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Computation of Error Bounds for P-matrix Linear Complementarity Problems

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Abstract We give new error bounds for the linear complementarity problem where the involved matrix is a P-matrix. Computation of rigorous error bounds can be turned into a P-matrix linear interval system. Moreover, for the involved matrix being an H-matrix with positive diagonals, an error bound can be found by solving a linear system of equations, which is sharper than the Mathias-Pang error bound. Preliminary numerical results show that the proposed error bound is efficient for verifying accuracy of approximate solutions.

Keywords accuracy · error bounds · linear complementarity problems

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1 Introduction

The linear complementarity problem is to find a vector $x \in R^n$ such that

$$Mx + q \geq 0, \quad x \geq 0, \quad x^T(Mx + q) = 0,$$

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or to show that no such vector exists, where $M \in R^{n\times n}$ and $q \in R^n$. We denote this problem by LCP(M,q) and its solution by x^*. Recall the following definitions for an $n \times n$ matrix.

M is called a P-matrix if $\max_{1\le i\le n} x_i(Mx)_i > 0 \quad$ for all $x \neq 0$.

M is called an M-matrix, if $M^{-1} \geq 0$ and $M_{ij} \leq 0$ $(i \neq j)$ for $i,j = 1,2,\ldots,n$.

M is called an H-matrix, if its comparison matrix is an M-matrix.

It is known that an H-matrix with positive diagonals is a P-matrix. Moreover, M is a P-matrix if and only if the LCP(M,q) has a unique solution x^* for any $q \in R^n$. See [4].

It is easy to verify that x^* solves the LCP(M,q) if and only if x^* solves

$$r(x) := \min(x, Mx+q) = 0,$$

where the min operator denotes the componentwise minimum of two vectors. The function r is called the natural residual of the LCP(M,q), and often used in error analysis. Error bounds for the LCP(M,q) have been studied extensively, see [3–6,8,9,13]. For M being a P-matrix, Mathias and Pang [6] present the following error bound

$$\|x - x^*\|_\infty \leq \frac{1+\|M\|_\infty}{c(M)}\|r(x)\|_\infty, \tag{1.1}$$

for any $x \in R^n$, where

$$c(M) = \min_{\|x\|_\infty=1}\left\{\max_{1\le i\le n} x_i(Mx)_i\right\}.$$

This error bound is well known and widely cited. However, the quantity $c(M)$ in (1.1) is not easy to find. For M being an H-matrix with positive diagonals, Mathias and Pang [6] gave a computable lower bound for $c(M)$,

$$c(M) \geq \frac{(\min_i b_i)(\min_i(\tilde{M}^{-1}b)_i)}{(\max_j(\tilde{M}^{-1}b)_j)^2} =: \tilde{c}(b), \tag{1.2}$$

for any vector $b > 0$, where $\tilde{M}$ is the comparison matrix of M, that is

$$\tilde{M}_{ii} = M_{ii} \qquad \tilde{M}_{ij} = -|M_{ij}| \quad \text{for} \quad i \neq j.$$

However, finding a large value of $\tilde{c}(b)$ is not easy. For some b, $\tilde{c}(b)$ can be very small, and thus the error coefficient

$$\mu(b) := \frac{1+\|M\|_\infty}{\tilde{c}(b)} \tag{1.3}$$

can be very large. See examples in Section 3.

Interval methods for validation of solution of the LCP(M,q) have been studied in [1,12]. When a numerical validation condition for the existence of a solution holds, a numerical error bound is provided. However, the numerical validation condition is not ensured to be held at every point x.

In this paper, for M being a P-matrix, we present a new error bound in $\|\cdot\|_p$ ($p \geq 1$, or $p=\infty$) norms,

$$\|x-x^*\|_p \leq \max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_p \|r(x)\|_p, \tag{1.4}$$

where D =diag$(d_1,d_2,\ldots,d_n)$. Moreover, for M being an H-matrix with positive diagonals, we show

$$\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_p \leq \|\tilde{M}^{-1}\max(\Lambda,I)\|_p \tag{1.5}$$

where Λ is the diagonal part of M, and the max operator denotes componentwise maximum of two matrices. This implies

$$\max(\Lambda,I)=\mathrm{diag}(\max(M_{11},1),\max(M_{22},1),\ldots,\max(M_{nn},1)).$$

In comparison with the Mathias-Pang error coefficients (1.1) and (1.3), we give the following inequalities.

If M is a P-matrix,

$$\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_\infty \leq \frac{\max(1,\|M\|_\infty)}{c(M)} = \frac{1+\|M\|_\infty}{c(M)} - \frac{\min(1,\|M\|_\infty)}{c(M)}. \tag{1.6}$$

If M is an H-matrix with positive diagonals,

$$\|\tilde{M}^{-1}\max(\Lambda,I)\|_\infty \leq \mu(b) - \|\tilde{M}^{-1}\min(\Lambda,I)\|_\infty, \qquad \text{for any}\, b>0. \tag{1.7}$$

If M is an M-matrix,

$$\|M^{-1}\max(\Lambda,I)\|_\infty \leq \frac{1+\|M\|_\infty}{c(M)} - \|M^{-1}\min(\Lambda,I)\|_\infty. \tag{1.8}$$

In addition, for M being an M-matrix, the optimal value

$$\|(I-D^*+D^*M)^{-1}\|_1 := \max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_1$$

can be found by solving a simple convex programming problem.

In Section 3, we use some numerical examples to illustrate these error bounds. In particular, for some cases, (1.4),(1.5),(1.6),(1.7), (1.8) hold with equalities, which indicate that they are tight estimates. Preliminary numerical results show that the new error bounds are much sharper than existing error bounds.

Notations: Let $N=\{1,2,\ldots,n\}$. Let e denote the vector whose all elements are 1. The absolute matrix of an $n\times n$ matrix B is denoted by $|B|$. Let $\|\cdot\|$ denote the p-norm for $p\geq 1$ or $p=\infty$.

2 New error bounds

It is not difficult to find that for every $x, y \in R^n$,

$$\min(x_i, y_i) - \min(x_i^*, y_i^*) = (1 - d_i)(x_i - x_i^*) + d_i(y_i - y_i^*), \qquad i \in N \quad (2.1)$$

where

$$d_i = \begin{cases} 0 & \text{if } y_i \geq x_i,\ y_i^* \geq x_i^* \\ 1 & \text{if } y_i \leq x_i,\ y_i^* \leq x_i^* \\ \dfrac{\min(x_i, y_i) - \min(x_i^*, y_i^*) + x_i^* - x_i}{y_i - y_i^* + x_i^* - x_i} & \text{otherwise.} \end{cases}$$

Moreover, we have $d_i \in [0, 1]$. Hence putting $y = Mx + q$ and $y^* = Mx^* + q$ in (2.1), we obtain

$$r(x) = (I - D + DM)(x - x^*), \quad (2.2)$$

where D is a diagonal matrix whose diagonal elements are $d = (d_1, d_2, \ldots, d_n) \in [0, 1]^n$.

It is known that M is a P-matrix if and only if $I - D + DM$ is nonsingular for any diagonal matrix $D = \text{diag}(d)$ with $0 \leq d_i \leq 1$ [7]. This together with (2.2) yields upper and lower error bounds,

$$\frac{\|r(x)\|}{\max\limits_{d \in [0,1]^n} \|I - D + DM\|} \leq \|x - x^*\| \leq \max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\| \|r(x)\|. \quad (2.3)$$

Moreover, it is not difficult to verify that if M is a P-matrix and $D = \text{diag}(d)$ with $d \in [0, 1]^n$, we have

$$\max_{1 \leq i \leq n} x_i((I - D + DM)x)_i > 0, \quad \text{for all}\, x \neq 0,$$

that is, $(I - D + DM)$ is a P-matrix. Therefore, computation of rigorous error bounds can be turned into $\|\cdot\|$ optimization problems over a P-matrix interval set, which is related to linear P-matrix interval systems.

The linear interval system has been studied intensively and some highly efficient numerical methods have been developed, see [10, 12] for references. In the rest part of this section, we give some simple upper bounds for

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|.$$

Lemma 2.1 *If M is an M-matrix, then $I - D + DM$ is an M-matrix for $d \in [0, 1]^n$.*

Proof From I_{27} of Theorem 2.3, Chap. 6 in [2], there is $u > 0$ such that $Mu > 0$. It is easy to verify that $(I - D + DM)u > 0$. Applying the theorem again, we find that $I - D + DM$ is an M-matrix.

Theorem 2.1 *Suppose that M is an H-matrix with positive diagonals. Then we have*

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\| \leq \|\tilde{M}^{-1} \max(\Lambda, I)\|. \quad (2.4)$$

Proof Let $M = \Lambda - B$. We can write

$$(I - D + DM)^{-1} = (I - (I - D + D\Lambda)^{-1}DB)^{-1}(I - D + D\Lambda)^{-1}. \quad (2.5)$$

We first prove (2.4) for M being an M-matrix. Note that $B \geq 0$ with zero diagonal entries, and for any $d \in [0,1]^n$, Lemma 2.1 ensures that $I - D + DM$ and $(I - (I - D + D\Lambda)^{-1}DB)$ are M-matrices.

For each ith diagonal element of the diagonal matrix $(I - D + D\Lambda)^{-1}D$, we consider the function

$$\phi(t) = \frac{t}{1 - t + tM_{ii}}, \quad \text{for} \quad t \in [0,1].$$

It is easy to verify that $\phi(t) \geq 0$ and monotonically increasing for $t \in [0,1]$. Hence, we have

$$\Lambda^{-1} \geq (I - D + D\Lambda)^{-1}D \geq 0, \quad \text{for} \quad d \in [0,1]^n.$$

Since B is nonnegative, we get

$$\Lambda^{-1}B \geq (I - D + D\Lambda)^{-1}DB \geq 0, \quad \text{for} \quad d \in [0,1]^n.$$

By Theorem 5.2, Chap. 7 and Corollary 1.5, Chap. 2 in [2], the spectral radius satisfies

$$1 > \rho(\Lambda^{-1}B) \geq \rho((I - D + D\Lambda)^{-1}DB), \quad \text{for} \quad d \in [0,1]^n.$$

Therefore, we find that

$$\begin{aligned}
&(I - (I - D + D\Lambda)^{-1}DB)^{-1} \\
&= I + (I - D + D\Lambda)^{-1}DB + \cdots + ((I - D + D\Lambda)^{-1}DB)^k + \cdots \\
&\leq I + \Lambda^{-1}B + \cdots + (\Lambda^{-1}B)^k + \cdots \\
&= (I - \Lambda^{-1}B)^{-1} \\
&= (\Lambda - B)^{-1}\Lambda \\
&= M^{-1}\Lambda.
\end{aligned}$$

Now for each ith diagonal element of the diagonal matrix $(I - D + D\Lambda)^{-1}$, we consider the function

$$\psi(t) = \frac{1}{1 - t + tM_{ii}}.$$

For $t \in [0,1]$, $\psi(t) > 0$, and $\psi'(t) \geq 0$ if $M_{ii} < 1$ otherwise $\psi'(t) \leq 0$. Hence, we obtain

$$\max_{t \in [0,1]} \psi(t) = \begin{cases} 1/M_{ii} & \text{if } M_{ii} < 1 \\ 1 & \text{otherwise.} \end{cases}$$

This implies

$$(I - D + D\Lambda)^{-1} \leq \max(\Lambda^{-1}, I), \quad \text{for} \quad d \in [0,1]^n. \quad (2.6)$$

Therefore, the upper bound (2.4) for M being an M-matrix can be derived by (2.6), (2.5) and that for all $d \in [0,1]^n$, $(I-(I-D+D\Lambda)^{-1}DB)^{-1}$ and $(I-D+D\Lambda)^{-1}$ are nonnegative and

$$\begin{aligned}&(I-(I-D+D\Lambda)^{-1}DB)^{-1}(I-D+D\Lambda)^{-1}\\&\leq M^{-1}\Lambda\max(\Lambda^{-1},I)\\&=M^{-1}\max(\Lambda,I).\end{aligned}$$

Now we show (2.4) for M being an H-matrix with positive diagonals. Since for any $n\times n$ matrix A, $\rho(A)\leq\rho(|A|)$, we have that for all $d\in[0,1]^n$,

$$\rho((I-D+D\Lambda)^{-1}DB)\leq\rho((I-D+D\Lambda)^{-1}D|B|)\leq\rho(\Lambda^{-1}|B|)<1.$$

Therefore, we have

$$\begin{aligned}&|(I-(I-D+D\Lambda)^{-1}DB)^{-1}|\\&=|I+(I-D+D\Lambda)^{-1}DB+\cdots+((I-D+D\Lambda)^{-1}DB)^k+\cdots|\\&\leq I+(I-D+D\Lambda)^{-1}D|B|+\cdots+((I-D+D\Lambda)^{-1}D|B|)^k+\cdots\\&\leq I+\Lambda^{-1}|B|+\cdots+(\Lambda^{-1}|B|)^k+\cdots\\&=(I-\Lambda^{-1}|B|)^{-1}\\&=(\Lambda-|B|)^{-1}\Lambda\\&=\tilde{M}^{-1}\Lambda.\end{aligned}$$

This together with (2.5) and (2.6) gives

$$\|(I-D+DM)^{-1}\|\leq\|\tilde{M}^{-1}\Lambda\max(\Lambda^{-1},I)\|=\|\tilde{M}^{-1}\max(\Lambda,I)\|.$$

Remark 1. Since $\tilde{M}^{-1}\max(\Lambda,I)\geq 0$, we have

$$\|\tilde{M}^{-1}\max(\Lambda,I)\|_\infty=\|\tilde{M}^{-1}\max(\Lambda,I)e\|_\infty$$

and

$$\|\tilde{M}^{-1}\max(\Lambda,I)\|_1=\|(e^T\tilde{M}^{-1}\max(\Lambda,I))^T\|_\infty.$$

The upper error bound in (2.4) with $\|\cdot\|_\infty$ or $\|\cdot\|_1$ can be computed by solving a linear system of equations $\min(\Lambda^{-1},I)\tilde{M}x=e$ or $\tilde{M}^T\min(\Lambda^{-1},I)x=e$.

Theorem 2.2 *Suppose that M is an M-matrix. Let $V=\{v\,|\,M^Tv\leq e, v\geq 0\}$ and $f(v)=\max\limits_{1\leq i\leq n}(e+v-M^Tv)_i$. Then we have*

$$\max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_1=\max_{v\in V}f(v). \tag{2.7}$$

Proof From Lemma 2.1, we have that

$$(I-D+DM)^{-1}\geq 0, \qquad \text{for all } d\in[0,1]^n.$$

This implies that

$$\|(I-D+DM)^{-1}\|_1=\|(e^T(I-D+DM)^{-1})^T\|_\infty=\|(I-D+M^TD)^{-1}e\|_\infty.$$

Therefore,

$$\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_1 = \begin{array}{ll} \max\limits_{u} \max\limits_{1\le i\le n} & u_i \\ \text{s.t.} & u - Du + M^T Du = e \\ & 0 \le d \le e. \end{array} \tag{2.8}$$

Let $v = Du$, then from $u \geq 0$ and $d \in [0,1]^n$, we have $0 \leq v \leq u = v - M^T v + e$. This implies $v \in V$. Hence we obtain

$$\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_1 \leq \max_{v\in V} f(v).$$

Conversely, suppose that v is a maximum solution of $f(v)$ in V. We set $u = v - M^T v + e$ and

$$d_i = \begin{cases} v_i/u_i & \text{if } u_i \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \in N$. Then $d \in [0,1]^n$ and $u - Du + M^T Du = e$. This implies that u is a feasible point of the maximization problem (2.8). Thus,

$$\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_1 \geq \max_{v\in V} f(v).$$

Furthermore, the feasible set V is convex and bounded, and the objective function f is convex. Thus, $\max\limits_{v\in V} f(v)$ always has an optimal value. The proof is completed.

Now we show that error bounds given in this paper are sharper than the Mathias-Pang error bounds.

Theorem 2.3 *If M is a P-matrix, then for any $x \in R^n$, the following inequalities hold.*

$$\begin{aligned} & \frac{1}{1+\|M\|_\infty}\|r(x)\|_\infty \qquad (\textit{Mathias-Pang } [6]) \\ \leq\ & \frac{1}{\max(1,\|M\|_\infty)}\|r(x)\|_\infty \qquad (\textit{Cottle-Pang-Stone } [4]) \\ =\ & \frac{1}{\max\limits_{d\in[0,1]^n}\|I-D+DM\|_\infty}\|r(x)\|_\infty \\ \leq\ & \|x-x^*\|_\infty \\ \leq\ & \max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty\|r(x)\|_\infty \\ \leq\ & \frac{\max(1,\|M\|_\infty)}{c(M)}\|r(x)\|_\infty \\ =\ & \frac{1+\|M\|_\infty}{c(M)}\|r(x)\|_\infty - \frac{\min(1,\|M\|_\infty)}{c(M)}\|r(x)\|_\infty \\ \leq\ & \frac{1+\|M\|_\infty}{c(M)}\|r(x)\|_\infty \qquad (\textit{Mathias-Pang } [6])). \end{aligned}$$

Proof The first inequality is obvious. For the next equality, we set $D^* = \text{diag}(d_1^*, \ldots, d_n^*)$ to be an optimal point such that

$$\|I - D^* + D^*M\|_\infty = \max_{d\in[0,1]^n} \|I - D + DM\|_\infty.$$

From $M_{ii} > 0$, we have

$$\begin{aligned}
\|I - D^* + D^*M\|_\infty &= \max_{1\le i\le n} \left\{ |1 - d_i^* + d_i^* M_{ii}| + d_i^* \sum_{\substack{j=1\\ j\neq i}}^{n} |M_{ij}| \right\} \\
&= \max_{1\le i\le n} \left\{ 1 - d_i^* + d_i^* \sum_{j=1}^{n} |M_{ij}| \right\} \\
&=: 1 - d_{i_0}^* + d_{i_0}^* \sum_{j=1}^{n} |M_{i_0 j}|.
\end{aligned}$$

Hence the value $d_{i_0}^*$ must be a boundary point of $[0, 1]$. Moreover, it is easy to find

$$\|I - D^* + D^*M\|_\infty = \begin{cases} \|M\|_\infty & \text{if } \|M\|_\infty > 1 \\ 1 & \text{otherwise} \end{cases}$$

which implies

$$\max(1, \|M\|_\infty) = \max_{d\in[0,1]^n} \|I - D + DM\|_\infty. \tag{2.9}$$

The second and third inequalities follows from (2.3).

For the fourth inequality, we first prove that for any nonsingular diagonal matrix $D = \text{diag}(d)$ with $d \in (0, 1]^n$,

$$\|(I - D + DM)^{-1}\|_\infty \le \frac{\max(1, \|M\|_\infty)}{c(M)}. \tag{2.10}$$

Let $H = (I - D + DM)^{-1}$ and i_0 be the index such that $\sum_{j=1}^n |H_{i_0 j}| = \|(I - D + DM)^{-1}\|_\infty$. Define $y = (I - D + DM)^{-1}p$, where $p = (\text{sgn}(H_{i_0 1}), \cdots, \text{sgn}(H_{i_0 n}))^T$. Then $p = (I - D + DM)y$, $My = D^{-1}p + y - D^{-1}y$, and

$$\|(I - D + DM)^{-1}\|_\infty = \|y\|_\infty.$$

Furthermore, by the definition of $c(M)$, we have

$$0 < c(M)\|y\|_\infty^2 \le \max_i y_i(My)_i = \max_i y_i \left(\frac{p_i}{d_i} + y_i - \frac{y_i}{d_i} \right).$$

Let j be the index such that $y_j \left(\frac{p_j}{d_j} + y_j - \frac{y_j}{d_j} \right) = \max_i y_i \left(\frac{p_i}{d_i} + y_i - \frac{y_i}{d_i} \right)$.

(i) If $|y_j| \le 1$, then we have

$$c(M)\|y\|_\infty^2 \le |My|_j \le \|My\|_\infty \le \|M\|_\infty \|y\|_\infty.$$

This implies
$$\|(I-D+DM)^{-1}\|_\infty \le \|y\|_\infty \le \frac{\|M\|_\infty}{c(M)}.$$

(ii) If $y_j > 1$, then $\frac{p_j+d_jy_j-y_j}{d_j} > 0$ and $p_j > y_j - d_jy_j \ge 0$. Thus $p_j = 1$ and $d_j > 1 - \frac{1}{y_j}$. Hence, we obtain
$$0 < \frac{p_j+d_jy_j-y_j}{d_j} \le 1.$$

This implies $0 < (My)_j \le 1$. Thus $c(M)\|y\|_\infty^2 \le y_j \le \|y\|_\infty$ and $\|(I-D+DM)^{-1}\|_\infty \le \frac{1}{c(M)}$.

(iii) If $y_j < -1$, then $\frac{p_j+d_jy_j-y_j}{d_j} < 0$ and $p_j < y_j - d_jy_j \le 0$. Thus $p_j = -1$ and $d_j \ge 1 + \frac{1}{y_j}$. Similarly, we obtain
$$0 > \frac{p_j+d_jy_j-y_j}{d_j} \ge -1.$$

This implies $-1 \le (My)_j < 0$. Thus $c(M)\|y\|_\infty^2 \le -y_j \le \|y\|_\infty$ and $\|(I-D+DM)^{-1}\|_\infty \le \frac{1}{c(M)}$.

Combining the three cases, we claim that (2.10) holds for any nonsingular matrix D=diag(d) with $d \in (0,1]^n$.

Now we consider $d \in [0,1]^n$. Let $d_\epsilon = \min(d+\epsilon e, e)$, where $\epsilon \in (0,1]$. Then, we have
$$\|(I-D+DM)^{-1}\|_\infty = \lim_{\epsilon\downarrow 0}\|(I-D_\epsilon+D_\epsilon M)^{-1}\|_\infty \le \frac{\max(1,\|M\|_\infty)}{c(M)}.$$

Since D is arbitrarily chosen, we obtain the fourth inequality.

The next equality and inequality are trivial.

Theorem 2.4 *If M is an H-matrix with positive diagonals, then for any $x, b \in R^n$, $b > 0$, the following inequalities hold.*
$$\begin{aligned}
&\|x-x^*\|_\infty \\
&\le \max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty\|r(x)\|_\infty \\
&\le \|\tilde{M}^{-1}\max(\Lambda, I)\|_\infty\|r(x)\|_\infty \\
&\le (\mu(b) - \|\tilde{M}^{-1}\min(\Lambda, I)\|_\infty)\|r(x)\|_\infty \\
&\le \mu(b)\|r(x)\|_\infty \qquad (\textit{Mathias-Pang} \quad [6]).
\end{aligned}$$

In addition, if M is an M-matrix, then for any $x \in R^n$, the following inequalities hold.
$$\begin{aligned}
&\|x-x^*\|_\infty \\
&\le \|M^{-1}\max(\Lambda, I)\|_\infty\|r(x)\|_\infty
\end{aligned}$$

$$\leq (\frac{1+\|M\|_\infty}{c(M)} - \|M^{-1}\min(\Lambda, I)\|)\|r(x)\|_\infty$$
$$\leq \frac{1+\|M\|_\infty}{c(M)}\|r(x)\|_\infty \qquad (\textit{Mathias-Pang} \quad [6]).$$

Proof We first consider that M is an H-matrix with positive diagonals. The first and second inequalities follow (2.3) and Theorem 2.1. Now we show the third inequality.

For any $b \in R^n$, $b > 0$, let $b_0 = \min_{1\leq i\leq n} b_i$. Then $b \geq b_0 e$, and $\tilde{M}^{-1}b \geq \tilde{M}^{-1}b_0 e = b_0\tilde{M}^{-1}e$. Moreover, for every $j \in N$

$$((\tilde{M}^{-1}b)_j)^2 \geq (\tilde{M}^{-1}b)_j b_0 (\tilde{M}^{-1}e)_j \geq (\min_{1\leq i\leq n}(\tilde{M}^{-1}b)_i)(\min_{1\leq i\leq n} b_i)(\tilde{M}^{-1}e)_j.$$

Hence from $\|\tilde{M}^{-1}e\|_\infty = \|\tilde{M}^{-1}\|_\infty$, we obtain

$$(\max_j(\tilde{M}^{-1}b)_j)^2 \geq (\min_{1\leq i\leq n}(\tilde{M}^{-1}b)_i)(\min_{1\leq i\leq n} b_i)\|\tilde{M}^{-1}\|_\infty.$$

Therefore, from the following inequalities

$$1+\|M\|_\infty \geq \|I+M\|_\infty \geq \|I+\Lambda\|_\infty \geq \|\max(\Lambda, I)\|_\infty + \|\min(\Lambda, I)\|_\infty$$

we find

$$\begin{aligned}\mu(b) &= \frac{1+\|M\|_\infty}{\tilde{c}(b)} \\ &\geq \|\tilde{M}^{-1}\|_\infty(1+\|M\|_\infty) \\ &\geq \|\tilde{M}^{-1}\|_\infty(\|\max(\Lambda, I)\|_\infty + \|\min(\Lambda, I)\|_\infty) \\ &\geq \|\tilde{M}^{-1}\max(\Lambda, I)\|_\infty + \|\tilde{M}^{-1}\min(\Lambda, I)\|_\infty.\end{aligned}$$

Now, we consider that M is an M-matrix.

Let $\|M^{-1}w\|_\infty = \max_{\|y\|_\infty=1}\|M^{-1}y\|_\infty = \|M^{-1}\|_\infty$. From the definition of $c(M)$, we have

$$c(M)\|M^{-1}\|_\infty^2 \leq \max_{1\leq i\leq n}(M^{-1}w)_i(MM^{-1}w)_i \leq \|M^{-1}\|_\infty.$$

By the similar argument above, we find

$$\frac{1+\|M\|_\infty}{c(M)} \geq \|M^{-1}\|_\infty(1+\|M\|_\infty) \geq \|M^{-1}\max(\Lambda, I)\|_\infty + \|M^{-1}\min(\Lambda, I)\|_\infty.$$

Applying Theorem 2.1, we obtain the following relative error bounds

Corollary 2.1 *Suppose M is an H-matrix with positive diagonals. For any $x \in R^n$, we have*

$$\frac{\|r(x)\|}{(1+\|M\|)\|\tilde{M}^{-1}\max(\Lambda, I)\|\|(-q)_+\|} \leq \frac{\|x-x^*\|}{\|x^*\|} \leq \frac{\|M\|\|\tilde{M}^{-1}\max(\Lambda, I)\|\|r(x)\|}{\|(-q)_+\|}.$$

Proof . Set $x = 0$ in (2.3). From Theorem 2.1, we get

$$\|x^*\| \le \|\tilde{M}^{-1} \max(\Lambda, I)\| \|r(0)\| = \|\tilde{M}^{-1} \max(\Lambda, I)\| \|(-q)_+\|.$$

Moreover, from $Mx^* + q \ge 0$, we deduce $(-q)_+ \le (Mx^*)_+ \le |Mx^*|$. This implies $\|(-q)_+\| \le \|Mx^*\|$, and

$$\frac{\|(-q)_+\|}{\|M\|} \le \|x^*\|.$$

Combining (2.3) with the bounds for $\|x-x^*\|$ and $\|x^*\|$, we obtain the desired error bounds.

Remark 2. Note that for M being an H-matrix with positive diagonals, x^* solves LCP$(\Lambda^{-1}M, \Lambda^{-1}q)$ if and only if x^* solves LCP(M, q). Let $\bar{r}(x) = \min(\Lambda^{-1}Mx + \Lambda^{-1}q, x)$. From (2.3) and

$$\|I - D + D\Lambda^{-1}M\| \le \||I - D(I - \Lambda^{-1}M)|\| \le \|\Lambda^{-1}|M|\|,$$

we have

$$\frac{\|\bar{r}(x)\|}{\|\Lambda^{-1}|M|\|} \le \|x - x^*\| \le \|\tilde{M}^{-1}\Lambda\| \|\bar{r}(x)\|$$

for every $x \in R^n$. Moreover, from

$$\|\Lambda^{-1}M\|_p = \|\Lambda^{-1}|M|\|_p,$$

with $p = 1$ or $p = \infty$, we obtain

$$\frac{\|\bar{r}(x)\|_p}{\text{cond}_p(\Lambda^{-1}\tilde{M})\|(-q)_+\|_p} \le \frac{\|x - x^*\|_p}{\|x^*\|_p} \le \frac{\text{cond}_p(\Lambda^{-1}\tilde{M})\|\bar{r}(x)\|_p}{\|(-q)_+\|_p}, \qquad (2.11)$$

for $p = 1$ or $p = \infty$.

3 Numerical examples

In this section, we first use examples to illustrate error bounds derived in the last section. Next we report numerical results obtained by using Matlab 6.1 on an IBM PC.

Example 3.1 In [12], Schäfer considered an application of P-matrix linear complementarity problems, which arises from computing interval enclosure of the solution set of an interval linear system [10]. The following P-matrix is from [12]

$$M = \begin{pmatrix} 1 & -4 \\ 5 & 7 \end{pmatrix}.$$

This matrix is not an H-matrix. It is not difficult to find that

$$\max_{d \in [0,1]^2} \|(I - D + DM)^{-1}\|_\infty = \max_{d \in [0,1]^2} \frac{1 + 6d_2 + 4d_1}{1 + 6d_2 + 20d_1d_2} = 5,$$

and

$$\frac{1+\|M\|_\infty}{c(M)} \geq \frac{13}{\min(M_{ii})} = 13.$$

Example 3.2 Consider the following H-matrix with positive diagonals. (Example 5.10.4 in [4])

$$M = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

where $|t| \geq 1$. It is easy to show that $c(M) \leq 1/t^2$. Hence the error bound (1.1) has

$$\frac{1+\|M\|_\infty}{c(M)} \geq t^2(2+|t|) = O(t^3).$$

For $b = e$, we have

$$c(M) \geq \tilde{c}(b) = \frac{(\min_i b_i)(\min_i (\tilde{M}^{-1}b)_i)}{(\max_j (\tilde{M}^{-1}b)_j)^2} = 1/(1+|t|)^2$$

and

$$\mu(b) = \frac{1+\|M\|_\infty}{\tilde{c}(b)} = (1+|t|)^2(2+|t|) = O(t^3).$$

The error coefficients given in the last section satisfy for $p = 1, \infty$

$$\max_{d\in[0,1]^2} \|(I-D(I-M))^{-1}\|_p = \max_{d_1\in[0,1]} (1+d_1|t|) = \|\tilde{M}^{-1}\max(I,\Lambda)\|_p = 1+|t|.$$

Hence, the new error bounds are much smaller than the Mathias-Pang error bounds, especially when $t \to \infty$. Moreover, we can show that the new error bounds are tight. Let $t = -1$ and $q = (1, -1)^T$. Then the LCP(M, q) has a unique solution $x^* = (0, 1)^T$. For $x = (4, 3)^T$,

$$\|x^* - x\|_\infty = 4, \quad \|\tilde{M}^{-1}\max(I,\Lambda)\|_\infty = 2, \quad \|r(x)\|_\infty = 2.$$

Hence (1.4) and (1.5) hold with equality.

For $M = I$, (1.6),(1.7) with $b = e$ and (1.8) hold with equality.

Now we report some numerical results to compare these error coefficients.

Example 3.3 Let M be a tri-diagonal $n \times n$ matrix

$$M = \begin{pmatrix} b+\alpha\sin(\frac{1}{n}) & c & & & \\ a & b+\alpha\sin(\frac{2}{n}) & c & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c \\ & & & a & b+\alpha\sin(1) \end{pmatrix}$$

For $b = 2, a = c = -1, \alpha = 0$, the LCP$(M, q)$ with various q in an interval vector arises from the finite difference method for free boundary problems [11].

Table 1 Example 3.3, $n = 400$, $\kappa_1 = \max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_1$

α	a	b	c	κ_1	$\|M^{-1}\max(\Lambda, I)\|_\infty$	$\mu(e)$
0	-1	2	-1	2.0100E4	4.0200E4	2.0201E7
n^{-2}	-1.5	2	-0.5	3.9920E2	7.8832E2	1.5536E6
n^{-2}	-1.5	2.2	-0.5	6.3910E0	1.0999E1	3.6557E2
1	-1.5	3.0	-1.5	2.4399E1	7.3936E1	1.8060E4

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References

1. Alefeld, G.E., Chen, X., Potra, F.A.: Numerical validation of solutions of linear complementarity problems, Numer. Math. 83, 1-23(1999)
2. Berman, A., Plemmons, R.J.: Nonnegative Matrix in the Mathematical Sciences, SIAM Publisher, Philadelphia(1994)
3. Chen, B.: Error bounds for R_0-type and monotone nonlinear complementarity problems, J. Optim. Theory Appl. 108, 297–316(2001)
4. Cottle, R.W., Pang, J.-S., Stone, R.E.: The Linear Complementarity Problem, Academic Press, Boston, MA(1992)
5. Ferris, M.C., Mangasarian, O.L.: Error bounds and strong upper semicontinuity for monotone affine variational inequalities, Ann. Oper. Res. 47, 293-305(1993)
6. Mathias, R., Pang, J.-S.: Error bounds for the linear complementarity problem with a P-matrix, Linear Algebra Appl. 132, 123-136(1990)
7. Gabriel, S.A., Moré, J.J.: Smoothing of mixed complementarity problems. In: M.C.Ferris and J.-S.Pang (ed.) Complementarity and Variational Problems: State of the Art, 105-116. SIAM Publications, Philadelphia, PA(1997)
8. Mangasarian, O.L., Ren, J.: New improved error bounds for the linear complementarity problem, Math. Programming 66, 241-257(1994)
9. Pang, J.-S.: Error bounds in mathematical programming, Math. Programming 79, 299-332(1997)
10. Rohn, J.: Systems of linear interval equations, Linear Algebra Appl. 126, 39-78(1989)
11. Schäfer, U.: An enclosure method for free boundary problems based on a linear complementarity problem with interval data, Numer. Func. Anal. Optim. 22, 991-1011(2001)
12. Schäfer, U.: A linear complementarity problem with a P-matrix, SIAM Review 46, 189-201(2004)
13. Xiu, N., Zhang, J.: A characteristic quantity of P-matrices, App. Math. Lett. 15, 41-46(2002)