Finite Difference Smoothing Solution of Nonsmooth Constrained Optimal Control Problems

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Abstract The finite difference method and smoothing approximation for a nonsmooth constrained optimal control problem are considered. Convergence of solutions of discretized smoothing optimal control problems is proved. Error estimates of finite difference smoothing solution are given. Numerical examples are used to test a smoothing SQP method for solving the nonsmooth constrained optimal control problem.

Key words. Optimal control, nondifferentiability, finite difference method, smoothing approximation

AMS subject classifications. 49K20, 35J25

1 Introduction

Recently significant progress has been made in studying elliptic optimal control problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\int_\Omega (y-z_d)^2 dx + \frac{\alpha}{2}\int_\Omega (u-u_d)^2 dx \\ \text{subject to} & -\triangle y = f(x,y,u) \quad \text{in} \quad \Omega, \quad y=0 \quad \text{on} \quad \Gamma \\ & u \in \mathcal{U} \end{array} \tag{1.1}$$

where $z_d, u_d \in L^2(\Omega)$, $f \in C(\Omega \times R^2)$, $\alpha > 0$ is a constant, Ω is an open, bounded convex subset of $R^N, N \leq 3$, with smooth boundary Γ, and

$$\mathcal{U} = \{u \in L^2(\Omega) \,|\, u(x) \leq q(x) \text{ a.e in}\, \Omega\},$$

$q \in L^\infty(\Omega)$.

If f is linear with respect to the second and third variables, (1.1) is equivalent to its first order optimality system. Based on the equivalence, the primal-dual active set strategy [2] can solve problem (1.1) efficiently. Moreover, Hintermüller, Ito and

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Kunisch [14] proved that the optimality system is slantly differentiable [7] and the primal-dual active set strategy is a specific semismooth Newton method for the first order optimality system. The nondifferentiable term in the optimality system arises from the constraint $u \in U$. Without this constraint, the optimality system is a linear system. For such a case, Borzí, Kunisch and Kwak [3] gave convergence properties of the finite difference multigrid solution for the optimality system.

For the case where f is nonlinear with respect to the second and third variables, the first order optimality system and second order optimality system have been studied in order to obtain error estimates in discretization approximations and convergence theory in sequential quadratic programming algorithms. See for instance [4] and the references therein. Most of papers assume that f is of class C^2 with respect to the second and third variables. However, this assumption does not hold for the following optimal control problem

$$\begin{aligned} &\text{minimize} && \frac{1}{2}\int_\Omega (y-z_d)^2 d\omega + \frac{\alpha}{2}\int_\Omega (u-u_d)^2 d\omega \\ &\text{subject to} && -\triangle y + \lambda \max(y,0) = u + g \quad \text{in} \quad \Omega, \quad y=0 \quad \text{on}\,\Gamma \\ & && u \in \mathcal{U}, \end{aligned} \tag{1.2}$$

where $\lambda > 0$ is a constant and $g \in C(\Omega)$. The nonsmooth elliptic equations can be found in equilibrium analysis of confined MHD(magnetohydrodynamics) plasmas [6, 7, 18], thin stretched membranes partially covered with water[15], or reaction-diffusion problems [1].

The discretized nonsmooth constrained optimal control problems derived from a finite difference approximation or a finite element approximation of (1.2) with mass lump has the form:

$$\begin{aligned} &\text{minimize} && \frac{1}{2}(Y-Z_d)^T H (Y-Z_d) + \frac{\alpha}{2}(U-U_d)^T M (U-U_d) \\ &\text{subject to} && AY + \lambda D \max(0,Y) = NU + c \\ & && U \le b. \end{aligned} \tag{1.3}$$

Here $Z_d, c \in R^n$, $U_d, b \in R^m$, $H \in R^{n\times n}$, $M \in R^{m\times m}$, $A \in R^{n\times n}$,$D \in R^{n\times n}$, $N \in R^{n\times m}$, and $\max(0,\cdot)$ is understood componentwise. Moreover H, M, A, D are symmetric positive definite matrices, and D =diag$(d_1, \ldots, d_n)$ is a diagonal matrix.

It is aware that for the discretized nonsmooth constrained optimal control problem, a solution of the first order optimality system is not necessarily a solution of the optimal control problem. Also a solution of the optimal control problem is not necessarily a solution of the first order optimality system, see Example 2.1 in the next section. In [5], a sufficient condition for the two problems to be equivalent is given. In this paper, we consider the following smoothing approximation of (1.3)

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}(Y - Z_d)^T H(Y - Z_d) + \frac{\alpha}{2}(U - U_d)^T M(U - U_d) \\ \text{subject to} & AY + \Phi_\epsilon(Y) = NU + c \\ & \quad U \leq b, \end{array} \tag{1.4}$$

where $\Phi_\epsilon : R^n \to R^n$ is defined by a smoothing function $\phi_\epsilon : R \to R$ as

$$\Phi_\epsilon(Y) = \lambda D \begin{pmatrix} \phi_\epsilon(Y_1) \\ \phi_\epsilon(Y_2) \\ \vdots \\ \phi_\epsilon(Y_n) \end{pmatrix}.$$

Here ϵ is called a smoothing parameter. For $\epsilon > 0$, ϕ_ϵ is continuously differentiable and its derivative satisfies $\phi'_\epsilon \geq 0$. Moreover, it holds

$$|\phi_\epsilon(t) - \max(0, t)| \leq \kappa\epsilon$$

with a constant $\kappa > 0$ for all $\epsilon \geq 0$. We can find many smoothing functions having such properties. In this paper, we simply choose [12]

$$\phi_\epsilon(t) = \frac{1}{2}(t + \sqrt{t^2 + \epsilon^2}).$$

It is not difficult to verify that for a fixed $\epsilon > 0$, ϕ'_ϵ is continuously differentiable and

$$\phi'_\epsilon(t) = \frac{1}{2}(1 + \frac{t}{\sqrt{t^2 + \epsilon^2}}) > 0.$$

Moreover, for any t and $\epsilon > 0$,

$$|\phi_\epsilon(t) - \max(0, t)| \leq \frac{1}{2}\epsilon.$$

and

$$\phi^o(t) := \lim_{\epsilon \downarrow 0} \phi'_\epsilon(t) = \frac{1}{2} \begin{cases} 2 & \text{if } t > 0 \\ 1 & \text{if } t = 0 \\ 0 & \text{if } t < 0. \end{cases}$$

It was shown in [8] that $\phi^o(t)$ is an element of the subgradient [9] of $\max(0, \cdot)$. Following these properities of ϕ_ϵ, the smoothing approximation function Φ_ϵ is continuously differentiable for $\epsilon > 0$ and satisfies

$$||\Phi_\epsilon(Y) - \lambda D \max(0, Y)|| \leq \lambda\sqrt{n}||D||\epsilon \tag{1.5}$$

for $\epsilon \geq 0$. Here $\|\cdot\|$ denotes the Euclidean norm. In particular, for $\epsilon = 0$, $\Phi_0(Y) = \lambda D \max(0, Y)$. Moreover, the matrix

$$\Phi^o(Y) := \lambda D \text{diag}(\phi^o(Y_1), \phi^o(Y_2), \ldots, \phi^o(Y_n))$$

is an element of the Clarke generalized Jacobian [9] of $\lambda D \max(0, Y)$.

In this paper we investigate the convergence of the discretized smoothing problem (1.4) derived from the five-point difference method and smoothing approximation. In section 2, we describe the finite difference discretization of the optimal control problem. We prove that a sequence of optimal solutions of the discretized smoothing problem (1.4) converges to a solution of the discretized nonsmooth optimal control problem (1.3) as $\epsilon \to 0$. Moreover, under certain conditions, we prove that the distance between the two solution sets of (1.3) and (1.4) is $O(\epsilon)$. We show that the difference between the optimal objective value of (1.3) and the optimal objective value of the continuous problem (1.2) is $O(h^\gamma)$, where $0 < \gamma < 1$. In section 3 we use numerical examples to show that the nonsmooth optimal control problem can be solved by a finite difference smoothing SQP method efficiently.

2 Convergence of smoothing discretized problem

In this section, we investigate convergence of the smoothing discretized optimal control problem (1.4) derived from the smoothing approximation and the five-point finite difference method with $\Omega = (0,1) \times (0,1)$. To simplify our discussion, we use the same mesh size for discretization approximation of y and u.

Let ν be a positive integer. Set $h = 1/(\nu+1)$. Denote the set of grids by

$$\Omega_h = \{(ih, jh) \,|\, i, j = 1, \ldots, \nu\}.$$

The Dirichlet problem in the constraints of (1.2) is approximated by the five-point finite difference approximation with uniform mesh size h. For grid functions W and V defined on Ω, we denote the discrete L_h^2-scalar product

$$(W, V)_{L_h^2} = h^2 \sum_{i,j=1}^{\nu} W(ih, jh) V(ih, jh).$$

Let P be the restriction operator from $L^2(\Omega)$ to $L_h^2(\Omega)$. Let $Z_d = P z_d$, $U_d = P u_d$ and $b = Pq$. Then we obtain the discretized optimal control problem (1.3).

Denote the objective functions of the continuous problem and the discretized problems

$$J(y, u) = \frac{1}{2} \int_\Omega (y - z_d)^2 dx + \frac{\alpha}{2} \int_\Omega (u - u_d)^2 dx$$

and

$$J_h(Y,U) = \frac{1}{2}(Y - Z_d)^T H(Y - Z_d) + \frac{\alpha}{2}(U - U_d)^T M(U - U_d),$$

respectively. Let $\mathcal{S}, \mathcal{S}_h$, and $\mathcal{S}_{h,\epsilon}$ denote the solution sets of (1.2), (1.3), and (1.4), respectively.

2.1 Problems (1.4) and (1.3)

For a fixed h, we investigate the limiting behaviour of optimal solutions of (1.4) as the smoothing parameter $\epsilon \to 0$.

Theorem 2.1 *For any mesh size h and smoothing parameter $\epsilon \geq 0$, the solution sets $\mathcal{S}_h$ and $\mathcal{S}_{h,\epsilon}$ are nonempty and bounded. Moreover, there exists a constant $\eta > 0$ such that*

$$\mathcal{S}_{h,\epsilon} \subseteq \{(Y,U) \mid J_h(Y,U) \leq \eta\} \tag{2.1}$$

for all $\epsilon \in [0,1]$.

Proof: First we observe that for fixed $U \leq b$ and $\epsilon \geq 0$, the system of equations

$$AY + \Phi_\epsilon(Y) = NU + c$$

is equivalent to the strongly convex unconstrained minimization problem

$$\min_Y \frac{1}{2} Y^T AY + \sum_{i=1}^{n} \int_0^{Y_i} \phi_\epsilon(t)dt - Y^T(NU + c).$$

By the strong convexity, the problem has a unique solution Y_ϵ. Hence the feasible sets of (1.3) and (1.4) are nonempty. Moreover, we notice that the objective function J_h of (1.3) and (1.4) is strongly convex, which implies that the solution sets of (1.3) and (1.4) are nonempty and bounded.

Now we prove (2.1).

In (1.3), the constraint

$$AY + \lambda D \max(0, Y) = NU + c$$

can be written as

$$(A + \lambda DE(Y))Y = NU + c,$$

where $E(Y)$ is a diagonal matrix whose diagonal elements are

$$E_{ii}(Y) = \begin{cases} 1 & \text{if} \quad Y_i > 0 \\ 0 & \text{if} \quad Y_i \leq 0. \end{cases}$$

Since A is an M-matrix, from the structure of $\lambda DE(Y)$, we have that $A+\lambda DE(Y)$ is also an M-matrix and

$$\|Y\| \le \|(A+\lambda DE(Y))^{-1}(NU+c)\| \le \|A^{-1}\|(\|N\|\|U\|+\|c\|). \qquad (2.2)$$

See Theorem 2.4.11 in [16].

Moreover, for $\epsilon > 0$, let Y_ϵ satisfy

$$AY_\epsilon + \Phi_\epsilon(Y_\epsilon) = NU + c.$$

By the mean value theorem, we find

$$\begin{aligned} 0 &= A(Y_\epsilon - Y) + \Phi_\epsilon(Y_\epsilon) - \lambda D \max(0, Y) \\ &= A(Y_\epsilon - Y) + \Phi_\epsilon(Y_\epsilon) - \Phi_\epsilon(Y) + \Phi_\epsilon(Y) - \lambda D \max(0, Y) \\ &= (A + \frac{\lambda}{2}\tilde{D})(Y_\epsilon - Y) + \frac{\lambda}{2}\hat{D}\epsilon, \end{aligned}$$

where

$$\tilde{D} = D\mathrm{diag}(1 + \frac{\tilde{Y}_1}{\sqrt{\tilde{Y}_1^2 + \epsilon^2}}, \ldots, 1 + \frac{\tilde{Y}_n}{\sqrt{\tilde{Y}_n^2 + \epsilon^2}})$$

and

$$\hat{D} = D\mathrm{diag}(\frac{\hat{\epsilon}}{\sqrt{Y_1^2 + \hat{\epsilon}^2}}, \ldots, \frac{\hat{\epsilon}}{\sqrt{Y_n^2 + \hat{\epsilon}^2}}).$$

Here $\hat{\epsilon} \in (0, \epsilon]$ and $\tilde{Y}_i$ lies between $(Y_\epsilon)_i$ and Y_i.

Using that all diagonal elements of $\tilde{D}$ are positive and

$$0 < \frac{\hat{\epsilon}}{\sqrt{Y_i^2 + \hat{\epsilon}^2}} \le 1, \quad i = 1, \ldots, n$$

we obtain

$$\|Y_\epsilon - Y\| \le \frac{\lambda}{2}\|(A + \frac{\lambda}{2}\tilde{D})^{-1}\hat{D}\|\epsilon \le \frac{\lambda}{2}\|A^{-1}\|\|D\|\epsilon. \qquad (2.3)$$

Therefore, using (2.2), we find that for $\epsilon \in (0, 1]$,

$$\|Y_\epsilon\| \le \frac{\lambda}{2}\|A^{-1}\|\|D\|\epsilon + \|Y\| \le \|A^{-1}\|(\frac{\lambda}{2}\|D\| + \|N\|\|U\| + \|c\|).$$

Let $(Y_\epsilon^*, U_\epsilon^*) \in \mathcal{S}_{h,\epsilon}$ and $(Y^*, U^*) \in \mathcal{S}_h$. Let Y_ϵ be the solution of

$$AY + \Phi_\epsilon(Y) = NU^* + c,$$

that is, (Y_ϵ, U^*) is a feasible point of (1.4). By the argument above, we have

$$\begin{aligned}
& J_h(Y_\epsilon^*, U_\epsilon^*) \\
\leq\ & J_h(Y_\epsilon, U^*) \\
\leq\ & \frac{1}{2}\|H\|\|Y_\epsilon - Z_d\|^2 + \frac{\alpha}{2}\|M\|\|U^* - U_d\|^2 \\
\leq\ & \frac{1}{2}\|H\|(\|Y_\epsilon\| + \|Z_d\|)^2 + \frac{\alpha}{2}\|M\|(\|U^*\| + \|U_d\|)^2 \\
\leq\ & \frac{1}{2}\|H\|(\|A^{-1}\|(\frac{\lambda}{2}\|D\| + \|N\|\|U^*\| + \|c\|) + \|Z_d\|)^2 + \frac{\alpha}{2}\|M\|(\|U^*\| + \|U_d\|)^2.
\end{aligned}$$

In the first part of the proof, we have shown that the solution set $\mathcal{S}_h$ is bounded, that is, $\|U^*\|$ is smaller than a positive constant. Hence there exists a constant $\eta > 0$ such that

$$\frac{1}{2}\|H\|(\|A^{-1}\|(\frac{\lambda}{2}\|D\| + \|N\|\|U^*\| + \|c\|) + \|Z_d\|)^2 + \frac{\alpha}{2}\|M\|(\|U^*\| + \|U_d\|)^2 \leq \eta.$$

This completes the proof. ∎

Theorem 2.2 *Letting $\epsilon \to 0$, any accumulation point of a sequence of optimal solutions of (1.4) is an optimal solution of (1.3), that is,*

$$\{\lim_{\substack{(Y,U)\in\mathcal{S}_{h,\epsilon} \\ \epsilon\downarrow 0}} (Y, U)\} \subseteq \mathcal{S}_h.$$

Proof: Let $(Y_\epsilon^*, U_\epsilon^*) \in \mathcal{S}_{h,\epsilon}$, $(Y^*, U^*) \in \mathcal{S}_h$ and (Y_ϵ, U^*) satisfy

$$AY_\epsilon + \Phi_\epsilon(Y_\epsilon) = NU^* + c.$$

Then the following inequality holds

$$J_h(Y_\epsilon^*, U_\epsilon^*) \leq J_h(Y_\epsilon, U^*). \tag{2.4}$$

Using the Talyor expansion, we find

$$J_h(Y_\epsilon, U^*) = J_h(Y^*, U^*) + (Y_\epsilon - Y^*)^T H(Y^* - Z_d) + \frac{1}{2}(Y_\epsilon - Y^*)^T H(Y_\epsilon - Y^*).$$

Following the argument on (2.3), we have

$$\|Y_\epsilon - Y^*\| \leq \frac{\lambda}{2}\|A^{-1}\|\|D\|\epsilon.$$

Since the solution set of (1.3) is bounded, there is a constant $\kappa > 0$ such that for $\epsilon \leq 2/(\lambda\|A^{-1}\|\|D\|)$,

$$\begin{aligned}
& (Y_\epsilon - Y^*)^T H(Y^* - Z_d) + \frac{1}{2}(Y_\epsilon - Y^*)^T H(Y_\epsilon - Y^*) \\
& \leq \kappa(\|Y_\epsilon - Y^*\| + \|Y_\epsilon - Y^*\|^2) \\
& \leq \lambda\kappa\|A^{-1}\|\|D\|\epsilon.
\end{aligned}$$

Combining this with (2.4), the Talyor expansion gives

$$J_h(Y_\epsilon^*, U_\epsilon^*) \le J_h(Y^*, U^*) + \lambda\kappa \|A^{-1}\| \|D\| \epsilon. \tag{2.5}$$

Moreover, from Theorem 2.1, there is a bounded closed set $\mathcal{L}$ such that

$$\mathcal{S}_{h,\epsilon} \subseteq \mathcal{L}, \text{ for all } \epsilon \in [0,1].$$

Hence, without loss of generality, we may assume that

$$(Y_\epsilon^*, U_\epsilon^*) \to (\bar{Y}, \bar{U}) \in \mathcal{L} \qquad \text{as} \quad \epsilon \to 0.$$

Now, we show that $(\bar{Y}, \bar{U})$ is a feasible point of (1.3). Obviously $\bar{U} \le b$ as $U_\epsilon^* \le b$ for all $\epsilon > 0$. The other constraint also holds, since

$$\begin{aligned}
&\|A\bar{Y} + \lambda D \max(0, \bar{Y}) - N\bar{U} - c\| \\
&= \lim_{\epsilon \downarrow 0} \|AY_\epsilon^* + \lambda D \max(0, \bar{Y}) - N\bar{U} - c\| \\
&= \lim_{\epsilon \downarrow 0} \|NU_\epsilon^* - \Phi_\epsilon(Y_\epsilon^*) + \lambda D \max(0, \bar{Y}) - N\bar{U}\| \\
&\le \lim_{\epsilon \downarrow 0} \|\lambda D \max(0, \bar{Y}) - \Phi_\epsilon(Y_\epsilon^*)\| \\
&\le \lim_{\epsilon \downarrow 0} (\lambda \|D\| \|\max(0, \bar{Y}) - \max(0, Y_\epsilon^*)\| + \|\lambda D \max(0, Y_\epsilon^*) - \Phi_\epsilon(Y_\epsilon^*)\|) \\
&\le \lambda \|D\| (\lim_{\epsilon \downarrow 0} \|V\| \|\bar{Y} - Y_\epsilon^*\| + \sqrt{n}\epsilon) \\
&= 0.
\end{aligned}$$

Here V is a diagonal matrix whose diagonal elements are

$$V_{ii} = \begin{cases} 0 & \text{if} \quad (\bar{Y} - Y_\epsilon^*)_i = 0 \\ \dfrac{\max(0, \bar{Y})_i - \max(0, Y_\epsilon^*)_i}{(\bar{Y} - Y_\epsilon^*)_i} & \text{otherwise} \end{cases}$$

Obviously, we have $0 \le V_{ii} \le 1$.

Now let $\epsilon \to 0$ in (2.5), we get

$$J_h(\bar{Y}, \bar{U}) \le J_h(Y^*, U^*).$$

Hence $(\bar{Y}, \bar{U})$ is a solution of (1.3). ∎

To estimate the distance between the two solution sets $\mathcal{S}_h$ and $\mathcal{S}_{h,\epsilon}$ we have to consider the first order optimality system for (1.3). We say (Y, U) satisfies the first order conditions of (1.3), or (Y, U) is a KKT (Karush-Kuhn-Tucker) point of (1.3), if it together with some $(s, t) \in R^n \times R^m$ satisfies

$$\begin{pmatrix} H(Y - Z_d) + As + \lambda DE(Y)s \\ \alpha M(U - U_d) - N^T s + t \\ AY + \lambda D \max(0, Y) - NU - c \\ \min(t, b - U) \end{pmatrix} = 0. \tag{2.6}$$

The vectors $s \in R^n$ and $t \in R^m$ are referred to as Lagrange multipliers. It was shown in [5] that for any U, the system of nonsmooth equations

$$AY + \lambda D \max(0, Y) - NU - c = 0$$

has a unique solution, and it defines a solution function $Y(U)$. Moreover, (2.6) is equivalent to the following system

$$\begin{pmatrix} ((A + \lambda DE(Y(U)))^{-1}N)^T H(Y(U) - Z_d) + \alpha M(U - U_d) + t \\ \min(t, b - U) \end{pmatrix} = 0. \qquad (2.7)$$

However, for the discretized nonsmooth constrained optimal control problem (1.3), a KKT point of (1.3) is not necessarily a solution of the optimal control problem (1.3). Also a solution of the optimal control problem (1.3) is not necessarily a KKT point of (1.3).

Example 2.1 Let $n = 2, m = 1, M = \alpha = \lambda = b = 1, H = D = I, c = 0$,

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

1. For $U_d = 1$ and $Z_d = (0, -3)^T$, $(\tilde{Y}, \tilde{U}) = (0, 0, 0)^T$ is a KKT point of (1.3), but $(\tilde{Y}, \tilde{U})$ is not a solution of (1.3).

2. For $U_d = 0$ and $Z_d = (0, 1)$, $(Y^*, U^*) = (0, 0, 0)^T$ is a solution of (1.3), but (Y^*, U^*) is not a KKT point of (1.3).

Let $A_{\mathcal{K}(Y)}$ be the submatrix of A whose entries lie in the rows of A indexed by the set

$$\mathcal{K}(Y) = \{i \mid Y_i = 0, \ i = 1, 2, \ldots, n\}.$$

Lemma 2.1 *[5] Suppose that (Y^*, U^*) is a local optimal solution of (1.3), and either $\mathcal{K}(Y^*) = \emptyset$ or $((A + \lambda DE(Y^*))^{-1}N)_{\mathcal{K}(Y^*)} = 0$, then (Y^*, U^*) is a KKT point of (1.3).*

Theorem 2.3 *Let $(Y^*_\epsilon, U^*_\epsilon) \in \mathcal{S}_{h,\epsilon}$ and $(Y^*, U^*) \in \mathcal{S}_h$. Under assumptions of Lemma 2.1, we have*

$$\|Y^*_\epsilon - Y^*\| + \|U^*_\epsilon - U^*\| \leq O(\epsilon).$$

Proof: Let us set

$$W = (A + \lambda DE(Y^*))^{-1}N.$$

By Theorem 2.1 in [5], the assumptions implies that in a neighborhood of U^*, the solution function $Y(\cdot)$ can be expressed by

$$Y(U) = WU.$$

Moreover, $Y(\cdot)$ is differentiable at U^* and $Y'(U^*) = W$. In such a neighborhood, we define a function

$$F(U,t) = \begin{pmatrix} W^T H(Y(U) - Z_d) + \alpha M(U - U_d) + t \\ \min(t, b - U) \end{pmatrix}.$$

From Lemma 2.1, there is $t^* \in R^m$ such that

$$F(U^*, t^*) = 0.$$

The Clarke generalized Jacobian $\partial F(U,t)$ [9] of F at (U^*, t^*) is the set of matrices that have the version

$$\begin{pmatrix} W^T HW + \alpha M & I \\ T & I + T \end{pmatrix}$$

where T is a diagonal matrix whose diagonal elements are

$$T_{ii} = \begin{cases} -1 & \text{if} \quad (b - U^*)_i < t_i^* \\ 0 & \text{if} \quad (b - U^*)_i > t_i^* \\ \tau_i & \text{if} \quad (b - U^*)_i = t_i^*, \qquad \tau_i \in [-1, 0]. \end{cases}$$

This is easy to see that all matrices in $\partial F(U^*, t^*)$ are nonsingular. By Proposition 3.1 in [17], there is a neighborhood $\mathcal{N}$ of (U^*, t^*) and a constant $\beta > 0$ such that for any $(U,t) \in \mathcal{N}$ and any $V \in \partial F(U,t)$, V is nonsingular and $\|V^{-1}\| \le \beta$.

Now we consider a function of F_ϵ defined by the first order condition of the smoothing problem (1.4) as

$$F_\epsilon(U,t) = \begin{pmatrix} ((A + \Phi'(Y_\epsilon(U)))^{-1} N)^T H(Y_\epsilon(U) - Z_d) + \alpha M(U - U_d) + t \\ \min(t, b - U) \end{pmatrix}.$$

Here $Y_\epsilon(U)$ is the unique solution of the system of smoothing equations

$$AY + \Phi_\epsilon(Y) - NU - c = 0.$$

Since (1.4) is a smoothing problem, $(Y_\epsilon^*, U_\epsilon^*) \in \mathcal{S}_{h,\epsilon}$ implies that there is $t_\epsilon^* \in R^m$ such that

$$F_\epsilon(U_\epsilon^*, t_\epsilon^*) = 0.$$

Applying the mean value theorem for Lipschitz continuous functions in [9], we have

$$F(U^*,t^*) - F(U^*_\epsilon,t^*_\epsilon) = \mathrm{co}\partial F(\overline{U^*U^*_\epsilon},\overline{t^*t^*_\epsilon}) \begin{pmatrix} U^* - U^*_\epsilon \\ t^* - t^*_\epsilon \end{pmatrix},$$

where $\mathrm{co}\partial F(\overline{U^*U^*_\epsilon},\overline{t^*t^*_\epsilon})$ denotes the convex hull of all matrices $V \in \partial F(Z)$ for Z in the line segment bewteen (U^*,t^*) and $(U^*_\epsilon,t^*_\epsilon)$.

Therefore, we can claim that

$$\left\| \begin{pmatrix} U^* - U^*_\epsilon \\ t^* - t^*_\epsilon \end{pmatrix} \right\| \le 2\beta \|F(U^*,t^*) - F(U^*_\epsilon,t^*_\epsilon)\|. \tag{2.8}$$

Note that

$$\lim_{\epsilon\downarrow 0} \Phi'_\epsilon(Y^*) = \Phi^o(Y^*)$$

and $\Phi^o(Y^*) \in \partial\lambda D \max(0,Y^*)$. By Lemma 2.2 in [5]

$$(A + \Phi^o(Y^*))^{-1}N = W.$$

From the nonsingularity of $\partial F(Y^*,U^*)$, (Y^*,U^*) is the unique solution of (1.3). From Theorem 2.2, $Y^*_\epsilon \to Y^*$ as $\epsilon \to 0$. Hence there are constants $\nu_1 > 0$, $\nu_2 > 0$ and $\bar\epsilon > 0$ such that for all $\epsilon \in (0,\bar\epsilon]$,

$$\|H(Y^*_\epsilon - Z_d)\| \le \nu_1,$$

and for $i \notin \mathcal{K}(Y^*)$, $i \notin \mathcal{K}(Y^*_\epsilon)$ and

$$|\Phi^o(Y^*) - \Phi'_\epsilon(Y^*_\epsilon)|_i = \frac{1}{2}\frac{\sqrt{(Y^*_\epsilon)^2_i + \epsilon^2} - |Y^*_\epsilon|_i}{\sqrt{(Y^*_\epsilon)^2_i + \epsilon^2}} \le \nu_2\epsilon. \tag{2.9}$$

Therefore, from $F(U^*,t^*) = 0$ and $F_\epsilon(U^*_\epsilon,t^*_\epsilon) = 0$, we find

$$\begin{aligned}
&\|F(U^*,t^*) - F(U^*_\epsilon,t^*_\epsilon)\| \\
&= \|F_\epsilon(U^*_\epsilon,t^*_\epsilon) - F(U^*_\epsilon,t^*_\epsilon)\| \\
&= \|(((A + \Phi^o(Y^*))^{-1} - (A + \Phi'_\epsilon(Y^*_\epsilon))^{-1})N)^T H(Y^*_\epsilon - Z_d)\| \\
&\le \nu_1 \|((A + \Phi^o(Y^*))^{-1} - (A + \Phi'_\epsilon(Y^*_\epsilon))^{-1})N\| \\
&\le \nu_1 \|(A + \Phi'_\epsilon(Y^*_\epsilon))^{-1}(\Phi'_\epsilon(Y^*_\epsilon) - \Phi^o(Y^*))(A + \Phi^o(Y^*))^{-1}N\| \\
&\le \nu_1 \|A^{-1}\| \|(\Phi'_\epsilon(Y^*_\epsilon) - \Phi^o(Y^*))(A + \Phi^o(Y^*))^{-1}N\| \\
&\le \nu_1\nu_2\sqrt{n}\|A^{-1}\|^2\|N\|\epsilon.
\end{aligned}$$

The last inequality uses $\|(A + \Phi^o(Y^*))^{-1}\| \le \|A^{-1}\|$, (2.9) and

$$((A + \Phi^o(Y^*))^{-1}N)_{\mathcal{K}(Y^*)} = 0.$$

This, together with (2.8), gives

$$\|U^* - U_\epsilon^*\| \le O(\epsilon).$$

Furthermore, from the convergence of Y_ϵ^* and the assumptions, we have that for sufficiently small ϵ,

$$\|Y^* - Y_\epsilon^*\| = \|W(U^* - U_\epsilon^*)\| \le O(\epsilon).$$

This completes the proof. ∎

2.2 Problems (1.3) and (1.2)

Note that $L_h^2(\Omega)$-scalar product $(Y, Y)_{L_h^2}$ associated with $Y = Py$ can be considered as the Riemann sum for the multidimensional integral $\int_\Omega y^2 dx$. By the error bound (5.5.5) in [10], we have

$$|\int_\Omega y^2 dx - (Y, Y)_{L_h^2}| \le \frac{2V(y^2)}{\sqrt{n}},$$

where

$$V(y^2) = \max_{\substack{x,z \in \Omega_h \\ x \ne z}} \frac{|y^2(x) - y^2(z)|}{\|x - z\|}.$$

If y is Lipschitz continuous in Ω with a Lipschitz constant K, then there is β such that $\beta \ge \max_{x\in\Omega} |y(x)|$ and

$$|y^2(x) - y^2(z)| = |y(x) + y(z)||y(x) - y(z)| \le 2\beta K\|x - z\|.$$

Hence the Lipschitzan continuity of f yields an error bound for the Riemann sum

$$|\int_\Omega y^2 dx - (Y, Y)_{L_h^2}| \le \frac{4\beta K}{\sqrt{n}} = O(h). \tag{2.10}$$

For a given function u, error bounds for the five-point finite difference method to solve the nonsmooth Dirichlet problem

$$-\triangle y + \lambda \max(0, y) = u + g \quad \text{in}\, \Omega, \quad y = 0 \text{ on}\, \Gamma. \tag{2.11}$$

can be found in [6].

Lemma 2.2 *[6] Let $y \in C^{2,\gamma}(\bar\Omega)$ be a solution of (2.11), and let Y be the finite difference solution of (2.11). Then we have*

$$A(Py) + \lambda D \max(0, Py) = Nu + c + O(h^\gamma)$$

and

$$\|Py - Y\| \le O(h^\gamma).$$

Here γ stands for the exponent of Hölder-continuity, and $0 < \gamma < 1$.

Theorem 2.4 *Suppose that (1.2) has a Lipschitz continuous solution* (y^*, u^*) *and* $y^* \in C^{2,\gamma}$. *Let* (Y^*, U^*) *be a solution of (1.3). Then we have*

$$J_h(Y^*, U^*) \leq J_h(Py^*, Pu^*) + O(h^\gamma). \tag{2.12}$$

Moreover, if there exists $\check{u} \in C^{0,\gamma}$, *together with* $\check{y} \in C^{2,\gamma}$, *satisfies the constraints of (1.2) and*

$$\|P\check{u} - U^*\| \leq O(h^\gamma),$$

then we have

$$J_h(Y^*, U^*) \geq J_h(Py^*, Pu^*) - O(h^\gamma). \tag{2.13}$$

Proof: By Lemma 2.2, the truncation error of the finite difference method yields

$$A(Py^*) + \lambda D \max(0, Py^*) = N(Pu^*) + c + O(h^\gamma). \tag{2.14}$$

We enlarge the feasible set of (1.3) and consider a relaxing problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}(Y - Z_d)^T H(Y - Z_d) + \frac{\alpha}{2}(U - U_d)^T M(U - U_d) \\ \text{subject to} & AY + \lambda D \max(0, Y) = NU + c \\ & U \leq b + \nu h^\gamma e, \end{array} \tag{2.15}$$

where $e = (1, 1, \ldots, 1)^T \in R^n$, and ν is a positive constant such that (Py^*, Pu^*) is a feasible point of (2.15).

Let $(\tilde{Y}, \tilde{U})$ be a solution of (2.15). Then it holds

$$J_h(\tilde{Y}, \tilde{U}) \leq J_h(Py^*, Pu^*). \tag{2.16}$$

Moreover, since the feasible set of (1.3) is contained in that of (2.15), we have

$$J_h(\tilde{Y}, \tilde{U}) \leq J_h(Y^*, U^*).$$

Take a point $\hat{U} = \min(b, \tilde{U})$, together with $\hat{Y}$ satisfying

$$A\hat{Y} + \lambda D \max(0, \hat{Y}) = N\hat{U} + c.$$

Then $(\hat{Y}, \hat{U})$ is a feasible point of (1.3). Moreover, from $\tilde{U} \leq b + \nu h^\gamma e$, we have

$$\|\hat{U} - \tilde{U}\| \leq O(h^\gamma)$$

and

$$\|\hat{Y} - \tilde{Y}\| \leq \|A^{-1}\| \|N\| \|\hat{U} - \tilde{U}\| \leq O(h^\gamma).$$

Therefore, we find

$$J_h(Y^*, U^*) \le J_h(\hat{Y}, \hat{U}) \le J_h(\tilde{Y}, \tilde{U}) + O(h^\gamma).$$

This, together with (2.16), implies

$$J_h(Y^*, U^*) \le J_h(Py^*, Pu^*) + O(h^\gamma).$$

To prove (2.13), we let $\check{Y}$ be the finite difference solution of the Diriclet problem

$$-\triangle y + \lambda \max(0, y) = \check{u} + g \quad \text{in}\, \Omega, \quad y = 0 \ \text{on}\, \Gamma. \tag{2.17}$$

By Lemma 2.2, we have

$$\|P\check{y} - \check{Y}\| \le O(h^\gamma). \tag{2.18}$$

Moreover, from

$$A\check{Y} + \lambda D \max(0, \check{Y}) = N(P\check{u}) + c$$

and

$$AY^* + \lambda D \max(0, Y^*) = NU^* + c$$

we find

$$\|\check{Y} - Y^*\| = \|(A+V)^{-1} N(P\check{u} - U^*)\| \le \|A^{-1}\| \|N\| \|P\check{u} - U^*\| \le O(h^\gamma).$$

Here V is a nonnegative diagonal matrix. (See the proof of Theorem 2.2.) This, together with (2.18), implies that

$$\|P\check{y} - Y^*\| \le O(h^\gamma).$$

Therefore, we obtain

$$J_h(P\check{y}, P\check{u}) \le J_h(Y^*, U^*) + O(h^\gamma). \tag{2.19}$$

From the assumption that $y^*, u^*, \check{y}$ and $\check{u}$ are Lipschitz continuous functions, we can estimate the errors of the integrals in J and get

$$J_h(Py^*, Pu^*) - O(h) \le J(y^*, u^*) \tag{2.20}$$

and

$$J(\check{y}, \check{u}) \le J_h(P\check{y}, P\check{u}) + O(h). \tag{2.21}$$

Finally, using the optimality of (y^*, u^*), that is,

$$J(y^*, u^*) \le J(\check{y}, \check{u}),$$

we obtain (2.13) from (2.19), (2.20),(2.21). ∎

From Theorem 2.3 and Theorem 2.4, we find a nice relation between the solution (y^*, u^*) of the nonsmooth optimal control problem (1.2) and the solution $(Y_\epsilon^*, U_\epsilon^*)$ of the finite difference smoothing approximation (1.4) as follows:

$$\|J_h(Y_\epsilon^*, U_\epsilon^*) - J_h(Py^*, Pu^*)\| \le O(h^\gamma) + O(\epsilon).$$

3 Numerical Examples

Convergence analysis and error estimates in Section 2 suggest that the discretized smoothing constrained optimal control problem (1.4) is a good approximation of the nonsmooth optimal control problem (1.2). In this section, we propose a smoothing SQP (sequential quadratic programming) method for solving (1.4) and report numerical results. Examples are generated by adding the nonsmooth term $\lambda \max(0, y)$ to examples in [2]. Several tests for different values of λ were performed. The tests were carried out on a IBM workstation using Matlab.

Smoothing SQP method(SSQP)

Choose parameters $\epsilon > 0$, $\sigma > 0$ and a feasible point (Y^0, U^0) of (1.3). For $k \geq 0$ we solve the quadratic program

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}(Y - Z_d)^T H (Y - Z_d) + \frac{\alpha}{2}(U - U_d)^T M (U - U_d) \\ \text{subject to} \quad & AY + \Phi_\epsilon(Y^k) + \Phi'_\epsilon(Y^k)(Y - Y^k) = NU + c \\ & U \leq b \end{aligned}$$

and let the optimal solution be (Y^{k+1}, U^{k+1}). We stop the iteration when

$$|J_h(Y^{k+1}, U^{k+1}) - J_h(Y^k, U^k)| \leq \sigma.$$

The SSQP method is a standard SQP method for solving the smoothing optimization problem (1.4). Convergence analysis can be found in [11]. Furthermore, the quadratic program at each step can be solved by an optimization toolbox, for example, *quadprog* in MATLAB.

In the numerical test, we chose $\Omega = (0,1) \times (0,1)$, $n = m = 200$, $\epsilon = 10^{-6}$, $\sigma = 10^{-8}$, $g = 0$, and $(Y^0, U^0) = (0, \dots, 0)^T$. In Tables 1-3, k is the number of iterations,

$$L(Y^k, U^k) = \| \min(-((A + \lambda D E(Y^k))^{-1} N)^T H (Y^k - Z_d) - \alpha M (U^k - U_d), b - U^k) \|_\infty,$$

$$|J_h^k - J_h^{k-1}| = |J_h(Y^k, U^k) - J_h(Y^{k-1}, U^{k-1})|$$

and

$$r_\epsilon = \|AY^k + \lambda D \max(0, Y^k) - NU^k - c\|_\infty.$$

Example 3.1 Let $q(x) \equiv 0$ and

$$z_d(x) = \frac{1}{6}\exp(2x_1)\sin(2\pi x_1)\sin(2\pi x_2).$$

Table 1: Example 3.1(a) $u_d \equiv 0$, $\alpha = 10^{-2}$

λ	k	$L(Y^k, U^k)$	$J_h(Y^k, U^k)$	$\lvert J_h^k - J_h^{k-1}\rvert$	r_ϵ
0.2	4	8.4e-9	0.0419	1.1e-12	8.9e-13
0.8	4	2.5e-7	0.0419	3.1e-11	3.6e-12
1.6	4	2.1e-7	0.0419	2.6e-11	7.1e-12
3.2	4	6.1e-8	0.0419	5.3e-12	1.4e-11
6.4	4	1.6e-7	0.0419	2.7e-11	2.6e-11
12.8	4	2.9e-7	0.0419	2.8e-11	3.2e-11

Table 2: Example 3.1(b) $u_d \equiv 1$, $\alpha = 10^{-6}$

λ	k	$L(Y^k, U^k)$	$J_h(Y^k, U^k)$	$\lvert J_h^k - J_h^{k-1}\rvert$	r_ϵ
0.2	4	8.0e-9	0.0302	8.7e-13	6.6e-14
0.8	4	1.1e-6	0.0302	7.6e-11	2.7e-13
1.6	4	7.0e-11	0.0303	5.4e-15	5.3e-13
3.2	4	6.0e-7	0.0302	8.6e-11	1.1e-12
6.4	4	7.8e-9	0.0302	5.4e-15	2.1e-12
12.8	4	1.0e-7	0.0302	7.6e-12	4.0e-12

Table 3: Example 3.2 $\alpha = 10^{-6}$

λ	k	$L(Y^k, U^k)$	$J_h(Y^k, U^k)$	$\lvert J_h^k - J_h^{k-1}\rvert$	r_ϵ
0.2	4	1.5e-15	0.0584	2.4e-11	3.4e-11
0.8	4	1.5e-14	0.0584	5.6e-16	5.8e-12
1.6	4	3.0e-14	0.0584	8.3e-16	9.9e-12
3.2	4	6.3e-14	0.0584	2.1e-16	1.5e-11
6.4	4	1.3e-13	0.0584	1.2e-17	1.3e-10
12.8	5	3.1e-10	0.0584	2.1e-16	1.2e-10

Example 3.2 Let $q(x) \equiv 1$, $u_d \equiv 0$, $\alpha = 1.0^{-6}$ and

$$z_d(x) = \begin{cases} 200x_1x_2(x_1 - \frac{1}{2})^2(1 - x_2) & \text{if} \quad 0 < x_1 \le 1/2 \\ 200x_2(x_1 - 1)(x_1 - \frac{1}{2})^2(1 - x_2) & \text{if} \quad 1/2 < x_1 \le 1 \end{cases}$$

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Appendix: Proof of Example 2.1 The solution function $Y(\cdot)$ can be given explicitly as

$$Y(U) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} U & \text{if} \quad U \geq 0 \\ \begin{pmatrix} 5/3 \\ 1/3 \end{pmatrix} U & \text{if} \quad U < 0. \end{cases}$$

1. Since $\tilde{Y} = Y(\tilde{U}) = (0,0)$ and $E(Y(\tilde{U}))$ is a zero matrix, we have

$$(A^{-1}N)^T H(\tilde{Y} - Z_d) + \alpha M(\tilde{U} - U_d) + t = \frac{1}{3}(5,1)\begin{pmatrix} 0 \\ 3 \end{pmatrix} - 1 + t = 0$$

with $t = 0$, and

$$\min(t, b - \tilde{U}) = 0.$$

Hence $(\tilde{U}, \tilde{U})$ is a KKT point of (1.3). However,

$$J_h(\tilde{Y}, \tilde{U}) = \frac{1}{2}(0, 3)\begin{pmatrix} 0 \\ 3 \end{pmatrix} + \frac{1}{2} = 5,$$

and

$$J_h(\bar{Y}, \bar{U}) = \frac{1}{2}(\frac{1}{2}, 3)\begin{pmatrix} \frac{1}{2} \\ 3 \end{pmatrix} + \frac{1}{2}(\frac{1}{2} - 1)^2 = \frac{39}{8} < J_h(\tilde{Y}, \tilde{U})$$

where $\bar{U} = 1/2$ and $\bar{Y} = (1/2, 0)^T$. Hence $(\tilde{Y}, \tilde{U})$ is not a solution of (1.3).

2. For $U \geq 0$,

$$J_h(Y(U), U) = \frac{1}{2}(U^2 + 1) + \frac{1}{2}U^2.$$

For $U < 0$,

$$J_h(Y, U) = \frac{1}{2}(\frac{25}{9}U^2 + (\frac{U}{3} - 1)^2) + \frac{1}{2}U^2.$$

Hence $(Y^*, U^*) = 0$ is the solution of (1.3). However

$$(A^{-1}N)^T(Y^* - Z_d) + t = \frac{1}{3}(5, 1)\begin{pmatrix} 0 \\ -1 \end{pmatrix} + t = 0$$

implies $t = 1/3$ and $\min(t, b - U^*) = \min(1/3, 1) = 1/3$, that is, (Y^*, U^*) is not a KKT point.