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Linear convergence of an algorithm for computing the largest eigenvalue of a nonnegative tensor

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SUMMARY

An iterative method for finding the largest eigenvalue of a nonnegative tensor was proposed by Ng, Qi, and Zhou in 2009. In this paper, we establish an explicit linear convergence rate of the Ng–Qi–Zhou method for essentially positive tensors. Numerical results are given to demonstrate linear convergence of the Ng–Qi–Zhou algorithm for essentially positive tensors. Copyright © 2011 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Consider an m-order n-dimensional tensor $\mathcal{A}$ consisting of n^m entries in the real field $\Re$:

$$\mathcal{A} = \left(a_{i_1\cdots i_m}\right), \quad a_{i_1\cdots i_m} \in \Re, \quad 1 \leqslant i_1, \ldots, i_m \leqslant n.$$

An m-order n-dimensional tensor $\mathcal{A}$ is called nonnegative (or, respectively, positive) if $a_{i_1\cdots i_m} \geqslant 0$ (or, respectively, $a_{i_1\cdots i_m} > 0$). A tensor $\mathcal{A}$ is called reducible, if there exists a nonempty proper index subset $J \subset \{1, 2, \ldots, n\}$ such that

$$a_{i_1\cdots i_m} = 0, \quad \forall i_1 \in J, \quad \forall i_2, \ldots, i_m \notin J.$$

If $\mathcal{A}$ is not reducible, then we say that $\mathcal{A}$ is irreducible. This definition was used in [1–6]. In [7, 8], this property is called indecomposable.

To an n-dimensional column vector $x = (x_1; x_2; \ldots; x_n)$, real or complex, and any complex number α, we define n-dimensional column vectors $\mathcal{A}x^{m-1}$ and $x^{[\alpha]}$:

$$\mathcal{A}x^{m-1} := \left(\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2}\ldots x_{i_m}\right)_{1\leqslant i\leqslant n}, \quad x^{[\alpha]} := \left(x_i^{\alpha}\right)_{1\leqslant i\leqslant n}.$$

Let C be the complex field. A pair $(\lambda, x) \in \mathrm{C} \times (\mathrm{C}^n \backslash \{0\})$ is called an eigenvalue–eigenvector pair of $\mathcal{A}$, if they satisfy:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{1}$$

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This definition was introduced by Qi [9] when m is even and $\mathcal{A}$ is symmetric and extended to the general case in [1]. Independently, Lim [10] gave such a definition but restricted x and λ to be real. Unlike matrices, the eigenvalue problem for tensors is nonlinear.

Recently, the largest eigenvalue problem for a nonnegative tensor has attracted much attention because it has many important applications such as multilinear pagerank [11], hypergraphs [12], higher-order Markov chains [4, 13], and positive definiteness of a multivariate form [3]. Chang, Pearson, and Zhang [1] generalized the Perron–Frobenius theorem from nonnegative matrices to irreducible nonnegative tensors. Yang and Yang [6] generalized the weak Perron–Frobenius theorem to general nonnegative tensors. Bulò and Pelillo [12] gave new bounds on the clique number of graphs on the basis of spectral hypergraph theory. The calculation of these new bounds relies on finding the largest eigenvalue of a $\{0, 1\}$ nonnegative tensor. Ng, Qi, and Zhou [4] proposed an iterative method for finding the largest eigenvalue of an irreducible nonnegative tensor, which is an extension of the Collatz method [14] for calculating the spectral radius of an irreducible nonnegative matrix. Pearson [5] introduced the notion of essentially positive tensors and proved that the unique positive eigenvalue is real geometrically simple when the tensor is essentially positive with even order. Here, 'real geometrically simple' means that the corresponding real eigenvector is unique up to a scaling constant. Liu, Zhou, and Ibrahim [3] modified the Ng–Qi–Zhou method such that the modified algorithm is always convergent for finding the largest eigenvalue of an irreducible nonnegative tensor. Chang, Pearson, and Zhang [2] introduced primitive tensors. An essentially positive tensor is a primitive tensor, and a primitive tensor is an irreducible nonnegative tensor but not vice versa. Chang, Pearson, and Zhang [2] established convergence of the Ng–Qi–Zhou method for primitive tensors. Friedland, Gaubert, and Han [7] pointed out that the Perron–Frobenius theorem for nonnegative tensors has a very close link with the Perron–Frobenius theorem for homogeneous monotone maps, initiated by Nussbaum [15] and further studied by Gaubert and Gunawardena [8]. Friendland, Gaubert, and Han [7] introduced weakly irreducible nonnegative tensors and established the Perron–Frobenius theorem for them.

Denote $\Re^n_+ = \{x \in \Re^n : x \geqslant 0\}$ and $\Re^n_{++} = \{x \in \Re^n : x > 0\}$. A map $F : \Re^n_+ \to \Re^n_+$ is called a homogeneous monotone map if $F(tx) = tF(x)$ for any $x \in \Re^n_+$ and any positive number t and if $F(x) \leqslant F(y)$ for any $x, y \in \Re^n_+$ with $x \leqslant y$. For an m-order n-dimensional tensor $\mathcal{A}$, define $F : \Re^n_+ \to \Re^n_+$ by $F(x) = \left(\mathcal{A}x^{m-1}\right)^{[\frac{1}{m-1}]}$ for any $x \in \Re^n_+$. Then F is a homogeneous monotone map. Hence, in $\Re_+ \times \Re^n_+$, eigenvalues and eigenvectors of nonnegative tensors, discussed here, fall in the class of eigenvalues and eigenvectors of homogeneous monotone maps. Eigenvalues and eigenvectors of nonnegative tensors are defined in $\mathrm{C} \times (\mathrm{C}^n \backslash \{0\})$. In Corollary 4.3 of [7], Friendland, Gaubert, and Han showed how to extend the Perron–Frobenius result for a nonnegative tensor from $\Re_+ \times \Re^n_+$ to $\mathrm{C} \times (\mathrm{C}^n \backslash \{0\})$.

In this paper, we establish an explicit linear convergence rate of the Ng–Qi–Zhou method for essentially positive tensors. Such a result has not appeared in the literature of eigenvalues of homogeneous monotone maps and nonnegative tensors.

This paper is organized as follows. In Section 2, we recall some preliminary results. In Section 3, we rewrite the Ng–Qi–Zhou method in the spirit of the earlier work of Hall and Porsching [16, 17]. An explicit linear convergence rate of the Ng–Qi–Zhou method is established for essentially positive tensors, with a specified starting point, in Section 4. Finally, in Section 5, we report some numerical results.

Note that Chang, Pearson, and Zhang [2] showed that linear convergence rate did not hold for some primitive but not essentially positive tensors for the Ng–Qi–Zhou method. This shows that our result is sharp.

2. PRELIMINARIES

First, we state the Perron–Frobenius theorem for nonnegative tensors given in [1, Theorem 1.4] and the minimax theorem for irreducible nonnegative tensors given in [1, Theorem 4.2]. These two results, as we stated in the introduction, may be derived from the earlier results on homogeneous monotone maps.

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Theorem 2.1
If $\mathcal{A}$ is an irreducible nonnegative tensor of order m and dimension n, then there exist $\lambda_0 > 0$ and $x_0 \in \Re^n_{++}$ such that

$$\mathcal{A}x_0^{m-1} = \lambda_0 x_0^{[m-1]}.$$

Moreover, if λ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \lambda_0$. If λ is an eigenvalue of $\mathcal{A}$, then $|\lambda| \leqslant \lambda_0$.

Theorem 2.2
Assume that $\mathcal{A}$ is an irreducible nonnegative tensor of order m and dimension n. Then

$$\min_{x\in\Re^n_{++}} \max_{1\leqslant i\leqslant n} \frac{\left(\mathcal{A}x^{m-1}\right)_i}{x_i^{m-1}} = \lambda_0 = \max_{x\in\Re^n_{++}} \min_{1\leqslant i\leqslant n} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}},$$

where λ_0 is the unique positive eigenvalue corresponding to a positive eigenvector.

On the basis of Theorem 2.2, the Ng–Qi–Zhou method presented in [4] works as follows. Choose $x^{(0)} \in \Re^n_{++}$ and let $y^{(0)} = \mathcal{A}\left(x^{(0)}\right)^{m-1}$. For $k = 0, 1, 2, \ldots$, compute

$$x^{(k+1)} = \frac{\left(y^{(k)}\right)^{[\frac{1}{m-1}]}}{\left\|\left(y^{(k)}\right)^{[\frac{1}{m-1}]}\right\|}, \quad y^{(k+1)} = \mathcal{A}\left(x^{(k+1)}\right)^{m-1},$$

$$\underline{\lambda}_{k+1} = \min_{x_i^{(k+1)}>0} \frac{\left(y^{(k+1)}\right)_i}{\left(x_i^{(k+1)}\right)^{m-1}}, \quad \overline{\lambda}_{k+1} = \max_{x_i^{(k+1)}>0} \frac{\left(y^{(k+1)}\right)_i}{\left(x_i^{(k+1)}\right)^{m-1}}. \tag{2}$$

It is shown in [4] that the obtained sequences $\{\underline{\lambda}_k\}$ and $\{\overline{\lambda}_k\}$ converge to some numbers $\underline{\lambda}$ and $\overline{\lambda}$, respectively, and we have $\underline{\lambda} \leqslant \lambda_0 \leqslant \overline{\lambda}$, where λ_0 is the largest eigenvalue of $\mathcal{A}$, defined in Theorem 2.1. If $\underline{\lambda} = \overline{\lambda}$, that is, the gap is zero, then both the sequences $\{\underline{\lambda}_k\}$ and $\{\overline{\lambda}_k\}$ converge to λ_0. However, a positive gap may happen, which can be seen in an example given in [4]. That example is irreducible but not primitive. Chang, Pearson, and Zhang [2] established convergence of the Ng–Qi–Zhou method for primitive tensors and gave an example, which is primitive but not essentially positive, such that linear convergence fails for this example with the Ng–Qi–Zhou method.

We thus intend to establish an explicit linear convergence rate of the Ng–Qi–Zhou method for essentially positive tensors. For this purpose, we need the following definition given in [5].

Definition 2.1
A nonnegative m-order n-dimensional tensor $\mathcal{A}$ is essentially positive if $\mathcal{A}x^{m-1} \in \Re^n_{++}$ for any nonzero $x \in \Re^n_+$.

By Theorem 3.2 and Definition 2.1 in [5], it is easy to obtain the following result.

Proposition 2.1
A nonnegative m-order n-dimensional tensor $\mathcal{A}$ is essentially positive if and only if $a_{i\,j\ldots j} > 0$ for $i, j \in \{1, 2, \ldots, n\}$.

For the remainder of this paper, denote

$$I = \{(i_2, \ldots, i_m) \mid i_2, \ldots, i_m \in \{1, \ldots, n\}\}.$$

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3. ALGORITHM

We first give inclusion bounds for the largest eigenvalue of an irreducible nonnegative tensor. The following lemma was given in [1, Lemma 2.2].

Lemma 3.1
If a nonnegative tensor $\mathcal{A}$ of order m and dimension n is irreducible, then

$$R_i := \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} > 0, \quad i = 1, 2, \ldots, n.$$

On the basis of Theorem 2.2 and Lemma 3.1, we obtain the following inclusion bounds.

Proposition 3.1
Let $\mathcal{A}$ be an irreducible nonnegative tensor of order m and dimension n, and let λ_0 be the largest eigenvalue of $\mathcal{A}$. Then

$$\min_{1\leqslant i\leqslant n} M_i \leqslant \lambda_0 \leqslant \max_{1\leqslant i\leqslant n} M_i,$$

where

$$M_i = \frac{1}{R_i} \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} R_{i_2}^{\frac{1}{m-1}} \cdots R_{i_m}^{\frac{1}{m-1}}, \quad i = 1, 2, \ldots, n.$$

Proof
Define an n-dimensional column vector $R := (R_i)_{1\leqslant i\leqslant n}$, where R_i is defined in Lemma 3.1. By Lemma 3.1, the vector $R^{[\frac{1}{m-1}]} \in \Re^n_{++}$. Taking $x = R^{[\frac{1}{m-1}]}$ into the equalities in Theorem 2.2, we immediately get the lower and upper bounds. □

Let

$$\overline{R} = \max_{1\leqslant i\leqslant n} R_i, \quad \underline{R} = \min_{1\leqslant i\leqslant n} R_i. \tag{3}$$

For $i = 1, 2, \ldots, n$, we have

$$\underline{R} = \frac{\underline{R}}{R_i} \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} \leqslant M_i \leqslant \frac{\overline{R}}{R_i} \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} = \overline{R}.$$

This shows that the lower and upper bounds given in Proposition 3.1 are better than the bounds in [6, Lemma 5.6].

On the basis of Proposition 3.1 and in the spirit of the earlier work of Hall and Porsching [16,17], we rewrite the Ng–Qi–Zhou method as follows.

Algorithm 3.1

Step 0. Let $\mathcal{A}^{(0)} = \mathcal{A}$ and $S_i^{(0)} = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m}$ for $i = 1, \ldots, n$. Let the accurate tolerance $\varepsilon > 0$ be a sufficiently small number and set $k := 0$.

Step 1. Set

$$\mathcal{A}^{(k+1)} = \left(a_{i_1\ldots i_m}^{(k+1)}\right), \quad S_i^{(k+1)} = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m}^{(k+1)}$$

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for $i = 1, \ldots, n$, where

$$a_{ii_2\ldots i_m}^{(k+1)} = a_{ii_2\ldots i_m}^{(k)} \frac{\left(S_{i_2}^{(k)}\right)^{\frac{1}{m-1}} \cdots \left(S_{i_m}^{(k)}\right)^{\frac{1}{m-1}}}{S_i^{(k)}}. \tag{4}$$

Step 2. Let

$$\overline{\lambda}_k = \max_{1\leqslant i\leqslant n} S_i^{(k+1)}, \quad \underline{\lambda}_k = \min_{1\leqslant i\leqslant n} S_i^{(k+1)}.$$

If $\overline{\lambda}_k - \underline{\lambda}_k < \varepsilon$, stop. Output the maximal eigenvalue $\lambda_0 = \frac{1}{2}(\overline{\lambda}_k + \underline{\lambda}_k)$. Otherwise, set $k := k + 1$ and go to Step 1.

Algorithm 3.1 is well defined. By Lemma 3.1, for $i = 1, \ldots, n$, $S_i^{(0)} = R_i > 0$ and there exists at least one subindex array $(i_2, \ldots, i_m) \in I$ such that $a_{ii_2\ldots i_m} > 0$. Hence, $a_{ii_2\ldots i_m}^{(1)} > 0$ and $S_i^{(1)} > 0$. By induction, we have $a_{ii_2\ldots i_m}^{(k)} > 0$ and $S_i^{(k)} > 0$ for $k = 2, 3, \ldots$.

We now prove that Algorithm 3.1 is actually the Ng–Qi–Zhou method with a specified starting point.

Proposition 3.2
Let $\mathcal{A}$ be an irreducible nonnegative tensor of order m and dimension n. Then Algorithm 3.1 is just the Ng–Qi–Zhou method with the starting point $x^{(0)} = (1; 1; \ldots; 1) \in \Re^n$.

Proof
For $k = 0, 1, \ldots$, define an n-dimensional column vector

$$x^{(k)} := \left(\prod_{j=0}^{k} S_i^{(j)}\right)_{1\leqslant i\leqslant n}.$$

Because $\mathcal{A}$ is irreducible, by Lemma 3.1, $x^{(k)} \in \Re_{++}^n$ for $k = 0, 1, \ldots$. Set

$$y^{(k)} := \frac{\left(x^{(k)}\right)^{[\frac{1}{m-1}]}}{\left\|\left(x^{(k)}\right)^{[\frac{1}{m-1}]}\right\|}, \quad k = 0, 1, \ldots. \tag{5}$$

By the definition of $\mathcal{A}^{(k)}$ in Step 1, we obtain for $1 \leqslant i \leqslant n$,

$$\begin{aligned}
\frac{\left(\mathcal{A}\left(y^{(k)}\right)^{m-1}\right)_i}{\left(y_i^{(k)}\right)^{m-1}} &= \frac{1}{x_i^{(k)}} \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} \left(x_{i_2}^{(k)}\right)^{\frac{1}{m-1}} \cdots \left(x_{i_m}^{(k)}\right)^{\frac{1}{m-1}} \\
&= \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m}^{(1)} \frac{\left(\prod_{j=1}^{k} S_{i_2}^{(j)}\right)^{\frac{1}{m-1}} \cdots \left(\prod_{j=1}^{k} S_{i_m}^{(j)}\right)^{\frac{1}{m-1}}}{\prod_{j=1}^{k} S_i^{(j)}} \\
&= \cdots = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m}^{(k)} \frac{\left(S_{i_2}^{(k)}\right)^{\frac{1}{m-1}} \cdots \left(S_{i_m}^{(k)}\right)^{\frac{1}{m-1}}}{S_i^{(k)}} \\
&= S_i^{(k+1)},
\end{aligned} \tag{6}$$

which concludes that $\underline{\lambda}_k$ is just $\underline{\lambda}_{k+1}$ in (2) for $k = 0, 1, 2, \ldots$. This also holds for the sequence $\{\overline{\lambda}_k\}$. Hence, we conclude the statement of this proposition. □

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Numerical results reported in [3,4] imply that Algorithm 3.1 is efficient. In particular, Liu, Zhou, and Ibrahim [3] applied a modification of Algorithm 3.1 to study the positive definiteness of a multivariate form. Testing positive definiteness of a multivariate form is an important problem in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control. Researchers in automatic control studied the conditions of such positive definiteness intensively [9]. For $n \geqslant 3$ and $m \geqslant 4$, this problem is a hard problem in mathematics. There are only a few methods to answer the question, and these methods are computationally expensive when $n > 3$ [3]. Numerical results in [3] show the method of using the largest eigenvalue of nonnegative tensors is effective in some cases.

4. LINEAR CONVERGENCE

By Proposition 3.2, it suffices to establish an explicit linear convergence rate of Algorithm 3.1 in the case of essentially positive tensors. By some straightforward computations, we immediately obtain the following two propositions.

Proposition 4.1
For $k = 0, 1, 2, \ldots,$

$$S_i^{(k+1)} = \sum_{i_2,\ldots,i_m=1}^{n} a_{i i_2 \ldots i_m}^{(k)} \frac{\left(S_{i_2}^{(k)}\right)^{\frac{1}{m-1}} \cdots \left(S_{i_m}^{(k)}\right)^{\frac{1}{m-1}}}{S_i^{(k)}}, \quad i = 1, 2, \ldots, n.$$

Proof
For $i = 1, 2, \ldots, n$, by (4), we have

$$S_i^{(k+1)} = \sum_{i_2,\ldots,i_m=1}^{n} a_{i i_2 \ldots i_m}^{(k+1)} = \sum_{i_2,\ldots,i_m=1}^{n} a_{i i_2 \ldots i_m}^{(k)} \frac{\left(S_{i_2}^{(k)}\right)^{\frac{1}{m-1}} \cdots \left(S_{i_m}^{(k)}\right)^{\frac{1}{m-1}}}{S_i^{(k)}}.$$

□

Proposition 4.2
For $k = 0, 1, 2, \ldots,$

$$a_{i\,i\ldots i}^{(k)} = a_{i\,i\ldots i}, \quad i = 1, 2, \ldots, n.$$
$$a_{i\,j\ldots j}^{(k)} a_{j\,i\ldots i}^{(k)} = a_{i\,j\ldots j} a_{j\,i\ldots i}, \quad i, j \in \{1, 2, \ldots, n\}.$$

Proof
For $i = 1, 2, \ldots, n$, it follows from (4) that

$$a_{i\,i\ldots i}^{(k)} = a_{i\,i\ldots i}^{(k-1)} \frac{\left(S_i^{(k-1)}\right)^{\frac{1}{m-1}} \cdots \left(S_i^{(k-1)}\right)^{\frac{1}{m-1}}}{S_i^{(k-1)}} = a_{i\,i\ldots i}^{(k-1)} = \cdots = a_{i\,i\ldots i}.$$

For $i, j \in \{1, 2, \ldots, n\}$, according to (4), by a direct computation,

$$\begin{aligned} a_{i\,j\ldots j}^{(k)} a_{j\,i\ldots i}^{(k)} &= a_{i\,j\ldots j}^{(k-1)} \frac{S_j^{(k-1)}}{S_i^{(k-1)}} \, a_{j\,i\ldots i}^{(k-1)} \frac{S_i^{(k-1)}}{S_j^{(k-1)}} \\ &= a_{i\,j\ldots j}^{(k-1)} a_{j\,i\ldots i}^{(k-1)} = \cdots = a_{i\,j\ldots j} a_{j\,i\ldots i}. \end{aligned}$$

□

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Let $\mathcal{A}$ be an irreducible nonnegative tensor of order m dimension n. By Theorem 2.1, there exists a positive eigenvalue of $\mathcal{A}$, which is the largest in modulus among all the eigenvalues of $\mathcal{A}$. Algorithm 3.1 yields two sequences of lower and upper bounds for this largest eigenvalue.

Theorem 4.1
Let $\mathcal{A}$ be an irreducible nonnegative tensor of order m and dimension n and λ_0 be its largest eigenvalue. Assume that $\{\overline{\lambda}_k\}$ and $\{\underline{\lambda}_k\}$ are two sequences generated by Algorithm 3.1. Then

$$\underline{R} \leqslant \underline{\lambda}_0 \leqslant \underline{\lambda}_1 \leqslant \cdots \leqslant \underline{\lambda}_k \leqslant \cdots \leqslant \lambda_0 \leqslant \cdots \leqslant \overline{\lambda}_k \leqslant \cdots \leqslant \overline{\lambda}_1 \leqslant \overline{\lambda}_0 \leqslant \overline{R}.$$

Proof
For $k = 0, 1, \ldots$, let $y^{(k)}$ be defined by (5). Taking the vector $y^{(k)}$ into the two equalities in Theorem 2.2, we obtain from (6),

$$\underline{\lambda}_k = \min_{1\leqslant i\leqslant n} S_i^{(k+1)} \leqslant \lambda_0 \leqslant \max_{1\leqslant i\leqslant n} S_i^{(k+1)} = \overline{\lambda}_k, \quad k = 0, 1, \ldots.$$

We now prove for any $k \geqslant 0$,

$$\underline{\lambda}_k \leqslant \underline{\lambda}_{k+1} \quad \text{and} \quad \overline{\lambda}_{k+1} \leqslant \overline{\lambda}_k.$$

We assume, without loss of generality, that $\underline{\lambda}_{k+1} = S_p^{(k+2)}$ and $\overline{\lambda}_{k+1} = S_q^{(k+2)}$ where $p, q \in \{1, 2, \ldots, n\}$. We have by Proposition 4.1,

$$\begin{aligned}
\underline{\lambda}_{k+1} &= S_p^{(k+2)} \\
&= \frac{1}{S_p^{(k+1)}} \sum_{i_2,\ldots,i_m=1}^{n} a_{pi_2\ldots i_m}^{(k+1)} \left(S_{i_2}^{(k+1)}\right)^{\frac{1}{m-1}} \cdots \left(S_{i_m}^{(k+1)}\right)^{\frac{1}{m-1}} \\
&\geqslant \min_{1\leqslant i\leqslant n} S_i^{(k+1)} \\
&= \underline{\lambda}_k.
\end{aligned}$$

Similarly, we can prove that $\overline{\lambda}_{k+1} \leqslant \overline{\lambda}_k$. This, together with (3), completes our proof. □

Theorem 4.1 indicates that $\{\overline{\lambda}_k\}$ and $\{\underline{\lambda}_k\}$ converge, and the sequences $\{\overline{\lambda}_k - \underline{\lambda}_k\}$ is nonnegative and monotonically decreasing. Hence, $\{\overline{\lambda}_k - \underline{\lambda}_k\}$ has a limit. We now show that in the case of essentially positive tensors, $\{\overline{\lambda}_k - \underline{\lambda}_k\}$ linearly converges to zero with an explicit convergence rate. This establishes an explicit linear converge rate of the Ng–Qi–Zhou method for essentially positive tensors.

Theorem 4.2
Let $\mathcal{A}$ be a nonnegative tensor of order m and dimension n. If $\mathcal{A}$ is essentially positive, then

$$\overline{\lambda}_k - \underline{\lambda}_k \leqslant \alpha\left(\overline{\lambda}_{k-1} - \underline{\lambda}_{k-1}\right), \quad k = 1, 2, \ldots, \tag{7}$$

where

$$\alpha := 1 - \frac{\beta}{\overline{R}} \in (0, 1), \tag{8}$$

$$\beta := \min_{i,j\in\{1,2,\ldots,n\}} a_{i\,j\ldots j},$$

$$\overline{R} = \max_{1\leqslant i\leqslant n} R_i$$

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and

$$R_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m}.$$

Proof
By Proposition 2.1, the nonnegative tensor $\mathcal{A}$ is irreducible.

Without loss of generality, assume that $\overline{\lambda}_k = S_p^{(k+1)}$ and $\underline{\lambda}_k = S_q^{(k+1)}$. Then by Proposition 4.1, we have

$$\begin{aligned}\overline{\lambda}_k - \underline{\lambda}_k &= S_p^{(k+1)} - S_q^{(k+1)} \\ &= \sum_{i_2,\ldots,i_m=1}^{n} \left(\frac{a_{pi_2\ldots i_m}^{(k)}}{S_p^{(k)}} - \frac{a_{qi_2\ldots i_m}^{(k)}}{S_q^{(k)}} \right) \left(S_{i_2}^{(k)}\right)^{\frac{1}{m-1}} \cdots \left(S_{i_m}^{(k)}\right)^{\frac{1}{m-1}}. \end{aligned} \tag{9}$$

Define

$$I(k) = \left\{ (i_2,\ldots,i_m) \in I \;\middle|\; \frac{a_{pi_2\ldots i_m}^{(k)}}{S_p^{(k)}} \geqslant \frac{a_{qi_2\ldots i_m}^{(k)}}{S_q^{(k)}} \right\}.$$

By the definition of $S_i^{(k)}$, for $i = 1, \cdots, n$, we have

$$1 = \sum_{(i_2,\ldots,i_m)\in I(k)} \frac{a_{ii_2\ldots i_m}^{(k)}}{S_i^{(k)}} + \sum_{(i_2,\ldots,i_m)\in I\setminus I(k)} \frac{a_{ii_2\ldots i_m}^{(k)}}{S_i^{(k)}}.$$

Letting $i = p$ and q and combining these two equalities, we have

$$\sum_{(i_2,\ldots,i_m)\in I(k)} \left(\frac{a_{pi_2\ldots i_m}^{(k)}}{S_p^{(k)}} - \frac{a_{qi_2\ldots i_m}^{(k)}}{S_q^{(k)}} \right) = - \sum_{(i_2,\ldots,i_m)\in I\setminus I(k)} \left(\frac{a_{pi_2\ldots i_m}^{(k)}}{S_p^{(k)}} - \frac{a_{qi_2\ldots i_m}^{(k)}}{S_q^{(k)}} \right). \tag{10}$$

Combining (9), (10), and the definitions of $\overline{\lambda}_{k-1}$ and $\underline{\lambda}_{k-1}$, by Theorem 4.1, we obtain

$$\begin{aligned}\overline{\lambda}_k - \underline{\lambda}_k &= \sum_{(i_2,\ldots,i_m)\in I(k)} \left(\frac{a_{pi_2\ldots i_m}^{(k)}}{S_p^{(k)}} - \frac{a_{qi_2\ldots i_m}^{(k)}}{S_q^{(k)}} \right) \left(S_{i_2}^{(k)}\right)^{\frac{1}{m-1}} \cdots \left(S_{i_m}^{(k)}\right)^{\frac{1}{m-1}} \\ &\quad + \sum_{(i_2,\ldots,i_m)\in I\setminus I(k)} \left(\frac{a_{pi_2\ldots i_m}^{(k)}}{S_p^{(k)}} - \frac{a_{qi_2\ldots i_m}^{(k)}}{S_q^{(k)}} \right) \left(S_{i_2}^{(k)}\right)^{\frac{1}{m-1}} \cdots \left(S_{i_m}^{(k)}\right)^{\frac{1}{m-1}} \\ &\leqslant \overline{\lambda}_{k-1} \sum_{(i_2,\ldots,i_m)\in I(k)} \left(\frac{a_{pi_2\ldots i_m}^{(k)}}{S_p^{(k)}} - \frac{a_{qi_2\ldots i_m}^{(k)}}{S_q^{(k)}} \right) + \underline{\lambda}_{k-1} \sum_{(i_2,\ldots,i_m)\in I\setminus I(k)} \left(\frac{a_{pi_2\ldots i_m}^{(k)}}{S_p^{(k)}} - \frac{a_{qi_2\ldots i_m}^{(k)}}{S_q^{(k)}} \right) \\ &= \left(\overline{\lambda}_{k-1} - \underline{\lambda}_{k-1}\right) \sum_{(i_2,\ldots,i_m)\in I(k)} \left(\frac{a_{pi_2\ldots i_m}^{(k)}}{S_p^{(k)}} - \frac{a_{qi_2\ldots i_m}^{(k)}}{S_q^{(k)}} \right) \\ &= \left(\overline{\lambda}_{k-1} - \underline{\lambda}_{k-1}\right) \left(1 - \left(\sum_{(i_2,\ldots,i_m)\in I\setminus I(k)} \frac{a_{pi_2\ldots i_m}^{(k)}}{S_p^{(k)}} + \sum_{(i_2,\ldots,i_m)\in I(k)} \frac{a_{qi_2\ldots i_m}^{(k)}}{S_q^{(k)}} \right) \right) \\ &\leqslant \left(\overline{\lambda}_{k-1} - \underline{\lambda}_{k-1}\right) \left(1 - \frac{\Delta_1 + \Delta_2}{\overline{\lambda}_{k-1}} \right) \\ &\leqslant \left(\overline{\lambda}_{k-1} - \underline{\lambda}_{k-1}\right) \left(1 - \frac{\Delta_1 + \Delta_2}{\overline{R}} \right), \end{aligned} \tag{11}$$

DOI: 10.1002/nla

where

$$\Delta_1 = \sum_{(i_2,\ldots,i_m)\in I\setminus I(k)} a^{(k)}_{pi_2\ldots i_m}, \quad \Delta_2 = \sum_{(i_2,\ldots,i_m)\in I(k)} a^{(k)}_{qi_2\ldots i_m}.$$

Because $\mathcal{A}$ is essentially positive, we have from Proposition 2.1,

$$a_{i\,j\ldots j} > 0, \quad i, j \in \{1, 2, \ldots, n\}. \tag{12}$$

Because $\mathcal{A}$ is irreducible, from (3) and (12), we have

$$0 < \beta < \overline{R}, \tag{13}$$

where β is defined in the statement of the theorem. We now consider the subindex array $(p, \ldots, p) \in I$. If $(p, \ldots, p) \in I\setminus I(k)$, then the summation Δ_1 must include $a^{(k)}_{p\,p\ldots p}$. If $(p, \ldots, p) \in I(k)$, there are two possibilities:

(1) $(q, \ldots, q) \in I(k)$. In this case, the summation Δ_2 must include $a^{(k)}_{q\,q\ldots q}$.
(2) $(q, \ldots, q) \in I\setminus I(k)$. In this case, the summation Δ_1 must include $a^{(k)}_{p\,q\ldots q}$, and the summation Δ_2 must include $a^{(k)}_{q\,p\ldots p}$.

From the discussion, by Proposition 4.2, we can obtain

$$\begin{aligned}\Delta_1 + \Delta_2 &\geqslant \min\left\{a^{(k)}_{p\,p\ldots p}, a^{(k)}_{q\,q\ldots q}, a^{(k)}_{p\,q\ldots q} + a^{(k)}_{q\,p\ldots p}\right\}\\ &\geqslant \min\left\{a_{p\,p\ldots p}, a_{q\,q\ldots q}, 2\sqrt{a_{p\,q\ldots q}\,a_{q\,p\ldots p}}\right\}\\ &\geqslant \beta.\end{aligned} \tag{14}$$

Take

$$\alpha := 1 - \frac{\beta}{\overline{R}}.$$

Then $0 < \alpha < 1$ follows from (13). Combining (11) and (14), we obtain (7) for $k = 1, 2, \ldots$. □

Theorem 4.2 achieved the target of this paper, namely, to establish an explicit linear convergence rate of the Ng–Qi–Zhou method for essentially positive tensors.

In the following corollary, we give an explicit asymptotic estimate on the number of iterations.

Corollary 4.1
Let α and $\overline{R}, \underline{R}$ be defined by (8) and (3), respectively. Let $\varepsilon > 0$ be the sufficiently small number given in Algorithm 3.1. If $\mathcal{A}$ is essentially positive, then Algorithm 3.1 terminates in at most

$$K = \left\lceil \frac{\log\left(\frac{\varepsilon}{\overline{R}-\underline{R}}\right)}{\log(\alpha)} \right\rceil + 1 \tag{15}$$

iterations with

$$\overline{\lambda}_K - \underline{\lambda}_K < \varepsilon.$$

Proof
By (7) in Theorem 4.2, we have for $k = 1, 2, \ldots$,

$$\overline{\lambda}_k - \underline{\lambda}_k \leqslant \alpha^k(\overline{R} - \underline{R}). \tag{16}$$

It follows from (15) and $\alpha \in (0, 1)$ that

$$\log(\alpha^K) = K\log(\alpha) < \log(\alpha)\frac{\log\left(\frac{\varepsilon}{\overline{R}-\underline{R}}\right)}{\log(\alpha)} = \log\left(\frac{\varepsilon}{\overline{R}-\underline{R}}\right),$$

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which yields

$$\alpha^K < \frac{\varepsilon}{\overline{R} - \underline{R}}.$$

This, together with (16), implies

$$\overline{\lambda}_K - \underline{\lambda}_K \leq \alpha^K(\overline{R} - \underline{R}) < \varepsilon.$$

This completes the proof. □

5. NUMERICAL EXPERIMENTS

To demonstrate the linear convergence of the Ng–Qi–Zhou algorithm (Algorithm 3.1) for essentially positive tensors, we made numerical experiments on some numerical examples with the stop rule $\varepsilon = 10^{-8}$.

We consider the following five classes of nonnegative tensors of order $m = 3$ and dimension n.

$$\mathcal{A}1: \quad A1_{ijj} = i \times j \ \text{ for } i, j = 1, 2, \ldots, n, \text{ and zero elsewhere.}$$

$$\mathcal{A}2: \quad A2_{ijj} = \frac{1}{i} \ \text{ for } i, j = 1, 2, \ldots, n, \text{ and zero elsewhere.}$$

$$\mathcal{A}3: \quad A3_{ijj} = \frac{i}{j} \ \text{ for } i, j = 1, 2, \ldots, n, \text{ and zero elsewhere.}$$

Table I. Numerical results for Algorithm 3.1

Tensor	Dimension	Eig	No. Iter	Ratio	(Max, Min) Ratio
$\mathcal{A}1$	5	55	6	0.00179, 0.00179, 0.00179	(0.00461, 0.00179)
	30	9455	4	0.0001, 0.0001, 0.0001	(0.0015, 0.0001)
	60	73810	3	0.136e−4, 0.136e−4, 0.136e−4	(3.671e−4, 0.136e−4)
$\mathcal{A}2$	5	2.2833	17	0.3046, 0.3046, 0.3046	(0.3327, 0.2736)
	30	3.9950	14	0.2002, 0.2002, 0.2002	(0.2448, 0.0806)
	60	4.6799	13	0.1761, 0.1761, 0.1761	(0.2246, 0.0466)
$\mathcal{A}3$	5	5.0000	12	0.1667, 0.1667, 0.1667	(0.1780, 0.1472)
	30	30.0000	7	0.0323, 0.0323, 0.0323	(0.0514, 0.0323)
	60	60.0000	7	0.0164, 0.0164, 0.0164	(0.0381, 0.0164)
$\mathcal{A}4$	5	3.2143	21	0.3792, 0.3796, 0.3800	(0.5635, 0.3333)
	30	19.3565	18	0.2900, 0.2588, 0.2858	(0.4823, 0.1301)
	60	38.4843	19	0.2998, 0.3016, 0.3003	(0.4471, 0.1877)

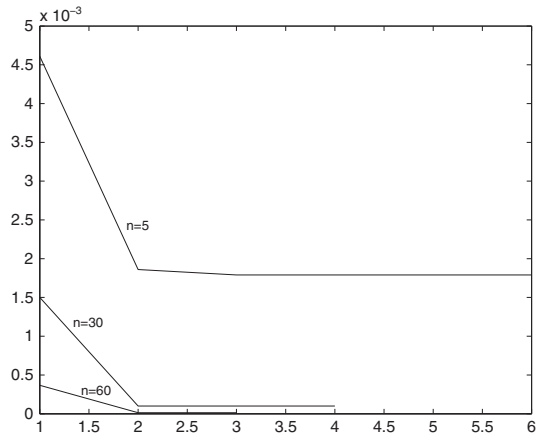

Figure 1. The curves of ratio for $\mathcal{A}1$.

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$\mathcal{A}4$: $A4_{1jj} = 1$ for $j = 2, \ldots, n$, $A4_{ijj} = 1$ for $i = 2, \ldots, n$, $j = 1, \ldots, n - i + 1$, and zero elsewhere.

$\mathcal{A}5$: $A5_{122} = 4$, $A5_{211} = 1$, and zero elsewhere.

Note that $\mathcal{A}1$, $\mathcal{A}2$, and $\mathcal{A}3$ are all essentially positive. $\mathcal{A}4$ is primitive but not essentially positive. $\mathcal{A}5$ is irreducible but not primitive and not essentially positive. We apply Algorithm 3.1 to find the

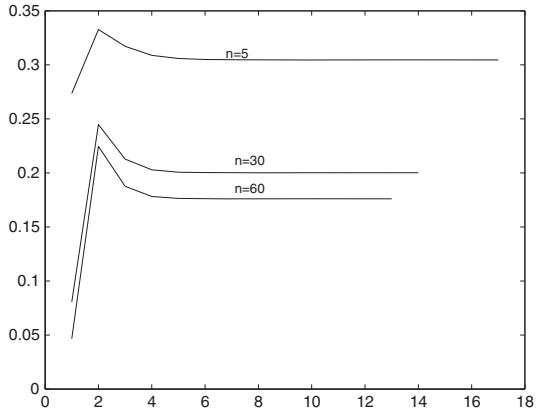

Figure 2. The curves of ratio for $\mathcal{A}2$.

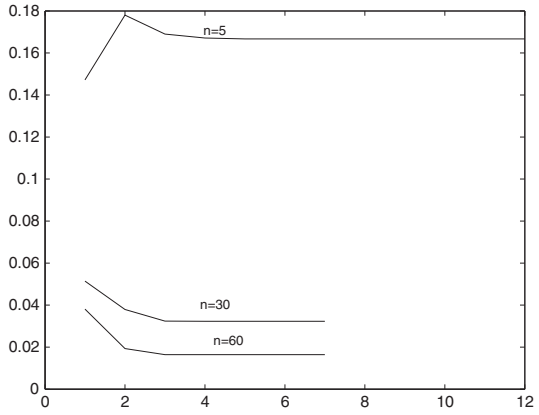

Figure 3. The curves of ratio for $\mathcal{A}3$.

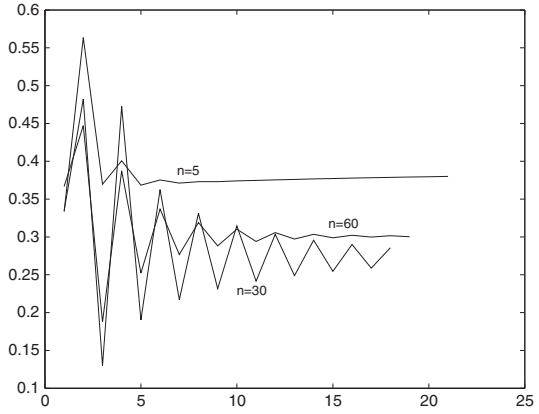

Figure 4. The curves of ratio for $\mathcal{A}4$.

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largest eigenvalues of all the five tensors with different dimensions $n = 5, 30, 60$. We summarize the numerical results in the following table, where *No.Iter* denotes the number of iterations, *Eig* denotes the largest eigenvalue, and *Ratio* denotes the ratio of $\overline{\lambda}_{k+1} - \underline{\lambda}_{k+1}$ to $\overline{\lambda}_k - \underline{\lambda}_k$ at the last three iterations, *(Max, Min) Ratio* denotes the maximum ratio and the minimum ratio. We also draw the curve of the convergence ratio to describe the linear convergence in the following figures.

For tensor $\mathcal{A}5$, the Ng–Qi–Zhou algorithm is divergent. From Table I and Figures 1–3, we see that the Ng–Qi–Zhou algorithm is linearly convergent for tensors $\mathcal{A}1$, $\mathcal{A}2$, and $\mathcal{A}3$. From Table I and Figure 4, we see that the Ng–Qi–Zhou algorithm is also convergent for $\mathcal{A}4$, but it is not linearly convergent. This echoes the results in [2]. Hence, numerical examples show that the Ng–Qi–Zhou algorithm is linearly convergent for essentially positive tensors, and this result is sharp.

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