Dynamic Stochastic Variational Inequalities and Convergence of Discrete Approximation

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Abstract. This paper studies dynamic stochastic variational inequalities (DSVIs) to deal with uncertainties in dynamic variational inequalities (DVIs). We show the existence and uniqueness of a solution for a class of DSVIs in $C^1 \times \mathcal{Y}$, where C^1 is the space of continuously differentiable functions and $\mathcal{Y}$ is the space of measurable functions, and discuss non-Zeno behavior. We use the sample average approximation (SAA) and time-stepping schemes as discrete approximation for the uncertainty and dynamics of the DSVIs. We then show the uniform convergence and an exponential convergence rate of the SAA of the DSVI. A time-stepping EDIIS method is proposed to solve the DVI arising from the SAA of DSVI; its convergence is established. Our results are illustrated by a point-queue model for an instantaneous dynamic user equilibrium in traffic assignment problems.

Key words. Dynamic stochastic variational inequalities, sample average approximation, time-stepping method, Anderson acceleration.

AMS subject classifications. 90C39, 90C33, 90C15

1. Introduction.

Consider the following dynamic stochastic variational inequality (DSVI)

$$\dot{x}(t) = \gamma \cdot \Big\{ \Pi_X \Big(x(t) - \mathbb{E}[\Phi(t, \xi, x(t), y(t,\xi))] \Big) - x(t) \Big\}, \tag{1.1}$$

$$x(0) = x_0, \tag{1.2}$$

$$0 \in \Psi(t, \xi, x(t), y(t,\xi)) + \mathcal{N}_{C_\xi}(y(t,\xi)), \quad \text{for a.e. } \xi \in \Xi. \tag{1.3}$$

Here γ is a nonzero real number, $X \subseteq \mathbb{R}^n$ is a nonempty closed convex set, $\xi : \Omega \to \mathbb{R}^d$ is a random vector defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose probability distribution $P = \mathbb{P} \circ \xi^{-1}$ is supported on the set $\Xi := \xi(\Omega) \subseteq \mathbb{R}^d$, $\Phi : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $\Psi : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, $\Pi_X : \mathbb{R}^n \to X$ denotes the Euclidean projection operator onto X, and $\mathcal{N}_{C_\xi}(y(t,\xi))$ is the normal cone to C_ξ at $y(t,\xi)$, where C_ξ is a nonempty closed convex set in $\mathbb{R}^m$ for each ξ and is $\mathcal{A}$-measurable. We make the following assumption through this paper (unless otherwise stated):

A.0 Given $\xi \in \Xi$, the functions $\Phi(\cdot, \xi, \cdot, \cdot)$ and $\Psi(\cdot, \xi, \cdot, \cdot)$ are Lipschitz continuous in (t, x, y) with Lipschitz moduli $\kappa_\Phi(\xi)$ and $\kappa_\Psi(\xi)$ with respect to a norm (e.g., $\|\cdot\|_2$ or $\|\cdot\|_\infty$), respectively, where $\kappa_\Phi(\cdot)$ and $\kappa_\Psi(\cdot)$ are measurable.

Further, let $\mathcal{Y}$ denote the space of measurable functions from Ξ to $\mathbb{R}^m$. For a given (t, x), let $\mathrm{SOL}(t, x, \xi(\cdot)) : \Omega \rightrightarrows \mathcal{Y}$ denote the solution set of the variational inequality or VI (1.3), which is a random set-valued mapping. Let $y_x(t, \cdot)$ or simply $y(t, \cdot)$ be a measurable selection of solutions in $\mathrm{SOL}(t, x, \xi(\cdot))$ of the VI (1.3) such that the expected value in (1.1) is well defined, i.e., each element of $\mathbb{E}[\Phi(t, \xi, x, y(t,\xi))]$ attains a finite value for any (t, x). Specific conditions ensuring these assumptions to hold will be given in the following development.

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The DSVI (1.1)-(1.3) includes the deterministic differential variational inequality (DVI) as a special case. In fact, if $\gamma = -1$, $X = \mathbb{R}^n$, and $y(t,\cdot)$ is deterministic, then the DSVI becomes

$$\dot{x}(t) = \Phi(t, x(t), y(t)), \quad x(0) = x_0, \tag{1.4}$$

$$0 \in \Psi(t, x(t), y(t)) + \mathcal{N}_C(y(t)), \tag{1.5}$$

which is the deterministic DVI [2, 11, 22, 23, 24]. The DSVI also reduces to the functional evolutionary VI [2] if $\gamma = 1$ and Φ is deterministic and independent of y.

A class of the bimodal piecewise affine system [14] can be written as the dynamic linear complementarity problem (DLCP)

$$\dot{x}(t) = Ax(t) - e\max(c^T x(t), 0) + f + by(t), \quad x(0) = x_0, \tag{1.6}$$

$$0 \le y(t) \perp N(t)x(t) + M(t)y(t) + q(t) \ge 0, \tag{1.7}$$

where $A, e, c, f, b, N(t), M(t), q(t)$ are given vectors or matrices. When the data b, N, M, q have uncertainties, we consider the following model

$$\dot{x}(t) = Ax(t) - e\max(c^T x(t), 0) + \mathbb{E}[B(\xi)y(t,\xi)] + f, \quad x(0) = x_0 \tag{1.8}$$

$$0 \le y(t,\xi) \perp N(t,\xi)x(t) + M(t,\xi)y(t,\xi) + q(t,\xi) \ge 0, \quad \text{for a.e. } \xi \in \Xi. \tag{1.9}$$

Here $A \in \mathbb{R}^{n\times n}$, $c, f \in \mathbb{R}^n$, $B(\cdot) : \mathbb{R}^d \to \mathbb{R}^{n\times m}$, $M(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{m\times m}$, $N(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{m\times n}$, and $q(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}^l \to \mathbb{R}^m$ are continuous matrix valued mappings, and $e \in \mathbb{R}^n$ is the vector with all elements 1. The above model is a special case of the DSVI (1.1)-(1.3) when $X = \mathbb{R}^n$, $\gamma = 1$, and $\Phi(t, \xi, x(t), y(t,\xi)) = -[Ax(t) - e\max(c^T x(t), 0) + B(\xi)y(t,\xi) + f]$ so that $\mathbb{E}[\Phi(t, \xi, x(t), y(t,\xi))] = -[Ax(t) - e\max(c^T x(t), 0) + \mathbb{E}[B(\xi)y(t,\xi)] + f]$.

Consider the case where the functions Φ and Ψ are independent of t, namely, they are *time invariant*. Hence, we write them as $\Phi(\xi, x, y)$ and $\Psi(\xi, x, y)$ respectively. Suppose this DSVI is well-posed, i.e., its solution $(x(t), y(t,\xi))$ exists and is unique for any $t \ge 0$ and any initial condition x_0. Then $(x^e, y^e(\xi)) \in \mathbb{R}^n \times \mathcal{Y}$ is called an *equilibrium* of the DSVI if for a.e. $\xi \in \Xi$,

$$0 = \Pi_X(x^e - \mathbb{E}[\Phi(\xi, x^e, y^e(\xi))]) - x^e, \quad \text{and} \quad 0 \in \Psi(\xi, x^e, y^e(\xi)) + \mathcal{N}_{C_\xi}(y^e(\xi)). \tag{1.10}$$

Clearly, $(x(t), y(t,\xi)) = (x^e, y^e(\xi))$ for all $t \ge 0$ provided that $x(0) = x^e$. Note that the value of the nonzero constant γ on the right-hand side of (1.1) does not affect such an equilibrium although it does affect the dynamics of the DSVI.

The first equation of (1.10) is defined by the natural mapping associated with the VI: $-F(v) \in \mathcal{N}_X(v)$, and is known to be an equivalent formulation of this VI [15, Section 1.5.2]. Therefore, $(x^e, y^e(\xi))$ is an equilibrium of the DSVI if and only if it is a solution to the following (static) two-stage stochastic variational inequality (SVI) extensively studied recently [5, 6, 7, 26, 27]:

$$0 \in \mathbb{E}[\Phi(\xi, x, y(\xi))]) + \mathcal{N}_X(x), \tag{1.11}$$

$$0 \in \Psi(\xi, x, y(\xi)) + \mathcal{N}_{C_\xi}(y(\xi)), \quad \text{for a.e. } \xi \in \Xi. \tag{1.12}$$

Moreover, as far as the equilibria of the DSVI (or the solutions of the two-stage SVI) are concerned, we may replace the right-hand side of (1.1) by any function (or even a set-valued mapping) whose zero set, along with (1.3), gives rise to the same SVI (1.11)-(1.12) for its equilibrium. This leads to different formulations of the DSVI using various equation formulations of the VIs or complementarity problems. For example, in view of $u = \Pi_X(x - G(t,x))$ if and only if $0 \in u - (x - G(t,x)) + \mathcal{N}_X(u)$, the DSVI (1.1)-(1.3) can be equivalently written as

$$\dot{x}(t) = \gamma \cdot \big(u(t) - x(t)\big), \qquad x(0) = x_0, \tag{1.13}$$

$$0 \in u(t) - x(t) + \mathbb{E}[\Phi(t,\xi,x(t),y_x(t,\xi))] + \mathcal{N}_X(u(t)), \tag{1.14}$$

$$0 \in \Psi(t,\xi,x(t),y(t,\xi)) + \mathcal{N}_{C_\xi}(y(t,\xi)), \quad \text{for a.e. } \xi \in \Xi. \tag{1.15}$$

Moreover, when $X = \mathbb{R}^n_+$, many equation formulations can be obtained from the NCP-functions and residual functions of nonlinear complementarity problems [15].

The main contributions of this paper are two-fold. (i) We show under certain conditions that DSVI (1.1)-(1.3) has a unique solution of a pair $x \in C^1[0,T]$ and $y \in C^0[0,T] \times \mathcal{Y}$, where C^1 is the space of continuously differentiable functions and $\mathcal{Y}$ is the space of measurable functions. Moreover, we provide sufficient conditions for the non-Zeno behavior of the solution x. (ii) We establish the uniform convergence and an exponential convergence rate of the sample average approximation (SAA) of DSVI. We propose a time-stepping EDIIS method to solve the DVI arising from the SAA of the DSVI, and provide a convergence theorem. It worth noting that the analysis for DSVI requires not only the existing results for DVI and SVI but also new techniques for dynamic equilibrium problems in an uncertain environment.

This paper is organized as follows. In Section 2, we discuss solution existence, uniqueness, and non-Zenoness of the DSVI (1.1)-(1.3). Section 3 establishes the uniform convergence and an exponential convergence rate of the SAA of the DSVI. In Section 4, we propose a time-stepping EDIIS method. Section 5 considers a point-queue model for the instantaneous dynamic user equilibrium.

2. Fundamental Solution Properties.

This section is concerned with the solution existence and uniqueness (i.e., well-posedness) and other basic solution properties of the initial-value problem of the DSVI (1.1)-(1.3). Toward this end, we introduce the following assumptions:

- **A.1** For any given $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, the stochastic VI: $0 \in \Psi(t,\xi,x,\cdot) + \mathcal{N}_{C_\xi}(\cdot)$ a.e. $\xi \in \Xi$ has a solution $y_x(t,\xi) \in \mathcal{Y}$;
- **A.2** The function $G(t,x) := \mathbb{E}[\Phi(t,\xi,x,y(t,\xi))]$ is (locally) Lipschitz continuous at any given $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ for some measurable selection of solutions $y_x(t,\xi) \in$ SOL(t,x,ξ) at each (t,x).

In Section 3, we give sufficient conditions on Φ and Ψ such that **A.1**-**A.2** hold.

LEMMA 2.1. *Under assumptions* **A.1**-**A.2***, for any $T > 0$, the DSVI (1.1)-(1.3) has a solution $(x(t,x_0), y(t,\xi))$ for any $t \in [0,T]$ and any initial condition x_0 with $x(t,x_0)$ being unique and C^1. Further, if $y(t,\xi)$ in* **A.1** *is also unique for any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$, then the DSVI solution $(x(t,x_0), y(t,\xi))$ is also unique. Besides, $x(t,x_0)$ is continuous in x_0 at each t.*

Proof. It suffices to prove that the time-varying ODE: $\dot{x}(t) = \gamma \cdot [\Pi_X(x(t) - G(t,x(t))) - x(t)]$ with $x(0) = x_0$ has a unique C^1 solution. Since $\Pi_X(\cdot)$ is globally Lipschitz with the Lipschitz constant one with respect to $\|\cdot\|_2$, the right-hand side of this ODE is locally Lipschtiz at any (t,x). It follows from the Picard-Lindelöf Theorem that there exists a unique C^1 solution $x(t)$ for all $t \in [-\delta,\delta]$ for a positive number $\delta > 0$ with the initial value $x(0) = x_0$ [12]. Since δ is independent of the initial point and T, we can repeat the argument on each interval $[t, t+\delta]$ and show that for any $T > 0$ and any initial condition, the DSVI (1.1)-(1.3) has a solution

$(x(t,x_0), y(t,\xi))$ with $x(t,x_0)$ being unique and C^1. The rest of the statement follows readily. ∎

LEMMA 2.2. *Suppose* **A.1**-**A.2** *hold. Let $x(t,x_0)$ denote the solution of the ODE (1.1): $\dot{x}(t) = \gamma \cdot [\Pi_X(x(t) - G(t,x(t))) - x(t)]$ from the initial condition x_0. The following statements hold:*

(i) Let $\gamma \geq 0$. Then $x_0 \in X \Longrightarrow x(t,x_0) \in X, \forall\, t \geq 0$.

(ii) Let X be an affine set. Then for any $\gamma \in \mathbb{R}$, $x_0 \in X \Longrightarrow x(t,x_0) \in X, \forall\, t \geq 0$.

Proof. (i) This proof is similar to [2, Proposition 5.8]. We provide essential details to be self-contained. Since $\dot{x}(t) = \gamma \cdot [\Pi_X(x(t) - G(t,x(t))) - x(t)]$ and $x(0) = x_0$, we have, for any $t \geq 0$,

$$x(t,x_0) \,=\, e^{-\gamma t}x_0 + \int_0^t e^{-\gamma(t-\tau)}\gamma\Pi_X\big[\underbrace{x(\tau,x_0) - G(\tau, x(\tau,x_0))}_{h(\tau)}\big]d\tau.$$

Letting $s := t > 0$ and $\tau' = \tau$, we have

$$x(s,x_0) \,=\, e^{-\gamma s}x_0 + \Big(1 - e^{-\gamma s}\Big)\underbrace{\frac{\int_0^s e^{\gamma\tau'}\Pi_X(h(\tau'))d\tau'}{\int_0^s e^{\gamma\tau'}d\tau'}}_{z}.$$

Since X is a closed convex set, it follows from the proof of [2, Proposition 5.8] that $z \in X$. Further, because $\gamma \geq 0$ and $s > 0$, we see that $x(s,x_0)$ is a convex combination of $x_0 \in X$ and $z \in X$. Therefore, $x(t,x_0) = x(s,x_0) \in X$.

(ii) When X is an affine set, we see from the proof for (i) that for any γ, $x(s,x_0)$ is an affine combination of $x_0 \in X$ and $z \in X$. Hence, $x(s,x_0) \in X$. ∎

When $\gamma < 0$, statement (i) may fail. For example, let $X = \mathbb{R}_+$. This yields $\dot{x} = -\gamma \cdot \min(x, G(t,x))$. Suppose $G(t,x) = x - 1 - t$ whose associated LCP: $0 \leq x \perp x - 1 - t \geq 0$ has a unique solution $x_*(t) = 1 + t$. Since $G(0,0) < 0$ and $\gamma < 0$, then for $x_0 = 0$, $\dot{x}(0) = -\gamma G(0,0) < 0$ so that $x(t) < 0$ for all $t > 0$ sufficiently small.

2.1. Mode Switching and non-Zeno Properties of the DSVI. When X is a proper subset of $\mathbb{R}^n$ and/or G is nonsmooth in x, the right-hand side of the DSVI (1.1) is defined by a nonsmooth function due to the projection operator Π_X. Further, along with nonsmooth properties of the stochastic VI in (1.3), the right-hand side of the DSVI (1.1) may be cast as a piecewise continuous (or smooth) function such that the solution $x(t,x_0)$ demonstrates mode switching behaviors, which lead to the so-called Zeno or non-Zeno behaviors [17, 29, 31]. In what follows, we discuss Zeno-free cases; these results are useful for numerical computation and analysis of the DSVI.

To characterize the non-Zeno behavior, we introduce several notions. Consider the ODE $\dot{x} = f(x)$ with $x(0) = x_0$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and piecewise affine. Hence, f attains a polyhedral subdivision of $\mathbb{R}^n$ given by $\{\mathcal{X}_i\}_{i=1}^p$ [15, Section 4.2]. For a solution $x(t,x_0)$ starting from the initial condition x_0, a time t_* is *not* a switching time along $x(t,x_0)$ if there exist $\mathcal{X}_i$ and a constant $\varepsilon > 0$ such that $x(t,x_0) \in \mathcal{X}_i$ for all $t \in [t_* - \varepsilon, t_* + \varepsilon]$; otherwise, the ODE has a mode switching at t_*. For a given constant $T > 0$ and a given x_0, $x(t,x_0)$ is *non-Zeno* if there are finitely many switchings on the time interval $[0,T]$. The ODE is *robust non-Zeno* if there is a uniform bound on the number of switchings on $[0,T]$ regardless of x_0's [30]. Other mode switching and non-Zeno notions for DVIs can be found in [3, 17, 22, 29, 31, 32].

LEMMA 2.3. *Suppose X is polyhedral, Φ and Ψ are time invariant, and $\widetilde{G}(x) := \mathbb{E}[\Phi(\xi, x, y(\xi))]$ is piecewise affine (and continuous). Then the ODE (1.1) is robust non-Zeno in the above sense.*

Proof. Since X is a polyhedral set, its Euclidean projection operator $\Pi_X(\cdot)$ is continuous and piecewise affine [15, Proposition 4.1.4]. As $\widetilde{G}$ is continuous and piecewise affine, we deduce that the right-hand side function of (1.1) given by $\gamma \cdot [\Pi_X(x - \widetilde{G}(x)) - x]$ is also continuous and piecewise affine. Hence, it follows from [30, Theorem 2.19] that the ODE (1.1) is robust non-Zeno. ∎

We apply the above lemma to a specific example. Consider the stochastic linear complementarity problem (SLCP) with $C_\xi = \mathbb{R}^m_+$ for all $\xi \in \Xi$. Then the DSVI becomes the following DSLCP:

$$\text{(2.1)} \qquad \begin{aligned} &\dot{x} = \gamma\Big\{\Pi_X\Big(x - \big(Ax + \mathbb{E}[B(\xi)y_x(\xi)] + q_1\big)\Big) - x\Big\}, \\ &0 \le y(\xi) \perp M(\xi)y(\xi) + N(\xi)x + q_2(\xi) \ge 0, \quad \text{a.e. } \xi \in \Xi. \end{aligned}$$

Suppose the solution set $\mathrm{SOL}(M(\xi), N(\xi)x + q_2(\xi))$ of the SLCP in (2.1) is nonempty for any $\xi \in \Omega$ and x, and $B(\xi)\mathrm{SOL}(M(\xi), N(\xi)x + q_2(\xi))$ is singleton. This condition holds, for example, when $M(\xi)$ is a P-matrix; see [32] for other examples where $\mathrm{SOL}(M(\xi), N(\xi)x + q_2(\xi))$ is non-singleton. It is known that for each ξ, $B(\xi)\mathrm{SOL}(M(\xi), N(\xi)x + q_2(\xi))$ is continuous and piecewise affine in x [32]. Further, if ξ has a discrete and finite distribution, then $\mathbb{E}[B(\xi)\mathrm{SOL}(M(\xi), N(\xi)x + q_2(\xi))]$ is continuous and piecewise affine in x. Therefore, when X is polyhedral, the DSLCP (2.1) is robust non-Zeno.

REMARK 2.1. It is worth pointing out that if ξ has a continuous distribution, then $\mathbb{E}[B(\xi)\mathrm{SOL}(M(\xi), N(\xi)x + q_2(\xi))]$ is not necessarily piecewise affine although it remains continuous in x. For example, let $x, y(\cdot) \in \mathbb{R}$, ξ be uniformly distributed on $\Omega := [0, 1] \subset \mathbb{R}$, $M(\xi) \equiv 1$, $N(\xi) = \xi$, and $q_2(\xi) \equiv 1$, which yields that $0 \le y(\xi) \perp y(\xi) + [\xi x - 1] \ge 0$ has a unique solution $y_x(\xi) = -\min(\xi x - 1, 0)$. Suppose $B(\xi) \equiv 1$. Then $\mathbb{E}[B(\xi)y_x(\xi)] = -\mathbb{E}[\min(\xi x - 1, 0)]$, where

$$\mathbb{E}[\min(\xi x - 1, 0)] = \begin{cases} \displaystyle\int_0^1 (\xi x - 1)d\xi, & \text{if } x \le 1 \\ \displaystyle\int_0^{1/x} (\xi x - 1)d\xi, & \text{if } x \ge 1 \end{cases} = \begin{cases} \frac{x}{2} - 1, & \text{if } x \le 1 \\ -\frac{1}{2x}, & \text{if } x \ge 1 \end{cases}$$

which is not piecewise affine for $x \ge 1$. Hence, the right-hand side of (2.1) is not piecewise affine when $X = \mathbb{R}$ (although it is piecewise affine when $X \subset (-\infty, 1]$ by Lemma 2.3). However, it is seen that the right-hand side of (2.1) is piecewise analytic in the following sense [29].

We introduce the concept of piecewise analytic systems treated in [33] as follows. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a *piecewise analytic function*, namely, there exists a finite family of selection functions $\{f^i\}_{i=1}^m$ such that $f(x) \in \{f^i(x)\}_{i=1}^m$ for each $x \in \mathbb{R}^n$, and that the following conditions hold:

(H1) For each f^i, there exists a nonempty subanalytic set $\mathcal{X}_i \subseteq \mathbb{R}^n$ such that $f(x) = f^i(x)$, $\forall\, x \in \mathcal{X}_i$, and $\{\mathcal{X}_i\}_{i=1}^m$ forms a finite partition of $\mathbb{R}^n$;

(H2) For each $\mathcal{X}_i$, there exists an open set $\Omega_i \subseteq \mathbb{R}^n$ such that $\mathrm{cls}\, \mathcal{X}_i \subseteq \Omega_i$ and f^i is real analytic on Ω_i, where cls stands for the closure of a set;

(H3) The continuity of f holds, i.e., $x \in \operatorname{cls}\mathcal{X}_i \cap \operatorname{cls}\mathcal{X}_j \implies f^i(x) = f^j(x)$ for any $i, j \in \{1, \dots, m\}$.

Consider the ODE system whose right-hand side f satisfies (H1)–(H3):

$$\dot{x} = f(x). \tag{2.2}$$

Given $T > 0$, let $x(t, x_0)$ be a solution of (2.2) on $[0, T]$ with the initial condition x_0. We say that $x(t, x_0)$ has no switching at a time instant t_* [33] if there exist $i \in \{1, \dots, m\}$ and a constant $\varepsilon > 0$ such that $x(t, x_0) \in \mathcal{X}_i, \forall\, t \in [t_* - \varepsilon, t_* + \varepsilon]$; otherwise, $x(t, x_0)$ has a *mode switching* at t_*.

Theorem 2.1. *[33, Theorem II] Consider the system (2.2) satisfying (H1)-(H3). For a compact set $\mathcal{V} \subseteq \mathbb{R}^n$ and a constant $T > 0$, there exists $N(\mathcal{V}, T) \in \mathbb{N}$ such that for any time interval $I \subseteq [0, T]$, if $x(t, x_0)$ satisfies $\{x(t, x_0) \,|\, t \in I\} \subseteq \mathcal{V}$, then $x(t, x_0)$ has at most $N(\mathcal{V}, T)$ mode switchings on I.*

Motivated by the example in Remark 2.1, we consider the following case.

Lemma 2.4. *Let $I = [a, b]$ with $a, b \in \mathbb{R}$ and $a < b$, $q \in \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ be a strictly monotone and analytic function such that $g(0) = 0$ and $g'(\xi) \neq 0$ for all $\xi \neq 0$. Let h be a real-valued, analytic function over an open set containing I. Define $G(x) := \int_{\xi \in I} \min(0, g(\xi)x + q)h(\xi)d\xi$, $\forall\, x \in \mathbb{R}$. Then G satisfies the conditions (H1)-(H3) and is piecewise analytic on $\mathbb{R}$.*

The above setting includes the case where $g'(0) \neq 0$, e.g., $g(\xi) = 2\xi$ or $g(\xi) = -\xi$.

Proof. Since I is compact and the integrand of G is continuous in (x, ξ), $G(x) \in \mathbb{R}$ for each $x \in \mathbb{R}$ and G is continuous on $\mathbb{R}$. Clearly, g is a homeomorphism such that its inverse function g^{-1} is strictly monotone and continuous on $\mathbb{R}$. Since $g(0) = 0$ and g is strictly monotone, $g(\xi) \neq 0$ for all $\xi \neq 0$. Further, since $g'(\xi) \neq 0$ for all $\xi \neq 0$, we deduce via the Inverse Function Theorem that g^{-1} is analytic at each $g(\xi)$ with $\xi \neq 0$. Hence, $g^{-1}(z)$ is analytic at any $z \neq 0$. By the definition of G,

$$G(x) = \begin{cases} \displaystyle\int_a^{\min(b, g^{-1}(-\frac{q}{x}))} [g(\xi)x + q]h(\xi)d\xi, & \text{if} \quad x > 0; \\ \displaystyle\int_{\max(a, g^{-1}(-\frac{q}{x}))}^b [g(\xi)x + q]h(\xi)d\xi, & \text{if} \quad x < 0. \end{cases}$$

When $q = 0$, it is easy to see that G is a piecewise linear function and thus satisfies (H1)-(H3). In what follows, we consider $q < 0$ only since $q > 0$ follows from the similar argument. Further, we assume, without loss of generality, that g is strictly increasing, since otherwise $g(\xi)x$ can be written as $[-g(\xi)](-x)$ and the desired result will follow.

Let $q < 0$. Since g and g^{-1} are strictly increasing, $\min(b, g^{-1}(s)) = g^{-1} \circ \min(g(b), s)$ and $\max(a, g^{-1}(s)) = g^{-1} \circ \max(g(a), s)$ for any $s \in \mathbb{R}$. Using this result and letting $f(x, \xi) := [g(\xi)x + q]h(\xi)$, we obtain: for $x > 0$,

$$G(x) = \begin{cases} \displaystyle\int_a^b f(x, \xi)d\xi, & \text{if} \quad g(b) \leq 0; \\ \displaystyle\int_a^{g^{-1}(-\frac{q}{x})} f(x, \xi)d\xi, & \text{if} \quad g(b) > 0 \ \text{ and } \ x \geq -\frac{q}{g(b)}; \\ \displaystyle\int_a^b f(x, \xi)d\xi, & \text{if} \quad g(b) > 0 \ \text{ and } \ 0 < x \leq -\frac{q}{g(b)} \end{cases}$$

and for $x < 0$,

$$G(x) = \begin{cases} \int_a^b f(x,\xi)d\xi, & \text{if} \quad g(a) \geq 0; \\ \int_{g^{-1}(-\frac{q}{x})}^b f(x,\xi)d\xi, & \text{if} \quad g(a) < 0 \quad \text{and} \quad x \leq -\frac{q}{g(a)}; \\ \int_a^b f(x,\xi)d\xi, & \text{if} \quad g(a) < 0 \quad \text{and} \quad 0 > x \geq -\frac{q}{g(a)}. \end{cases}$$

Consequently, we have the following results for G:

Case (1): $g(b) \leq 0$, which implies $g(a) < 0$ as g is strictly increasing. In this case,

$$G(x) = \begin{cases} \int_a^b f(x,\xi)d\xi, & \text{if} \quad x \geq -\frac{q}{g(a)}; \\ \int_{g^{-1}(-\frac{q}{x})}^b f(x,\xi)d\xi, & \text{if} \quad x \leq -\frac{q}{g(a)}. \end{cases}$$

Case (2): $g(b) > 0$ and $g(a) \geq 0$. In this case,

$$G(x) = \begin{cases} \int_a^{g^{-1}(-\frac{q}{x})} f(x,\xi)d\xi, & \text{if} \quad x \geq -\frac{q}{g(b)}; \\ \int_a^b f(x,\xi)d\xi, & \text{if} \quad x \leq -\frac{q}{g(b)}. \end{cases}$$

Case (3): $g(b) > 0$ and $g(a) < 0$. In this case,

$$G(x) = \begin{cases} \int_a^{g^{-1}(-\frac{q}{x})} f(x,\xi)d\xi, & \text{if} \quad x \geq -\frac{q}{g(b)}; \\ \int_a^b f(x,\xi)d\xi, & \text{if} \quad -\frac{q}{g(a)} \leq x \leq -\frac{q}{g(b)}; \\ \int_{g^{-1}(-\frac{q}{x})}^b f(x,\xi)d\xi, & \text{if} \quad x \leq -\frac{q}{g(a)}. \end{cases}$$

Consider Case (3) first. The domain of each selection function in G is a closed interval in $\mathbb{R}$. In fact, $\mathcal{X}_1 = [-\frac{q}{g(b)}, \infty)$, $\mathcal{X}_2 = [-\frac{q}{g(a)}, -\frac{q}{g(b)}]$, and $\mathcal{X}_3 = (-\infty, -\frac{q}{g(a)}]$, which are clearly subanalytic and form a partition of $\mathbb{R}$. As $q < 0$, $g(b) > 0$ and $g(a) < 0$, we have $-\frac{q}{g(b)} > 0$ and $-\frac{q}{g(a)} < 0$. Hence, there exists a sufficiently small constant $\varepsilon > 0$ such that the open interval $\Omega_1 := (-\frac{q}{g(b)} - \varepsilon, \infty)$ contains $\mathcal{X}_1$ and $-\frac{q}{x} > 0$ for all $x \in \Omega_1$. Since $g^{-1}(z)$ is analytic at each $z \neq 0$ and h is analytic on an open set containing I, it is easy to verify that the selection function $f^1(x) := \int_a^{g^{-1}(-\frac{q}{x})} f(x,\xi)d\xi$ is analytic on Ω_1. Similarly, f^3 is analytic on an open set Ω_3 containing $\mathcal{X}_3$. Further, since $f^2(x) := \int_a^b f(x,\xi)d\xi$ is an affine function, it is analytic on an open interval containing $\mathcal{X}_2$. Consequently, G satisfies (H1)-(H3) and is piecewise analytic on $\mathbb{R}$. The similar argument can be used to show the desired results for Cases (1)-(2). $\square$

REMARK 2.2. The above lemma can be extended to a strictly increasing and analytic function g satisfying the following conditions: there exists some $c \in \mathbb{R}$ such that $g'(x) \neq 0$ for all $x \neq c$, and either one of the following holds: (1) $g(c) \notin [g(a), 0)$ if $g(a) < g(b) \leq 0$; (2) $g(c) \notin (0, g(b)]$ if $g(b) > g(a) \geq 0$; and (3) $g(c) \notin [g(a), 0) \cup (0, g(b)]$ if $g(b) > 0 > g(a)$. The similar extension can be made for a strictly decreasing and analytic function g.

PROPOSITION 2.1. *Consider the DSLCP (2.1), where $m = n = d$. Suppose that X is a polyhedral set, M is a constant diagonal matrix with positive diagonal entries, $\big(N(\xi)x\big)_i = g_i(\xi_i)x_i$ for each i, where g_i satisfies the assumption on g in Lemma 2.4 or Remark 2.2, and q_2 is a constant vector. Further, assume that the support Ξ is a compact box constraint, and the probability density function $\rho(\cdot)$ and $B(\cdot)$ are analytic over an open set containing Ξ. Then the right hand side of the DSLCP (2.1) is piecewise analytic on $\mathbb{R}^n$ and is non-Zeno in the sense of Theorem 2.1.*

Proof. Let $m_{ii}, i = 1, \ldots, n$ be the positive diagonal entries of M. Then for each j, $0 \le y_j(\xi) \perp m_{jj} y_j(\xi) + g_j(\xi_j)x_j + (q_2)_j \ge 0$ has a unique solution $\widehat{y}_j(x, \xi) = -\min(0, \frac{1}{m_{jj}}[g_j(\xi_j)x_j + (q_2)_j])$. Let $\Xi = [a_1, b_1] \times \cdots \times [a_n, b_n]$, where $-\infty < a_i < b_i < \infty$ for $i = 1, \ldots, n$. For each i, j, let

$$f_{i,j}(x_j, \xi) \;:=\; -B_{ij}(\xi) \min\Big(0, \frac{1}{m_{jj}}[g_j(\xi_j)x_j + (q_2)_j]\Big)\rho(\xi).$$

Hence,

$$\begin{aligned}\mathbb{E}[B_{ij}(\xi)\widehat{y}_j(x, \xi)] &= \int_{\xi\in\Xi} f_{i,j}(x_j, \xi) d\xi_1 \cdots d\xi_n \\ &= \int_{a_1}^{b_1} \cdots \int_{a_{j-1}}^{b_{j-1}} \int_{a_{j+1}}^{b_{j+1}} \cdots \int_{a_n}^{b_n} \left(\int_{a_j}^{b_j} f_{i,j}(x_j, \xi) d\xi_j \right) d\xi_1 \cdots d\xi_{j-1} d\xi_{j+1} \cdots d\xi_n.\end{aligned}$$

By Lemma 2.4, it is easy to show that $\mathbb{E}[B_{ij}(\xi)\widehat{y}_j(x, \xi)]$ satisfies the conditions (H1)-(H3) and is piecewise analytic in x_j on $\mathbb{R}$. Hence, $\mathbb{E}[B(\xi)\widehat{y}(x, \xi)]$ is piecewise analytic on $\mathbb{R}^n$. Since X is polyhedral, Π_X is piecewise affine. Since the composition of two piecewise analytic functions remains piecewise analytic, we see that the right-hand side of (2.1) is piecewise analytic and is therefore non-Zeno in the sense of Theorem 2.1. ∎

We comment that the results in Proposition 2.1 can be generalized to other DSVIs. For example, the non-Zeno result remains to hold if the term $Ax + q_1$ in the DSLCP (2.1) is replaced by a piecewise analytic function in x.

2.2. Strongly Regular DSVI: Local Solution Existence and Uniqueness. We have focused on the global solution existence and uniqueness at the beginning of this section. In what follows, we discuss a case where local solution existence and uniqueness can be obtained. Consider the time-invariant DSVI of the following form:

$$\text{(2.3)} \quad \dot{x} = \gamma\Big\{\Pi_X\big(x - \mathbb{E}[\Phi(\xi, x, y_x(\xi))]\big) - x\Big\}, \;\; 0 \le y(\xi) \perp H(x, y(\xi), \xi) \ge 0, \;\; \text{a.e. } \xi \in \Xi.$$

Consider the stochastic NCP: $0 \le u \perp H(x, u, \xi) \ge 0$, where we assume that $H(\cdot, \cdot, \xi)$ is continuously differentiable for any given ξ. Given $\xi \in \Xi$, define the three fundamental index sets $(\alpha_0, \beta_0, \gamma_0)$ corresponding to the solution pair $(x_0, u_0(\xi))$. (We write $u_0(\xi)$ as u_0 below for notational simplicity.)

$$\alpha_0(x_0, u_0, \xi) = \{i \,:\, (u_0)_i > 0 = H_i(x_0, u_0, \xi)\},$$
$$\beta_0(x_0, u_0, \xi) = \{i \,:\, (u_0)_i = 0 = H_i(x_0, u_0, \xi)\},$$
$$\gamma_0(x_0, u_0, \xi) = \{i \,:\, (u_0)_i = 0 < H_i(x_0, u_0, \xi)\}.$$

The Jacobian $J_u H(x_0, u_0, \xi)$ is given by

$$J_u H(x_0, u_0, \xi) = \begin{bmatrix} J_{u_{\alpha_0}} H_{\alpha_0}(x_0, u_0, \xi) & J_{u_{\beta_0}} H_{\alpha_0}(x_0, u_0, \xi) & J_{u_{\gamma_0}} H_{\alpha_0}(x_0, u_0, \xi) \\ J_{u_{\alpha_0}} H_{\beta_0}(x_0, u_0, \xi) & J_{u_{\beta_0}} H_{\beta_0}(x_0, u_0, \xi) & J_{u_{\gamma_0}} H_{\beta_0}(x_0, u_0, \xi) \\ J_{u_{\alpha_0}} H_{\gamma_0}(x_0, u_0, \xi) & J_{u_{\beta_0}} H_{\gamma_0}(x_0, u_0, \xi) & J_{u_{\gamma_0}} H_{\gamma_0}(x_0, u_0, \xi) \end{bmatrix}.$$

For a given ξ, $u_0(\xi)$ is a *strongly regular* solution of x_0 [22, 25] if (i) $J_{u_{\alpha_0}} H_{\alpha_0}(x_0, u_0, \xi)$ is invertible, and (ii) the following Schur complement is a P-matrix:

$$\begin{aligned} &M(x_0, u_0, \xi) \\ &:= J_{u_{\beta_0}} H_{\beta_0}(x_0, u_0, \xi) - J_{u_{\alpha_0}} H_{\beta_0}(x_0, u_0, \xi)\big[J_{u_{\alpha_0}} H_{\alpha_0}(x_0, u_0, \xi)\big]^{-1} J_{u_{\beta_0}} H_{\alpha_0}(x_0, u_0, \xi). \end{aligned}$$

We make the following assumption on the stochastic NCP at x_0:

H For a.e. $\xi \in \Xi$, $u_0(\xi)$ is a strongly regular solution of x_0, $u_0(\xi)$ is measurable, and the following conditions hold: there exist a constant $c_1 > 0$ and two measurable functions $c_i(\xi) > 0$ with $i = 2, 3$ such that for a.e. $\xi \in \Xi$, $c(M(x_0, u_0(\xi), \xi)) \geq c_1$, $\|J_x H(x_0, u_0(\xi), \xi)\|_\infty \leq c_2(\xi)$, and

$$\|K(\xi) \cdot J_{u_{\beta_0}} H_{\alpha_0}(x_0, u_0(\xi), \xi)\|_\infty \max(\|J_{u_{\alpha_0}} H_{\beta_0}(x_0, u_0(\xi), \xi) \cdot K(\xi)\|_\infty, 1) + c_1 \cdot \|K(\xi)\|_\infty$$

$\leq c_3(\xi)$, where

$c(M) := \min_{\|z\|_\infty = 1} \max_{1 \leq i \leq m} z_i (Mz)_i$ and $K(\xi) := -\big[J_{u_{\alpha_0}} H_{\alpha_0}(x_0, u_0(\xi), \xi)\big]^{-1}$.

The following example illustrates the conditions given in **H**. Suppose Ξ is a compact support, and the stochastic NCP corresponding to a solution pair $(x_0, u_0(\xi))$ in (2.3) is such that $u_0(\xi)$ is continuous in ξ, $J_u H(x_0, u_0(\xi), \xi)$ is a P-matrix for each given $\xi \in \Xi$, and $J_u H(x_0, u_0(\xi), \xi)$ and $J_x H(x_0, u_0(\xi), \xi)$ are continuous in ξ on Ξ. Then $(x_0, u_0(\xi))$ is a strongly regular solution of x_0 for each ξ as the Schur complement of a P-matrix remains a P-matrix. Further, $K(\xi)$ defined above is continuous in ξ. Along with the continuity of $J_x H$ and $J_u H$ in ξ at $(x_0, u_0(\xi))$ and the compactness of Ξ, we see that there exists $c_1 > 0$ such that $c(M(x_0, u_0(\xi), \xi)) \geq c_1$ and the desired c_2, c_3 can be chosen as certain positive constants. Hence, **H** holds.

LEMMA 2.5. *Suppose* **H** *holds. Then for any given constant $\varepsilon > 0$ and a.e. $\xi \in \Xi$, there exist two neighborhoods $\mathcal{V}_\xi$ of x_0 and $\mathcal{U}_\xi$ of $u_0(\xi)$ and a Lipschitz continuous function $u_\xi : \mathcal{V}_\xi \to \mathcal{U}_\xi$ with the Lipschitz constant $(c_2(\xi) + \varepsilon)[\max(c_3(\xi)/c_1, 1/c_1) + \varepsilon]$ with respect to $\|\cdot\|_\infty$ such that for any $x \in \mathcal{V}_\xi$, $u_\xi(x) \in \mathcal{U}_\xi$ is a solution of the stochastic NCP corresponding to x and ξ.*

Note that the stochastic NCP may attain multiple solutions at $x \in \mathcal{V}_\xi$, and $u_\xi(x) \in \mathcal{U}_\xi$ is one of these solutions indicated in the above lemma.

Proof. Fix a constant $\varepsilon > 0$ and a $\xi \in \Xi$ where $u_0(\xi)$ is a strongly regular solution at x_0. Then there exist two neighborhoods $\mathcal{V}_\xi$ of x_0 and $\mathcal{U}_\xi$ of $u_0(\xi)$ and a Lipschitz function $u_\xi : \mathcal{V}_\xi \to \mathcal{U}_\xi$ such that for any $x \in \mathcal{V}_\xi$, $u_\xi(x) \in \mathcal{U}_\xi$ is a solution of the NCP corresponding to x and ξ [22, 25]. To establish the desired Lipschitz constant of u_ξ, consider the following LCP in v obtained from the linearization of the NCP at $(x_0, u_0(\xi))$:

$$0 \leq (u_0(\xi) + v) \perp H(x_0, u_0(\xi), \xi) + J_u H(x_0, u_0(\xi), \xi) v + p \geq 0,$$

where the vector $p = (p_{\alpha_0}, p_{\beta_0}, p_{\gamma_0})$, and we write its solution as $v_\xi(p)$. Denote $M(x_0, u_0(\xi), \xi)$ by $M(\xi)$ for notational simplicity. For any p of sufficiently small magnitude, we have

$$v_{\xi,\alpha_0}(p) = K'(\xi) \cdot v_{\beta_0}(p) + K(\xi) p_{\alpha_0}, \;\; 0 \leq v_{\xi,\beta_0}(p) \perp M(\xi) v_{\xi,\beta_0}(p) + K''(\xi) p_{\alpha_0} + p_{\beta_0} \geq 0,$$

and $v_{\xi,\gamma_0} = 0$, where the matrices $K(\xi) := -\big[J_{u_{\alpha_0}} H_{\alpha_0}(x_0, u_0(\xi), \xi)\big]^{-1}$, and

$$K'(\xi) := -K(\xi) \cdot J_{u_{\beta_0}} H_{\alpha_0}(x_0, u_0(\xi), \xi), \quad K''(\xi) := -J_{u_{\alpha_0}} H_{\beta_0}(x_0, u_0(\xi), \xi) \cdot K(\xi).$$

Since $M(\xi)$ is a P-matrix, we have, for all p, q of sufficiently small magnitude,

$$\|v_{\xi,\beta_0}(p) - v_{\xi,\beta_0}(q)\|_\infty \leq \frac{\max(\|K''(\xi)\|_\infty, 1)}{c_1}\|p-q\|_\infty,$$

and

$$\begin{aligned}\|v_{\xi,\alpha_0}(p) - v_{\xi,\alpha_0}(q)\|_\infty &\leq \Big(\|K'(\xi)\|_\infty \cdot \frac{\max(\|K''(\xi)\|_\infty, 1)}{c_1} + \|K(\xi)\|_\infty\Big)\|p-q\|_\infty \\ &\leq \frac{c_3(\xi)}{c_1}\|p-q\|_\infty.\end{aligned}$$

This yields the local Lipschitz constant $\max(c_3(\xi), 1)/c_1$ of $v_\xi(\cdot)$ with respect to $\|\cdot\|_\infty$. Finally, given $u_0(\xi)$ for a fixed ξ, we have, for all $x, x' \in \mathcal{V}_\xi$ by possibly restricting $\mathcal{V}_\xi$,

$$\begin{aligned}\|H(x, u_0(\xi), \xi) - H(x', u_0(\xi), \xi)\|_\infty &\leq \big[\|J_x H(x_0, u_0(\xi), \xi)\|_\infty + \varepsilon\big] \cdot \|x - x'\|_\infty \\ &\leq (c_2(\xi) + \varepsilon) \cdot \|x - x'\|_\infty.\end{aligned}$$

By [25, Corollary 2.2], $(c_2(\xi)+\varepsilon)[\max(c_3(\xi)/c_1, 1/c_1)+\varepsilon]$ is the local Lipschitz constant of u_ξ. ∎

Suppose that there exist an open set $\mathcal{V}_0$ of x_0 with $\mathcal{V}_0 \subseteq \mathcal{V}_\xi$ a.e. $\xi \in \Xi$ and another open set $\mathcal{U}_0$ with $\mathcal{U}_0 \subseteq \mathcal{U}_\xi$ a.e. $\xi \in \Xi$. (Clearly, such $\mathcal{V}_0$ and $\mathcal{U}_0$ exist if ξ has a finite discrete distribution.) Furthermore, suppose $\mathbb{E}[\kappa_\Phi(\xi)] < \infty$, $\mathbb{E}[\kappa_\Phi(\xi)\max(c_3(\xi), 1)] < \infty$ and $\mathbb{E}[\kappa_\Phi(\xi)c_2(\xi)\max(c_3(\xi), 1)] < \infty$. For a given $\varepsilon > 0$, define $G(x) := \mathbb{E}[\Phi(\xi, x, u_\xi(x)]$ for $x \in \mathcal{V}_0$ and $u_\xi(x) \in \mathcal{U}_0$. Then for any $x, x' \in \mathcal{V}_0$, we have, via assumption **A.0**, that

$$\begin{aligned}\|G(x) - G(x')\|_\infty &\leq \mathbb{E}\big[\kappa_\Phi(\xi)\|(x, u_\xi(x)) - (x', u_\xi(x'))\|_\infty\big] \\ &\leq \underbrace{\mathbb{E}\big[\kappa_\Phi(\xi)\big(1 + (c_2(\xi)+\varepsilon)(\max(c_3(\xi)/c_1, 1/c_1) + \varepsilon)\big)\big]}_{:=\kappa_G} \cdot \|x - x'\|_\infty.\end{aligned}$$

By the given assumptions, $0 < \kappa_G < \infty$ such that $G(\cdot)$ is Lipschitz continuous on the neighborhood $\mathcal{V}_0$ of x_0. This shows that there exists a constant $\varphi > 0$ such that the DSVI (2.3) has a unique solution $x(t) := x(t, x_0) \in \mathcal{V}_0$ on the time interval $[-\varphi, \varphi]$ with $x(0) = x_0$ and $\widehat{y}(x(t), \xi) := u_\xi(x(t)) \in \mathcal{U}_0$ for all $t \in [-\varphi, \varphi]$.

3. Sample Average Approximation of the DSVI.

In this section, we concentrate on two cases. The first case is when the underlying VI in the second stage defined by Ψ is strongly monotone, whereas in the second case, we consider a special non-monotone VI given by a box-constrained linear VI satisfying the P-property.

ASSUMPTION 3.1. **Case (i)** *The function Ψ is (uniformly) strongly monotone on C_ξ respect to y for any $t, x \in \mathbb{R}^n$, a.e. $\xi \in \Xi$ in the sense that there is a constant $\eta > 0$, independent of t, x and ξ, such that*

$$(3.1)\quad (z - z')^\top\Big(\Psi(t, \xi, x, z) - \Psi(t, \xi, x, z')\Big) \geq \eta\|z - z'\|_2^2, \quad \forall\, z, z' \in C_\xi, \text{ a.e. } \xi \in \Xi.$$

Case (ii) *The set $C_\xi = [l_\xi, u_\xi]$ a.e. $\xi \in \Xi$, where $l_\xi \in \{\mathbb{R} \cup \{-\infty\}\}^n$, $u_\xi \in \{\mathbb{R} \cup \{\infty\}\}^n$, and $l_\xi < u_\xi$, and $\Psi(t, \xi, x, y) = M(\xi)y + \psi(t, \xi, x)$, where $M(\xi) \in \mathbb{R}^{m\times m}$ is a P-matrix and there is a constant $\widetilde{\eta} > 0$ independent of ξ such that*

$$(3.2)\qquad \min_{\|z\|_\infty = 1}\Big(\max_{1\leq i\leq m} z_i(M(\xi)z)_i\Big) \geq \widetilde{\eta}, \qquad \text{a.e. } \xi \in \Xi,$$

and the function $\psi(\cdot, \xi, \cdot)$ is Lipschitz continuous a.e. $\xi \in \Xi$.

We make two comments on Case (ii) as follows.

(ii.1) Clearly, the Lipschitz continuity of the function $\psi(\cdot,\xi,\cdot)$ a.e. $\xi\in\Xi$ follows from the Lipschitz continuity of Ψ in assumption **A.0**. Conversely, if $\psi(\cdot,\xi,\cdot)$ is Lipschitz in (t,x) a.e. $\xi\in\Xi$ with the measurable Lipschitz modulus and $\|M(\xi)\|$ is measurable, then $\Psi(\cdot,\xi,\cdot,\cdot)$ is Lipschitz in (t,x,y) with the measurable Lipschitz modulus.

(ii.2) When Ξ is a compact support and $M(\cdot)$ is continuous, there exists a constant $\widetilde{\eta}>0$ independent of ξ such that (3.2) holds for all $\xi\in\Xi$. In fact, let $f(\xi,z):=\max_{i=1,\dots,n}\big(z_i(M(\xi)z)_i\big)$, which is continuous in (ξ,z). Hence, f attains a minimizer (ξ^*,z^*) on the compact set $\Xi\times\{z\,|\,\|z\|_\infty=1\}$. Since $M(\xi^*)$ is a P-matrix and $z^*\neq 0$, $\widetilde{\eta}:=f(\xi^*,z^*)>0$. Thus $\min_{\|z\|_\infty=1} f(\xi,z)\geq\widetilde{\eta}$ for all $\xi\in\Xi$.

In either case of Assumption 3.1, the VI (1.3) has a unique solution $\widehat{y}_x(t,\xi)$ [15, Theorem 2.3.3, Proposition 3.5.10] for any $t\geq 0, x\in\mathbb{R}^n$, a.e. $\xi\in\Xi$. We assume that $\widehat{y}_x(t,\cdot)$ is measurable for any given (t,x) so that assumptions **A.1** holds. Sufficient conditions for the measurability of $\widehat{y}_x(t,\cdot)$ can be established. For example, in Case (i), if $C_\xi\equiv C$ for a closed convex set C and for any fixed (t,x) and any given $y\in C$, $\Psi(t,\cdot,x,y)$ is continuous on Ξ and $\kappa_\Psi(\cdot)$ is bounded on any small neighborhood of each $\xi\in\Xi$, then by the similar argument in (3.5), the unique solution $\widehat{y}_x(t,\cdot)$ is continuous at any $\xi\in\Xi$ and thus measurable. This result can be extended to the case when the closed, convex-valued set-valued mapping C_ξ is continuous in ξ; see [15, Corollary 5.1.5] and [15, Proposition 5.4.1] for the related results.

Consider Case (ii). Let $M\in\mathbb{R}^{m\times m}$, $q\in\mathbb{R}^m$, $l\in(\mathbb{R}\cup\{-\infty\})^m$, $u\in(\mathbb{R}\cup\{+\infty\})^m$ with $l<u$, and $K=\{v\in\mathbb{R}^m\,|\,l\leq v\leq u\}$. The box-constrained linear VI, denoted by LVI(M,q,l,u), is to find $v\in\mathbb{R}^m$ such that

$$0\in Mv+q+\mathcal{N}_K(v).$$

Let mid denote the componentwise median operator, i.e., for any $a,b,c\in\mathbb{R}$, $\text{mid}(a,b,c):=a+b+c-\max(a,b,c)-\min(a,b,c)$. When M is a P-matrix, it is shown in [8, 10] that the solution of the LVI is Lipschitz continuous in (M,q); the following lemma shows the continuity in (M,q,l,u).

LEMMA 3.1. *Suppose M^* is a P-matrix. Then the unique solution of this LVI is continuous in (M,q,l,u) at (M^*,q^*,l^*,u^*) for any $q^*\in\mathbb{R}^m$, $l^*\in(\mathbb{R}\cup\{-\infty\})^m$, $u^*\in(\mathbb{R}\cup\{+\infty\})^m$ with $l^*<u^*$.*

Proof. Let $\{(M^k,q^k,l^k,u^k)\}$ be a sequence that converges to (M^*,q^*,l^*,u^*). Since M^* is a P-matrix, we may assume without of generality that each M^k is a P-matrix such that the LVI attains a unique solution v^k for each k. Therefore, v^k satisfies the equation $\text{mid}(v^k-l^k,v^k-u^k,M^kv^k+q^k)=0$ for each k [8]. We first consider the case where both $l^*,u^*\in\mathbb{R}^m$. Clearly, $\{l^k\}$ and $\{u^k\}$ are bounded such that $\{v^k\}$ is bounded and hence has a convergent subsequence. Let $\{v^{k'}\}$ be an arbitrary convergent subsequence of $\{v^k\}$, and let its limit be $v^\diamond$. Since the median operator is continuous, it can be seen by passing the limit that $v^\diamond$ satisfies $\text{mid}(v^\diamond-l^*,v^\diamond-u^*,M^*v^\diamond+q^*)=0$. Since the LVI$(M^*,q^*,l^*,u^*)$ has the unique solution v^*, we have $v^\diamond=v^*$. This shows that any convergent subsequence of $\{v^k\}$ has the same limit v^*. Hence, $\{v^k\}$ converges to v^*. This shows that the solution of the LVI is continuous in (M,q,l,u) at (M^*,q^*,l^*,u^*).

Next, we consider the case where some l_i or u_i takes an extended real-value. Let $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$ be three disjoint index subsets of $\{1,\dots,m\}$ such that $l_i^*=-\infty$ and $u_i^*\in\mathbb{R}$ for all $i\in\mathcal{I}$, $u_i^*=+\infty$ and $l_i^*\in\mathbb{R}$ for all $i\in\mathcal{J}$, and $l_i^*=-\infty$ and $u_i^*=+\infty$

for all $i \in \mathcal{K}$. Hence, for any $v \in \mathbb{R}^m$,

$$\mathrm{mid}(v_{\mathcal{I}} - l^*_{\mathcal{I}}, v_{\mathcal{I}} - u^*_{\mathcal{I}}, (M^*v)_{\mathcal{I}} + q^*_{\mathcal{I}}) = \max(v_{\mathcal{I}} - u^*_{\mathcal{I}}, (M^*v)_{\mathcal{I}} + q^*_{\mathcal{I}}),$$
$$\mathrm{mid}(v_{\mathcal{J}} - l^*_{\mathcal{J}}, v_{\mathcal{J}} - u^*_{\mathcal{J}}, (M^*v)_{\mathcal{J}} + q^*_{\mathcal{J}}) = \min(v_{\mathcal{J}} - l^*_{\mathcal{J}}, (M^*v)_{\mathcal{J}} + q^*_{\mathcal{J}}),$$
$$\mathrm{mid}(v_{\mathcal{K}} - l^*_{\mathcal{K}}, v_{\mathcal{K}} - u^*_{\mathcal{K}}, (M^*v)_{\mathcal{K}} + q^*_{\mathcal{K}}) = (M^*v)_{\mathcal{K}} + q^*_{\mathcal{K}}.$$

Besides, for each $i \notin \mathcal{I} \cup \mathcal{J} \cup \mathcal{K}$, we have $\mathrm{mid}(v^*_i - l^*_i, v^*_i - u^*_i, (M^*v^*)_i + q^*_i) = 0$. We claim that (v^k) is bounded. Suppose not. Without loss of generality, we let $\|v^k\| \to \infty$, $\frac{v^k}{\|v^k\|} \to \widetilde{v} \neq 0$, and for all large k, $l^k_i = -\infty$ for all $i \in \mathcal{I} \cup \mathcal{K}$, and $u^k_i = +\infty$ for all $i \in \mathcal{J} \cup \mathcal{K}$. Since, for all large k,

$$\frac{\max(v^k_{\mathcal{I}} - u^k_{\mathcal{I}}, (M^kv^k)_{\mathcal{I}} + q^k_{\mathcal{I}})}{\|v^k\|} = 0, \quad \frac{\min(v^k_{\mathcal{J}} - l^k_{\mathcal{J}}, (M^kv^k)_{\mathcal{J}} + q^k_{\mathcal{J}})}{\|v^k\|} = 0,$$

$$\frac{(M^kv^k)_{\mathcal{K}} + q^k_{\mathcal{K}}}{\|v^k\|} = 0, \text{ and } \frac{\mathrm{mid}(v^k_i - l^k_i, v^k_i - u^k_i, (M^kv^k)_i + q^k_i)}{\|v^k\|} = 0, \text{for} i \notin \mathcal{I} \cup \mathcal{J} \cup \mathcal{K},$$

we have, by passing the limit, that $\max(\widetilde{v}_{\mathcal{I}}, (M^*\widetilde{v})_{\mathcal{I}}) = 0$, $\min(\widetilde{v}_{\mathcal{J}}, (M^*\widetilde{v})_{\mathcal{J}}) = 0$, $(M^*\widetilde{v})_{\mathcal{K}} = 0$, and for each $i \notin \mathcal{I} \cup \mathcal{J} \cup \mathcal{K}$, $\mathrm{mid}(\widetilde{v}_i, \widetilde{v}_i, (M^*\widetilde{v})_i) = 0$. This implies that $\widetilde{v}_i(M^*\widetilde{v})_i = 0$ for all $i = 1, \ldots, n$. Since M^* is a P-matrix, we have $\widetilde{v} = 0$, yielding a contradiction. Hence, (v^k) is bounded. It follows from the similar argument for the first case and the continuity of min, max and mid that any convergent subsequence of $\{v^k\}$ has the limit v^*, leading to the desired continuity. ◻

In what follows, let $\mathcal{P}$ be the set of all P-matrices in $\mathbb{R}^{m \times m}$, and $\mathcal{W} := \{(l, u) \in (\mathbb{R} \cup \{-\infty\})^m \times (\mathbb{R} \cup \{+\infty\})^m \,|\, l < u\}$. Clearly, $\mathcal{P}$ and $\mathcal{W}$ are open.

Corollary 3.1. *In Case (ii), if each entry of $(M(\xi), l(\xi), u(\xi)) \in \mathcal{P} \times \mathcal{W}$ is a measurable function on Ξ, and each entry of $\psi(t, \cdot, x)$ is measurable for any (t, x), then $y^*_x(t, \cdot)$ is measurable for any (t, x).*

Proof. Fix (t, x). Let $y^*(\xi) \in \mathbb{R}^m$ be the unique solution of the LVI in Case (ii) (we omit (t, x) in y^* as it is fixed). Let $q(\xi) := \psi(t, \cdot, x)$, which is measurable on Ξ. By Lemma 3.1, y^* viewed as a function of (M, q, l, u) is continuous on the open set $\mathcal{P} \times \mathbb{R}^m \times \mathcal{W}$. Since each entry of $M(\cdot), q(\cdot), l(\cdot), u(\cdot)$ is measurable, we see that for each $i = 1, \ldots, m$, the real-valued function $y^*_i(\cdot)$ is a composition of a continuous function and finitely many measurable functions. Hence, each $y^*_i(\cdot)$ is measurable so that $y^*(\cdot)$ is measurable. ◻

The next lemma provides sufficient conditions for assumption **A.2** being fulfilled in each of the two cases of Assumption 3.1. As $G(t, x) = \mathbb{E}[\Phi(t, \xi, x, y(t, \xi))]$, the DSVI (1.1)-(1.3) can be written as

$$\dot{x}(t) = \gamma \cdot \Big\{\Pi_X\Big(x(t) - G(t, x(t))]\Big) - x(t)\Big\}, \tag{3.3}$$
$$x(0) = x_0. \tag{3.4}$$

For notation simplicity, we write $\widehat{y}_x(t, \xi)$ as $\widehat{y}(x, t, \xi)$ in the subsequent development.

Lemma 3.2. *Suppose that $\mathbb{E}[\kappa_\Phi(\xi)] < \infty$, $\mathbb{E}[\kappa_\Phi(\xi)\kappa_\Psi(\xi)] < \infty$, and $\mathbb{E}[\kappa_\Phi(\xi)\kappa^2_\Psi(\xi)] < \infty$. In either of the two cases in Assumption 3.1, the function G is globally Lipschitz continuous, and for any initial condition x_0, the DSVI (1.1)-(1.3) has a unique solution $(x^*(t), y^*(t, \xi))$ with $x^* \in C^1[0, \infty)$ and $y^*(\cdot, \xi)$ being (locally) Lipschitz continuous in $[0, \infty)$ a.e. $\xi \in \Xi$.*

Proof. By [15, Theorem 2.3.3, Proposition 3.5.10], given any $t \geq 0, x \in \mathbb{R}^n$, a.e. $\xi \in \Xi$, the VI (1.3) has a unique solution measurable on Ξ. To show that G is (globally) Lipschitz continuous, let $v = \widehat{y}(x, t, \xi)$ and $v' = \widehat{y}(x', t', \xi)$ for a fixed $\xi \in \Xi$, where $(x, t), (x', t') \in \mathbb{R}^n \times \mathbb{R}$. Clearly, $v, v' \in C_\xi$.

Case (i) It follows from (3.1) that for almost every $\xi \in \Xi$,

$$\begin{aligned} \|v - v'\|_2 &\leq \eta'(\xi)\|v - \Pi_{C_\xi}(v - \Psi(t', \xi, x', v))\|_2 \\ &\leq \eta'(\xi)\|v - \Pi_{C_\xi}(v - \Psi(t', \xi, x', v)) - v + \Pi_{C_\xi}(v - \Psi(t, \xi, x, v))\|_2 \\ &\leq \eta'(\xi)\|\Psi(t', \xi, x', v) - \Psi(t, \xi, x, v)\|_2 \\ &\leq \eta'(\xi)\kappa_\Psi(\xi)\|(t, x) - (t', x')\|_2, \end{aligned} \tag{3.5}$$

where the first inequality is from [15, Theorem 2.3.3] with $\eta'(\xi) = (1+\kappa_\Psi(\xi))/\eta$ [1], the second inequality is due to $v - \Pi_{C_\xi}(v - \Phi(t, \xi, x, v)) = 0$, and the third inequality follows from the fact that the Euclidean projection is Lipschitz continuous with Lipschitz constant 1. Hence we obtain

$$\begin{aligned} \|G(t, x) - G(t', x')\|_2 &= \big\|\mathbb{E}\big[\Phi(t, \xi, x, \widehat{y}(x, t, \xi)) - \Phi(t', \xi, x', \widehat{y}(x', t', \xi))\big]\big\|_2 \\ &\leq \mathbb{E}\big[\|\Phi(t, \xi, x, \widehat{y}(x, t, \xi)) - \Phi(t', \xi, x', \widehat{y}(x', t', \xi))\|_2\big] \\ &\leq \mathbb{E}\big[\kappa_\Phi(\xi) \cdot \|(t, x, \widehat{y}(x, t, \xi)) - (t', x', \widehat{y}(x', t', \xi))\|_2\big] \\ &\leq \mathbb{E}\big[\kappa_\Phi(\xi)\big(1 + \eta'(\xi)\kappa_\Psi(\xi)\big)\big] \cdot \|(t, x) - (t', x')\|_2, \end{aligned} \tag{3.7}$$

where the first inequality follows from the Jensen's inequality. By $\eta'(\xi) = (1 + \kappa_\Psi(\xi))/\eta$, we obtain

$$\begin{aligned} \kappa_G &:= \mathbb{E}\big[\kappa_\Phi(\xi)\big(1 + \eta'(\xi)\kappa_\Psi(\xi)\big)\big] \;=\; \mathbb{E}[\kappa_\Phi(\xi)] + \mathbb{E}[\kappa_\Phi(\xi)\eta'(\xi)\kappa_\Psi(\xi)] \\ &= \mathbb{E}[\kappa_\Phi(\xi)] + \frac{1}{\eta}\Big(\mathbb{E}[\kappa_\Phi(\xi)\kappa_\Psi(\xi)] + \mathbb{E}[\kappa_\Phi(\xi)\kappa_\Psi^2(\xi)]\Big) < \infty, \end{aligned} \tag{3.8}$$

where the last inequality follows from the given assumption on expectations. Hence, G is (globally) Lipschitz continuous with the Lipschitz constant κ_G.

By Lemma 2.1, (3.3) and (3.4) has a unique solution $x^* \in C^1[0, \infty)$. From (3.5), $y^*(t, \xi) := \widehat{y}(x^*(t), t, \xi)$ is (locally) Lipschitz continuous in $[0, \infty)$ a.e. $\xi \in \Xi$.

Case (ii) Since $M(\xi)$ is a P-matrix, the box-constrained linear VI has a unique solution for any fixed t, x, ξ [15, Section 3.5.2]. For any given (t, x) and (t', x'), the unique solutions v and v' can be expressed in terms of the median operator $\mathrm{mid}(\cdot)$ respectively:

$$v - \mathrm{mid}\big(l_\xi, u_\xi, x - \Psi(t, \xi, x, v)\big) = 0, \quad v' - \mathrm{mid}\big(l_\xi, u_\xi, x' - \Psi(t', \xi, x', v')\big) = 0, \tag{3.9}$$

[1] There is a minor mistake in the proof of [15, Theorem 2.3.3(ii)]. Here we give a modified proof of [15, Theorem 2.3.3(ii)] and derive the following inequality

$$\|v - v^*\|_2 \leq \Big(\frac{\kappa + 1}{c}\big\|v - \Pi_C(v - F(v))\big\|_2\Big)^{\frac{1}{\varsigma - 1}}, \qquad \forall\, v \in C, \tag{3.6}$$

where $C \subset \mathbb{R}^n$ is a closed convex set, $\varsigma \geq 2$, $c > 0, \kappa > 0$, $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$(u - v)^\top(F(u) - F(v)) \geq c\|u - v\|_2^\varsigma, \ \forall\, u, v \in C, \quad \text{and} \quad \|F(u) - F(v)\|_2 \leq \kappa\|u - v\|_2,$$

and v^* is the unique solution of the VI: $0 \in F(u) + \mathcal{N}_C(u)$. For a given $v \in C$, let $r = v - \Pi_C(v - F(v))$. Following the same argument as in [15, Theorem 2.3.3(iii)], we have $(v^* - v + r)^\top(F(x) - r) \geq 0$ and $(v - r - v^*)^\top F(v^*) \geq 0$. Adding these inequalities and using the conditions on F, we deduce

$$c\|v - v^*\|_2^\varsigma \leq (v - v^*)^\top(F(v) - F(v^*)) \leq r^\top(F(v) - F(v^*)) - r^\top r - (v^* - v)^\top r \leq \|r\|_2 \cdot \kappa \cdot \|v - v^*\|_2 + \|r\|_2 \cdot \|v - v^*\|_2.$$

This gives rise to (3.6).

where we recall that $\Psi(t,\xi,x,y) = M(\xi)y + \psi(t,\xi,x)$. Following the same argument in the proof of [8, Lemma 2.1], there exists a vector $\widehat{d} \in [0,1]^m$ (depending on v and v') such that $(I-D)(v-v') + D\Big(M(\xi)(v-v') + \psi(t,\xi,x) - \psi(t',\xi,x')\Big) = 0$, where $D := \mathrm{diag}(\widehat{d})$. This implies $\Big(I - D + DM(\xi)\Big)(v-v') = -D\big(\psi(t,\xi,x) - \psi(t',\xi,x')\big)$. Since $M(\xi)$ is a P-matrix a.e. $\xi \in \Xi$, it is known that $I - D + DM(\xi)$ is also a P-matrix [10, Theorem 2.2] and thus invertible a.e. $\xi \in \Xi$. Define $\beta_\infty(M(\xi)) := \max_{\widehat{d}\in[0,1]^m} \|(I - D + DM(\xi))^{-1}D\|_\infty$, and $c(M(\xi)) := \min_{\|z\|_\infty=1}\big(\max_{1\le i\le m} z_i(M(\xi)z)_i\big)$. It is known that $\beta_\infty(M(\xi)) \le \frac{1}{c(M(\xi))}$ [10, Theorem 2.2]. Hence, by (3.2), $\beta_\infty(M(\xi)) \le \frac{1}{\widetilde{\eta}}$ a.e. $\xi \in \Xi$. Further, it follows from [8, Lemma 2.1] and [13, Lemma 7.3.10] that

$$
\begin{aligned}
\|v - v'\|_\infty &\le \beta_\infty(M(\xi))\big\|\psi(t,\xi,x) - \psi(t',\xi,x')\big\|_\infty \le \frac{1}{c(M(\xi))}\big\|\psi(t,\xi,x) - \psi(t',\xi,x')\big\|_\infty \\
&\le \frac{1}{\widetilde{\eta}}\big\|\psi(t,\xi,x) - \psi(t',\xi,x')\big\|_\infty = \frac{1}{\widetilde{\eta}}\big\|\Psi(t,\xi,x,v') - \Psi(t',\xi,x',v')\big\|_\infty \qquad (3.10)\\
&\le \frac{\kappa_{\Psi(\xi)}}{\widetilde{\eta}}\big\|(t,x) - (t',x')\big\|_\infty, \quad \text{a.e. } \xi \in \Xi.
\end{aligned}
$$

Therefore, G is (globally) Lipschitz continuous with the Lipschitz constant $\kappa_G := \mathbb{E}[\kappa_\Phi(\xi)(1 + \frac{\kappa_\Psi(\xi)}{\widetilde{\eta}})] < \infty$ (with respect to $\|\cdot\|_\infty$), by the same argument in the proof for Case (i). ∎

Remark 3.1. If $\eta = 0$ in (3.1) or $\widetilde{\eta} = 0$ in (3.2), the solution set of each second stage problem may be empty or has multiple solutions. In the latter case, we can use the regularization approach by $\Psi_\epsilon(t,\xi,x,z) = \Psi(t,\xi,x,z) + \epsilon z$ with $\epsilon > 0$ (see for example [9]). The function Ψ_ϵ satisfies Assumption 3.1 and each second stage problem has a unique solution $y_\epsilon(t,\xi)$ for any $\epsilon > 0$, which converges to a solution of the original problem as $\epsilon \downarrow 0$ for any fixed t, ξ.

Let $\{\xi^i\}$ with $\xi^i = \xi^i(\omega), \forall\, i \in \mathbb{N}$ be an independent identically distributed (iid) sequence of d-dimensional random vectors defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider the sample average approximation (SAA) of (3.3)-(3.4) as follows:

$$
\dot{x}(t) = \gamma \cdot \Big\{\Pi_X\Big(x(t) - G^N(t, x(t))]\Big) - x(t)\Big\}, \qquad (3.11)
$$

$$
x(0) = x_0, \qquad (3.12)
$$

where

$$
G^N(t, x(t)) = \frac{\sum_{i=1}^N \Phi(t,\xi^i, x(t), \widehat{y}(x(t), t, \xi^i))}{N}
$$

with $\widehat{y}(x(t), t, \xi^i)$ being the unique solution of the variational inequality

$$
0 \in \Psi(t, \xi^i, x(t), y) + \mathcal{N}_{C_{\xi^i}}(y).
$$

Since all $\xi^i = \xi^i(\omega)$ are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we view $G^N(t,x)$ as the random function $G^N(t,x,\omega)$ on $\mathbb{R}\times\mathbb{R}^n\times\Omega$. By a similar argument in Lemma 3.2, G^N is (globally) Lipschitz continuous in (t,x). Hence, the DSVI (3.11)-(3.12) has a unique solution $x^N \in C^1[0,\infty)$.

In what follows, we prove the uniform convergence of $\{x^N\}$ to the solution of (3.3)-(3.4) with probability 1 for either of the two cases of Assumption 3.1. Toward this end, we recall some results and introduce more notions. For either case of Assumption 3.1, let $\widehat{\Phi}(t,x,\xi) := \Phi(t,\xi,x,\widehat{y}(x,t,\xi))$. It is shown in the proof of Lemma 3.2 that in either case, there exists a measurable function $\kappa_c : \Xi \to \mathbb{R}_+$ such that

$$\left\|\widehat{\Phi}(t,x,\xi) - \widehat{\Phi}(t',x',\xi)\right\| \le \kappa_c(\xi)\,\|(t,x)-(t',x')\|, \quad \text{a.e.} \quad \xi \in \Xi. \tag{3.13}$$

In particular, for case (i), $\kappa_c(\xi) := \kappa_\Phi(\xi)\big(1+\eta'(\xi)\kappa_\Psi(\xi)\big)$ with respect to $\|\cdot\|_2$, where $\eta'(\xi) = (1+\kappa_\Psi(\xi))/\eta$; for case (ii), $\kappa_c(\xi) := \kappa_\Phi(\xi)(1+\frac{\kappa_\Psi(\xi)}{\widetilde{\eta}})$ with respect to $\|\cdot\|_\infty$. Under the assumptions of Lemma 3.2, $\mathbb{E}[\kappa_c(\xi)] < \infty$ for both the cases.

We define moment generating functions for κ_c and $\widehat{\Phi}_i, i=1,\dots,n$ as follows. Let

$$M_{\kappa_c}(\tau) := \mathbb{E}[\exp(\tau\kappa_c(\xi))], \qquad M^i_{(t,x)}(\tau) := \mathbb{E}[\exp(\tau(\widehat{\Phi}_i(t,x,\xi)], \quad i=1,\dots,n.$$

Recall that the moment generating function $M_\chi(\tau) := \mathbb{E}[e^{\tau\chi}]$ of a (real-valued) random variable χ is finite-valued in a neighborhood of zero if there exists a constant $\varepsilon > 0$ such that for any $\tau \in (-\varepsilon,\varepsilon)$, $M_\chi(\tau) < \infty$. We make the following assumption: on M_{κ_c} and $M^i_{(t,x)}$ for any $(t,x) \in [0,T] \times X$:

$$(\mathrm{M}): \quad M_{\kappa_c} \text{ and all } M^i_{(t,x)} \text{ are finite valued in a neighborhood of zero.}$$

REMARK 3.2. Obviously, if Ξ is a compact support and κ_c, $\widehat{\Phi}_i, i=1,\dots,n$ are continuous in $\xi \in \Xi$ for any given (t,x), then the condition (M) holds; see Example 4.1 for an example. For a general case where the support Ξ is unbounded, one may establish the decay rate of moments using the probability density function of ξ and properties of κ_c and $\widehat{\Phi}_i$'s to show that their moment generating functions are finite valued near zero. Further, one can approximate an unbounded support by a compact support and show that the error between the original DSVI solution and its approximate solution can be made arbitrarily small by choosing a suitable approximating compact support; see [7] for the related results.

We first consider a convex compact set X.

THEOREM 3.1. *Suppose that the assumptions of Lemma 3.2 hold, X is a convex compact set, $x(0) \in X$, $T > 0$, and $\gamma > 0$. Let x^* be the unique solution of (3.3)-(3.4) and $\theta = \frac{1+\kappa_G}{\exp(\gamma(1+\kappa_G)T)-1}$. Then the following statements hold for either of the two cases in Assumption 3.1:*

(i) $\{x^N\}$ converges to x^* uniformly on $[0,T]$ w.p. 1;

(ii) Suppose, in addition, that the assumption (M) holds. Then for any constant $\epsilon > 0$, there exist positive constants $\rho(\theta\epsilon)$ and $\sigma(\theta\epsilon)$, independent of N, such that

$$\mathbb{P}\Big\{\sup_{t\in[0,T]} \|x^N(t) - x^*(t)\| \ge \epsilon\Big\} \le \rho(\theta\epsilon)\exp\big(-N\sigma(\theta\epsilon)\big). \tag{3.14}$$

Proof. (i) We first show that $G^N(\cdot,\cdot)$ converges uniformly to $G(\cdot,\cdot)$ on $[0,T]\times X$ with probability 1. For this purpose, we establish the following two claims.

Claim (a): $\widehat{\Phi}(t,x,\xi)$ is continuous in (t,x) at each (t,x) a.e. $\xi \in \Xi$.

To prove Claim (a), note that in both cases of Assumption 3.1, Φ is Lipschitz continuous in (t,x,y) and $\widehat{y}(x,t,\xi)$ is Lipschitz in (x,t) as shown in Lemma 3.2 a.e. $\xi \in \Xi$. Hence, $\widehat{\Phi}(t,x,\xi) := \Phi(t,\xi,x,\widehat{y}(x,t,\xi))$ is continuous in (t,x) a.e. $\xi \in \Xi$.

Claim (b): Each element of $\widehat{\Phi}(t,x,\xi)$ is dominated by a nonnegative integrable function $h(\xi)$, i.e., $h(\xi)$ is a nonnegative measurable function with $\mathbb{E}[h(\xi)] < +\infty$ such that for any $(t,x) \in [0,T]\times X$, $|\widehat{\Phi}_i(t,x,\xi)| \le h(\xi)$ for each $i=1,\dots,n$.

To show Claim (b), consider case (i) of Assumption 3.1 first. It follows from (3.7) that for any $(t,x),(t',x') \in [0,T]\times X$, $\|\widehat{\Phi}(t,x,\xi)-\widehat{\Phi}(t',x',\xi)\|_2 \le \kappa_c(\xi)\cdot\|(t,x)-(t',x')\|_2$. Since X and $[0,T]$ are bounded, there exists a constant $\nu>0$ such that for any $(t,x),(t',x')\in[0,T]\times X$, $\|\widehat{\Phi}(t,x,\xi)-\widehat{\Phi}(t',x',\xi)\|_2\le \nu\kappa_c(\xi)$. Furthermore, choose an arbitrary $(t^\diamond,x^\diamond)\in[0,T]\times X$. Since $\widehat{\Phi}(t^\diamond,x^\diamond,\xi)$ is measurable and its expectation is of finite value, $\|\widehat{\Phi}(t^\diamond,x^\diamond,\xi)\|_2$ is also measurable and $\mathbb{E}[\|\Phi(t^\diamond,x^\diamond,\xi)\|_2]<+\infty$. Define the nonnegative measurable function $h(\xi):=\|\widehat{\Phi}(t^\diamond,x^\diamond,\xi)\|_2+\nu\kappa_c(\xi)$. Clearly, for any $(t,x)\in[0,T]\times X$, we have

$$\|\widehat{\Phi}(t,x,\xi)\|_2 \le \|\widehat{\Phi}(t^\diamond,x^\diamond,\xi)\|_2+\|\widehat{\Phi}(t,x,\xi)-\widehat{\Phi}(t^\diamond,x^\diamond,\xi)\|_2\le h(\xi), \qquad \text{a.e. } \xi\in\Xi.$$

From the assumptions of Lemma 3.2, we have $\mathbb{E}[h(\xi)]=\mathbb{E}[\|\widehat{\Phi}(t^\diamond,x^\diamond,\xi)\|_2]+\nu\kappa_G<\infty$, where κ_G is given in (3.8). Consequently, each element of $\widehat{\Phi}(t,x,\xi)$ is dominated by the nonnegative integrable function $h(\xi)$. The same result can be shown for case (ii) of Assumption 3.1 using the similar argument in Lemma 3.2.

In view of the above two claims and the fact that the sample $\{\xi^1,\dots,\xi^N\}$ is iid, we deduce via [28, Theorem 7.48] that for each $i=1,\dots,n$, $G_i^N(t,x)$ converges uniformly to $G_i(t,x)$ on $[0,T]\times X$ with probability 1, i.e., $\sup_{(s,x)\in[0,T]\times X}|G_i^N(s,x)-G_i(s,x)|\to 0$ w.p. 1. Hence, $\sup_{(s,x)\in[0,T]\times X}\|G^N(s,x)-G(s,x)\|\to 0$ w.p. 1.

Next, we use the above results to establish the uniform convergence of $\{x^N\}$ to x^*. It follows from Lemma 3.2 that $x^N\in C^1[0,T]$ and from (i) of Lemma 2.2 that $x^N(t)\in X$ for all $t\in[0,T]$ and N. Further, by Lemma 2.2, we have, for each N,

$$x^N(t)=e^{-\gamma t}x_0+\int_0^t e^{-\gamma(t-\tau)}\gamma\,\Pi_X\big[x^N(\tau)-G^N(\tau,x^N(\tau))\big]\,d\tau,$$
$$x^*(t)=e^{-\gamma t}x_0+\int_0^t e^{-\gamma(t-\tau)}\gamma\,\Pi_X\big[x^*(\tau)-G(\tau,x^*(\tau))\big]\,d\tau.$$

Therefore, using the κ_G derived in the proof of Lemma 3.2 for either of the two cases in Assumption 3.1, we have, for any $t\in[0,T]$,

$$\begin{aligned}
&\big\|x^N(t)-x^*(t)\big\|\\
&\le \int_0^t e^{-\gamma(t-\tau)}\gamma\big\|x^N(\tau)-G^N(\tau,x^N(\tau))-x^*(\tau)-G(\tau,x^*(\tau))\big\|d\tau\\
&\le \gamma\int_0^t\Big(\big\|x^N(\tau)-x^*(\tau)\big\|+\big\|G(\tau,x^N(\tau))-G(\tau,x^*(\tau))\big\|+\big\|G^N(\tau,x^N(\tau))-G(\tau,x^N(\tau))\big\|\Big)d\tau\\
&\le \gamma\int_0^t\Big((1+\kappa_G)\big\|x^N(\tau)-x^*(\tau)\big\|+\sup_{(s,x)\in[0,T]\times X}\big\|G^N(s,x)-G(s,x)\big\|\Big)d\tau.
\end{aligned}$$

Since $\sup_{(s,x)\in[0,T]\times X}\|G^N(s,x)-G(s,x)\|\to 0$ w.p. 1, we have that for all sufficiently large N, $\sup_{(s,x)\in[0,T]\times X}\|G^N(s,x)-G(s,x)\|<\infty$ a.e. $\xi\in\Xi$. Using [9, Lemma 2.6] and the Grönwall inequality [12, pp. 146], we obtain that for all large N and for any $t\in[0,T]$,

$$\big\|x^N(t)-x^*(t)\big\| \le \frac{\exp(\gamma(1+\kappa_G)t)-1}{1+\kappa_G}\sup_{(s,x)\in[0,T]\times X}\big\|G^N(s,x)-G(s,x)\big\|.$$

Recalling that $\theta=\frac{1+\kappa_G}{\exp(\gamma(1+\kappa_G)T)-1}$, we thus have, for all large N,

$$\theta\sup_{t\in[0,T]}\|x^N(t)-x^*(t)\|\le\sup_{(s,x)\in[0,T]\times X}\big\|G^N(s,x)-G(s,x)\big\|. \tag{3.15}$$

Since $\sup_{(s,x)\in[0,T]\times X}\|G^N(s,x)-G(s,x)\|\to 0$ w.p. 1, we conclude that $\{x^N\}$ uniformly converges to x^* on $[0,T]$ w.p. 1.

(ii) In view of the above proof for part (i), it suffices to establish the uniform exponential bound

$$\mathbb{P}\Big\{\sup_{(t,x)\in[0,T]\times X}\|G^N(t,x)-G(t,x)\|\ge\epsilon\Big\}$$

for any constant $\epsilon>0$. Toward this end, consider Case (i) of Assumption 3.1 first. Under the condition (M), M_{κ_c} and all $M^i_{(t,x)}$ are finite valued in a neighborhood of zero at any $(t,x)\in[0,T]\times X$. Since each $G_i(t,x)$ is finite valued at any $(t,x)\in[0,T]\times X$, it is easy to see that for any $(t,x)\in[0,T]\times X$ and each $i=1,\dots,n$, the moment generating function $\mathbb{E}[\exp(\tau(\widehat{\Phi}_i(t,x,\xi)-G_i(t,x))]$ is finite valued in a neighborhood of zero. Further, for each $i=1,\dots,n$,

$$\big|\widehat{\Phi}_i(t,x,\xi)-\widehat{\Phi}_i(t',x',\xi)\big|\;\le\;\big\|\widehat{\Phi}(t,x,\xi)-\widehat{\Phi}(t',x',\xi)\big\|_2\;\le\;\kappa_c(\xi)\,\|(t,x)-(t',x')\|_2$$

for all $\xi\in\Xi$ and any $(t,x),(t',x')\in[0,T]\times X$. Consequently, it follows from [28, Theorem 7.65] that for any constant $\epsilon>0$, there exist positive constants $\rho(\epsilon)$ and $\sigma(\epsilon)$, independent of N, such that

$$\mathbb{P}\Big\{\sup_{(t,x)\in[0,T]\times X}\|G^N(t,x)-G(t,x)\|_2\ge\epsilon\Big\}\;\le\;\rho(\epsilon)\exp(-N\sigma(\epsilon)). \tag{3.16}$$

In light of (3.15), we obtain

$$\mathbb{P}\Big\{\sup_{t\in[0,T]}\|x^N(t)-x^*(t)\|_2\ge\epsilon\Big\}\;\le\;\rho(\theta\epsilon)\exp(-N\sigma(\theta\epsilon)).$$

The similar result can be established for Case (ii) of Assumption 3.1 where $\|\cdot\|_\infty$ is used. ◻

Using (ii) of Lemma 2.2 and Theorem 3.1, we have the following corollary.

COROLLARY 3.2. *If X is a bounded affine set and $x(0)\in X$, then Theorem 3.1 holds with* $\theta=\frac{1+\kappa_G}{\exp(|\gamma|(1+\kappa_G)T)-1}$.

To handle an unbounded closed convex set X, we make the following assumption:

A.3 (i) There exist constants $L_\Phi>0$ and $L_\Psi>0$ such that $\kappa_\Phi(\xi)\le L_\Phi$ and $\kappa_\Psi(\xi)\le L_\Psi$ a.e. $\xi\in\Xi$; and
(ii) there exist $t^\diamond, x^\diamond$ and a constant $\beta>0$ such that $\|\Phi(t^\diamond,\xi,x^\diamond,\widehat{y}(x^\diamond,t^\diamond,\xi))\|\le\beta$ a.e. $\xi\in\Xi$, where $\widehat{y}(x^\diamond,t^\diamond,\xi)$ is a solution of the VI: $0\in\Psi(t^\diamond,\xi,x^\diamond,y)+\mathcal{N}_{C_\xi}(y)$.

By **A.3**, κ_Ψ,κ_Φ and $\|\Phi(t^\diamond,\cdot,x^\diamond,\widehat{y}(x^\diamond,t^\diamond,\cdot))\|$ are essentially bounded. Furthermore, $\mathbb{E}[\kappa_\Phi(\xi)]\le L_\Phi<\infty$, $\mathbb{E}[\kappa_\Phi(\xi)\kappa_\Psi(\xi)]\le L_\Phi\cdot L_\Psi<\infty$, and $\mathbb{E}[\kappa_\Phi(\xi)\kappa^2_\Psi(\xi)]\le L_\Phi\cdot(L_\Psi)^2<\infty$. Hence, Lemma 3.2 holds.

REMARK 3.3. Sufficient conditions for **A.3** to hold can be established for specific classes of DSVIs. For example, consider the DSLCP in (2.1). We show below that **A.3** holds if $\|B(\xi)\|$, $\|M(\xi)\|$, $\|N(\xi)\|$ and $\|q_2(\xi)\|$ are essentially bounded and Case (ii) of Assumption 3.1 holds. Clearly, if $\|B(\xi)\|,\|M(\xi)\|,\|N(\xi)\|$ are essentially bounded, then κ_Φ and κ_Ψ are essentially bounded such that (i) of **A.3** holds. We next show that (ii) of **A.3** holds. Let $x^\diamond=0$. The SLCP in (2.1) becomes: $0\le y\perp M(\xi)y+q_2(\xi)\ge 0$.

Since $M(\xi)$ is a P-matrix a.e. $\xi \in \Xi$, the SLCP has a unique solution $y(\xi)$ for a given $q_2(\xi)$. Particularly, the solution $y(\xi) = 0$ when $q_2(\xi) = 0$. Therefore, by (3.10), $\|y(\xi) - 0\|_\infty \leq \frac{1}{\widetilde{\eta}}\|q_2(\xi) - 0\|_\infty$ a.e. $\xi \in \Xi$, where $\widetilde{\eta} > 0$ is a constant independent of ξ given in (3.2). Hence, $\|\Phi(\xi, x^\diamond, \widehat{y}(x^\diamond, \xi))\|_\infty = \|B(\xi)\widehat{y}(x^\diamond, \xi) + q_1\|_\infty \leq \|B(\xi)\|_\infty \cdot \frac{1}{\widetilde{\eta}}\|q_2(\xi)\|_\infty + \|q_1\|_\infty$ a.e. $\xi \in \Xi$. Thus $\|\Phi(\xi, x^\diamond, \widehat{y}(x^\diamond, \xi))\|_\infty$ is essentially bounded such that (ii) of **A.3** holds. Consequently, **A.3** holds. This result also holds when the assumptions of Case (ii) of Assumption 3.1 are replaced by those of Case (i). In fact, when Case (i) holds for the DSLCP, $M(\xi)$ satisfies $z^T M(\xi) z \geq \eta \|z\|_2^2$ a.e. $\xi \in \Xi$. In view of $\max_{i=1,\ldots,m} z_i(M(\xi)z)_i \geq \frac{z^T M(\xi) z}{m}$, we see that (3.2) in Case (ii) holds with $\widetilde{\eta} := \frac{\eta}{m} > 0$. Hence, the desired result follows. Furthermore, consider the DSVI satisfying the conditions in Case (i). Suppose Ξ is a compact support. If κ_Ψ, κ_Φ are continuous in ξ, then they are essentially bounded on Ξ. Besides, as indicated below Comment (ii.2), if $C_\xi \equiv C$ for a closed convex set C and Ψ, Φ are continuous in ξ on Ξ for any fixed (t, x, y), then the unique solution $\widehat{y}(x, t, \cdot)$ is continuous in ξ using the techniques for parametric VIs [15, Section 5.1]. Thus for any fixed $(x^\diamond, t^\diamond)$, $\|\Phi(t^\diamond, \xi, x^\diamond, \widehat{y}(x^\diamond, t^\diamond, \xi))\|$ is continuous in ξ and attains a uniform upper bound on the compact support Ξ. Therefore, **A.3** holds.

Under **A.3** and Case (i) of Assumption 3.1 (i.e., Φ is strongly monotone on C_ξ uniformly in ξ, where $\eta > 0$ is independent of ξ), equation (3.5) shows that for any (t, x) and (t', x') and a.e. $\xi \in \Xi$,

$$\big\|\widehat{y}(x, t, \xi) - \widehat{y}(x', t', \xi)\big\|_2 \leq \eta'(\xi)\kappa_\Psi(\xi)\,\|(t, x) - (t', x')\|_2,$$

where $\eta'(\xi) := (1 + \kappa_\Psi(\xi))/\eta$. Hence, $\eta'(\xi) \leq (1 + L_\Psi)/\eta$ a.e. $\xi \in \Xi$. Moreover, for any iid sample $\{\xi^1, \ldots, \xi^N\}$ of the random vector $\xi \in \Xi$,

$$(3.17)\quad \|G^N(t, x) - G^N(t', x')\|_2 \leq \frac{\sum_{i=1}^N \kappa_\Phi(\xi^i)[1 + \eta'(\xi^i)\kappa_\Psi(\xi^i)]}{N}\,\big\|(t, x) - (t', x')\big\|_2.$$

Let $L := L_\Phi \times [1 + \frac{1+L_\Psi}{\eta} L_\Psi] > 0$. By **A.3**, we see that $\|G^N(t, x) - G^N(t', x')\|_2 \leq L\|(t, x) - (t', x')\|_2$ independent of N. Similar results can be obtained for Case (ii) of Assumption 3.1.

THEOREM 3.2. *Suppose that* **A.3** *and the assumptions of Lemma 3.2 hold, and* $\gamma > 0$*. Let* x^* *be the unique solution of (3.3)-(3.4). Then for any given* $T > 0$ *and any initial condition* $x_0 \in \mathbb{R}^n$*, the sequence* $\{x^N\}$ *that converges to* x^* *uniformly on* $[0, T]$ *with probability 1 for either of the two cases in Assumption 3.1.*

Proof. We consider Case (i) of Assumption 3.1 only, since Case (ii) follows from an almost identical argument. Consider an arbitrary constant $T > 0$ and an arbitrary initial condition $x_0 \in \mathbb{R}^n$. Let $f^N(t, x)$ denote the right hand side of (3.11) for each N, i.e.,

$$f^N(t, x) := \gamma \cdot \big\{\Pi_X\big[x - G^N(t, x)\big] - x\big\}.$$

Similar to $G^N(t, x)$, we view $f^N(t, x)$ as the random function $f^N(t, x, \omega)$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Since $G^N(\cdot, \cdot)$ has the uniform Lipschitz constant $L > 0$ independent of N with probability 1, it is easy to see that $f^N(t, x)$ has a uniform Lipschitz constant $\widetilde{L} > 0$ regardless of N with probability 1. Further, since

$$\begin{aligned} x^N(t, x_0) &= x_0 + \int_0^t f^N(\tau, x^N(\tau, x_0))d\tau \\ &= x_0 + \int_0^t f^N(0, x_0)d\tau + \int_0^t \big[f^N(\tau, x^N(\tau, x_0)) - f^N(0, x_0)\big]d\tau, \end{aligned}$$

we have for each $t \in [0,T]$,

$$\|x^N(t,x_0) - x_0\|_2 \leq \|f^N(0,x_0)\|_2 \times T + \widetilde{L}\int_0^t \|(x^N(\tau,x_0),\tau) - (x_0,0)\|_2 d\tau$$

$$\leq \big(\|f^N(0,x_0)\|_2 + \widetilde{L}\big) \times T + \widetilde{L}\int_0^t \|x^N(\tau,x_0) - x_0\|_2 d\tau. \tag{3.18}$$

We claim that $\|f^N(0,x_0)\|_2$ is uniformly bounded regardless of N with probability 1. To show it, we first show that $\|G^N(0,x_0)\|_2$ is uniformly bounded regardless of N with probability 1, where

$$G^N(0,x_0) \;=\; \frac{\sum_{i=1}^N \Phi(0,\xi^i,x_0,\widehat{y}(x_0,0,\xi^i))}{N}.$$

In fact, due to (i) of **A.3**, we have that a.e. $\xi \in \Xi$,

$$\begin{aligned}
&\big\|\Phi(0,\xi,x_0,\widehat{y}(x_0,0,\xi)) - \Phi(t^\diamond,\xi,x^\diamond,\widehat{y}(x^\diamond,t^\diamond,\xi))\big\|_2 \\
&\quad\leq L_\Phi\big\|(-t^\diamond, x_0 - x^\diamond, \widehat{y}(x_0,0,\xi) - \widehat{y}(x^\diamond,t^\diamond,\xi))\big\|_2 \\
&\quad\leq L_\Phi\Big(|t^\diamond| + \|x_0 - x^\diamond\|_2 + \|\widehat{y}(x_0,0,\xi) - \widehat{y}(x^\diamond,t^\diamond,\xi)\|_2\Big) \\
&\quad\leq L_\Phi\Big(|t^\diamond| + \|x_0 - x^\diamond\|_2 + \eta'(\xi)\kappa_\Psi(\xi)\big(\|x_0 - x^\diamond\|_2 + |t^\diamond|\big)\Big) \\
&\quad\leq L_\Phi\Big(|t^\diamond| + \|x_0 - x^\diamond\|_2 + \frac{1+L_\Psi}{\eta}L_\Psi(\|x_0 - x^\diamond\|_2 + |t^\diamond|)\Big),
\end{aligned}$$

where the second to the last inequality follows from (3.5).

By (ii) of **A.3**, $\|\Phi(t^\diamond,\xi,x^\diamond,\widehat{y}(x^\diamond,t^\diamond,\xi))\| \leq \beta$ a.e. $\xi \in \Xi$. Hence, there exists a constant $\beta' > 0$ such that $\|\Phi(0,\xi,x_0,\widehat{y}(x_0,0,\xi))\|_2 \leq \beta'$ a.e. $\xi \in \Xi$. This shows that $\|G^N(0,x_0)\|_2 \leq \beta'$ regardless of N with probability 1. Further, for an arbitrary but fixed $z \in \mathbb{R}^n$, it is easy to see that

$$\begin{aligned}
\|\Pi_X(x_0 - G^N(0,x_0))\|_2 &\leq \|\Pi_X(x_0 - z)\|_2 + \|\Pi_X(x_0 - G^N(0,x_0)) - \Pi_X(x_0 - z)\|_2 \\
&\leq \|\Pi_X(x_0 - z)\|_2 + \|z - G^N(0,x_0)\|_2 \\
&\leq \|\Pi_X(x_0 - z)\|_2 + \|z\|_2 + \beta'
\end{aligned}$$

regardless of N and $\{\xi^i\}_{i=1}^N$. Hence, $\|f^N(0,x_0)\|_2$ is uniformly bounded regardless of N. Consequently, applying the Grönwall inequality [12, pp. 146] to (3.18), we see that there exists a constant $\gamma > 0$ such that $\|x^N(t,x_0) - x_0\|_2 \leq \gamma, \forall\, t \in [0,T]$ for all N with probability 1.

Let $\mathcal{D}$ be the closed 2-ball centered at x_0 with the radius γ. It is easy to show via a similar argument that $x^*(t,x_0) \in \mathcal{D}$ for all $t \in [0,T]$. Therefore, the sequence $\{x^N(t,x_0)\}_N$ is uniformly bounded in $C[0,T]$ with probability 1. By the similar argument for part (i) of Theorem 3.1, we have that

$$\sup_{(s,x)\in[0,T]\times\mathcal{D}} \|G^N(s,x) - G(s,x)\|_2 \to 0, \qquad \text{w.p. } 1,$$

and, for all large N,

$$\theta \sup_{t\in[0,T]} \|x^N(t) - x^*(t)\|_2 \;\leq\; \sup_{(s,x)\in[0,T]\times\mathcal{D}} \big\|G^N(s,x) - G(s,x)\big\|_2,$$

where $\theta = \frac{1+\kappa_G}{\exp(\gamma(1+\kappa_G)T)-1}$. This leads to the desired result. ∎

4. The Time-stepping EDIIS Method. In this section, we propose a time-stepping Energy Direct Inversion on the Iterative Subspace (EDIIS) method [4] for solving (3.11)-(3.12) on $[0, T]$ under Assumption 3.1.

Let the step size be $h = T/\nu$ for a positive integer ν, and $t_j = jh$, $j = 1, \ldots, \nu$. The time-stepping method in a backward Euler type for (3.11) on $[0, T]$ yields the following scheme: for each $j = 1, \ldots, \nu$,

$$x_j = x_{j-1} + h\gamma\Big(\Pi_X(x_j - G^N(t_j, x_j)) - x_j\Big), \tag{4.1}$$

where, for a given sample $\{\xi^1, \ldots, \xi^N\}$,

$$G^N(t_j, x_j) = \tfrac{1}{N}\textstyle\sum_{i=1}^N \Phi(t_j, \xi^i, x_j, \widehat{y}(x_j, t_j, \xi^i))$$

and $\widehat{y}(x_j, t_j, \xi^i)$ is the unique solution of the VI

$$0 \in \Psi(t_j, \xi^i, x_j, v) + \mathcal{N}_{C_{\xi^i}}(v),$$

and $x_0 = x(0)$. Let $\bar{x} = \frac{1}{1+h\gamma}x_{j-1}$, and $\mu = \frac{h\gamma}{1+h\gamma}$. At each $\bar{t} = t_j$, $(x_j^\top, \widehat{y}(x_j, t_j, \xi^1)^\top, \ldots, \widehat{y}(x_j, t_j, \xi^N)^\top)^\top \in \mathbb{R}^{n+mN}$ is a solution of the following VI:

$$x = \bar{x} + \mu\Pi_X(x - G^N(\bar{t}, x)), \tag{4.2}$$

$$0 \in \Psi(\bar{t}, \xi^i, x, y_i) + \mathcal{N}_{C_{\xi^i}}(y_i), \quad i = 1, \ldots, N. \tag{4.3}$$

Problem (4.2) can be treated as a fixed point problem as shown shortly, and problem (4.3) can be solved in parallel to obtain $\widehat{y}(x_j, t_j, \xi^i)$, $i = 1, \ldots, N$ once x_j is found. The EDIIS algorithm [4] is a modification of Anderson acceleration and widely used in quantum chemistry. Since the most computational cost is to get the function value $G^N(\bar{t}, x)$, we use the EDIIS algorithm to optimize the utility of computed function values $G^N(\bar{t}, x^k)$ in the last few steps. We present the EDIIS(ℓ) algorithm for the VI (4.2)-(4.3) in Algorithm 4.1, where ℓ is the depth of iterations.

Recall that for any iid sample $\{\xi^1, \ldots, \xi^N\}$ of the random variable $\xi \in \Xi$, it is shown in (3.17) that for Case (i) of Assumption 3.1,

$$\|G^N(t, x) - G^N(t', x')\|_2 \le \kappa_{G^N}\big\|(t, x) - (t', x')\big\|_2,$$

where $\kappa_{G^N} := \frac{\sum_{i=1}^N \kappa_\Phi(\xi^i)[1+\eta'(\xi^i)\kappa_\Psi(\xi^i)]}{N}$. Similarly, for Case (ii) of Assumption 3.1,

$$\|G^N(t, x) - G^N(t', x')\|_\infty \le \kappa_{G^N}\big\|(t, x) - (t', x')\big\|_\infty,$$

where $\kappa_{G^N} := \frac{\sum_{i=1}^N \kappa_\Phi(\xi^i)[1+\frac{\kappa_\Psi(\xi^i)}{\widetilde{\eta}}]}{N}$.

THEOREM 4.1. *Assume that one of (i) and (ii) in Assumption 3.1 holds, $\gamma > 0$, $\mu(1 + \kappa_{G^N}) < 1$, and $x_0 \in X$. Then the following statements hold.*

(i) The VI (4.2)-(4.3) has a unique solution $(x_j^\top, \widehat{y}(x_j, t_j, \xi^1)^\top, \ldots, \widehat{y}(x_j, t_j, \xi^N)^\top)^\top \in \mathbb{R}^{n+mN}$;

(ii) The sequence $\{((x^k)^\top, (y_1^k)^\top, \ldots, (y_N^k)^\top)^\top\}$ generated by Algorithm 4.1 converges to the unique solution of the VI (4.2)-(4.3);

(iii) The time-stepping method (4.1) converges to the unique solution x^N of (3.11)-(3.12) as $h \to 0$ in the sense that $\|x_j - x^N(jh)\| = O(h)$ for all $j = 1, \ldots, \nu$.

Algorithm 4.1 EDIIS for the VI (4.2)-(4.3)
Initial step Choose $x^0 = x_{j-1} \in X$, $\bar{x} = \frac{1}{1+h\gamma} x_{j-1}$ and $\bar{t} = t_j$.

$$\text{Find } y_i^0 \text{ such that } 0 \in \Psi(\bar{t}, \xi^i, x^0, y_i^0) + \mathcal{N}_{C_{\xi^i}}(y_i^0), \quad i = 1, \dots, N. \tag{4.4}$$

$$\text{Set} \quad \begin{aligned} &G^N(\bar{t}, x^0) = \tfrac{1}{N} \textstyle\sum_{i=1}^N \Phi(\bar{t}, \xi^i, x^0, y_i^0), \\ &x^1 = \bar{x} + \mu \Pi_X\big(x^0 - G^N(\bar{t}, x^0)\big), \quad F_0 = x^1 - x^0. \end{aligned} \tag{4.5}$$

EDIIS For $k \geq 1$: choose $\ell_k \leq \min\{\ell, k\}$.

$$\text{Find} \quad \alpha \in \ \operatorname{argmin} \|\textstyle\sum_{\tau=0}^{\ell_k} \alpha_\tau F_{k-\ell+\tau}\| \ \text{ s.t.} \textstyle\sum_{\tau=0}^{\ell_k} \alpha_\tau = 1, \ \alpha_\tau \geq 0, \tau = 0, \dots, \ell_k. \tag{4.6}$$

$$\text{Set} \quad \begin{aligned} x^{k+1} &= \bar{x} + \mu \textstyle\sum_{\tau=0}^{\ell_k} \alpha_\tau^k \Pi_X\big(x^{k-\ell+\tau} - G^N(\bar{t}, x^{k-\ell+\tau})\big), \\ F_k &= x^{k+1} - x^k. \end{aligned} \tag{4.7}$$

$$\text{Find } y_i^{k+1} \text{ such that} \quad 0 \in \Psi(\bar{t}, \xi^i, x^{k+1}, y_i^{k+1}) + \mathcal{N}_{C_{\xi^i}}(y_i^{k+1}), \quad i = 1, \dots, N. \tag{4.8}$$

$$\text{Set} \quad G^N(\bar{t}, x^{k+1}) = \tfrac{1}{N} \textstyle\sum_{i=1}^N \Phi(\bar{t}, \xi^i, x^{k+1}, y_i^{k+1}).$$

Proof. (i) Since X is a convex set and $x_0 \in X$, it can be proved by induction that for any $j = 1, \dots, \nu$ and any x, $\frac{1}{1+h\gamma} x_{j-1} + \frac{h\gamma}{1+h\gamma} \Pi_X(x - G^N(\bar{t}, x)) \in X$. Consider a fixed j. Then from Lemma 3.2, for any $x, v \in X$, we have

$$\big\|\bar{x} + \mu \Pi_X(x - G^N(\bar{t}, x)) - \bar{x} - \mu \Pi_X(v - G^N(\bar{t}, v))\big\| \ \leq \ \mu(1 + \kappa_{G^N}) \|x - v\|.$$

By the assumption that $\mu(1 + \kappa_{G^N}) < 1$, the mapping $\bar{x} + \mu \Pi_X(x - G^N(\bar{t}, x))$ is a contractive mapping in x on X. Hence (4.2) has a unique fixed point x_j in X. Therefore, by Lemma 3.2, $(x_j^\top, \widehat{y}(x_j, t_j, \xi^1)^\top, \dots, \widehat{y}(x_j, t_j, \xi^N)^\top)^\top$ is the unique solution of the VI (4.2)-(4.3) for each j.

(ii) From the construction of Algorithm 4.1, we have $\{x^k\} \subset X$. By the contraction property of $\bar{x} + \mu \Pi_X(x - G^N(\bar{t}, x))$ and [4, Theorem 2.1], we have that $\{x^k\}$ converges to the unique solution x_j of (4.2). From Lemma 3.2, y_i^k is the unique solution of (4.4) for $k = 0$ and (4.8) for $k \geq 1$. Moreover, there is a constant $c > 0$ such that $\|y_i^k - \widehat{y}(x_j, t_j, \xi^i)\| \leq c\|x^k - x_j\|$ for $i = 1, \dots, N$. Hence $\{y_i^k\}$ converges to $\widehat{y}(x_j, t_j, \xi^i)$, for $i = 1, \dots, N$.

(iii) Since $\widehat{y}(\cdot, t, \xi^i)$ is Lipschitz continuous [10, 11], the right hand side of (3.11) is Lipschitz continuous in (t, x). Hence it has a unique solution x^N. Moreover, it follows from the standard argument [9] that the time-stepping method (4.1) converges to the unique solution x^N of (3.11) as $h \to 0$ in the sense that $\|x_j - x^N(jh)\| = O(h)$ for all $j = 1, \dots, \nu$. ∎

For each $\nu \in \mathbb{N}$, let $x^{N,\nu}(\cdot)$ be a piecewise continuous function in t generated by linear interpolations of $x_j, j = 1, \dots, \nu$. By (iii) of the above theorem, it can be shown that the sequence $(x^{N,\nu})$ converges uniformly to the unique solution $x^N(\cdot)$ of (3.11)-(3.12) on $[0, T]$ as $\nu \to \infty$.

REMARK 4.1. If $\ell = 0$, Algorithm 4.1 is the Picard or fixed point method. Using $\ell > 0$ can accelerate the convergence [4]. Any norm can be used in the optimization problem in (4.6) without changes in (ii) of Theorem 4.1. If the 1-norm, ∞-norm or 2-norm is used, the optimization problem is either a linear programming or a quadratic programming, which can be solved easily and efficiently. If the function $\begin{pmatrix} \Phi(t_j,\xi^i,\cdot,\cdot) \\ \Psi(t_j,\xi^i,\cdot,\cdot) \end{pmatrix}$ is monotone, the progressive hedging method can be applied to solve (4.3) under the assumptions in Case (i) of Assumption 3.1 and $\gamma > 0$ [7, 27]. Comparing with the monotone assumption, $\mu(1+\kappa_{G^N}) < 1$ is much weaker. In fact, since $\mu \to 0$ as $h \to 0$, we have $\mu(1+\kappa_{G^N}) < 1$ for all sufficiently small h.

The following example illustrates the SAA and time-stepping EDIIS method.

EXAMPLE 4.1. Let $\gamma = 1$, $X = [-1,1]\times[-1,1] \subset \mathbb{R}^2$, $C_\xi = \mathbb{R}^3_+$, $x_0 = (0,1)^T \in X$, $\xi = (\xi_1,\xi_2)^T$, and

$$\Phi(t,\xi,x,y) = Ax + B(\xi)y + f(t),$$
$$\Psi(t,\xi,x,y) = M(\xi)y + Q(\xi)x + q(t,\xi),$$

where $A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, $B(\xi) = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & \xi_1 \end{pmatrix}$,

$$M(\xi) = \begin{pmatrix} 1 & 0 & 0 \\ \xi_1 & 1 & 0 \\ -1 & -1 & 0.1 \end{pmatrix}, \quad Q(\xi) = \begin{pmatrix} \xi_1 & 0 \\ 1 & \xi_2 \\ 1 & 1 \end{pmatrix},$$

$f(t) = (t,1)^T$ and $q(t,\xi) = (t\xi_1,\, \xi_2,\, 1)^T$.

Let $\Xi_N := \{\xi^1,\dots,\xi^N\}$ be independent identically distributed (i.i.d.) samples of $\xi = (\xi_1,\xi_2)^T$, where each ξ_i, $i = 1,2$, follows truncated normal distribution over $[-1,1]$, which is constructed from normal distribution with mean 0 and standard deviation σ independently. Since $\Xi = [-1,1]\times[-1,1]$ is a compact support and $M(\cdot)$ is continuous, it follows from the comment below (3.2) that there exists a constant $\widetilde{\eta} > 0$ such that (3.2) holds for all $\xi \in \Xi$. Further, it follows from [15, Proposition 5.10.11] with $p = (1,1,1)$ that $\widetilde{\eta} \ge \frac{1}{40^2} = \frac{1}{1600}$.

It is easy to verify that Φ and Ψ are globally Lipschitz continuous in (x,y,t) with respect to $\|\cdot\|_\infty$ for each $\xi \in \Xi$, where the Lipschitz constants $\kappa_\Phi(\xi) = \max(\|A\|_\infty, \|B(\xi)\|_\infty, 1)$ and $\kappa_\Psi(\xi) = \max(\|M(\xi)\|_\infty, \|Q(\xi)\|_\infty, \xi_1)$. Since Ξ is a compact support and κ_Φ and κ_Ψ are continuous in ξ, $\mathbb{E}[\kappa_\Phi(\xi)] < \infty$ and $\mathbb{E}[\kappa_\Phi(\xi)\kappa_\Psi(\xi)] < \infty$ such that assumptions for Case (ii) of Assumption 3.1 and Lemma 3.2 hold. Therefore, by Lemma 3.2, the DSVI

$$\text{(4.9)} \qquad \dot{x}(t) = \Pi_X\Big(x(t) - \mathbb{E}[\Phi(t,\xi,x(t),y(t,x(t),\xi))\Big) - x(t), \quad x(0) = x_0,$$

$$0 \le y(t,x(t),\xi) \perp \Psi(t,\xi,x(t),y(t,x(t),\xi)) \ge 0, \quad \text{a.e. } \xi \in \Xi.$$

and its SAA

$$\text{(4.10)} \qquad \dot{x}(t) = \Pi_X\Big(x(t) - \frac{1}{N}\sum_{i=1}^N \Phi(t,\xi^i,x(t),y(t,x(t),\xi))\Big) - x(t), \quad x(0) = x_0,$$

$$0 \le y(t,x(t),\xi^i) \perp \Psi(t,\xi^i,x(t),y(t,x(t),\xi^i)) \ge 0, \quad i = 1,\dots,N$$

have unique solutions $x^* \in C^1[0,T]$ and $x^N \in C^1[0,T]$, respectively.

As discussed below (3.13), the Lipschitz constant $\kappa_c(\xi) := \kappa_\Phi(\xi)(1 + \frac{\kappa_\Psi(\xi)}{\widetilde{\eta}})$ with respect to $\|\cdot\|_\infty$ is continuous in ξ since $\kappa_\Phi(\xi)$ and $\kappa_\Psi(\xi)$ are continuous. Further, for given t, x, ξ, the solution $\widehat{y}(t,x,\xi) \in \mathbb{R}^3$ of the VI in (4.9) has the following closed-form expressions: letting $w_i := [Q(\xi)x + q(t,\xi)]_i$ for $i = 1,2,3$,

$$\widehat{y}_1(t,x,\xi) = \begin{cases} 0, & w_1 \geq 0 \\ -w_1, & \text{otherwise,} \end{cases}$$
$$\widehat{y}_2(t,x,\xi) = \begin{cases} 0, & w_2 + \xi_1 \widehat{y}_1(t,x,\xi) \geq 0 \\ -w_2 - \xi_1 \widehat{y}_1(t,x,\xi), & \text{otherwise,} \end{cases}$$

and

$$\widehat{y}_3(t,x,\xi) = \begin{cases} 0, & w_3 - \widehat{y}_1(t,x,\xi) - \widehat{y}_2(t,x,\xi) \geq 0 \\ 10[-w_3 + \widehat{y}_1(t,x,\xi) + \widehat{y}_2(t,x,\xi)], & \text{otherwise.} \end{cases}$$

Since $Q(\cdot)$ and $q(\cdot,\cdot)$ are continuous in (t,ξ), we see from the above closed-form expressions of $\widehat{y}$ that $\widehat{y}(t,x,\xi)$ is also continuous. Hence, $\widehat{\Phi}(t,x,\xi) := \Phi(t,\xi,x,\widehat{y}(t,x,\xi))$ is continuous in (t,x,ξ). Since Ξ is a compact support, we see from Remark 3.2 that the moment generating functions $M_{\kappa_c}(\tau)$ and $M^i_{(t,x)}(\tau), i = 1,2,3$ have finite values for all τ in a neighborhood of zero. Consequently, it follows from Theorem 3.1 that $\{x^N\}$ converges to the solution x^* of (4.9) w.p. 1 and for any constant $\epsilon > 0$, there exist positive constants $\rho(\theta\epsilon)$ and $\sigma(\theta\epsilon)$, independent of N, such that

$$\mathbb{P}\Big\{ \sup_{t\in[0,T]} \big\|x^N(t) - x^*(t)\big\|_\infty \geq \epsilon \Big\} \;\leq\; \rho(\theta\epsilon)\exp\big(-N\sigma(\theta\epsilon)\big),$$

where $\theta = \frac{1+\kappa_G}{\exp(\gamma(1+\kappa_G)T)-1}$.

Given $N \in \mathbb{N}$, the time-stepping scheme for the SAA (4.10) is given by

$$x_j = x_{j-1} + h\Pi_X\Big(x_j - \frac{1}{N}\sum_{i=1}^N \Phi(t_j,\xi^i,x_j,y(t_j,x_j,\xi^i))]\Big) - hx_j, \quad j = 1,\ldots,\nu,$$
$$(4.11) \qquad 0 \leq y(t_j,x_j,\xi^i) \perp \Psi(t_j,\xi^i,x_j,y(t_j,x_j,\xi^i)) \geq 0, \quad i = 1,\ldots,N.$$

Once x_j is known, the VI solution $\widehat{y}(t_j,x_j,\xi^i)$ in (4.11) has a closed form expression as before by setting $t = t_j$, $x = x_j$ and $\xi = \xi^i$. Problem (4.10) is a DVI with a Lipschitz continuous right-hand side function in the ODE. The convergence of the time-stepping method (4.11) follows from Theorem 4.1, which means that $\{x_j\}$ converges to x^N as $h = T/\nu \to 0$ in the sense that $\|x_j - x^N(jh)\| = O(h)$ for all $j = 1,\ldots,\nu$.

We use the EDIIS(1) method with the 2-norm in (4.6). In this case, the solution of minimization problem (4.6) has the closed-form expressions

$$\alpha_0 = 1 - \alpha_1, \quad \alpha_1 = \text{mid}\left\{0, \frac{F_k^T(F_k - F_{k-1})}{\|F_{k-1} - F_k\|^2}, 1\right\}.$$

Moreover (4.7) reduces to

$$x^{k+1} = \bar{x} + \mu(1-\alpha_1)\Pi_X\big(x^{k-1} - G^N(\bar{t},x^{k-1})\big) + \mu\alpha_1\Pi_X\big(x^k - G^N(\bar{t},x^k)\big).$$

In our numerical experiments, we let $T = 1$, $\bar{x}$ be a computed solution with $h = 10^{-3}$ and $N = 2000$. We stop EDIIS(1) once $\|x^{k+1} - x^k\| \leq 10^{-6}$. For the fixed constant $h = 10^{-3}$, we carry out tests with sample size $N = 100, 200, 400, 800, 1200, 1500$

and the standard deviation 0.5, 1, 1.5, 2 of the truncated normal distribution over the compact support Ξ. We compute x^N and

$$Er_1 = 10^{-3}\sum_{i=1}^{10^3}\left\|\bar{x}_1(ih) - x_1^N(ih)\right\| \quad \text{and} \quad Er_2 = 10^{-3}\sum_{i=1}^{10^3}\left\|\bar{x}_2(ih) - x_2^N(ih)\right\|$$

60 times and average them. Figure 1 depicts the decreasing tendencies of Er_1 and Er_2 as N increases and σ decreases.

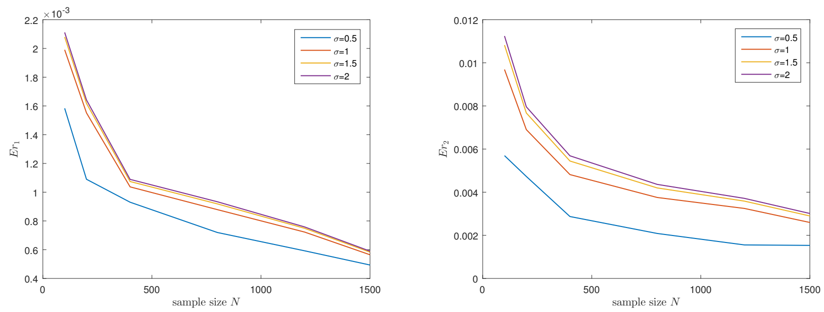

FIG. 1. *Decreasing tendencies of Er_1 and Er_2.*

5. A Modified Point-queue Model for the Instantaneous Dynamic User Equilibrium in Traffic Assignment Problems. Stochastic variational inequalities and dynamic variational inequalities have been extensively studied for traffic assignment problems [5, 16, 20, 36]. Since the travel demand and travel cost are often uncertain and subject to stochastic uncertainties, it is natural to study dynamic traffic assignment problems via DSVIs. We formulate such a problem as a DSVI as follows.

Consider the α-point-queue model for the instantaneous dynamic user equilibrium (IDUE) problem proposed in [19, 20]. We focus on the single destination case treated in [20, Section 3.1], and we introduce the following notation:

$\mathcal{N}$	the set of nodes
$\mathcal{L}$	the set of links given by (i,j) with $i,j \in \mathcal{N}$
$d_i(t)$	the travel demand from node $i \in \mathcal{N}$ to the destination, a given (nonnegative) function of t
$q_{ij}(t)$	the queue length of traffic on link $(i,j) \in \mathcal{L}$
$p_{ij}(t)$	the (nonnegative) rate of entry flow on link $(i,j) \in \mathcal{L}$
$\eta_i(t)$	the (nonnegative) instantaneous minimum travel time from node $i \in \mathcal{N}$ to the destination
τ_{ij}^0	the positive free flow travel time on link $(i,j) \in \mathcal{L}$
$\overline{C}_{ij}$	the positive capacity of exit flow on link $(i,j) \in \mathcal{L}$
α_{ij}	the positive constant associated with the queue length dynamic $q_{ij}(t)$ on link $(i,j) \in \mathcal{L}$

In the case of single destination [20, Section 3.1], the queue length of traffic on each link $(i,j) \in \mathcal{L}$ satisfies

$$\dot{q}_{ij}(t) = \begin{cases} 0, & \text{if } \ t \in [0, \tau_{ij}^0] \\ \max\Big(p_{ij}(t-\tau_{ij}^0) - \overline{C}_{ij}, -\alpha_{ij} q_{ij}(t)\Big), & \text{if } \ t > \tau_{ij}^0. \end{cases}$$

The other quantities are defined by the complementarity conditions:

$$0 \le p_{ij}(t) \perp \tau_{ij}^0 + \frac{q_{ij}(t)}{\overline{C}_{ij}} + \eta_j(t) - \eta_i(t) \ge 0, \quad \forall\ (i,j) \in \mathcal{L}, \quad \forall\ t \in [0,T],$$

$$0 \le \eta_i(t) \perp \sum_{j:(i,j)\in\mathcal{L}} p_{ij}(t) - \sum_{k:(k,i)\in\mathcal{L}} \min\Big(\overline{C}_{ki}, p_{ki}(t-\tau_{ki}^0) + \alpha_{ki} q_{ki}(t)\Big) - d_i(t) \ge 0,$$

for all $i \in \mathcal{N}$ and all $t \ge \tau_{ij}^0$, with the following initial conditions: $q_{ij}(t) = 0$ and $\min\big(\overline{C}_{ij}, p_{ki}(t-\tau_{ij}^0) + \alpha_{ij} q_{ij}(t)\big) = 0$ for all $t \in [0, \tau_{ij}^0]$, where $d_i(t)$ is a given time-varying demand function for each i. Hence, for all $t \ge \tau_{ij}^0$, the above system can be formulated as a time-delayed linear dynamical complementary system.

The time delay in the above system yields many analytic and numerical challenges. To obtain a regular ODE model, we approximate the time-delay term $p_{ij}(t-\tau_{ij}^0)$ using ODE techniques. The Laplace operator of the time delay function with the delay constant $\tau > 0$ is given by $e^{-\tau s}$, where $s \in \mathbb{C}$. It can be approximated using the pole approximation, i.e., $e^{-\tau s} = \frac{1}{e^{\tau s}} = \frac{1}{1+\sum_{k=1}^{\infty} \frac{(\tau s)^k}{k!}} \approx \frac{1}{1+\tau s + \frac{\tau^2}{2} s^2}$. Therefore, for any $(i,j) \in \mathcal{L}$, $[z_{ij}(t)]_+ \approx p_{ij}(t-\tau_{ij}^0)$, where $[z_{ij}]_+$ imposes the non-negativeness of approximation of p_{ij}, and $z_{ij}(t)$ is the solution of the 2nd order ODE: $\frac{(\tau_{ij}^0)^2}{2}\ddot{z}_{ij}(t) + \tau_{ij}^0 \dot{z}_{ij}(t) + z_{ij}(t) = p_{ij}(t)$ or equivalently

$$\begin{pmatrix} \dot{z}_{ij}(t) \\ \ddot{z}_{ij}(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{2}{(\tau_{ij}^0)^2} & -\frac{2}{\tau_{ij}^0} \end{bmatrix} \begin{pmatrix} z_{ij}(t) \\ \dot{z}_{ij}(t) \end{pmatrix} + \frac{2}{(\tau_{ij}^0)^2} \begin{pmatrix} 0 \\ p_{ij}(t) \end{pmatrix}.$$

Using this approximation, we obtain the following dynamical complementarity problem: for each $(i,j) \in \mathcal{L}$ and all $t \ge \tau_{ij}^0$,

$$\begin{pmatrix} \dot{z}_{ij}(t) \\ \ddot{z}_{ij}(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{2}{(\tau_{ij}^0)^2} & -\frac{2}{\tau_{ij}^0} \end{bmatrix} \begin{pmatrix} z_{ij}(t) \\ \dot{z}_{ij}(t) \end{pmatrix} + \frac{2}{(\tau_{ij}^0)^2} \begin{pmatrix} 0 \\ p_{ij}(t) \end{pmatrix}$$

$$\dot{q}_{ij}(t) = -\alpha_{ij} q_{ij}(t) + \big[[z_{ij}(t)]_+ - \overline{C}_{ij} - \alpha_{ij} q_{ij}(t)\big]_+$$

$$0 \le p_{ij}(t) \perp \tau_{ij}^0 + \frac{q_{ij}(t)}{\overline{C}_{ij}} + \eta_j(t) - \eta_i(t) \ge 0, \qquad \forall\ (i,j) \in \mathcal{L},$$

$$0 \le \eta_i(t) \perp \sum_{j:(i,j)\in\mathcal{L}} p_{ij}(t) - \sum_{k:(k,i)\in\mathcal{L}} \big(\overline{C}_{ki} - u_{ki}(t)\big) - d_i(t) \ge 0, \quad \forall\ i \in \mathcal{N},$$

$$0 \le u_{ki}(t) \perp u_{ki}(t) - \big[\overline{C}_{ki} - [z_{ki}(t)]_+ - \alpha_{ki} q_{ki}(t)\big] \ge 0, \quad \forall\ k : (k,i) \in \mathcal{L},$$

where $u_{ki}(\cdot)$ is the (time-varying) slack variable for the link (k,i). Suppose the time dependent demand function is random and is given by $d_i(t,\xi)$ for each $i \in \mathcal{N}$, where ξ is a random variable. Then for all $t \ge \tau_{ij}^0$,

(5.1) $$\begin{pmatrix} \dot{z}_{ij}(t) \\ \ddot{z}_{ij}(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{2}{(\tau_{ij}^0)^2} & -\frac{2}{\tau_{ij}^0} \end{bmatrix} \begin{pmatrix} z_{ij}(t) \\ \dot{z}_{ij}(t) \end{pmatrix} + \frac{2}{(\tau_{ij}^0)^2} \begin{pmatrix} 0 \\ \mathbb{E}[p_{ij}(t,\xi)] \end{pmatrix},$$

(5.2) $$\dot{q}_{ij}(t) = -\alpha_{ij} q_{ij}(t) + \big[[z_{ij}(t)]_+ - \overline{C}_{ij} - \alpha_{ij} q_{ij}(t)\big]_+,$$

(5.3) $$0 \le u_{ki}(t) \perp u_{ki}(t) - \big[\overline{C}_{ki} - [z_{ki}(t)]_+ - \alpha_{ki} q_{ki}(t)\big] \ge 0, \quad \forall\ k : (k,i) \in \mathcal{L},$$

(5.4) $$0 \le p_{ij}(t,\xi) \perp \tau_{ij}^0 + \frac{q_{ij}(t)}{\overline{C}_{ij}} + \eta_j(t,\xi) - \eta_i(t,\xi) \ge 0, \qquad \forall\ (i,j) \in \mathcal{L},$$

(5.5) $$0 \le \eta_i(t,\xi) \perp \sum_{j:(i,j)\in\mathcal{L}} p_{ij}(t,\xi) - \sum_{k:(k,i)\in\mathcal{L}} \big(\overline{C}_{ki} - u_{ki}(t)\big) - d_i(t,\xi) \ge 0, \quad \forall\ i \in \mathcal{N}.$$

Let $d \in \mathcal{N}$ denote the (single) destination node. Then $\eta_d(t) \equiv 0$ and $d_d(t,\xi) \equiv 0$.

To formulate the system in (5.1)-(5.5) as a DSVI, let

$$x(t) := \Big(z_{ij}(t), \dot{z}_{ij}(t), q_{ij}(t)\Big)_{(i,j)\in\mathcal{L}} \in \mathbb{R}^n,$$
$$y(t,\xi) := \Big(p_{ij}(t,\xi), \eta_i(t,\xi), u_{ki}(t)\Big)_{(i,j)\in\mathcal{L}, i\in\mathcal{N}, k:(k,i)\in\mathcal{L}} \in \mathbb{R}^m,$$

for some suitable $n, m \in \mathbb{N}$. Let $X = \mathbb{R}^n$ and $\gamma = 1$. Define $t_0 := \max_{(i,j)\in\mathcal{L}} \tau^0_{ij}$. Then for all $t \ge t_0$, (5.1)-(5.5) can be expressed as the following DSVI:

$$\text{(5.6)} \qquad \begin{aligned} &\dot{x} = \gamma\Big\{\Pi_X\Big(x - \big(Ax + \mathbb{E}[By_x(\xi)] + (Cx+f)_+\big)\Big) - x\Big\},\\ &0 \le y(\xi) \perp My(\xi) + Nx + g(t,\xi) \ge 0, \quad \text{a.e. } \xi \in \Xi, \end{aligned}$$

for constant matrices A, B, C, M, N, a constant vector f, and a vector-valued function g. When $0 \le t \le \min_{(i,j)\in\mathcal{L}} \tau^0_{ij}$, the point-queue model is described by a static complementarity problem (without ODE dynamics), and when t is between $\min_{(i,j)\in\mathcal{L}} \tau^0_{ij}$ and t_0, it yields a mixed model of a DSVI and a static complementarity problem.

We discuss the analytic properties of the DSVI (5.6). First, if the DSVI (5.6) has a solution $x(t)$ and $q_{ij}(t_0) \ge 0, \forall\,(i,j) \in \mathcal{L}$, then it follows from (5.2) that $q_{ij}(t) = e^{-\alpha_{ij}(t-t_0)} q_{ij}(t_0) + \int_{t_0}^t e^{-\alpha_{ij}(t-s)}\big[[z_{ij}(s)]_+ - \overline{C}_{ij} - \alpha_{ij} q_{ij}(s)\big]_+ ds$ such that $q_{ij}(t) \ge 0$ for all $t \ge t_0$ along $x(t)$. Similarly, by this result and (5.3), $\overline{C}_{ki} - u_{ki}(t) \ge 0$ for all $t \ge t_0$ along $x(t)$. For notational simplicity, let $y = (p, \eta, u)$, where

$$p := (p_{ij})_{(i,j)\in\mathcal{L}} \in \mathbb{R}^{m_p}, \quad \eta := (\eta_i)_{i\in\mathcal{N}} \in \mathbb{R}^{m_\eta}, \quad u := (u_{ki})_{k:(k,i)\in\mathcal{L}} \in \mathbb{R}^{m_u}.$$

Then the matrix in the underlying LCP in (5.6) is $M = \begin{bmatrix} 0 & M_{p\eta} & 0 \\ M_{\eta p} & 0 & M_{\eta u} \\ 0 & 0 & I_{m_u} \end{bmatrix}$, where the submatrix $\begin{bmatrix} 0 & M_{p\eta} \\ M_{\eta p} & 0 \end{bmatrix}$ is copositive [1, Proposition 2]. Since $M_{\eta u}$ is nonnegative, M is copositive. In light of $\eta_d = 0$, it can be shown that $y^T M y = 0$, $My \ge 0$, and $y \ge 0$ imply that

$$u = 0, \quad \eta = 0, \quad y^T(Nx + g(t,\xi)) = \sum_{(i,j)\in\mathcal{L}} p^T_{ij}\Big(\tau^0_{ij} + \frac{q_{ij}}{\overline{C}_{ij}}\Big) \ge 0$$

provided that $q_{ij} \ge 0, \forall\,(i,j) \in \mathcal{L}$. By [13, Theorem 3.8.6], the underlying LCP in (5.6) has a (possibly non-unique) solution for any Nx and $g(t,\xi)$ satisfying $q_{ij} \ge 0$.

To further study the DSVI (5.6), we consider the case where each non-destination node has exactly one exit link, i.e., $(i,j) \in \mathcal{L}$ if and only if $j = i+1$ for $i \ne d$. Hence, $m_p = |\mathcal{L}| = |\mathcal{N}| - 1 = m_\eta - 1$, $M_{\eta p} = \begin{bmatrix} I_{m_p} \\ 0 \end{bmatrix}$ and $M_{p\eta} = \begin{bmatrix} M'_{p\eta} & e_{m_p} \end{bmatrix}$, where $M'_{p\eta}$ is a square matrix of order m_p whose diagonal entries are -1, $(M'_{p\eta})_{i,i+1} = 1$ and other entries are zero. Further, $e_{m_p} = (0,\ldots,0,1)^T \in \mathbb{R}^{m_p}$. It is easy to show that $\big(M'_{p\eta}\big)^{-1}$ is a non-positive matrix. Suppose $\eta_d = \eta_{m_\eta}$, and $\eta' := (\eta_1,\ldots,\eta_{m_p})^T \in \mathbb{R}^{m_p}$. It can be verified that the underlying LCP has the following solution: $u_{ki} = \big[\overline{C}_{ki} - [z_{ki}]_+ - \alpha_{ki} q_{ki}\big]_+ \le \overline{C}_{ki}$, $p = (p_{ij})_{(i,j)\in\mathcal{L}} = \big(\sum_{k:(k,i)\in\mathcal{L}}\big(\overline{C}_{ki} - u_{ki}\big) + d_i(t,\xi)\big)_{(i,j)\in\mathcal{L}}$, and $\eta' = -\big(M'_{p\eta}\big)^{-1} w$, where $w := (w_i) = \big(\tau^0_{i,i+1} + \frac{q_{i,i+1}}{\overline{C}_{i,i+1}}\big) \ge 0$ if $q_{i,i+1} \ge 0$. This particular LCP solution can be compactly written as $u = (N_u x + g_u^0)_+$ for a constant matrix N_u and a constant vector g_u^0, $p = F_p u + g_p^0 + \widehat{d}(t,\xi)$ for a constant matrix F_p and a constant vector g_p^0 with $\widehat{d}(t,\xi) = (d_i(t,\xi))_{i\in\mathcal{L}}$, and $\eta = F_\eta(Nx + g(t,\xi))$ for a constant matrix F_η. Thus for some constant matrix B_p, the ODE in (5.6) becomes

$$\dot{x} = -Ax - B_p\Big(F_p(N_u x + g_u^0)_+ + g_p^0 + \mathbb{E}[\widehat{d}(t,\xi)]\Big) - (Cx+f)_+.$$

Hence, the right-hand side of the ODE is piecewise affine in x. If $\mathbb{E}[\widehat{d}(t,\xi)]$ is Lipschitz continuous in t, then the ODE has a unique solution $x(t)$ for $t \geq t_0$. Therefore, all the assumptions are fulfilled. We summarize these results as follows.

PROPOSITION 5.1. *Consider the DSVI (5.6) for the α-point queue model whose non-destination node has exactly one exit link. Further, consider the particular LCP solution given above. If $\mathbb{E}[\widehat{d}(t,\xi)]$ is Lipschitz continuous in t and $q_{ij}(t_0) \geq 0$ for all $(i,j) \in \mathcal{L}$, then the DSVI has a unique solution $x(t)$ for all $t \geq t_0$.*

6. Conclusion. The dynamic stochastic variational inequality (DSVI) (1.1)-(1.3) encompasses the DVI (1.4)-(1.5) and the two-stage stochastic SVI (1.11)-(1.12), which can efficiently model dynamic equilibria subject to uncertainties. We show the solution existence and uniqueness for a class of DSVIs under some Lipschitz conditions. Moreover, we proposed a discretization scheme of the DSVI using the SAA and the time-stepping EDIIS method. We established the uniform convergence and an exponential convergence rate, and proved the convergence of the EDIIS method. We illustrated our results via a class of dynamic stochastic user equilibrium problems in traffic assignment problems. Future research topics include long-time dynamics of the DSVI, e.g., stability of its equilibria.

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