, $\dot{x}_i$ and x_i, are, respectively, the acceleration, velocity and displacement of the structures relative to the ground, f is the pounding force and has the form

$$f(x_1, x_2, \dot{x}_1, \dot{x}_2) = \begin{cases} \alpha(x_1 - x_2 - d) & \text{if } x_1 - x_2 > d, \\ 0 & \text{if } x_1 - x_2 \leqslant d. \end{cases}$$

Here d is the initial separation distance between the two structures and $\alpha > 0$. Introducing $x_3(t) = \dot{x}_1(t)$, $x_4(t) = \dot{x}_2(t)$ and $y = f(x_1, x_2)$, and considering the expression of the pounding force f, we can write

TABLE 1 *Values of* $\epsilon_x^{h,N+1}$ *for Example* 5.1 *with* $T = 1$

	h	10^{-4}	5×10^{-4}	10^{-3}	5×10^{-3}	10^{-2}
$\gamma = 0.2$	$\theta = 0$	1.0492×10^{-1}	5.3416×10^{-1}	$1.0929 \times 10^{+0}$	$6.5869 \times 10^{+0}$	$1.6871 \times 10^{+1}$
	$\theta = 0.3$	1.0483×10^{-1}	5.3198×10^{-1}	$1.0839 \times 10^{+0}$	$6.3108 \times 10^{+0}$	$1.5421 \times 10^{+1}$
	$\theta = 0.5$	1.0478×10^{-1}	5.3052×10^{-1}	$1.0780 \times 10^{+0}$	$6.1349 \times 10^{+0}$	$1.4542 \times 10^{+1}$
	$\theta = 0.7$	1.0472×10^{-1}	5.2908×10^{-1}	$1.0721 \times 10^{+0}$	$5.9654 \times 10^{+0}$	$1.3726 \times 10^{+1}$
	$\theta = 1$	1.0464×10^{-1}	5.2692×10^{-1}	$1.0633 \times 10^{+0}$	$5.7222 \times 10^{+0}$	$1.2609 \times 10^{+1}$
	$\theta = 0$	1.3726×10^{-1}	6.9913×10^{-1}	$1.4312 \times 10^{+0}$	$8.6687 \times 10^{+0}$	$2.2348 \times 10^{+1}$
	$\theta = 0.3$	1.3715×10^{-1}	6.9625×10^{-1}	$1.4194 \times 10^{+0}$	$8.3027 \times 10^{+0}$	$2.0413 \times 10^{+1}$
$\gamma = 0.5$	$\theta = 0.5$	1.3707×10^{-1}	6.9433×10^{-1}	$1.4116 \times 10^{+0}$	$8.0698 \times 10^{+0}$	$1.9241 \times 10^{+1}$
	$\theta = 0.7$	1.3700×10^{-1}	6.9243×10^{-1}	$1.4038 \times 10^{+0}$	$7.8452 \times 10^{+0}$	$1.8154 \times 10^{+1}$
	$\theta = 1$	1.3688×10^{-1}	6.8958×10^{-1}	$1.3923 \times 10^{+0}$	$7.5232 \times 10^{+0}$	$1.6667 \times 10^{+1}$
	$\theta = 0$	1.9662×10^{-1}	9.9848×10^{-1}	$2.0651 \times 10^{+0}$	$1.2897 \times 10^{+1}$	$3.4359 \times 10^{+1}$
	$\theta = 0.3$	1.9646×10^{-1}	9.9805×10^{-1}	$2.0476 \times 10^{+0}$	$1.2343 \times 10^{+1}$	$3.1333 \times 10^{+1}$
$\gamma = 1.0$	$\theta = 0.5$	1.9635×10^{-1}	9.9772×10^{-1}	$2.0361 \times 10^{+0}$	$1.1989 \times 10^{+1}$	$2.9503 \times 10^{+1}$
	$\theta = 0.7$	1.9624×10^{-1}	9.9492×10^{-1}	$2.0247 \times 10^{+0}$	$1.1649 \times 10^{+1}$	$2.7806 \times 10^{+1}$
	$\theta = 1$	1.9607×10^{-1}	9.9074×10^{-1}	$2.0077 \times 10^{+0}$	$1.1162 \times 10^{+1}$	$2.5487 \times 10^{+1}$
	$\theta = 0$	2.4641×10^{-1}	$1.2565 \times 10^{+0}$	$2.5759 \times 10^{+0}$	$1.5781 \times 10^{+1}$	$4.1278 \times 10^{+1}$
	$\theta = 0.3$	2.4621×10^{-1}	$1.2513 \times 10^{+0}$	$2.5545 \times 10^{+0}$	$1.5110 \times 10^{+1}$	$3.7680 \times 10^{+1}$
$\gamma = 1.5$	$\theta = 0.5$	2.4607×10^{-1}	$1.2478 \times 10^{+0}$	$2.5403 \times 10^{+0}$	$1.4683 \times 10^{+1}$	$1.4683 \times 10^{+1}$
	$\theta = 0.7$	2.4594×10^{-1}	$1.2444 \times 10^{+0}$	$2.5262 \times 10^{+0}$	$1.4271 \times 10^{+1}$	$3.3482 \times 10^{+1}$
	$\theta = 1$	2.4573×10^{-1}	$1.2392 \times 10^{+0}$	$2.5053 \times 10^{+0}$	$1.3682 \times 10^{+1}$	$3.0720 \times 10^{+1}$
	$\theta = 0$	3.0216×10^{-1}	$1.5411 \times 10^{+0}$	$3.1603 \times 10^{+0}$	$1.9406 \times 10^{+1}$	$5.0910 \times 10^{+1}$
	$\theta = 0.3$	3.0191×10^{-1}	$1.5347 \times 10^{+0}$	$3.1340 \times 10^{+0}$	$1.8581 \times 10^{+1}$	$4.6472 \times 10^{+1}$
$\gamma = 2.0$	$\theta = 0.5$	3.0174×10^{-1}	$1.5305 \times 10^{+0}$	$3.1166 \times 10^{+0}$	$1.8056 \times 10^{+1}$	$4.3785 \times 10^{+1}$
	$\theta = 0.7$	3.0157×10^{-1}	$1.5263 \times 10^{+0}$	$3.0993 \times 10^{+0}$	$1.7550 \times 10^{+1}$	$4.1294 \times 10^{+1}$
	$\theta = 1$	3.0132×10^{-1}	$1.5199 \times 10^{+0}$	$3.0737 \times 10^{+0}$	$1.6825 \times 10^{+1}$	$3.7887 \times 10^{+1}$

the seismic motion equation (5.1) as the following DLVI:

$$\begin{cases} \dot{x}_1(t) = x_3(t), \\ \dot{x}_2(t) = x_4(t), \\ \dot{x}_3(t) = -\dfrac{r_1}{m_1}x_3 - \dfrac{k_1}{m_1}x_1 - \dfrac{1}{m_1}y - \ddot{u}_g, \\ \dot{x}_4(t) = -\dfrac{r_2}{m_2}x_4 - \dfrac{k_2}{m_2}x_2 + \dfrac{1}{m_2}y - \ddot{u}_g, \\ 0 \leqslant y(t) \perp y(t) - \alpha(x_1(t) - x_2(t) - d) \geqslant 0, \\ x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0, \quad t \in [0, T]. \end{cases}$$

In our numerical experiment for Example 5.2 we set

$$\begin{aligned} &m_1 = m_2 = 7.8, \\ &r_1 = 16.34,\ r_2 = 8.17, \\ &k_1 = 3.4215,\ k_2 = 0.8554, \\ &d = 0.1,\ \alpha = 6. \end{aligned}$$

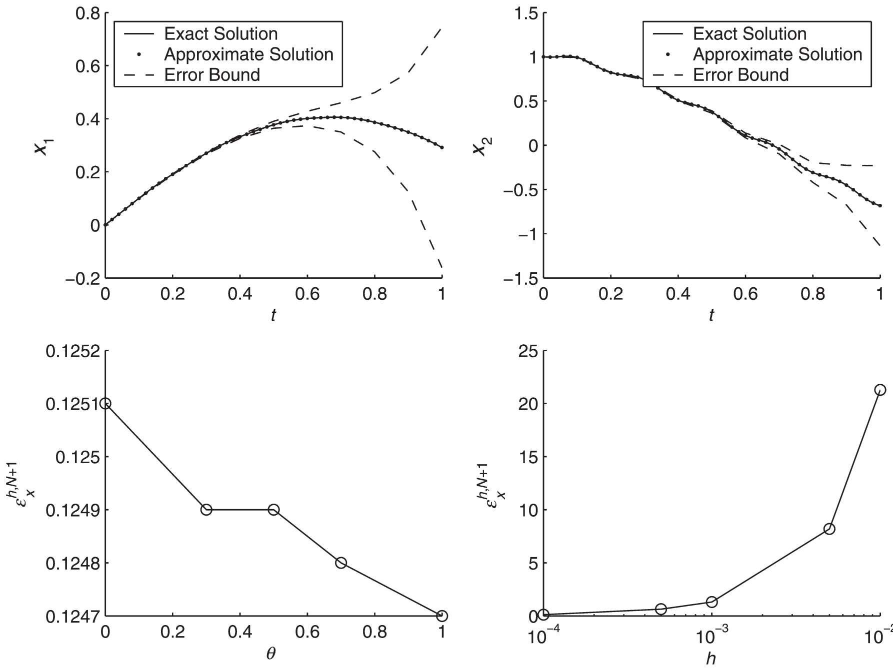

FIG. 1. Numerical results for Example 5.1 ($h = 0.002$, $\theta = 1$).

From the *PEER Strong Database* (http://peer.berkeley.edu/smcat/) we choose 27 ground motion records, with different what is PGA? The records are divided into three groups: I, II and III, according to their PGA levels. We report in Table 2 the error bound $\epsilon_x^{h,i}$. In our setting the error bounds are tight. As an illustration, of the record 'Chichi Taiwan 1999' (PGA $= 0.821$), we plot in Fig. 2 the displacement approximation $x_1^h(t)$ and $x_2^h(t)$, and the approximate pounding force $y^h(t)$; these are also accompanied, respectively, by the trajectories $x_1^h(t) \pm \epsilon_1^h(t)$, $x_2^h(t) \pm \epsilon_2^h(t)$ and $y^h(t) \pm \epsilon_y^h(t)$. Moreover, since the pounding force y is crucial in our model we enlarge the plot in order to show what it looks like.

EXAMPLE 5.3 (van Bokhoven, 1981). Consider an electrical network with (ideal) diodes (see Fig. 3 for its layout). This circuit is widely applied in the maximum gauge of AC voltage (see Pregla, 1998). We denote by V_{C_i} the voltage drop at the ith capacitor (with the capacity C_i), denote by z_i the current intensity and by y_i the voltage drop; denote $z = (z_i)$, $y = (y_i)$ and $V_C = (V_{C_i})$, where $i = 1, 2, 3, 4$. Denote by U and I the voltage and the current intensity of the voltage and current source, respectively. By Kirchhoff's voltage and current law we write the state–space equations as

$$\begin{cases} y = Gz + V_C + g, \\ \dot{V}_c = Dz, \end{cases}$$

along with the constraints for $i = 3, 4$,

$$y_i \geqslant 0,\ z_i \geqslant 0,\ y_i z_i = 0.$$

TABLE 2 *Numerical results for Example* 5.2

	Earthquake	PGA	$\epsilon_x^{h,i+1}$				
			$i = 1000$	$i = 2000$	$i = 3000$	$i = 4000$	$i = 5000$
	San Fernando, 1971	0.090	9.8237×10^{-5}	1.0500×10^{-3}	8.0970×10^{-3}	5.9660×10^{-2}	4.3547×10^{-1}
	Imperial Valley, 1979	0.100	1.3179×10^{-5}	1.4168×10^{-4}	1.1398×10^{-3}	8.4272×10^{-3}	6.1538×10^{-2}
	Morgen Hill, 1984	0.118	7.0535×10^{-6}	3.4782×10^{-5}	1.0650×10^{-4}	2.0345×10^{-4}	3.5515×10^{-4}
	Loma Prieta,1989	0.209	1.0054×10^{-4}	1.2775×10^{-3}	1.2308×10^{-2}	9.2882×10^{-2}	6.8054×10^{-1}
I	Cape Mendocino, 1992	0.042	9.0036×10^{-5}	8.7947×10^{-4}	6.5668×10^{-3}	4.8202×10^{-2}	3.5242×10^{-1}
	Landers, 1992	0.097	9.1433×10^{-5}	8.1844×10^{-4}	6.0942×10^{-3}	4.4675×10^{-2}	3.2638×10^{-1}
	Northridge, 1994	0.110	9.7803×10^{-5}	5.3114×10^{-4}	1.6072×10^{-3}	4.2805×10^{-2}	2.5606×10^{-0}
	Chichi Taiwan, 1999	0.183	4.3303×10^{-7}	5.1882×10^{-6}	4.2070×10^{-5}	3.1265×10^{-4}	2.2839×10^{-3}
	Duzce, 1999	0.038	1.8569×10^{-8}	2.9275×10^{-7}	2.4879×10^{-6}	1.8735×10^{-5}	1.3971×10^{-4}
	San Fernando, 1971	0.366	3.3344×10^{-4}	3.8624×10^{-3}	3.2346×10^{-2}	2.3862×10^{-1}	9.7446×10^{-0}
	Imperial Valley, 1979	0.588	6.2370×10^{-5}	6.3800×10^{-4}	5.1886×10^{-3}	3.8276×10^{-2}	2.8072×10^{-1}
	Morgen Hill, 1984	0.578	7.0785×10^{-5}	1.1123×10^{-3}	1.2612×10^{-2}	9.3965×10^{-2}	6.8688×10^{-1}
	Loma Prieta, 1989	0.473	7.7127×10^{-5}	8.6600×10^{-4}	8.0125×10^{-3}	6.1097×10^{-2}	4.4751×10^{-1}
II	Cape Mendocino, 1992	0.385	2.0741×10^{-4}	1.8095×10^{-3}	1.3533×10^{-2}	9.9240×10^{-2}	7.2445×10^{-1}
	Landers, 1992	0.417	5.5948×10^{-5}	6.0811×10^{-4}	4.8434×10^{-3}	3.5671×10^{-2}	2.6014×10^{-1}
	Northridge, 1994	0.514	1.5150×10^{-4}	1.5440×10^{-3}	1.1818×10^{-2}	8.7059×10^{-2}	6.3650×10^{-1}
	Chichi Taiwan, 1999	0.486	2.2929×10^{-5}	2.2463×10^{-4}	1.6975×10^{-3}	1.2426×10^{-2}	9.0597×10^{-2}
	Duzce, 1999	0.535	7.8271×10^{-5}	1.0003×10^{-3}	7.9244×10^{-3}	5.8562×10^{-2}	4.2870×10^{-1}
	San Fernando, 1971	1.226	2.0948×10^{-5}	1.8391×10^{-4}	5.5235×10^{-4}	1.4804×10^{-3}	3.3768×10^{-3}
	Imperial Valley, 1979	1.655	6.3339×10^{-5}	2.4829×10^{-4}	8.0819×10^{-4}	2.3885×10^{-3}	6.6604×10^{-3}
	Morgen Hill, 1984	1.298	4.1543×10^{-5}	3.6961×10^{-4}	1.5661×10^{-3}	5.9693×10^{-3}	2.1566×10^{-2}
	Loma Prieta, 1989	0.890	1.4509×10^{-4}	1.2984×10^{-3}	9.6219×10^{-3}	7.0334×10^{-2}	5.1260×10^{-1}
III	Cape Mendocino, 1992	1.497	1.0745×10^{-5}	4.3169×10^{-5}	1.2457×10^{-4}	3.1060×10^{-4}	6.5262×10^{-4}
	Landers, 1992	0.818	1.9945×10^{-6}	1.1160×10^{-4}	1.4938×10^{-3}	1.3751×10^{-2}	1.0320×10^{-1}
	Northridge, 1994	0.877	5.4387×10^{-5}	9.5559×10^{-4}	7.7831×10^{-3}	5.7511×10^{-2}	4.2019×10^{-1}
	Chichi Taiwan, 1999	0.821	1.4447×10^{-5}	1.9307×10^{-4}	1.1091×10^{-3}	4.5496×10^{-3}	1.6241×10^{-2}
	Duzce, 1999	0.822	6.1058×10^{-7}	6.5378×10^{-6}	5.5946×10^{-5}	4.2285×10^{-4}	3.0938×10^{-3}

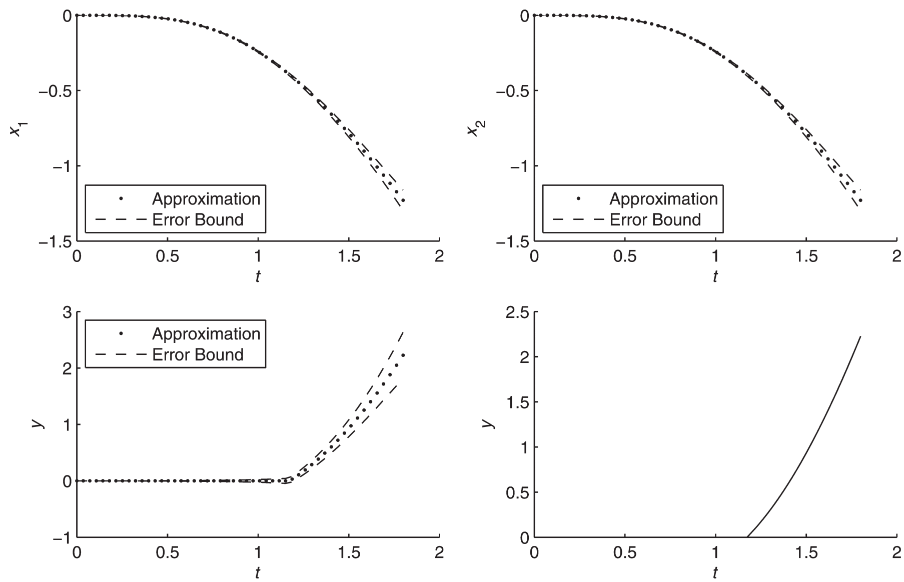

FIG. 2. Numerical results for Example 5.2 ($h = 0.006$, $\theta = 1$).

This complementarity condition arises from the appearance of the diodes (see van Bokhoven, 1981). On the general approach for deriving the state–space equations we refer to Anderson & Vongpanitlerd (1973), for example. Here

$$G = \begin{pmatrix} R_1 & R_1 & 0 & -R_1 \\ R_1 & R_1 + R_3 + R_4 & R_2 & -R_1 \\ 0 & R_3 & R_3 & 0 \\ -R_1 & -R_1 & 0 & R_1 + R_2 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ -IR_3 \\ -IR_3 \\ U \end{pmatrix}$$

and $D = \mathrm{diag}(1/C_i)$. G is called the impedance matrix. Rewriting the state-space equation and imposing the lower and upper bounds l_i and u_i on y_i for $i = 1, 2$, we can use the DLVI

$$\begin{cases} \dot{V}_C(t) = -DG^{-1}V_C(t) + DG^{-1}y(t) - DG^{-1}g, \\ y(t) \in \mathrm{SOL}(l, u, -G^{-1}V_C(t) - G^{-1}g, G^{-1}) \end{cases}$$

to simulate the circuit, where $l = (l_i)$ and $u = (u_i)$, $i = 1, 2, 3, 4$, and $l_3 = l_4 = 0$, $u_3 = u_4 = \infty$. The bounds l_1, l_2, u_1 and u_2 are determined by the physical parameters of the electrical elements.

In our numerical experiment for Example 5.3 we adopt the choices in van Bokhoven (1981, p. 42)

$$\begin{aligned} &R_1 = 50,\ R_2 = R_3 = R_4 = 100, \\ &C_1 = 20,\ C_2 = 10,\ C_3 = 30,\ C_4 = 20, \\ &U(t) = \cos(5t),\ I(t) = \sin(3t). \end{aligned}$$

These choices deliver a symmetric positive definite impedance matrix G, and so we have the estimate $\beta_{G^{-1}} \leqslant 2\|G\|_2$ (see Chen & Xiang, 2007). The bounds l and u relate to the appearance of the diodes and

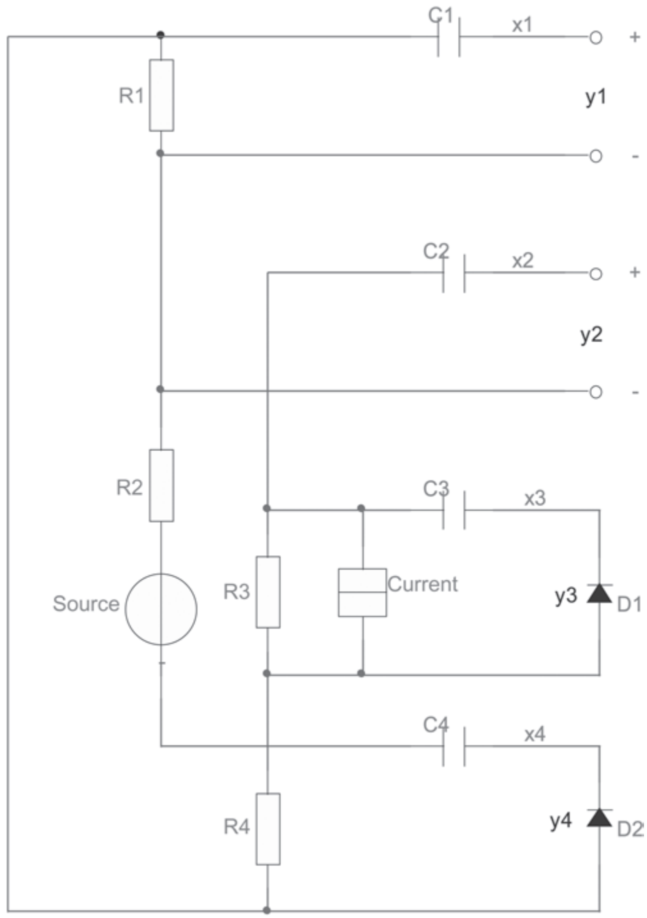

FIG. 3. Layout of circuit for Example 5.3.

to the disruption potential of the capacitors. We set $l = (-10, -10, 0, 0)^{\mathrm{T}}$ and $u = (10, 10, +\infty, \infty)^{\mathrm{T}}$. We choose the time span $[0, 2]$ and the initial value condition $V_C(0) = 0$. Here the voltage drop y and the current intensity z are of real interest, so we report $y^h(t)$ and $z^h(t) = G^{-1}(y^h(t) - V_C^h(t) - g)$. It is clear that in $[t_{h,i}, t_{h,i+1}]$ we have the estimate

$$\|z(t) - z^h(t)\| \leqslant \epsilon_z^{h,i} := \|G^{-1}\|(\epsilon_{V_C}^{h,i} + \epsilon_y^{h,i}).$$

Since the numerical results $(y_1^h(t), z_1^h(t))$ and $(y_3^h(t), z_3^h(t))$ are, respectively, very similar to $(y_2^h(t), z_2^h(t))$ and $(y_4^h(t), z_4^h(t))$, we plot the approximate solutions $y_1^h(t)$ and $y_3^h(t)$ in the first row of Fig. 4 and plot $z_1^h(t)$ and $z_3^h(t)$ in the first row of Fig. 5; they are all accompanied by the trajectories $y_1^h(t) \pm (\epsilon_y^h(t))_1$ and $y_3^h(t) \pm (\epsilon_y^h(t))_3$, $z_1^h(t) \pm \epsilon_1^h(t)$ and $z_3^h(t) \pm \epsilon_3^h(t)$, respectively. In the second row of Figs 4 and 5 we enlarge the plots to show what they look like.

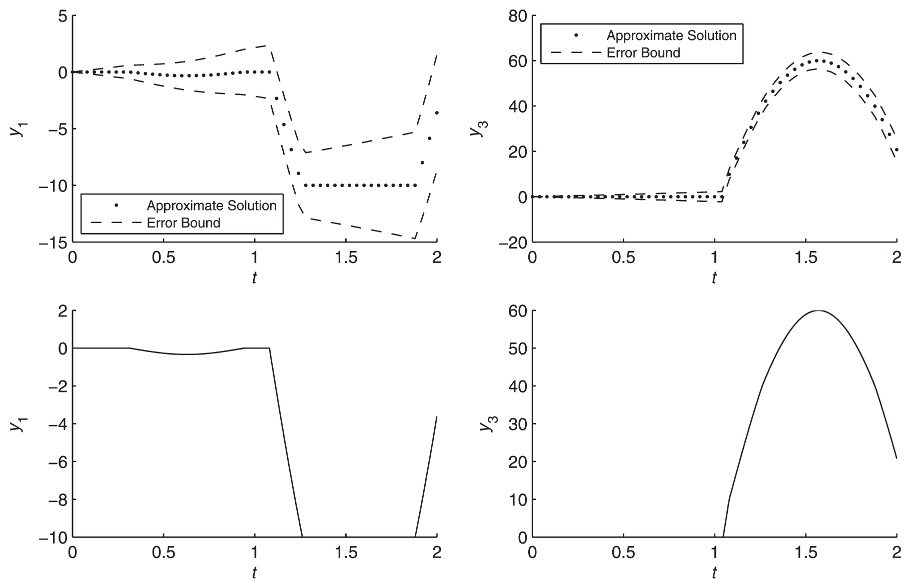

FIG. 4. Numerical results for Example 5.3 ($h = 0.002$, $\theta = 1$).

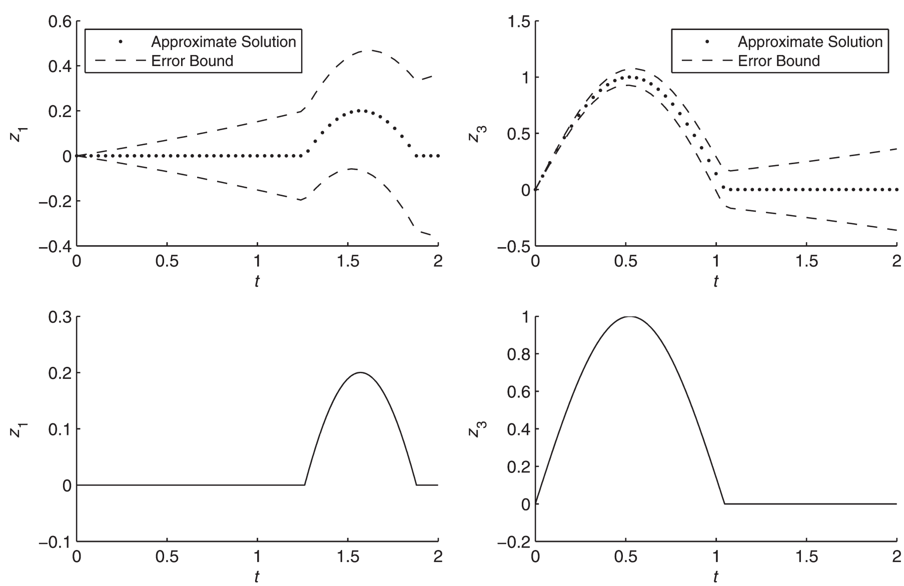

FIG. 5. Numerical results for Example 5.3 ($h = 0.002$, $\theta = 1$).

6. Final remarks

This paper first gives a numerical verification method for computing error bounds of approximate solutions generated by the time-stepping method. The novelty of this method is the use of computable error bounds for the variational inequality and the Euler method to define computable Lipschitz constants for the solution of the DLVI. The Lipschitz constants are necessary to derive computable and sharper error bounds for the approximate solutions generated by the time-stepping method. In many applications we cannot find an exact solution of the DLVI. Using this verification method we can determine the existence region of the exact solution. Moreover, the existence region tightly contains the exact solution and the error bounds go to zero as the step size h goes to zero.

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