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FULL LENGTH PAPER

Differential variational inequality approach to dynamic games with shared constraints

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Abstract The dynamic Nash equilibrium problem with shared constraints (NEPSC) involves a dynamic decision process with multiple players, where not only the players' cost functionals but also their admissible control sets depend on the rivals' decision variables through shared constraints. For a class of the dynamic NEPSC, we propose a differential variational inequality formulation. Using this formulation, we show the existence of solutions of the dynamic NEPSC, and develop a regularized smoothing method to find a solution of it. We prove that the regularized smoothing method converges to the least norm solution of the differential variational inequality, which is a solution of the dynamic NEPSC as the regularization parameter λ and smoothing parameter μ go to zero with the order $\mu = o(\lambda)$. Numerical examples are given to illustrate the existence and convergence results.

Keywords Generalized Nash game · Dynamic game · Monotone variational inequality · Smoothing · Regularization

Mathematics Subject Classification 90C30 · 90C33 · 90C39

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1 Introduction

The dynamic Nash equilibrium problem involves a decision process with multiple players, where each player solves an optimal control problem with his own cost function and strategy set. Each player's cost function is dependent on all players' variables but the strategy set is only dependent on the player's own variables described by the state dynamic and admissible control set. However, in many real world problems, each player's cost function and strategy set are both dependent on his rivals's variables, which yield a new model: the dynamic Nash equilibrium problem with shared constraints (NEPSC). Pang and Stewart [29] use the differential quasi variational inequality to study a class of dynamic NEPSC in which only control sets are coupled and the state dynamics are uncoupled across players. Motivated by the work of Pang and Stewart in [29], in this paper, we study such class of dynamic NEPSC that have a common constraint function for all players' control sets. To find certain solutions of the dynamic NEPSC, we consider the following differential variational inequality (DVI)

$$\begin{aligned} \dot{x}(t) &= F(t, x(t), u(t)) \\ u(t) &\in \mathrm{SOL}(U, \Psi(t, x(t), \cdot)) \\ 0 &= \Gamma(x(0), x(T)), \end{aligned} \tag{1}$$

where $F : R \times R^{2n} \times R^m \to R^{2n}, U \subseteq R^m, \Psi : R \times R^{2n} \times U \to R^m, \Gamma : R^{2n} \times R^{2n} \to R^{2n}$, and

$$\mathrm{SOL}(U, \Psi(t, x(t), \cdot)) := \{u \in U |\ (v - u)^T \Psi(t, x, u) \geq 0,\ \ \forall v \in U\},$$

is the solution set of the VI associated with the set U and the parameterized mapping $\Psi(t, x(t), \cdot) : U \to R^m$ and fixed $t \in [0, T]$ and $x \in R^{2n}$.

The DVI is a new modeling paradigm for many important applications in engineering and economics which presents dynamics, variational inequalities, equilibrium conditions in a systematic way [3,19,21,25,29,35]. To the best of our knowledge, the DVI formulation for the dynamic NEPSC has not yet been studied. Comparing with the differential quasi variational inequality formulation in [29], the DVI formulation is advantageous since it can be treated as a differential inclusion, or a system of differential algebraic equations (DAE), or more specifically as a system of ordinary differential equations (ODE), for which there are abundant theory and algorithms available.

The static NEPSC can be regarded as a special case of the dynamic NEPSC when the state variables are constant. Recently the static NEPSC has been intensively studied due to many important applications arising from engineering and economics, for instance, liberalized energy markets, global environment, traffic assignment with side constraints and oligopoly analysis [13,14,20,22,23,26,28,32]. It is known that a static NEPSC can have (possibly infinitely) many solutions, and some solutions can be found via a static variational inequality (VI) when the cost functions and the strategy sets are convex [13,14,22,23,26,34]. The VI approach for finding a solution of a static NEPSC has attracted growing attention because there are rich theory and efficient algorithms for solving VIs [16,31]. For instance, Wei and Smeers [34] formulated an oligopolistic electricity model as a static NEPSC and found a solution via its VI formulation, Facchinei et al. [14] proposed a semismooth Newton method for a static NEPSC via its VI formulation, and Nabetani et al. [26] proposed two parameterized VI approaches to solve a class of static NEPSC. Recently, Schiro et al. [33] presented a modified Lemake's method to solve a class of static affine NEPSC arising from a breadth of applications including environmental pollution games, rate allocation in communication networks and strategic behavior in power markets.

The dynamic NEPSC provides a fundamental generalization of the static NEPSC to consider some parameters and circumstances varying in the players' strategies. The dynamic NEPSC appears frequently in realistic applications. For example, let us consider a dynamic user equilibrium problem for traffic networks studied in [35]. In a traffic network, the travelers choose their departure times and routes to minimize their generalized travel costs under a traffic volume control scheme guaranteeing that the traffic volumes on specified links do not exceed preferred levels. This problem can be formulated as a special dynamic NEPSC: dynamic user equilibrium problem with side constraints where the side constraints characterize the restrictions on the traffic volumes.

Inspired by the VI approach for the static NEPSC, we apply the DVI approach to dynamic NEPSC where the cost functions and the strategy sets are convex. A classic solution of the DVI is a pair $(x(t), u(t))$ where x is continuously differentiable and u is continuous on $[0, T]$ such that the differential equations and the constraints in the DVI are fulfilled for all $t \in [0, T]$. However, in most cases, the DVI does not have a classic solution, and therefore we have to seek the weak solution $(x(t), u(t))$, where x is absolutely continuous and u is integrable on $[0, T]$ such that for all $0 \leq s \leq t \leq T$,

$$x(t) - x(s) = \int_s^t F(\tau, x(\tau), u(\tau))d\tau,$$

and for almost all $t \in [0, T]$, $u(t) \in \mathrm{SOL}(U, \Psi(t, x(t), \cdot))$. The latter implies $u(t) \in U$ holds almost everywhere and for any continuous functions $v : [0, T] \to U$ it holds

$$\int_0^T [v(\tau) - u(\tau)]^T \, \Psi(\tau, x(\tau), u(\tau))d\tau \geq 0.$$

Solving the DVI is a challenging problem because it involves at each grid a suitable selection of a set-valued mapping defining the dynamic, which actually needs solving a family of parameterized optimization problems without standard constraint qualifications. Another difficulty is that the solution of the DVI is usually non-smooth, and because of this we can not expect a high order convergence if the ODE-integrators are just extended to the DVI in a naive manner. Motivated by the availability of many powerful solvers for the ODEs with smooth dynamics, we propose a regularized smoothing method for solving the DVI. Namely, we use regularization and smoothing techniques for the DVI, which give a standard ODE that has a unique classical solution and can be efficiently solved by the existing high-order solvers. We establish the convergence of the solutions of the ODEs to a solution of the dynamic NEPSC when the regularization parameter λ and the smoothing parameter μ go to zero with the order $\mu = o(\lambda)$. Moreover, we present some desired properties of the limit solution.

The remaining of this paper is organized as follows. In Sect. 2, we present a detailed formulation of the dynamic NEPSC and reformulate it as a DVI. In Sect. 3, we study the solvability of the DVI. In Sect. 4, we introduce the regularized smoothing method and give the convergence analysis. In Sect. 5, we use the dynamic two-player zero sum game to illustrate the formulation and the convergence of the regularized smoothing method.

2 Problem formulation

Consider a dynamic Nash equilibrium problem with N players. We denote by $y_\nu \in R^{n_\nu}$ and $u_\nu \in R^{m_\nu}$ the ν-th player's state and strategy variables, respectively. The strategy is also called as action, decision or control. Collectively write $y = (y_\nu)_{\nu=1}^N \in R^n$, $u = (u_\nu)_{\nu=1}^N \in R^m$, $y_{-\nu} = (y_{\nu'})_{\nu' \neq \nu} \in R^{n-n_\nu}$ and $u_{-\nu} = (u_{\nu'})_{\nu' \neq \nu} \in R^{m-m_\nu}$, where $n = \sum_{\nu=1}^N n_\nu$ and $m = \sum_{\nu=1}^N m_\nu$. When we emphasize the ν-th player's state and strategy variables, we use $y = (y_\nu, y_{-\nu})$ and $u = (u_\nu, u_{-\nu})$ to represent y and u, respectively. For the ν-th player, we denote

– The strategy set (admissible control set) by

$$U_\nu(u_{-\nu}) = \{u_\nu \mid h_\nu(u_\nu) \leq 0, g(u_\nu, u_{-\nu}) \leq 0\},$$

where $h_\nu(\cdot) : R^{n_\nu} \to R^{l_\nu}$ and $g(\cdot, u_{-\nu}) : R^{n_\nu} \to R^\ell$;
– The initial state by $y_\nu^0 \in R^{n_\nu}$;
– The state dynamic by $\Theta_\nu(\cdot, \cdot, \cdot) : R^{1+n_\nu+m_\nu} \to R^{n_\nu}$;
– The cost functional by

$$\theta_\nu(y, u) = \psi_\nu(y(T)) + \int_0^T \varphi_\nu(t, y(t), u(t))dt,$$

where $\psi_\nu(\cdot) : R^n \to R$ and $\varphi_\nu(\cdot, \cdot, \cdot) : R^{1+n+m} \to R$, and $T > 0$ is the terminal time.

Writing $\theta_\nu(y, u) = \theta_\nu(y_\nu, y_{-\nu}, u_\nu, u_{-\nu})$, the solution (or called the equilibrium point) of the dynamic NEPSC is a state-control pair (y^*, u^*) satisfying: for fixed $y^*_{-\nu}$ and $u^*_{-\nu}$, (y^*_ν, u^*_ν) is a solution of the following optimal control problem

$$\begin{aligned} \min\ & \theta_\nu(y_\nu, y^*_{-\nu}, u_\nu, u^*_{-\nu}) \\ \text{s.t.}\ & \dot{y}_\nu(t) = \Theta_\nu(t, y_\nu, u_\nu) \\ & y_\nu(0) = y_\nu^0 \\ & u_\nu(t) \in U_\nu(u^*_{-\nu}(t)) \quad \text{for all most } t \in [0, T]. \end{aligned} \tag{2}$$

Note that without the shared constraint $g(u_\nu, u_{-\nu}) \le 0$ in $U_\nu(u_{-\nu})$, (2) reduces to the standard dynamic NEP. Write $\varphi_\nu(t, y, u) = \varphi_\nu(t, y_\nu, y_{-\nu}, u_\nu, u_{-\nu})$. Here we make the following blanket assumptions on the smoothness and convexity of functions in (2), which are fulfilled for many dynamic NEPSCs.

Assumption 1 For any $\nu \in \{1, \dots, N\}$ suppose that ψ_ν and each components of h_ν and $g(\cdot, u_{-\nu})$ are convex, and suppose that $\varphi_\nu(t, \cdot, y_{-\nu}, \cdot, u_{-\nu})$ and each component of $\Theta_\nu(t, y_\nu, \cdot)$ are convex and continuously differentiable for any fixed t, $y_{-\nu}$ and $u_{-\nu}$.

Define the Hamiltonian of player ν's by

$$H_\nu(t, v_\nu, y, u) = \varphi_\nu(t, y, u) + (v_\nu)^T \, \Theta_\nu(t, y_\nu, u_\nu)$$

where v_ν is the adjoint variable of the ODE constraint in player ν's control problem. By Bellman's principle of optimality, (2) yields the constrained Hamilton system

$$\begin{cases} \dot{v}_\nu(t) = -\nabla_{y_\nu} H_\nu(t, v_\nu(t), y(t), u(t)) \\ \dot{y}_\nu(t) = \ \ \Theta_\nu(t, y_\nu(t), u_\nu(t)) \\ u_\nu(t) \in \ \arg\min_z H_\nu(t, v_\nu(t), y(t), u_{-\nu}(t), z), \ \text{ s.t. } z \in U_\nu(u_{-\nu}) \\ y_\nu(0) = y_\nu^0 \ \text{ and } \ v_\nu(T) = \nabla_{y_\nu} \psi_\nu(y(T)). \end{cases} \tag{3}$$

Under Assumption 1, $H_\nu(t, v_\nu, y, u_{-\nu}, u_\nu)$ is convex in u_ν and the set $U_\nu(u_{-\nu})$ is convex, so the minimization problem in (3) is equivalent to the VI: find $u_\nu \in U_\nu(u_{-\nu})$ such that

$$(z - u_\nu)^T \nabla_{u_\nu} H_\nu(t, v_\nu, y, u_{-\nu}, u_\nu) \ \ge \ 0 \qquad \forall z \in U_\nu(u_{-\nu}). \tag{4}$$

We denote the solution set of (4) by $\text{SOL}(U_\nu(u_{-\nu}), \nabla_{u_\nu} H_\nu(t, v_\nu, y, u_{-\nu}, \cdot))$. Collectively write

$$\Psi(t, v, y, u) = \left(\nabla_{u_\nu} H_\nu(t, v_\nu, y, u)\right)_{\nu=1}^N$$

and

$$\Gamma(v(0), y(0), v(T), y(T)) = \begin{pmatrix} y_\nu(0) - y_\nu^0 \\ v_\nu(T) - \nabla_{y_\nu} \psi_\nu(y(T)) \end{pmatrix}_{\nu=1}^N .$$

Concatenating (3) with (4) for $\nu = 1, \dots, N$, we can formulate the dynamic NEPSC (2) as the following differential quasi VI [29]

$$\begin{aligned} \dot{v}(t) &= \left(-\nabla_{y_\nu} H_\nu(t, v_\nu(t), y(t), u(t))\right)_{\nu=1}^N , \\ \dot{y}(t) &= (\Theta_\nu(t, y_\nu(t), u_\nu(t)))_{\nu=1}^N , \\ u(t) &\in \text{SOL}(\tilde{U}(u(t)), \Psi(t, v(t), y(t), \cdot)) \\ 0 &= \Gamma(v(0), y(0), v(T), y(T)), \end{aligned} \tag{5}$$

where $\tilde{U}(u) = \prod_{\nu=1}^{N} U_\nu(u_{-\nu})$. Because of the complex structure of $\tilde{U}$, it is hard to analyze the solvability and the convergence of numerical algorithms for solving (5). Here we propose a DVI formulation of the dynamic NEPSC (2), instead of the quasi one. Define

$$U := \{u \in R^m \mid h_\nu(u_\nu) \le 0,\ \nu = 1, \ldots, N,\ g(u) \le 0\},$$

where $g(u) = g(u_\nu, u_{-\nu})$ for $\nu = 1, \ldots, N$.

The following lemma states that the solvability of the VI implies the solvability of the quasi VI, and justifies the DVI formulation of dynamic NEPSC [13,14,34].

Lemma 1 ([13]) *For any fixed t, $v(t)$ and $y(t)$, we have*

$$\mathrm{SOL}(U, \Psi(t, v(t), y(t)), \cdot)) \subseteq \mathrm{SOL}(\tilde{U}, \Psi(t, v(t), y(t), \cdot)).$$

In the remainder of this paper, we will study the DVI formulation (1) of the dynamic NEPSC (2), where

$$F(t, x(t), u(t)) = \begin{pmatrix} \left(-\nabla_{y_\nu} H_\nu(t, v_\nu, y, u)\right)_{\nu=1}^{N}, \\ (\Theta_\nu(t, y_\nu, u_\nu))_{\nu=1}^{N} \end{pmatrix} \quad \text{and} \quad x(t) = \begin{pmatrix} v(t) \\ y(t) \end{pmatrix}.$$

Here we call (y, u) as a *feasible pair* of the dynamic NEPSC (2) if $h_\nu(u_\nu) \le 0$, $g(u) \le 0$, and $\dot{y}_\nu(t) = \Theta_\nu(t, y_\nu(t), u_\nu(t))$ for $\nu = 1, \ldots, N$. The following theorem characterizes the relation between the DVI (1) and the dynamic NEPSC (2).

Theorem 1 *Suppose that Assumption 1 holds. Let (v^*, y^*, u^*) be a weak solution of (1), and let $\Theta_\nu(t, y_\nu, u_\nu)$ be linear with respect to (y_ν, u_ν). Then (y^*, u^*) is a solution of the dynamic NEPSC (2) in the following sense: for any feasible pair (y, u) of (2), we have*

$$\theta_\nu(y_\nu, y^*_{-\nu}, u_\nu, u^*_{-\nu}) \ \ge\ \theta_\nu(y^*_\nu, y^*_{-\nu}, u^*_\nu, u^*_{-\nu}), \quad \nu = 1, \ldots, N.$$

Proof Since ψ_ν is convex and $v^*_\nu(T) = \nabla_{y_\nu}\psi_\nu(y^*(T))$, we have

$$\begin{aligned} &\psi_\nu(y_\nu(T), y^*_{-\nu}(T)) - \psi_\nu(y^*_\nu(T), y^*_{-\nu}(T)) \\ &\quad \ge \langle \nabla_{y_\nu}\psi_\nu(y^*(T)), y_\nu(T) - y^*_\nu(T)\rangle \ = \ \langle v^*_\nu(T), y_\nu(T) - y^*_\nu(T)\rangle. \end{aligned} \tag{6}$$

By the linearity of Θ_ν we have

$$\Theta_\nu(t, y_\nu, u_\nu) - \Theta_\nu(t, y^*_\nu, u^*_\nu) = \left(\nabla_{y_\nu}\Theta_\nu(t, y^*_\nu, u^*_\nu), \nabla_{u_\nu}\Theta_\nu(t, y^*_\nu, u^*_\nu)\right) \begin{pmatrix} y_\nu - y^*_\nu \\ u_\nu - u^*_\nu \end{pmatrix},$$

this yields

$$\begin{aligned}\frac{d}{dt}\langle v_\nu^*, y_\nu - y_\nu^*\rangle &= \langle \dot{v}_\nu^*, y_\nu - y_\nu^*\rangle + \langle v_\nu^*, \dot{y}_\nu - \dot{y}_\nu^*\rangle \\ &= \langle \dot{v}_\nu^*, y_\nu - y_\nu^*\rangle + \langle v_\nu^*, \Theta_\nu(t, y_\nu, u_\nu) - \Theta_\nu(t, y_\nu^*, u_\nu^*)\rangle \\ &= \langle \dot{v}_\nu^*, y_\nu - y_\nu^*\rangle + \langle (v_\nu^*)^T \nabla_{y_\nu}\Theta_\nu(t, y_\nu^*, u_\nu^*), y_\nu - y_\nu^*\rangle \\ &\quad + \langle (v_\nu^*)^T \nabla_{u_\nu}\Theta_\nu(t, y_\nu^*, u_\nu^*), u_\nu - u_\nu^*\rangle. \end{aligned} \tag{7}$$

Noting

$$\begin{aligned}&\nabla_{y_\nu}\varphi_\nu(t, y_\nu^*, y_{-\nu}^*, u_\nu^*, u_{-\nu}^*) \\ &\quad = \nabla_{y_\nu} H_\nu(t, y_\nu^*, y_{-\nu}^*, u_\nu^*, u_{-\nu}^*) - (v_\nu^*)^T \nabla_{y_\nu}\Theta_\nu(t, y_\nu^*, u_\nu^*) \\ &\quad = -\dot{v}_\nu^* - (v_\nu^*)^T \nabla_{y_\nu}\Theta_\nu(t, y_\nu^*, u_\nu^*)\end{aligned}$$

and

$$\begin{aligned}&\nabla_{u_\nu}\varphi_\nu(t, y_\nu^*, y_{-\nu}^*, u_\nu^*, u_{-\nu}^*) \\ &\quad = \nabla_{u_\nu} H_\nu(t, y_\nu^*, y_{-\nu}^*, u_\nu^*, u_{-\nu}^*) - (v_\nu^*)^T \nabla_{u_\nu}\Theta_\nu(t, y_\nu^*, u_\nu^*),\end{aligned}$$

and noting that

$$\langle \nabla_{u_\nu} H_\nu(t, y_\nu^*, y_{-\nu}^*, u_\nu^*, u_{-\nu}^*), u_\nu - u_\nu^*\rangle \geq 0$$

holds for almost all $t \in [0, T]$ since (y, u) is feasible and (y^*, u^*) is a weak solution of (1), by using (7) and considering that $\varphi_\nu(t, \cdot, y_{-\nu}, \cdot, u_{-\nu})$ is convex and continuously differentiable, we have

$$\begin{aligned}&\varphi_\nu(t, y_\nu, y_{-\nu}^*, u_\nu, u_{-\nu}^*) - \varphi_\nu(t, y_\nu^*, y_{-\nu}^*, u_\nu^*, u_{-\nu}^*) \\ &\quad \geq \langle \nabla_{y_\nu}\varphi_\nu(t, y_\nu^*, y_{-\nu}^*, u_\nu^*, u_{-\nu}^*), y_\nu - y_\nu^*\rangle + \langle \nabla_{u_\nu}\varphi_\nu(t, y_\nu^*, y_{-\nu}^*, u_\nu^*, u_{-\nu}^*), u_\nu - u_\nu^*\rangle \\ &\quad = -\langle \dot{v}_\nu^*, y_\nu - y_\nu^*\rangle - \langle (v_\nu^*)^T \nabla_{y_\nu}\Theta_\nu(t, y_\nu^*, u_\nu^*), y_\nu - y_\nu^*\rangle \\ &\qquad + \langle \nabla_{u_\nu} H_\nu(t, y_\nu^*, y_{-\nu}^*, u_\nu^*, u_{-\nu}^*), u_\nu - u_\nu^*\rangle - \langle (v_\nu^*)^T \nabla_{u_\nu}\Theta_\nu(t, y_\nu^*, u_\nu^*), u_\nu - u_\nu^*\rangle \\ &\quad = -\frac{d}{dt}\langle v_\nu^*, y_\nu - y_\nu^*\rangle + \langle \nabla_{u_\nu} H_\nu(t, y_\nu^*, y_{-\nu}^*, u_\nu^*, u_{-\nu}^*), u_\nu - u_\nu^*\rangle \\ &\quad \geq -\frac{d}{dt}\langle v_\nu^*, y_\nu - y_\nu^*\rangle,\end{aligned}$$

and therefore

$$\begin{aligned}&\int_0^T \left[\varphi_\nu(t, y_\nu, y_{-\nu}^*, u_\nu, u_{-\nu}^*) - \varphi_\nu(t, y_\nu^*, y_{-\nu}^*, u_\nu^*, u_{-\nu}^*)\right] dt \\ &\quad \geq -\langle v_\nu^*(T), y_\nu(T) - y_\nu^*(T)\rangle.\end{aligned}$$

Now, by this inequality and (6), we obtain

$$\begin{aligned}&\theta_\nu(y_\nu, y^*_{-\nu}, u_\nu, u^*_{-\nu}) - \theta_\nu(y^*_\nu, y^*_{-\nu}, u^*_\nu, u^*_{-\nu})\\ &= \psi_\nu(y_\nu(T), y^*_{-\nu}(T)) - \psi_\nu(y^*_\nu(T), y^*_{-\nu}(T))\\ &\quad + \int_0^T \left[\varphi_\nu(t, y_\nu, y^*_{-\nu}, u_\nu, u^*_{-\nu}) - \varphi_\nu(t, y^*_\nu, y^*_{-\nu}, u^*_\nu, u^*_{-\nu})\right] dt \;\geq\; 0.\end{aligned}$$

This completes the proof. □

3 Solvability of the DVI formulation

The DVI (1) is solvable if we can find an initial value x^0 such that the initial value problem of the differential inclusion

$$\begin{cases} \dot{x}(t) = F(t, x(t), u(t)) \\ u(t) \in \mathrm{SOL}(U, \Psi(t, x, \cdot)) \\ x(0) = x^0 \end{cases}$$

has a solution $(x(t), u(t))$ fulfilling the boundary value condition of the DVI (1) formulated from the dynamic NEPSC: $\Gamma(x(0), x(T)) = 0$. We can see that when $T = 0$, this condition has the form

$$\Gamma(x, x) = \begin{pmatrix} y_\nu - y^0_\nu \\ v_\nu - \nabla_{y_\nu}\psi_\nu(y) \end{pmatrix}_{\nu=1}^N = 0,$$

which has a unique solution

$$\hat{x}^0 = \begin{pmatrix} (\nabla_{y_\nu}\psi_\nu(y^0))_{\nu=1}^N \\ y^0 \end{pmatrix},$$

where $y^0 = (y^0_\nu)_{\nu=1}^N$. Moreover, the Jacobian is of full rank:

$$\nabla_x \Gamma(\hat{x}^0, \hat{x}^0) = \begin{pmatrix} 0 & I \\ I & -(\nabla^2_{y_\nu y_{\nu'}}\psi_\nu(y^0))_{\nu,\nu'=1}^N \end{pmatrix}. \tag{8}$$

Denote $\mathcal{S}(t, x) := \mathrm{SOL}(U, \Psi(t, x, \cdot))$. We impose the following assumption for guaranteeing the solvability of (1).

Assumption 2 (A1) The solution set $\mathcal{S}(0, \hat{x}^0)$ is nonempty and bounded.
(A2) The function $\Psi(t, x, \cdot)$ is monotone.

Remark 1 Assumption (A1) is fulfilled in many practical settings. For instance, in a mixed strategy game with shared constraints, the admissible control set often has the form

$$U_\nu(u_{-\nu}) = \{u_\nu \in R^{m_\nu} \mid u_\nu \geq 0,\ e^T u = 1\},$$

where $e = (1, \ldots, 1)^T$, which gives $U = \{u \in R^m \mid u \geq 0,\ e^T u = 1\}$. In such a case, $\mathcal{S}(0, \hat{x}^0)$ is nonempty, convex and bounded.

Notice that $\mathcal{S}(t, x)$ is closed and convex for any fixed (t, x) when $\Psi(t, x, \cdot)$ is monotone. See Theorem 2.3.5 [16].

Remark 2 Assumption (A2) means that the DVI (1) is a differential monotone variational inequality, which has been used for many applied problems, and is called linear passive complementarity systems when the VI in (1) is a monotone linear complementarity problem [4,5,19,29]. An optimal control problem with joint control and state constraints can be formulated as a differential monotone VI, for which Han et al. [18] proposed a unified numerical scheme. For the dynamic NEPSC, Assumption 1 assumes that each diagonal block $\nabla^2_{u_\nu u_\nu} H_\nu(t, v_\nu, y, u)$ of the Jacobian $\nabla_u \Psi(t, v, y, u)$ is positive semi-definite, since $\Psi(t, x, u) = \left(\nabla_{u_\nu} H_\nu(t, v_\nu, y, u)\right)_{\nu=1}^N$. Assumption (A2) assumes the Jacobian $\nabla_u \Psi(t, v, y, u)$ is positive semi-definite. In general, Assumption 1 does not imply Assumption (A2). However, in many applications of the dynamic NEPSC, Assumption 1 implies Assumption (A2), that is, convexity of the objectives of individual players in their decision variables implies monotonicity of the VI.

Below we give some sufficient conditions imposed on the original dynamic NEPSC (2) for guaranteeing the monotonicity of the resulting DVI.

Proposition 1 *Suppose that Assumption 1 holds. Then the function $\Psi(t, x, \cdot)$ is monotone if the state dynamic $\Theta_\nu(t, y_\nu, u_\nu)$ is linear with respect to u_ν for $\nu = 1, \ldots, N$ and one of the following conditions on the cost functional θ_ν holds:*

(1) $\nabla^2_{u_\nu u_i} \varphi_\nu(t, y, u) = -\nabla^2_{u_i u_\nu} \varphi_i(t, y, u)$, for $\nu \neq i$, and $\nu, i = 1, \ldots, N$.

(2) $\varphi_\nu(t, y, u) = \phi_\nu(t, y, u_\nu) + u_\nu^T B \sum_{i=1}^N u_i + q_\nu(t, y)$ for $\nu = 1, \ldots, N$, where $u_\nu \in R^{m_1}$, $\phi_\nu : R^{1+n+m_1} \to R$ is convex, $B \in R^{m_1 \times m_1}$ is positive semi-definite, and $q_\nu : R \times R^n \to R$.

Proof (1) Assumption 1 implies that the matrices $\nabla^2_{u_\nu u_\nu} \varphi_\nu(t, y, u)$, $\nu = 1, \ldots, N$ are positive semi-definite. From the linearity of $\Theta_\nu(t, y_\nu, u_\nu)$ with respect to u_ν and condition (1), the Jacobian of $\Psi(t, x, u)$ has the form

$$\nabla_u \Psi(t, x, u) = \begin{pmatrix} \nabla^2_{u_1 u_1} \varphi_1 & \nabla^2_{u_1 u_2} \varphi_1 & \cdots & \nabla^2_{u_1 u_N} \varphi_1 \\ -\nabla^2_{u_1 u_2} \varphi_1 & \nabla^2_{u_2 u_2} \varphi_2 & \cdots & \nabla^2_{u_2 u_N} \varphi_2 \\ \vdots & \vdots & \cdots & \vdots \\ -\nabla^2_{u_1 u_N} \varphi_1 & -\nabla^2_{u_2 u_N} \varphi_2 & \cdots & \nabla^2_{u_N u_N} \varphi_N \end{pmatrix} (t, x, u)$$

which is positive semi-definite. Therefore, the mapping $\Psi(t, x, \cdot)$ is monotone.

(2) By simple calculations, we can find

$$\Psi(t, x, \cdot) = \left(\nabla_{u_\nu} \phi_\nu(u_\nu) + B^T u_\nu + B \sum_{i=1}^N u_i + \nabla_{u_\nu} \Theta_\nu(t, y_\nu, u_\nu)^T v_\nu \right)_{\nu=1}^N .$$

From the linearity of $\Theta_\nu(t, y_\nu, u_\nu)$ with respect to u_ν, the Jacobian of $\Psi(t, x, \cdot)$ has the form

$$\nabla_u \Psi(t, x, u) = \text{diag}(\nabla^2_{u_\nu u_\nu} \phi(t, y, u_\nu) + B^T) + B \otimes E,$$

where $E \in R^{N\times N}$ with all entries 1, $\otimes$ is the Kronecker tensor product:

$$E \otimes B = \begin{pmatrix} B & B & \cdots & B \\ B & B & \cdots & B \\ \vdots & \vdots & \cdots & \vdots \\ B & B & \cdots & B \end{pmatrix}.$$

The matrix $E \otimes B$ is positive semi-definite, since

$$z^T(E \otimes B)z = \sum_{j=1}^{N} \left(\sum_{i=1}^{N} z_i^T B \right) z_j = \left(\sum_{i=1}^{N} z_i \right)^T B \left(\sum_{j=1}^{N} z_j \right) \geq 0$$

for any $z = (u_1^T, \cdots, z_N^T)^T \in R^m$, $z_i \in R^{m_1}$ for $i = 1, \ldots, N$. From the positive semi-definite property of B and $\nabla^2_{u_\nu u_\nu} \phi(t, y, u_\nu)$ for $\nu = 1, \ldots, N$, $\nabla_u \Psi(t, x, u)$ is positive semi-definite, and hence $\Psi(t, x, \cdot)$ is monotone.

□

The following three examples show that the two conditions of Proposition 1 are from real applications. We assume the dynamic are linear in these examples.

Example 1 The two-player zero-sum game with shared constraints and linear dynamics satisfies condition (1) of Proposition 1. In such a game, we have two cost functionals $\varphi_1 = \varphi$ and $\varphi_2 = -\varphi$, one player seeks to minimize φ and the other seeks to maximize it. In this setting Assumption 1 is just the normal convex-concave assumption of φ, i.e., φ is assumed to be convex in u_1 and concave in u_2, which gives a monotone mapping $\Psi(t, x, \cdot)$ with the positive semi-definite Jacobian

$$\nabla_u \Psi(t, x, u) = \begin{pmatrix} \nabla^2_{u_1 u_1} \varphi & \nabla^2_{u_1 u_2} \varphi \\ -\nabla^2_{u_1 u_2} \varphi & -\nabla^2_{u_2 u_2} \varphi \end{pmatrix} (t, x, u).$$

Example 2 The dynamic NEPSC (2) with a separable cost function

$$\varphi_\nu(t, y, u) = \phi_\nu(t, y, u_\nu) + \hat{\phi}_\nu(t, y, u_{-\nu}), \quad \nu = 1, \ldots, N$$

satisfies condition (1) of Proposition 1. It is easy to see that condition (1) of Proposition 1 holds with $\nabla^2_{u_\nu u_i} \varphi_\nu(t, y, u) = 0$ for $\nu \neq i$, and $\nu, i = 1, \ldots, N$. By the definition of Ψ, the Jacobian of $\Psi(t, x, u)$ is a block diagonal matrix with the form

$$\nabla_u \Psi(t, x, u) = \text{diag}(\nabla^2_{u_1 u_1} \phi_1(t, y, u_1), \ldots, \nabla^2_{u_N u_N} \phi_N(t, y, u_N)).$$

Assumption 1 implies that $\nabla^2_{u_\nu u_\nu}\phi_\nu(t, y, u_\nu)$, $\nu = 1, \ldots, N$ are positive semi-definite. Hence $\Psi(t, x, \cdot)$ is monotone. Such separable cost function includes linear functions as a special case.

Example 3 Environmental pollution games with shared constraints, quadratic cost functionals and linear dynamics can be formulated as a dynamic monotone VI. The static river basin pollution game in [26,33] can be extended to a dynamic game. For the ν-th player, let the dynamic be linear with respect to $u_\nu \in R^{m_1}$, and let the cost function be

$$\varphi_\nu(t, y, u) = u_\nu^T \left(Q_\nu u_\nu + B \sum_{i=1}^{N} u_i + p_\nu \right) + q_\nu(y, t),$$

where $B, Q_\nu \in R^{m_1 \times m_1}$ are positive semi-definite, and $p_\nu \in R^{m_1}$, $\nu = 1, \ldots, N$. From (2) of Proposition 1 it follows that the function $\Psi(t, x, \cdot)$ is monotone.

In the case $Q_\nu = 0$, Assumption 1 implies Assumption (A2).

It is well known that the VI can be equivalently reformulated as a system of equations, namely, $u \in \mathrm{SOL}(U, \Psi(t, x, \dot{)})$ if and only if

$$G(t, x, u) := u - \Pi_U(u - \Psi(t, x, u)) = 0, \tag{9}$$

where $\Pi_U(\cdot)$ is the projection taken onto U in ℓ_2 norm. Now we study the solvability of the DVI (1) by equivalently rewriting it as a differential algebraic equation

$$\begin{aligned} \dot{x}(t) &= F(t, x(t), u(t)) \\ 0 &= G(t, x(t), u(t)) \\ 0 &= \Gamma(x(0), x(T)). \end{aligned} \tag{10}$$

From Assumption (A2) on the monotonicity of $\Psi(t, x, \cdot)$, it follows that the mapping $\hat{G}(u) := G(t, x, u)$ is weak univalent for any fixed t and x:

Definition 1 A mapping $\hat{G} : U \subseteq R^m \to R^m$ is said to be weakly univalent on its domain if it is continuous and there exists a sequence of univalent (i.e., continuous and injective) functions $\{G^k\}$ from U into R^m such that $\{G^k\}$ converges to $\hat{G}$ uniformly on bounded subsets of U.

Denote by $\mathcal{N}(x, r)$ the open ball centered by x with the radius of r in the ℓ_2 norm. The weakly univalent functions have the following properties which are useful for studying the solvability of the DVI. See Corollary 3.6.5 of [16].

Lemma 2 ([16]) *Let $\hat{G} : R^m \to R^m$ be weakly univalent. Suppose $\hat{G}^{-1}(0) \neq \emptyset$. If $\hat{G}^{-1}(0)$ is compact, then for every $\epsilon > 0$ there exists $\delta > 0$ such that for every weakly univalent function $\tilde{G} : R^m \to R^m$ satisfying*

$$\sup\{\|\hat{G}(u) - \tilde{G}(u)\| \mid x \in cl(\hat{G}^{-1}(0) + \mathcal{N}(0, \epsilon))\} \leq \delta,$$

where "cl" denotes the closure of a set, we have

$$\tilde{G}^{-1}(0) \subseteq \hat{G}^{-1}(0) + \mathcal{N}(0, \epsilon).$$

Denote

$$\mathcal{F}(t, x) := \{F(t, x, u) \,|\, u \in \mathcal{S}(t, x)\}$$

and

$$\Omega_\epsilon := \{u |\, \mathrm{dist}(u, \mathcal{S}(0, \hat{x}^0)) < \epsilon\}, \tag{11}$$

where $\mathrm{dist}(u, \mathcal{S}(0, \hat{x}^0)) = \min_{v \in \mathcal{S}(0, \hat{x}^0)} \|u - v\|_2$ is well defined if $\mathcal{S}(0, \hat{x}^0)$ is nonempty and bounded, and is closed because of the continuity of $G(t, x, \cdot)$.

By extending Lemma 2 we give the following properties of the set-valued mappings $\mathcal{S}(t, x)$ and $\mathcal{F}(t, x)$, serving as a preliminary of the solvability results for (10) and for the DVI (1).

Lemma 3 *Suppose that Assumption 2 holds, and $\Psi(\cdot, \cdot, u)$ is Lipschitzian near $(0, \hat{x}^0)$ for any $u \in \Omega_\epsilon$ with modular L_Ψ, where Ω_ϵ is defined by (11). Then the following statements hold:*

(i) $\exists\, \bar{T}, \bar{\delta} > 0$ *such that $\mathcal{S}(t, x)$ and $\mathcal{F}(t, x)$ are nonempty and bounded for any $(t, x) \in [0, \bar{T}] \times \mathcal{N}(\hat{x}^0, \bar{\delta})$;*
(ii) $\exists\, \bar{T}, \bar{\delta} > 0$ *such that $\mathcal{S}(t, x)$ and $\mathcal{F}(t, x)$ are upper semi-continuous in $(0, \bar{T}) \times \mathcal{N}(\hat{x}^0; \bar{\delta})$;*
(iii) $\exists\, T_0, \delta_0, \zeta > 0$ *such that $\mathcal{F}(\cdot, \cdot)$ maps $[0, T_0] \times \mathcal{N}(\hat{x}^0, \delta_0 + \zeta T_0)$ into $\mathcal{N}(0, \zeta)$.*

Remark 3 The proof for part (i) is similar to that of Theorem 2.4 in [9] for the P_0-function case. Here we give a simple proof for good readability.

Proof (i) From Lemma 2, it follows that there exists δ_1 such that

$$\sup_{u \in \Omega_\epsilon} \|G(t, x, u) - G(0, \hat{x}^0, u)\|_2 < \delta_1$$

implies

$$\emptyset \neq \mathcal{S}(t, x) \subseteq \Omega_\epsilon. \tag{12}$$

Choose $\bar{\delta}$ and $\bar{T}$ such that $L_\Psi(\bar{\delta} + \bar{T}) < \delta_1$. Then for any $(t, x) \in [0, \bar{T}] \times \mathcal{N}(\hat{x}^0, \bar{\delta})$, and any $u \in \mathcal{S}(0, \hat{x}^0) + \mathcal{N}(0, \epsilon)$, we have

$$\begin{aligned}
&\|G(t, x, u) - G(0, \hat{x}^0, u)\|_2 \\
&\quad \leq \|\Pi_U(u - \Psi(t, x, u)) - \Pi_U(u - \Psi(0, \hat{x}^0, u))\|_2 \\
&\quad \leq \|\Psi(t, x, u) - \Psi(0, \hat{x}^0, u)\|_2 \\
&\quad \leq L_\Psi(t + \|x - \hat{x}^0\|_2) < \delta_1.
\end{aligned} \tag{13}$$

Therefore, $\mathcal{S}(t,x)$ is nonempty and bounded, and so is $\mathcal{F}(t,x)$, which is due to the continuity of F.

(ii) Let $(t,x) \in (0,\bar{T}) \times \mathcal{N}(\hat{x}^0,\bar{\delta})$ be given, let Δt and Δx such that $(t+\Delta t, x+\Delta x) \in (0,\bar{T}) \times \mathcal{N}(\hat{x}^0,\bar{\delta})$, and let $\epsilon' > 0$ be small enough. Denote

$$\Omega_{\epsilon'} := \{u \mid \mathrm{dist}(u, \mathcal{S}(t,x)) < \epsilon'\}.$$

Again from Lemma 2, it follows that there is δ_2 such that $\emptyset \neq \mathcal{S}(t,x) \subseteq \Omega_{\epsilon'}$ if

$$\sup_{u\in\Omega_{\epsilon'}} \|G(t+\Delta t, x+\Delta x, u) - G(t,x,u)\|_2 < \delta_2.$$

Choose $\tilde{\delta}$ and $\tilde{T}$ such that $L_\Psi(\tilde{\delta}+\tilde{T}) < \delta_2$. Then if $|\Delta t| \leq \tilde{T}$ and $\|\Delta x\|_2 \leq \tilde{\delta}$ we have for any $u \in \Omega_{\epsilon'}$

$$\begin{aligned}
&\|G(t+\Delta t, x+\Delta x, u) - G(t,x,u)\|_2 \\
&\quad \leq \|\Pi_U(u - \Psi(t+\Delta t, x+\Delta x, u)) - \Pi_U(u - \Psi(t,x,u))\|_2 \\
&\quad \leq \|\Psi(t+\Delta t, x+\Delta x, u) - \Psi(t,x,u)\|_2 \\
&\quad \leq L_\Psi(\Delta t + \|\Delta x\|_2) < \delta_2.
\end{aligned}$$

Therefore, $\mathcal{S}(t+\Delta t, x+\Delta x) \subseteq \Omega_{\epsilon'}$, which gives the upper semi-continuity of $\mathcal{S}$ at (t,x). The upper semi-continuity of $\mathcal{F}$ is a direct consequence from that of $\mathcal{S}$.

(iii) Denote

$$\zeta_0 = \sup\{\|u\|_2 \mid u \in \mathcal{S}(0,\hat{x}^0) + \mathcal{N}(0,\epsilon)\}. \tag{14}$$

From (12), it follows $\mathcal{S}(t,x) \subseteq \mathcal{N}(0,\zeta_0)$ for any $(t,x) \in [0,\bar{T}] \times \mathcal{N}(\hat{x}^0,\bar{\delta})$, and so $\mathcal{F}(t,x) \subseteq \mathcal{N}(0,\zeta)$, where

$$\begin{aligned}
\zeta &:= \sup\{\|F(t,x,u)\|_2 \mid (t,x,u) \in [0,\bar{T}] \times \mathcal{N}(\hat{x}^0,\bar{\delta}) \times \mathcal{N}(0,\zeta_0)\} \\
&\geq \sup\{\|z\|_2 \mid z \in \mathcal{F}(t,x)\}.
\end{aligned} \tag{15}$$

Taking $\delta_0, T_0 > 0$ such that $\delta_0 + \zeta T_0 < \bar{\delta}$, we draw the conclusion.

□

If $\Psi(t,x,\cdot)$ is monotone and continuous, $\mathcal{S}(t,x) \neq \emptyset$ implies it is convex and closed. Therefore we can define the single-valued mapping

$$\mathcal{P}(t,x) = \Pi_{\mathcal{S}(t,x)}(0). \tag{16}$$

Clearly, $\mathcal{P}(t,x)$ is the least norm element of $\mathcal{S}(t,x)$. Below we give a solvability result of (1) by using the least norm solution.

Theorem 2 *Suppose that Assumption 2 holds, $\Psi(\cdot,\cdot,u)$ is Lipschitzian near $(0,\hat{x}^0)$ for any $u \in \Omega_\epsilon$ with modular L_Ψ, and $\Gamma(\cdot,\cdot)$ is Lipschitzian near $(\hat{x}^0,\hat{x}^0)$ with the modular L_Γ, where Ω_ϵ is defined by (11) with a fixed $\epsilon > 0$. If $\mathcal{S}(t,x)$ is lower semi-continuous near $(0,\hat{x}^0)$ or $\mathcal{F}(t,x)$ is singleton, then there exist $T,\delta_0,\zeta > 0$ such that the boundary value problem (1) has a solution (x,u) over $[0,T]$, where $x(t)$ is continuously differentiable, $x(0) \in \mathcal{N}(\hat{x}^0,\delta_0)$, $x(t) \in \mathcal{N}(\hat{x}^0,\delta_0+\zeta T)$ for any $t \in [0,T]$, and $u(t)$ is continuous and is the least norm element of $\mathcal{S}(t,x(t))$.*

Remark 4 The singleton assumption was imposed in [6].

Proof From Lemma 3 it follows that $\mathcal{S}(t,x)$ is upper semi-continuous in $(0,\bar{T}) \times \mathcal{N}(\xi,\bar{\delta})$. If it is moreover lower semi-continuous, then $\mathcal{S}(t,x)$ is continuous in $(0,\bar{T}) \times \mathcal{N}(\hat{x}^0,\bar{\delta})$. Assumption 2 (A2) implies $\mathcal{S}(t,x)$ is convex and $\mathcal{P}(t,x)$ is continuous. Hence $\mathcal{F}_\mathcal{P}(t,x) := F(t,x,\mathcal{P}(t,x))$ is continuous, by the continuity of $F(\cdot,\cdot,\cdot)$. Alternatively, if $\mathcal{F}(t,x)$ is singleton, then it is continuous because it is upper semi-continuous in $(0,\bar{T}) \times \mathcal{N}(\hat{x}^0,\bar{\delta})$. Moreover, $\mathcal{F}_\mathcal{P}(t,x) = \mathcal{F}(t,x)$ is continuous.

From (iii) of Lemma 3, and by noting $\mathcal{F}_\mathcal{P}(t,x) \in \mathcal{F}(t,x)$, we conclude that there exist $T_0,\delta_0 > 0$ such that $\mathcal{F}_\mathcal{P}(\cdot,\cdot)$ maps $[0,T_0] \times \mathcal{N}(\hat{x}^0,\delta_0+\zeta T_0)$ into $\mathcal{N}(0,\zeta)$. Applying the Peano existence theorem to

$$\begin{cases} \dot{x}(t) = \mathcal{F}_\mathcal{P}(t,x) \\ x(0) = \eta, \end{cases}$$

we know that for any $\eta \in \mathcal{N}(\hat{x}^0,\delta_0)$,

$$\begin{cases} \dot{x}(t) = F(t,x(t),u(t)) \\ u(t) \in \mathrm{SOL}(U,\Psi(t,x(t),\cdot)) \\ x(0) = \eta \end{cases} \tag{17}$$

has a solution (x,u) over $[0,T_0]$, where $x(t)$ is continuously differentiable, and $u(t)$ is the least norm element of $\mathrm{SOL}(U,\Psi(t,x(t),\cdot))$. Noting

$$x(t) = \eta + \int_0^t F(s,x(s),u(s))ds,$$

clearly, we have $x(t) \in \mathcal{N}(\hat{x}^0,\delta_0+\zeta T_0)$ for any $t \in [0,T_0]$. Therefore, for $(t,\eta) \in [0,T_0] \times \mathcal{N}(\hat{x}^0,\delta_0)$, we can define the operator

$$\mathcal{A}(t,\eta) = \{x(t) | \dot{x}(t) = \mathcal{F}_\mathcal{P}(t,x) \text{ with } x(0) = \eta\}. \tag{18}$$

From Theorem 2.2.1 [1, p. 104], it follows that $\mathcal{A}(t,\cdot)$ is continuous with $\mathcal{A}(0,\cdot)$ being identity. And for any $0 \le t < T_0$ and $\eta \in \mathcal{N}(\xi,\delta_0)$ we have

$$\|\mathcal{A}(t,\eta) - \eta\|_2 = \|x(t) - \eta\|_2 \le \int_0^t \|F(s,x(s),u(s))\|_2 ds \le t\zeta,$$

and so

$$\|\Gamma(\eta, \mathcal{A}(t, \eta)) - \Gamma(\eta, \eta)\|_2 \leq L_\Gamma \|\mathcal{A}(t, \eta) - \eta\|_2 \leq L_\Gamma \zeta t.$$

We remind us that $\Gamma(x, x)$ has a unique solution $\hat{x}^0$ with a nonsingular Jacobian, so $\Gamma(\hat{x}^0, \hat{x}^0) + \epsilon \nabla_x \Gamma(\hat{x}^0, \hat{x}^0) e^j$, $j = 1, \ldots, 2n$ will span a neighborhood $\mathcal{N}_\Gamma$ of $\Gamma(\hat{x}^0, \hat{x}^0)$, where ϵ is a small positive number and e^j is the jth column of the identity matrix [27, p. 148]. Hence, there must be a sufficiently small $0 < T < T_0$ such that $\Gamma(\eta, \mathcal{A}(\eta)) \in \mathcal{N}_\Gamma$ and $\Gamma(\eta, \eta) \in \mathcal{N}_\Gamma$ share the same degree near $\hat{x}^0$ [27, Theorem 6.2.1], which implies that the boundary value condition in (1) is fulfilled in $\mathcal{N}(\hat{x}^0, \delta_0)$. Obviously, we have $x(t) \in \mathcal{N}(\hat{x}^0, \delta_0 + \zeta T)$ for any $t \in [0, T]$. This completes the proof. □

Remark 5 In Theorem 2, the time span $[0, T]$ is required small enough. The locality of the existence of the DVI is typical in the existing work, see [19,29]. For the dynamic NEPSC with linear dynamics and quadratic cost functionals which are strictly convex, we can show that the initial value problem (17) has a unique solution over any time span [8]. However, for fulfilling the boundary value condition, additional assumptions are needed. For example, assuming that $\mathcal{F}(t, x)$ is a singleton for a general DVI with boundary value conditions in [30] and adding conditions on matrices involved in an affine DVI coming from the optimal control problems in [18].

4 Regularization and smoothing approximation

The formulated DVI (10) is a dynamical system over the non-smooth manifold defined by the system $G(t, x, u) = 0$, which may have no solution, or have multiple (possibly infinitely many) solutions, where $G(t, x, u)$ is defined in (9). Finding a solution of the system involves solving optimization problems without standard constraint qualifications at each grid.

In this section, we propose a regularized smoothing method to find a solution of (10). Our main idea is to replace $G(t, x, u)$ in (10) by the following regularized and smoothing function

$$G_{\lambda,\mu}(t, x, u) = \int_R [u - \Pi_U(u - \Psi(t, x, u) - \lambda u - \mu s e)]\, \rho(s) ds, \qquad (19)$$

where $\lambda > 0$ and $\mu > 0$ are the regularization and smoothing parameters. The integration is performed componentwise with $e = (1, 1, \ldots, 1)^T$ and $\rho(\cdot)$ is a density function with

$$\kappa = \int_R |s| \rho(s) ds < \infty.$$

For any fixed (t, x), the system

$$G_{\lambda,\mu}(t, x, u) = 0 \qquad (20)$$

has a unique solution u, which is continuously dependent on (t, x). Namely, we approximate (10) by the following differential algebraic system

$$\begin{aligned} \dot{x}(t) &= F(t, x(t), u(t)) \\ 0 &= G_{\lambda,\mu}(t, x, u) \\ 0 &= \Gamma(x(0), x(T)). \end{aligned} \tag{21}$$

In the following we will show that the system (21) has a classic solution $(x_{\lambda,\mu}(t), u_{\lambda,\mu}(t))$ and prove the convergence of the family of the classic solutions as $\lambda, \mu \downarrow 0$.

4.1 Regularization and smoothing for the static VI

When $\mu = 0$, the regularized system

$$G_{\lambda,0}(t, x, u) := u - \Pi_U(u - \Psi(t, x, u) - \lambda u) = 0$$

has a unique solution u for any fixed (t, x), but $G_{\lambda,0}$ and u may not be differentiable with respect to (t, x). To overcome the non-smoothness of the projection operator, we adopt the smoothing approximation. The regularized smoothing function $G_{\lambda,\mu}(t, x, u)$ has the following properties

$$\|G_{\lambda,0}(t, x, u) - G(t, x, u)\|_2 \le \lambda \|u\|_2$$

and

$$\|G_{\lambda,\mu}(t, x, u) - G_{\lambda,0}(t, x, u)\|_2 \le \kappa\sqrt{m}\mu. \tag{22}$$

For fixed $t \in R$, $x \in R^{2n}$, $\lambda > 0$ and $\mu > 0$ the mapping $G_{\lambda,\mu}(t, x, \cdot)$ is continuously differentiable and the system (20) has a unique solution $u_{\lambda,\mu}(t, x)$. For the properties of smoothing approximations, we refer to [7,16,17].

Smoothing approximation and regularization have been studied extensively in solving the static VI [16]. However, to the best of our knowledge, using both smoothing approximation and regularization to find the least norm solution of the monotone VI has not been studied. We derive sufficient conditions for the existence of the limit

$$\mathcal{S}_0(t, x) := \left\{ \lim_{\lambda,\mu\downarrow 0} u_{\lambda,\mu}(t, x) \right\}. \tag{23}$$

Moreover, we show that if $\mu = o(\lambda)$, then the limit of (23) is the least norm element of the solution set $\mathcal{S}(t, x) = \mathrm{SOL}(U, \Psi(t, x, \cdot))$. Note that finding the least norm solution is significant since it can provide a stable solution path of the DVI [10,11,19].

First of all, we use the following example to show that the relation of the two parameters λ, μ has a considerable impact on the behavior of the limit (23).

Example 4 Let $U = R^2_+$, and for a fixed (t, x) let

$$\Psi(t, x, u) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} u + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Obviously, we have $\mathcal{S}(t, x) = \{0\} \times [0, 1]$. Let us choose the following density function

$$\rho(s) = \frac{2}{(s^2 + 4)^{\frac{3}{2}}},$$

which has been used to define the so-called Chen-Harker-Kanzow-Smale smoothing function of $\max(0, u)$. The regularized smoothing function can be given by

$$G_{\lambda,\mu}(t, x, u) = \begin{pmatrix} (1 + \lambda)u_1 - u_2 + 1 - \sqrt{((1 - \lambda)u_1 + u_2 - 1)^2 + 4\mu^2} \\ (1 + \lambda)u_2 + u_1 - \sqrt{((1 - \lambda)u_2 - u_1)^2 + 4\mu^2} \end{pmatrix}.$$

For any fixed $\lambda > 0$ and $\mu > 0$, the solution of $G_{\lambda,\mu}(t, x, u) = 0$ satisfies $(\lambda u_1 - u_2 + 1)u_1 = \mu^2$ and $(u_1 + \lambda u_2)u_2 = \mu^2$. Since the solution set $\mathcal{S}(t, x)$ is bounded, the solution of $G_{\lambda,\mu}(t, x, u) = 0$ is bounded when $\lambda \to 0$ and $\mu \to 0$ [16]. Adding these two equations gives $\lambda(u_2^2 + u_1^2) + u_1 = 2\mu^2$, which, together with the boundedness of the solution, implies $(u_{\lambda,\mu})_1 \to 0$ as $\lambda, \mu \downarrow 0$. Moreover, from $(u_1 + \lambda u_2)u_2 = \mu^2$, we have $u_2 = (-u_1 + \sqrt{u_1^2 + 4\lambda\mu^2})/(2\lambda) \le \mu/\sqrt{\lambda}$. Figure 1 shows the trajectories of $u_{\lambda,\mu}(t, x)$ when $\lambda, \mu \to 0$ from 1 with different order, where the limit points are marked by "×". We see that $u_{\lambda,\mu}(t, x)$ converges to the least norm solution $(0, 0)$ as $\mu = \lambda^2 \to 0$, and converges to $(0, 0.1413)$, $(0, 0.3445)$ and $(0, 0.4534)$ as $\mu = \lambda^{0.8} \to 0$, $\mu = \lambda^{0.5} \to 0$ and $\mu = \lambda^{0.01} \to 0$, respectively.

Now we study the system (20) where we take $\mu = o(\lambda)$. It is obvious that $G_{\lambda,\mu}(t, x, \cdot)$ is continuously differentiable, univalent for any (t, x) and $\lambda \in [0, \bar{\lambda})$, and it holds

$$\|G_{\lambda,\mu}(t, x, u) - G(t, x, u)\|_2 \le \lambda\|u\|_2 + \kappa\sqrt{m}\mu \le (\|u\|_2 + \alpha)\lambda, \tag{24}$$

where $\alpha > 0$ is a constant independent of t, x, u and λ. We remind us that the system $G_{\lambda,\mu}(t, x, u) = 0$ has a unique solution $u_{\lambda,\mu}(t, x)$ for fixed (t, x) and $\lambda > 0$ and $\mu > 0$. We will study the convergence of $u_{\lambda,\mu}(t, x)$ to a certain element of the solution set $\mathcal{S}(t, x)$. The solution $u_{\lambda,\mu}(t, x)$ has the following properties.

Theorem 3 *Suppose that (A2) of Assumption 2 holds. If $\mathcal{S}(t, x)$ is nonempty and bounded and $u_{\lambda,\mu}(t, x) \in U$ for λ small enough with $\mu = o(\lambda)$, then* $\lim_{\lambda\downarrow 0} u_{\lambda,\mu}(t, x)$ *exists and is the least norm element of $\mathcal{S}(t, x)$.*

Proof Denote by $u_{\lambda,\mu}$ the unique solution of (20) and let $\tilde{u}$ be the least norm element of $\mathcal{S}(t, x)$. Denoting $u^* = u_{\lambda,\mu} - \Psi(t, x, u_{\lambda,\mu}) - \lambda u_{\lambda,\mu}$, we have

$$\begin{aligned}\|u_{\lambda,\mu} - \Pi_U(u^*)\|_2 &= \|u_{\lambda,\mu} - \Pi_U(u_{\lambda,\mu} - \Psi(t, x, u_{\lambda,\mu}) - \lambda u_{\lambda,\mu})\|_2 \\ &= \|G_{\lambda,\mu}(t, x, u_{\lambda,\mu}) - G_{\lambda,0}(t, x, u_{\lambda,\mu})\|_2 \le \mu\sqrt{m}\kappa.\end{aligned}$$

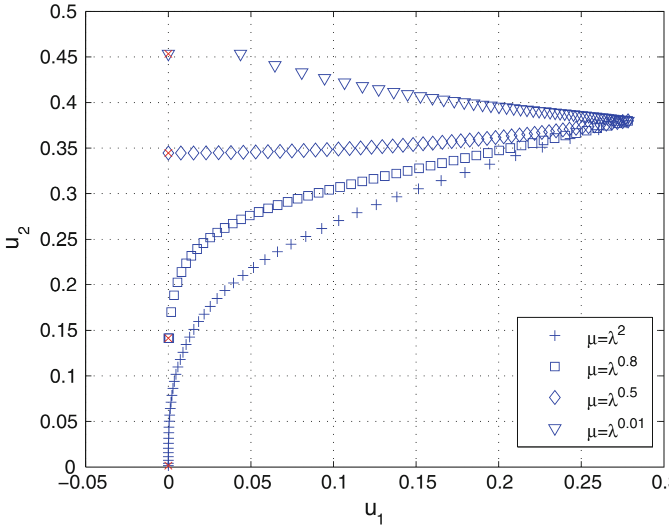

Fig. 1 Example 1, convergence of $u_{\lambda,\mu}(t,x)$ as $\lambda,\mu \downarrow 0$

Considering $\tilde{u} \in U$ and the well-known property of the projection

$$(\Pi_U(u^*) - \tilde{u})^T(u^* - \Pi_U(u^*)) \geq 0,$$

and noting $u^* - \Pi_U(u^*) = G_{\lambda,0}(t,x,u_{\lambda,\mu}) - \Psi(t,x,u_{\lambda,\mu}) - \lambda u_{\lambda,\mu}$, we have

$$\begin{aligned}&(u_{\lambda,\mu} - \tilde{u})^T(G_{\lambda,0}(t,x,u_{\lambda,\mu}) - \Psi(t,x,u_{\lambda,\mu}) - \lambda u_{\lambda,\mu})\\ &\quad \geq (u_{\lambda,\mu} - \Pi_U(u^*))^T(u^* - \Pi_U(u^*)) \ \geq\ -\mu\sqrt{m}\kappa\|u^* - \Pi_U(u^*)\|_2.\end{aligned}$$

Moreover noting

$$\begin{aligned}&(u_{\lambda,\mu} - \tilde{u})^T(G_{\lambda,\mu}(t,x,u_{\lambda,\mu}) - G_{\lambda,0}(t,x,u_{\lambda,\mu}))\\ &\quad \geq -\|u_{\lambda,\mu} - \tilde{u}\|_2\|G_{\lambda,\mu}(t,x,u_{\lambda,\mu}) - G_{\lambda,0}(t,x,u_{\lambda,\mu})\|_2\\ &\quad \geq -\|u_{\lambda,\mu} - \tilde{u}\|_2\mu\sqrt{m}\kappa,\end{aligned}$$

and from the monotonicity of $\Psi(t,x,\cdot)$, we have

$$\begin{aligned}0 &\geq (u_{\lambda,\mu} - \tilde{u})^T(G_{\lambda,\mu}(t,x,u_{\lambda,\mu}) - \Psi(t,x,\tilde{u}))\\ &= (u_{\lambda,\mu} - \tilde{u})^T(\Psi(t,x,u_{\lambda,\mu}) - \Psi(t,x,\tilde{u}) + G_{\lambda,\mu}(t,x,u_{\lambda,\mu}) - \Psi(t,x,u_{\lambda,\mu}))\\ &\geq (u_{\lambda,\mu} - \tilde{u})^T(G_{\lambda,\mu}(t,x,u_{\lambda,\mu}) - \Psi(t,x,u_{\lambda,\mu}))\\ &= (u_{\lambda,\mu} - \tilde{u})^T(\lambda u_{\lambda,\mu}) + (u_{\lambda,\mu} - \tilde{u})^T(G_{\lambda,0}(t,x,u_{\lambda,\mu}) - \Psi(t,x,u_{\lambda,\mu}) - \lambda u_{\lambda,\mu})\\ &\quad + (u_{\lambda,\mu} - \tilde{u})^T(G_{\lambda,\mu}(t,x,u_{\lambda,\mu}) - G_{\lambda,0}(t,x,u_{\lambda,\mu}))\\ &\geq (u_{\lambda,\mu} - \tilde{u})^T(\lambda u_{\lambda,\mu}) - \mu\sqrt{m}\kappa\|u^* - \Pi_U(u^*)\|_2 - \|u_{\lambda,\mu} - \tilde{u}\|_2\mu\sqrt{m}\kappa,\end{aligned}$$

therefore,

$$(u_{\lambda,\mu} - \tilde{u})^T (\lambda u_{\lambda,\mu}) \leq \mu\sqrt{m}\kappa \left(\|u^* - \Pi_U(u^*)\|_2 + \|u_{\lambda,\mu} - \tilde{u}\|_2\right),$$

and

$$\begin{aligned}(u_{\lambda,\mu}, u_{\lambda,\mu}) &\leq (\tilde{u}, u_{\lambda,\mu}) + \frac{\mu}{\lambda}\sqrt{m}\kappa \left(\|u^* - \Pi_U(u^*)\|_2 + \|u_{\lambda,\mu} - \tilde{u}\|_2\right) \\ &\leq \|\tilde{u}\|_2 \|u_{\lambda,\mu}\|_2 + \frac{\mu}{\lambda}\sqrt{m}\kappa \left(\|u^* - \Pi_U(u^*)\|_2 + \|u_{\lambda,\mu} - \tilde{u}\|_2\right).\end{aligned}$$

Let $\lambda_k \downarrow 0$ and $\mu_k \downarrow 0$ when $k \to \infty$. It can be readily shown that $\{u_{\lambda_k,\mu_k}\}$ is bounded [16] as $\mathcal{S}(t,x)$ is nonempty and bounded. Let u be an accumulation point of $\{u_{\lambda_k,\mu_k}\}$. Considering that $\mu = o(\lambda)$, we know $u = \tilde{u}$ is the least norm element of $\mathcal{S}(t,x)$. This completes the proof. □

Remark 6 For $\mu = 0$, we have $u_{\lambda,0}(t,x) \in U$. For $\mu \neq 0$, we can choose a suitable smoothing approximation to ensure $u_{\lambda,\mu}(t,x) \in U$. For example, for $U = R^m_+$ (the complementarity problem), if we use the Chen-Harker-Kanzow-Smale smoothing function (See Example 4) to define the smoothing approximation, then we have $u_{\lambda,\mu}(t,x) \in R^m_+$ for any $\lambda, \mu \geq 0$.

As shown in Example 4, the condition $\mu = o(\lambda)$ may be loosened for guaranteeing the convergence of $u_{\lambda,\mu}(t,x)$ to a solution, which, however, may not be the least norm solution.

4.2 Regularized smoothing DVI

Approximating $G(t,x,u)$ by $G_{\lambda,\mu}(t,x,u)$ defined by (19), we get the regularized smoothing system (21) of the DVI (1). In this section, we show that the system (21) has a unique classic solution $(x_{\lambda,\mu}(t), u_{\lambda,\mu}(t))$ for any $\lambda > 0, \mu > 0$ under certain conditions. Moreover, we derive the convergence analysis of the family $\{(x_{\lambda,\mu}(t), u_{\lambda,\mu}(t))\}_{\lambda>0}$ when $\lambda \downarrow 0$ and $\mu = o(\lambda)$.

Lemma 4 *Suppose that Assumption 2 holds, and $\Psi(\cdot,\cdot,u)$ is Lipschitzian near $(0,\hat{x}^0)$ for any $u \in \Omega_\epsilon$ with modular L_Ψ, where Ω_ϵ is defined by (11). Let*

$$\mathcal{F}_{\lambda,\mu}(t,x) = F(t,x,u_{\lambda,\mu}(t,x)).$$

Then there exist $\lambda_0, \mu_0, T_0, \delta_0, \zeta > 0$ such that $\forall\lambda \in [0,\lambda_0]$ and $\forall\mu \in [0,\bar{\mu}]$, $\mathcal{F}_{\lambda,\mu}(\cdot,\cdot)$ maps $[0,T_0] \times \mathcal{N}(\hat{x}^0, \delta_0 + \zeta T_0)$ into $\mathcal{N}(0,\zeta)$.

Proof From Lemma 2, it follows that there exists δ_1 such that

$$\sup_{u\in\Omega_\epsilon} \|G_{\lambda,\mu}(t,x,u) - G(0,\hat{x}^0,u)\|_2 < \delta_1$$

implies that there is $u_{\lambda,\mu}(t,x) \in \Omega_\epsilon$. Let ζ_0 be defined by (14). It is clear that $\Omega_\epsilon \subseteq \mathcal{N}(0,\zeta_0)$. Then by using inequalities (13) and (24), for any $(t,x) \in [0,\bar{T}] \times$

$\mathcal{N}(\hat{x}^0, \bar{\delta})$, $u \in \mathcal{S}(0, \hat{x}^0) + \mathcal{N}(0, \epsilon)$, $\lambda \in [0, \bar{\lambda}]$ and $\mu \in [0, \bar{\mu}]$, we have

$$\begin{aligned}&\|G_{\lambda,\mu}(t, x, u) - G(0, \hat{x}^0, u)\|_2 \\ &\quad \le \|G_{\lambda,\mu}(t, x, u) - G(t, x, u)\|_2 + \|G(t, x, u) - G(0, \hat{x}^0, u)\|_2 \\ &\quad \le \lambda\|u\|_2 + \kappa\sqrt{m}\mu + L_\Psi(t + \|x - \hat{x}^0\|_2).\end{aligned}$$

Choosing positive numbers $\bar{\delta}$, $\bar{T}$, $\bar{\lambda}$ and $\bar{\mu}$ such that

$$L_\Psi(\bar{\delta} + \bar{T}) < \frac{\delta_1}{3}, \qquad \bar{\lambda}\zeta_0 < \frac{\delta_1}{3}, \qquad \kappa\sqrt{m}\bar{\mu} < \frac{\delta_1}{3},$$

we obtain

$$\|G_{\lambda,\mu}(t, x, u) - G(0, \hat{x}^0, u)\|_2 \le \delta_1.$$

Hence $u_{\lambda,\mu}(t, x) \in \Omega_\epsilon \subseteq \mathcal{N}(0, \zeta_0)$ and $\mathcal{F}_{\lambda,\mu}(t, x) \in \mathcal{N}(0, \zeta)$, where ζ is defined by (15). Taking $\lambda_0, \delta_0, T_0 > 0$ such that $\delta_0 + \zeta T_0 < \bar{\delta}$, we draw the conclusion. □

Theorem 4 *Suppose that Assumption 2 holds, $\Psi(\cdot, \cdot, u)$ is Lipschitzian near $(0, \hat{x}^0)$ for any $u \in \Omega_\epsilon$ with modular L_Ψ, and $\Gamma(\cdot, \cdot)$ is Lipschitzian near $(\hat{x}^0, \hat{x}^0)$ with the modular L_Γ, where Ω_ϵ is defined by (11). If $\mathcal{S}(t, x)$ is lower semi-continuous near $(0, \hat{x}^0)$ or $\mathcal{F}(t, x)$ is singleton, then there exist $\lambda_0, T, \delta_0, \zeta > 0$ such that for any $0 < \lambda \le \lambda_0$ the regularized smoothing system (21) has a classical solution $(x_{\lambda,\mu}, u_{\lambda,\mu})$ over $[0, T]$, where $x_{\lambda,\mu}(0) \in \mathcal{N}(\hat{x}^0, \delta_0)$, and $x_{\lambda,\mu}(t) \in \mathcal{N}(\hat{x}^0, \delta_0 + \zeta T)$ for any $t \in [0, T]$.*

Proof From Theorem 3 and Lemma 4, it follows that there exist $\lambda_0, T_0, \delta_0, \zeta > 0$ such that $\forall\lambda \in [0, \lambda_0]$, $\mathcal{F}_{\lambda,\mu}(\cdot, \cdot)$ is continuous and maps $[0, T_0] \times \mathcal{N}(\hat{x}^0, \delta_0 + \zeta T_0)$ into $\mathcal{N}(0, \zeta)$. Then by Theorem 2.1.3 of [1], we know that

$$\begin{cases} \dot{x}(t) = F(t, x(t), u(t)) \\ 0 = G_{\lambda,\mu}(t, x(t), u(t)) \\ x(0) = \eta \end{cases} \tag{25}$$

has a solution $(x_{\lambda,\mu}, u_{\lambda,\mu})$ over $[0, T_0]$, where $x_{\lambda,\mu}(t)$ is continuously differentiable. The remaining part can be proved in the same manner as used in the proof for Theorem 2. □

Denote by $\mathcal{X}$ and $\mathcal{U}$ the spaces of the continuous functions and the square integrable functions over $[0, T]$, respectively, and denote for $x \in \mathcal{X}$

$$\|x\|_C := \sup_{t\in[0,T]} \|x(t)\|_2,$$

and denote for $u \in \mathcal{U}$

$$\|u\|_{L_2} := \langle u, u\rangle^{1/2}, \quad \text{where} \quad \langle u, v\rangle := \int_0^T u(t)^T v(t)dt.$$

We define the norm for $(x, u, \eta) \in \mathcal{W}_1 = \mathcal{X} \times \mathcal{U} \times R^n$:

$$\|(x, u, \eta)\|_{\mathcal{W}_1} = \|x\|_C + \|u\|_{L^2} + \|\eta\|_2. \tag{26}$$

Let $\mathcal{Z}$ denote the space of the continuous functions in $\mathcal{U}$. For $(x, u, \eta) \in \mathcal{W}_2 = \mathcal{X} \times \mathcal{Z} \times R^n$ we denote

$$\|(x, u, \eta)\|_{\mathcal{W}_2} := \|x\|_C + \|u\|_C + \|\eta\|_2. \tag{27}$$

It is clear that $\mathcal{W}_2 \subset \mathcal{W}_1$, and both are Banach spaces under the norm (26) and (27), respectively. Define

$$\Phi(x, u, \eta)(t) = \begin{pmatrix} x(t) - \eta - \int_0^t F(\tau, x(\tau), u(\tau))d\tau \\ G(t, x, u) \\ \Gamma(\eta, x(T)) \end{pmatrix}. \tag{28}$$

Obviously, $\Phi(x, u, \eta) \in \mathcal{W}_1$ for an $(x, u, \eta) \in \mathcal{W}_1$, and $\Phi(x, u, \eta) \in \mathcal{W}_2$ if moreover $(x, u, \eta) \in \mathcal{W}_2$. Then we can reformulate (1) as a minimization problem over $\mathcal{W}_1$:

$$\min_{(x,u,\eta)\in\mathcal{W}_1} \|\Phi(x, u, \eta)\|_{\mathcal{W}_1}.$$

Obviously, $\|\Phi(x, u, \eta)\|_{\mathcal{W}_1} = 0$ implies that (x, u) is a weak solution of (1). For a continuous u, then $\|\Phi(x, u, \eta)\|_{\mathcal{W}_2} = 0$ implies that (x, u) is a classic solution.

Let

$$\Phi_{\lambda,\mu}(x, u, \eta)(t) = \begin{pmatrix} x(t) - \eta - \int_0^t F(\tau, x(\tau), u(\tau))d\tau \\ G_{\lambda,\mu}(t, x, u) \\ \Gamma(\eta, x(T)) \end{pmatrix}, \tag{29}$$

where $G_{\lambda,\mu}$ is a smoothing regularization of G satisfying (24).

From Theorem 4 it follows that the regularized smoothing system (21) has a classic solution $(x_{\lambda,\mu}(t), u_{\lambda,\mu}(t))$ with $x_{\lambda,\mu}(0) = \eta_{\lambda,\mu}$. Then $(x_{\lambda,\mu}, u_{\lambda,\mu}, \eta_{\lambda,\mu})$ is a minimizer of the functional $\|\Phi_{\lambda,\mu}(x, u, \eta)\|_{\mathcal{W}_i}$. Here we study the convergence of $\{(x_{\lambda,\mu}, u_{\lambda,\mu}, \eta_{\lambda,\mu})\}_{k=1}^{\infty}$ by the so-called epigraphical convergence of the functional $\|\Phi_{\lambda,\mu}(x, u, \eta)\|_{\mathcal{W}_i}$ when $\lambda \downarrow 0$ and $\mu \downarrow 0$.

Let $\{\Phi^k\}_{k=1}^{\infty}$ be a sequence of approximate mappings of Φ. Taking $k \to \infty$, $\{\|\Phi^k\|_{\mathcal{W}_i}\}_{k=1}^{\infty}$ is said to be epigraphically convergent to $\|\Phi\|_{\mathcal{W}_i}$ if

(a) for any $\{(x^k, u^k, \eta^k)\}_{k=1}^{\infty}$ with $(x^k, u^k, \eta^k) \to (x, u, \eta)$

$$\liminf_{k\to\infty} \|\Phi^k(x^k, u^k, \eta^k)\|_{\mathcal{W}_i} \geq \|\Phi(x, u, \eta)\|_{\mathcal{W}_i};$$

(b) there is $\{(x^k, u^k, \eta^k)\}_{k=1}^{\infty}$ with $(x^k, u^k, \eta^k) \to (x, u, \eta)$ such that

$$\limsup_{k\to\infty} \|\Phi^k(x^k, u^k, \eta^k)\|_{\mathcal{W}_i} \leq \|\Phi(x, u, \eta)\|_{\mathcal{W}_i},$$

where the convergence of $(x^k, u^k, \eta^k) \to (x, u, \eta)$ is defined by the norm $\|\cdot\|_{\mathcal{W}_i}$, $i = 1, 2$. See [31], for example.

Taking sequences $\lambda_k \downarrow 0$ and $\mu_k \downarrow 0$ when $k \to \infty$, we have the following epigraphical convergence of the sequence of the functionals $\{\|\Phi_{\lambda_k,\mu_k}\|_{\mathcal{W}_i}\}_{k=1}^{\infty}$.

Lemma 5 *Let* $\{\lambda_k\}_{k=1}^{\infty} \downarrow 0$ *be given and* $\mu_k = o(\lambda_k)$. *Then* $\{\|\Phi_{\lambda_k,\mu_k}\|_{\mathcal{W}_i}\}_{k=1}^{\infty}$ *is epigraphically convergent to* $\|\Phi\|_{\mathcal{W}_i}$ *for* $i = 1, 2$.

Proof Let $(x^k, u^k, \eta^k) \to (x, u, \eta)$ in $\mathcal{W}_i$. Noting $\sqrt{a+b+c} \le \sqrt{a} + \sqrt{b} + \sqrt{c}$ for any nonnegative numbers a, b and c, we can see

$$
\begin{aligned}
&\|\Phi_{\lambda_k,\mu_k}(x^k, u^k, \eta^k) - \Phi(x^k, u^k, \eta^k)\|_{\mathcal{W}_1} \\
&= \left(\int_0^T \left\| G_{\lambda_k,\mu_k}(t, x^k(t), u^k(t)) - G(t, x^k(t), u^k(t)) \right\|_2^2 dt \right)^{1/2} \\
&\le \left(\int_0^T \left(\lambda_k \|u^k(t)\|_2 + \kappa\sqrt{m}\mu_k \right)^2 dt \right)^{1/2} \\
&= \left(\int_0^T \lambda_k^2 \|u^k(t)\|_2^2 dt + \int_0^T 2\kappa\sqrt{m}\mu_k\lambda_k\|u^k(t)\|_2 dt + \int_0^T \kappa^2 m \mu_k^2 dt \right)^{1/2} \\
&\le \left(\int_0^T \lambda_k^2 \|u^k(t)\|_2^2 dt \right)^{1/2} + \left(\int_0^T 2\kappa\sqrt{m}\mu_k\lambda_k\|u^k(t)\|_2 dt \right)^{1/2} + \kappa\mu_k\sqrt{mT},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\Phi_{\lambda_k,\mu_k}(x^k, u^k, \eta^k) - \Phi(x^k, u^k, \eta^k)\|_{\mathcal{W}_2} \\
&= \sup_{t\in[0,T]} \left\| G_{\lambda_k,\mu_k}(t, x^k(t), u^k(t)) - G(t, x^k(t), u^k(t)) \right\|_2 \\
&\le \sup_{t\in[0,T]} (\lambda_k\|u^k(t)\|_2 + \kappa\sqrt{m}\mu_k) = \lambda_k\|u\|_C + \kappa\sqrt{m}\mu_k.
\end{aligned}
$$

Now we have $\|\Phi_{\lambda_k,\mu_k}(x^k, u^k, \eta^k)\|_{\mathcal{W}_i} - \|\Phi(x^k, u^k, \eta^k)\|_{\mathcal{W}_i} \to 0$.

On the other hand we know $\|\Phi(x^k, u^k, \eta^k)\|_{\mathcal{W}_i} - \|\Phi(x, u, \eta)\|_{\mathcal{W}_i} \to 0$ since $\|\Phi\|_{\mathcal{W}_i}$ is continuous. Therefore we can conclude $\|\Phi_{\lambda_k,\mu_k}(x^k, u^k, \eta^k)\|_{\mathcal{W}_i} \to \|\Phi(x, u, \eta)\|_{\mathcal{W}_i}$, which implies the epigraphical convergence of $\{\|\Phi_{\lambda_k,\mu_k}\|_{\mathcal{W}_i}\}_{k=1}^{\infty}$ to $\|\Phi\|_{\mathcal{W}_i}$. This completes the proof. □

Using Lemma 5 we give the following result on the convergence of the solution $(x_{\lambda,\mu}, u_{\lambda,\mu}, \eta_{\lambda,\mu})$ of the regularized smoothing system (21).

Theorem 5 *Suppose that the conditions of Theorem 4 hold. Take* $\mu = o(\lambda)$. *There exist* $\{\lambda_k\}_{k=1}^{\infty} \downarrow 0$, $x \in \mathcal{X}$ *and* $u \in \mathcal{U}$ *such that* $x_{\lambda_k,\mu_k} \to x$ *uniformly and* $u_{\lambda_k,\mu_k} \to u$

weakly, where $(x_{\lambda_k,\mu_k}, u_{\lambda_k,\mu_k})$ *is the classic solution of (21) for* $\lambda = \lambda_k$*. Moreover, if* $u_{\lambda_k,\mu_k} \to u$ *with respect to* $\|\cdot\|_{L^2}$*, then* (x, u) *is a weak solution of (1); if* $u_{\lambda_k,\mu_k} \to u$ *uniformly, then* (x, u) *is a classic solution of (1).*

Proof From Theorem 4, it follows that there exist $\lambda_0, \delta_0, \zeta > 0$ such that for any $\lambda \in (0, \lambda_0)$ the regularized smoothing system (21) has a classical solution $(x_{\lambda,\mu}(t), u_{\lambda,\mu}(t))$ over $[0, T]$, where $x_{\lambda,\mu}(0) = \eta_{\lambda,\mu} \in \mathcal{N}(\hat{x}^0, \delta_0)$ and $x_{\lambda,\mu}(t) \in \mathcal{N}(\hat{x}^0, \delta_0 + \zeta T)$ for ant $t \in [0, T]$, where $\hat{x}^0$ is the solution of $\Gamma(x, x) = 0$ in Assumption 2. Hence $\{x_{\lambda_k,\mu_k}\}_{k=1}^{\infty}$ is uniformly bounded and $\{u_{\lambda_k,\mu_k}\}_{k=1}^{\infty} \subseteq \Omega_\epsilon$ is also uniformly bounded. Since

$$\dot{x}_{\lambda_k,\mu_k} = F(t, x_{\lambda_k,\mu_k}, u_{\lambda_k,\mu_k})$$

and $F(t, x, u)$ is continuous, the uniform boundedness of $\{x_{\lambda_k,\mu_k}\}_{k=1}^{\infty}$ and $\{u_{\lambda_k,\mu_k}\}_{k=1}^{\infty}$ follows that $\{\dot{x}_{\lambda_k,\mu_k}\}_{k=1}^{\infty}$ is also uniformly bounded. Then by the Arzelá-Ascoli theorem [24], we know that $\{x_{\lambda_k,\mu_k}\}$ is uniformly convergent to a continuous x. Since $\{u_{\lambda_k,\mu_k}\}_{k=1}^{\infty}$ is uniformly bounded and $\mathcal{U}$ is reflexive, by Alaoglu's theorem [24], there is a subsequence of $\{u_{\lambda_k,\mu_k}\}_{k=1}^{\infty}$ that is weakly convergent to $u \in \mathcal{U}$.

Because $(x_{\lambda,\mu}(t), u_{\lambda,\mu}(t))$ is a solution of the regularized smoothing system (21) and $x_{\lambda,\mu}(t)$ and $u_{\lambda,\mu}(t)$ are continuous with $\eta_{\lambda,\mu} = x_{\lambda,\mu}(0) \in \mathcal{N}(\hat{x}^0, \delta_0)$, we can see that it is a minimizer of $\|\Phi_{\lambda,\mu}\|_{\mathcal{W}_i}$ with

$$\|\Phi_{\lambda,\mu}(x_{\lambda,\mu}, y_{\lambda,\mu}, \eta_{\lambda,\mu})\|_{\mathcal{W}_1} = \|\Phi_{\lambda,\mu}(x_{\lambda,\mu}, y_{\lambda,\mu}, \eta_{\lambda,\mu})\|_{\mathcal{W}_2} = 0,$$

and $\{\eta_{\lambda,\mu}\}_{k=1}^{\infty}$ is bounded. Therefore there is a sequence $\{u_{\lambda_k,\mu_k}\}_{k=1}^{\infty}$ that is convergent to an η. If moreover $u_{\lambda_k,\mu_k} \to u$ with respect to $\|\cdot\|_{L^2}$, then the sequence $(x_{\lambda_k,\mu_k}, u_{\lambda_k,\mu_k}, \eta_{\lambda_k,\mu_k})$ is convergent to (x, u, η) with respect to $\|\cdot\|_{\mathcal{W}_1}$. Because $\{\|\Phi_{\lambda_k,\mu_k}\|_{\mathcal{W}_1}\}_{k=1}^{\infty}$ is epigraphically convergent to $\|\Phi\|_{\mathcal{W}_1}$, from the well known minima property of the epigraphically convergent functional sequence (see Proposition 7.18 of [12], for example), we conclude that (x, u, η) is a minimizer of $\Phi(x, u, \eta)$ in $\mathcal{W}_1$ with

$$\|\Phi(x, u, \eta)\|_{\mathcal{W}_1} = \limsup_{k\to\infty} \|\Phi_{\lambda_k,\mu_k}(x^k, u^k, \eta^k)\|_{\mathcal{W}_1} = 0.$$

Then (x, u) is a weak solution of (1).

If $u_{\lambda_k,\mu_k} \to u$ uniformly, then $u \in \mathcal{Z}$ is continuous, therefore the sequence $(x_{\lambda_k,\mu_k}, u_{\lambda_k,\mu_k}, \eta_{\lambda_k,\mu_k})$ is convergent to (x, u, η) with respect to $\|\cdot\|_{\mathcal{W}_2}$, and (x, u, η) is a minimizer of $\Phi(x, u, \eta)$ in $\mathcal{W}_2$ with

$$\|\Phi(x, u, \eta)\|_{\mathcal{W}_2} = \limsup_{k\to\infty} \|\Phi_{\lambda_k,\mu_k}(x^k, u^k, \eta^k)\|_{\mathcal{W}_1} = 0.$$

Then (x, u) is a classic solution of (1). □

We know that using regularization approximation for the static monotone VI can find the least norm solution [11,16]. The following theorem shows that this property can be extended to their dynamic cases.

Theorem 6 *Let $\Psi(t,x,u)$ be Lipschitzian in (t,x) for any u with modular L_Ψ, and let $x_{\lambda_k,\mu_k} \to x$ uniformly with*

$$\lim_{k\to\infty} \frac{\|x_{\lambda_k,\mu_k} - x\|_C}{\lambda_k} \le \varsigma, \tag{30}$$

and let $u_{\lambda_k,\mu_k} \to u$ with respect to $\|\cdot\|_{L^2}$. Then for any weak solution $(x,\tilde{u})$ of (1), we have

$$\langle u,u\rangle \le \langle \tilde{u},u\rangle + L_\Psi T\varsigma\|u-\tilde{u}\|_{L^2}. \tag{31}$$

Proof Denoting $u^* = u_{\lambda,\mu} - \Psi(t,x_{\lambda,\mu},u_{\lambda,\mu}) - \lambda u_{\lambda,\mu}$, in a similar manner as used in the proof for Theorem 3, we can show

$$\begin{aligned}&\left\langle u_{\lambda,\mu}-\tilde{u}, G_{\lambda,\mu}(t,x_{\lambda,\mu},u_{\lambda,\mu}) - \Psi(t,x_{\lambda,\mu},\tilde{u})\right\rangle\\ &\quad\ge \left\langle u_{\lambda,\mu}-\tilde{u}, \lambda u_{\lambda,\mu}\right\rangle - \mu\sqrt{m}\kappa\|u^*-\Pi_U(u^*)\|_{L^2} - \|u_{\lambda,\mu}-\tilde{u}\|_{L^2}\mu\sqrt{m}\kappa.\end{aligned}$$

As $u_{\lambda,\mu}(t)$ is continuous and $\tilde{u}(t)\in \mathrm{SOL}(U,\Psi(t,x(t),\cdot))$ for almost every $t\in[0,T]$, we have

$$(u_{\lambda,\mu}-\tilde{u})^T\Psi(t,x,\tilde{u}) \ge 0 \qquad \text{and} \qquad G_{\lambda,\mu}(t,x_{\lambda,\mu},u_{\lambda,\mu}) = 0.$$

Adding the above two inequalities and taking the integral over $[0,T]$, we can show

$$\begin{aligned}0 &\ge \left\langle u_{\lambda,\mu}-\tilde{u}, G_{\lambda,\mu}(t,x_{\lambda,\mu},u_{\lambda,\mu}) - \Psi(t,x,\tilde{u})\right\rangle\\ &\ge \left\langle u_{\lambda,\mu}-\tilde{u}, G_{\lambda,\mu}(t,x_{\lambda,\mu},u_{\lambda,\mu}) - \Psi(t,x_{\lambda,\mu},\tilde{u}) + \Psi(t,x_{\lambda,\mu},\tilde{u}) - \Psi(t,x,\tilde{u})\right\rangle\\ &\ge \left\langle u_{\lambda,\mu}-\tilde{u}, \lambda u_{\lambda,\mu}\right\rangle - \mu\sqrt{m}\kappa(\|u^*-\Pi_U(u^*)\|_{L^2} + \|u_{\lambda,\mu}-\tilde{u}\|_{L^2})\\ &\quad - L_\Psi\|x_{\lambda,\mu}-x\|_C\|u_{\lambda,\mu}-\tilde{u}\|_{L^2}.\end{aligned}$$

Now we have

$$\begin{aligned}\left\langle u_{\lambda,\mu}-\tilde{u}, \lambda u_{\lambda,\mu}\right\rangle &\le \mu\sqrt{m}\kappa(\|u^*-\Pi_U(u^*)\|_{L^2} + \|u_{\lambda,\mu}-\tilde{u}\|_{L^2})\\ &\quad + L_\Psi\|x_{\lambda,\mu}-x\|_C\|u_{\lambda,\mu}-\tilde{u}\|_{L^2},\end{aligned}$$

and therefore

$$\begin{aligned}\left\langle u_{\lambda,\mu}, u_{\lambda,\mu}\right\rangle - \left\langle \tilde{u}, u_{\lambda,\mu}\right\rangle &\le \frac{\mu}{\lambda}\sqrt{mT}\kappa(\|u^*-\Pi_U(u^*)\|_{L^2} + \|u_{\lambda,\mu}-\tilde{u}\|_{L^2})\\ &\quad + L_\Psi T\frac{\|x_{\lambda,\mu}-x\|_C}{\lambda}\|u_{\lambda,\mu}-\tilde{u}\|_{L^2}.\end{aligned}$$

Taking $(x_{\lambda_k,\mu_k}, u_{\lambda_k,\mu_k})$ converging to (x,u) with $\mu_k = o(\lambda_k)$ and (30), we draw the conclusion (31). □

We end this section by summarizing the results achieved in this section. Here the DVI is treated as a DAE (10), in which the function $G(t, x, u)$ defining the algebraic constraint is normally nonsmooth and weak univalent, the univalent property is given by Assumption (A2). By the regularization and smoothing techniques, we propose a regularized smoothing function $G_{\lambda,\mu}(t, x, u)$ to approximate $G(t, x, u)$ in the DAE (10), which yields the regularized smoothing system (21). In Theorem 4 we show that the system (21) has a classic solution $(x_{\lambda,\mu}, u_{\lambda,\mu})$, which can be efficiently solved by high order ODE-solvers. Then we show in Theorem 5 that $(x_{\lambda,\mu}, u_{\lambda,\mu})$ is convergent to a weak solution of the DVI (1), which, together with Theorem 1, gives an equilibrium point of the dynamic NEPSC.

5 Numerical illustration

We use the two-player zero-sum game with shared constraints to illustrate the differential monotone VI approach and the convergence of the regularized smoothing method. At first we show that if the cost functions of the two players are convex, then we can find a solution of the game via the differential monotone VI.

For $i = 1, 2$, we suppose that the i-th player's state dynamic is semi-linear:

$$\Theta_i(t, y_i, u_i) = f_i(t, y_i) + B_i u_i,$$

where $f_i : [0, T] \times R^{n_i} \to R^{n_i}$ and $B_i \in R^{n_i \times m_i}$ are given. Let the cost functional

$$\theta_1(y, u) = \psi_1(y(T)) + \int_0^T \varphi_1(t, y, u)dt$$

be given, where $T > 0$ is fixed, $y = (y_1^T, y_2^T)^T \in R^n, u = (u_1^T, u_2^T)^T \in R^m, n = n_1 + n_2, m = m_1 + m_2$, $\varphi_1(t, y, u)$ is convex in the control u_1 of Player 1, and concave in the control u_2 of Player 2. In the two-player zero-sum game, Player 1 minimizes the cost functional $\theta_1(y, u)$, while the other maximizes it. Then by the same manner as presented in Section 2, the dynamic NEPSC yields the DVI (1), where $x = (v, y)$ and $\Psi(t, x, u)$ has the following form

$$\Psi(t, x, u) = \begin{pmatrix} \nabla_{u_1}\varphi_1(t, y, u) + B_1^T v_1 \\ -\nabla_{u_2}\varphi_1(t, y, u) + B_2^T v_2 \end{pmatrix}.$$

Note that $\varphi_1(y, u)$ is convex in u_1 and concave in u_2. Hence the function $\Psi(t, v, y, \cdot)$ is monotone, and the dynamic NEPSC yields a monotone DVI when U is convex. Moreover, by Lemma 1, if $(x^*, u^*) = (v^*, y^*, u^*)$ is a solution of the monotone DVI, then (y^*, u^*) is a solution of two-player dynamic NEPSC.

We use a numerical example of the two-player zero-sum game with shared constraints to show the convergence of the regularized smoothing method. Let $n_1 = n_2 = 1, m_1 = m_2 = 2$. The players's state dynamics are

$$\Theta_1(t, y_1, u_1) = f_1(t, y_1) + B_1 u_1 \quad \text{with} \quad f_1(t, y_1) = -2 + 2y_1, \quad B_1 = (1, -2),$$
$$\Theta_2(t, y_2, u_2) = f_2(t, y_2) + B_2 u_2 \quad \text{with} \quad f_2(t, y_2) = -2t - y_2, \quad B_2 = (-6, 3).$$

The admissible control sets are

$$U_1(u_2) = \{u_1 \mid h_1(u_1) = -u_1 \leq 0, \quad g(u_1, u_2) = e^T(u_1 + u_2) - 1 \leq 0\},$$
$$U_2(u_1) = \{u_2 \mid h_2(u_2) = -u_2 \leq 0, \quad g(u_1, u_2) = e^T(u_1 + u_2) - 1 \leq 0\},$$

where $e = (1, 1)^T$. The initial states are $y_1^0 = -1$, $y_2^0 = 2$. The cost functional of Player 1 is defined by

$$\theta_1(y, u) = \psi_1(y(T)) + \int_0^T \varphi_1(t, y, u)dt$$

where

$$\begin{aligned} \psi_1(y(T)) &= y(T)^T \left[Ly(T) + c\right], \\ \varphi_1(t, y, u) &= y^T \left[Py + Su + h(t)\right] + u^T \left[Ru + d(t)\right], \\ L &= \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \\ P &= \begin{pmatrix} 3 & -6 \\ -1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} -1 & 0 & 2 & 3 \\ 6 & -9 & 0 & -2 \end{pmatrix}, \quad h(t) = \begin{pmatrix} -\sin(3t) \\ 1 \end{pmatrix}, \\ R &= \begin{pmatrix} 0 & -1 & -3 & 2 \\ 1 & 0 & 5 & -1 \\ 0 & -3 & -1 & 2 \\ 2 & 4 & -2 & 0 \end{pmatrix}, \quad d(t) = \begin{pmatrix} 0 \\ -1 \\ -\cos(t - \frac{\pi}{12}) \\ 0 \end{pmatrix}. \end{aligned}$$

Then the DVI, formulated from this two-player zero-sum game, has the form:

$$\begin{aligned} \dot{x}(t) &= q(t) + Ax(t) + Bu(t) \\ u(t) &\in \text{SOL}(U, p(t) + Qx(t) + M(\cdot)) \\ b &= Ex(0) + E_T x(T), \end{aligned} \tag{32}$$

where

$$A = \begin{pmatrix} -2 & 0 & -6 & 7 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -2 & -3 \\ 6 & -9 & 0 & -2 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -6 & 3 \end{pmatrix}, \quad q(t) = \begin{pmatrix} \sin(3t) \\ 1 \\ -2 \\ -2t \end{pmatrix},$$

and

$$Q = \begin{pmatrix} 1 & 0 & -1 & 6 \\ -2 & 0 & 0 & -9 \\ 0 & -6 & -2 & 0 \\ 0 & 3 & -3 & 2 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 & -3 & 4 \\ 0 & 0 & 2 & 3 \\ 3 & -2 & 2 & 0 \\ -4 & -3 & 0 & 0 \end{pmatrix}, \quad p(t) = \begin{pmatrix} 0 \\ -1 \\ \cos\left(t - \frac{\pi}{12}\right) \\ 0 \end{pmatrix},$$

and

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_T = \begin{pmatrix} 1 & 0 & -4 & -2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ -3 \\ -1 \\ 2 \end{pmatrix},$$

and $U = \{u \mid u \geq 0, e^T u \leq 1\} = \{u | Cu \geq c\}$, where

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

In practice, it is not easy to give the close form of the smoothing function $G_{\lambda,\mu}$ for a VI. For this example, we know however, that $u(t) \in \mathrm{SOL}(U, g(t) + Qx(t) + M(\cdot))$ if and only if there is a multiplier w such that the Karush-Kuhn-Tucker condition holds

$$p(t) + Qx(t) + Mu(t) - C^T w = 0 \quad \text{and} \quad 0 \leq w \perp Cu - c \geq 0.$$

Then the system (32) can be reformulated as the linear complementarity system

$$\begin{aligned} \dot{x}(t) &= q(t) + Ax(t) + Bu(t) \\ 0 &= p(t) + Qx(t) + Mu(t) - C^T w(t) \\ 0 &\leq w(t) \perp Cu(t) - c \geq 0 \\ b &= Ex(0) + E_T x(T). \end{aligned} \tag{33}$$

It is obvious that for this example, the algebraic system $\Gamma(x, x) = Ex + E_T x - b = 0$ has a unique solution $\hat{x}^0 = (-2, -1, -1, 2)^T = (E + E_T)^{-1} b$.

As the matrix M is positive semi-definite and the domain U is convex and compact, the problem $\mathrm{VI}(U, p(0) + Q\hat{x}^0 + M(\cdot))$ is solvable, and so is the VI problem

$$\begin{aligned} 0 &= p(0) + Q\xi + Mu - C^T w \\ 0 &\leq w \perp Cu - c \geq 0. \end{aligned} \tag{34}$$

We show that the solution set of the VI (34) is bounded. Let (34) have the solutions $\{(u^k, w^k)\}$. Obviously, the boundedness of U yields that $\{u^k\}$ is bounded. From the

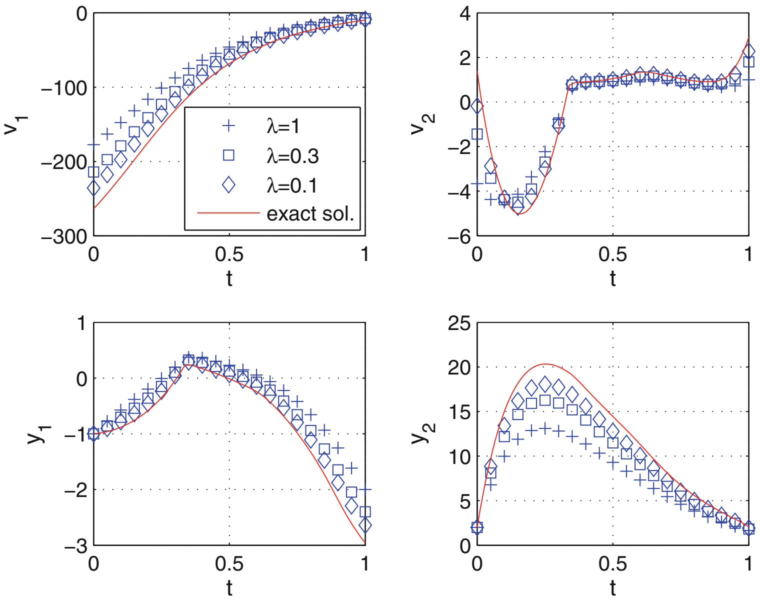

Fig. 2 Numerical results for $x(t)$ with $\mu = \lambda^2$

equality of (34) we have $(C^T w^k)_i = (p(0) + Q\xi + Mu^k)_i = w_i^k - w_5^k$, which means that $w_i^k - w_5^k$ is bounded for $i = 1, 2, 3, 4$. If $w_5^k \to \infty$, then $w_5^k > 0$ once k is large enough, then from $0 \le w \perp Cu - c \ge 0$, we know $u_1^k + u_2^k + u_3^k + u_4^k = 1$, which follows that there must be a component $u_i^k > 0$. Therefore we have $w_i^k = 0$ as $0 \le w_i^k \perp u_i^k \ge 0$, which yields the unboundedness of $\{w_i^k - w_5^k\}$, this gives a contradiction. Now we can conclude that $\{w^k\}$ is also bounded, and Assumption (A1) is fulfilled. Since M is positive semi-definite, it is obvious that Assumption (A2) is fulfilled.

Now we use the Chen-Harker-Kanzow-Smale smoothing function to give a smoothing regularization approximation of (33)

$$\begin{aligned} \dot{x}(t) &= q(t) + Ax(t) + Bu(t) \\ 0 &= p(t) + Qx(t) + (M + \lambda I)u(t) - C^T w(t) \\ \mu &= 4w_i(t)\,[Cu(t) - c + \lambda w(t)]_i \quad (1 \le i \le 5) \\ b &= Ex(0) + E_T x(T). \end{aligned} \tag{35}$$

This is a standard ODE. Here, on the platform of Matlab, we use the algebraic equation solver “fsolve.m” and the least square problem solver “lsqnonlin.m” to solve $G_{\lambda,\mu}(t, x, u) = 0$, for evaluating the right hand side of the ODE. For $\lambda = 1$, $\lambda = 0.3$ and $\lambda = 0.1$ with $\mu = \lambda^2$, by using the boundary value problem solver “bvp5c.m” to the ODE (35), we get the trajectories of $(x_{\lambda,\mu}(t), u_{\lambda,\mu}(t))$ of (35). Here we adopt $(E + E_T)^{-1} b$ to initialize the solver “bvp5c.m”.

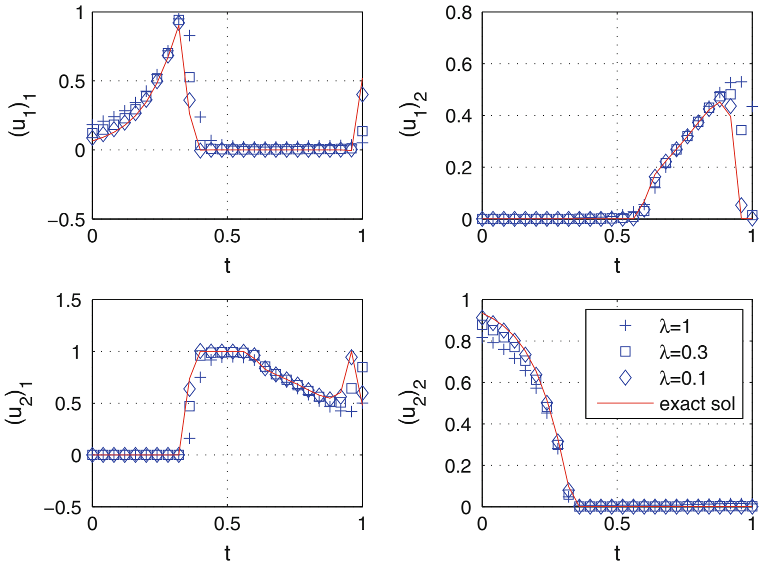

Fig. 3 Numerical results for $u(t)$ with $\mu = \lambda^2$

In Figs. 2, 3 we plot the trajectories of the adjoint variables v_1 and v_2, the state variables y_1 and y_2, the control variables $u_1 = ((u_1)_1, (u_1)_2)^T$ and $u_2 = ((u_2)_1, (u_2)_2)^T$. The numerical results strongly support the convergence of the regularization and smoothing approximation. From the figures we can observe that our method approximates the nonsmooth solution by the smooth one. In our method we use $\mu = o(\lambda)$ to get the least norm solution u in the solution set $\mathcal{S}(t, x)$.

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