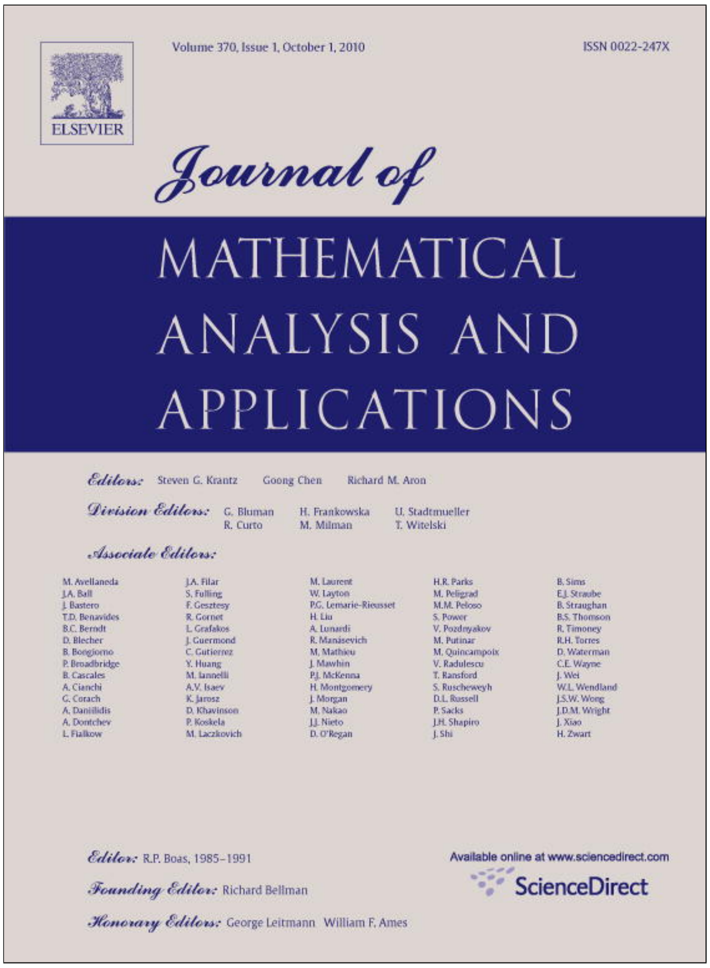
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Singular values of a real rectangular tensor

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ABSTRACT

Real rectangular tensors arise from the strong ellipticity condition problem in solid mechanics and the entanglement problem in quantum physics. In this paper, we systematically study properties of singular values of a real rectangular tensor, and give an algorithm to find the largest singular value of a nonnegative rectangular tensor. Numerical results show that the algorithm is efficient.

1. Introduction

Assume that M and N are positive integers, and $M, N \geqslant 2$. We call $\mathcal{B} = (b_{i_1 \cdots i_M})$, where $b_{i_1 \cdots i_M} \in \Re$, for $i_k = 1, \ldots, N$, $k = 1, \ldots, M$, a real Mth order N-dimensional **square tensor**, or simply a real square tensor. When $M = 2$, $\mathcal{B}$ is simply a real $N \times N$ square matrix. This justifies the word "square". We say that $\mathcal{B}$ is symmetric if $B_{i_1 \ldots i_M}$ is invariant under any permutation of indices $i_1, \ldots, i_M$. In the recent few years, eigenvalues of such square tensors have been introduced [16,12]. Nice properties such as the Perron–Frobenius theorem for eigenvalues of nonnegative square tensors [2] have been established. Applications of eigenvalues of square tensors include medical resonance imaging [1,21], higher-order Markov chains [14], positive definiteness of even-order multivariate forms in automatical control [15], and best-rank one approximation in data analysis [20], etc.

Very recently, a certain class of "rectangular" tensors attracted attention of the researchers. They arise from the strong ellipticity condition problem in solid mechanics [10,11,22,25] and the entanglement problem in quantum physics [6,7,23]. They have the form $\mathcal{A} = (a_{ijkl})$, where $i, j = 1, \ldots, m$, and $k, l = 1, \ldots, n$. Tensor $\mathcal{A}$ is partially symmetric, i.e., $a_{ijkl} = a_{jikl} = a_{ijlk}$ for all i, j, k and l. In [19] and [26], M-eigenvalues of such a tensor $\mathcal{A}$ are introduced. Let $x \in \Re^m$ and $y \in \Re^n$. Denote $\mathcal{A} \cdot xyy$ as a vector whose ith component is $\sum_{j=1}^m \sum_{k,l=1}^n a_{ijkl} x_j y_k y_l$, and $\mathcal{A}xxy\cdot$ as a vector whose lth component is $\sum_{i,j=1}^m \sum_{k=1}^n a_{ijkl} x_i x_j y_k$. Consider

$$\begin{cases} \mathcal{A} \cdot xyy = \lambda x, \\ \mathcal{A}xxy\cdot = \lambda y, \\ x^\top x = 1, \\ y^\top y = 1. \end{cases} \tag{1}$$

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If $\lambda \in \Re$, $x \in \Re^m$ and $y \in \Re^n$ satisfy (1), we call λ an M-eigenvalue of $\mathcal{A}$, and call x and y left and right M-eigenvectors of $\mathcal{A}$, associated with the M-eigenvalue λ. Algorithms for finding the largest M-eigenvalues are discussed in [9,13,26].

M-eigenvalues are parallel to Z-eigenvalues for square tensors [1,3,12,16–18,20]. A square tensor has the form $\mathcal{A} = (a_{i_1\ldots i_m})$, where $i_1, \ldots, i_m = 1, \ldots, n$. Let $x \in \Re^n$. Denote $\mathcal{A}x^{m-1}$ as a vector whose ith component is $\sum_{i_2,\ldots,i_m=1}^n a_{i_1\cdots i_m} x_{i_2} \cdots x_{i_m}$. Consider

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ x^\top x = 1. \end{cases} \tag{2}$$

If $\lambda \in \Re$ and $x \in \Re^n$ satisfy (2), we call λ a Z-eigenvalue of $\mathcal{A}$, and call x a Z-eigenvector of $\mathcal{A}$, associated with the Z-eigenvalue λ.

On the other hand, singular values of "non-square" tensors have been introduced in [12]. However, little exploration on properties of such singular values have been conducted.

In this paper, we systematically discuss properties of singular values of such rectangular tensors.

In the next section, we formally define singular values, H-singular values and N-singular values for a real rectangular tensor and study their properties. In Section 3, we study properties of singular values of a real partially symmetric rectangular tensor. Some properties are different from properties of eigenvalues of symmetric matrices. For example, we all know that a real symmetric matrix has only real eigenvalues, and it is positive definite if and only if all of its eigenvalues are positive. For a real even-order partially symmetric rectangular tensor, we show that it is positive definite if and only if all of its H-singular values are positive. This is similar to the matrix case. But we also show that in a certain case, such a positive definite partially symmetric tensor must have some N-singular values, and the sum of such N-singular values is negative. This shows that singular values of a rectangular tensor have their own structure.

Then, in Section 4, we extend the Perron–Frobenius theorem to singular values of nonnegative rectangular tensors. The crucial point is to define the irreducibility for rectangular tensors. We give an algorithm to find the largest singular value of a nonnegative rectangular tensor, in Section 5. Some numerical results are reported there. They show that our algorithm is efficient.

2. Singular values of a real rectangular tensor

Assume that p, q, m and n are positive integers, and $m, n \geqslant 2$. We call $\mathcal{A} = (a_{i_1\cdots i_p j_i \cdots j_q})$, where $a_{i_1\cdots i_p j_i \cdots j_q} \in \Re$, for $i_k = 1, \ldots, m$, $k = 1, \ldots, p$, and $j_k = 1, \ldots, n$, $k = 1, \ldots, q$, a real (p, q)th order $(m \times n)$-dimensional **rectangular tensor**, or simply a real rectangular tensor. When $p = q = 1$, $\mathcal{A}$ is simply a real $m \times n$ rectangular matrix. This justifies the word "rectangular".

Let

$$f(x, y) \equiv \mathcal{A}x^p y^q \equiv \sum_{i_1,\ldots,i_p=1}^m \sum_{j_1,\ldots,j_q=1}^n a_{i_1\cdots i_p j_1 \cdots j_q} x_{i_1} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}.$$

For example, let $p = q = 2$. Then $\mathcal{A}$ is a $(2, 2)$th order rectangular tensor. If $a_{1212} = 1$ and other $a_{ijkl} = 0$, then

$$f(x, y) = x_1 x_2 y_1 y_2.$$

When $p = q = 1$, $f(x, y)$ is simply a bilinear form of x and y.

For any vector x and any real number α, denote $x^{[\alpha]} = [x_1^\alpha, x_2^\alpha, \ldots, x_n^\alpha]^T$.

Let $\mathcal{A}x^{p-1}y^q$ be a vector in $\Re^m$ such that

$$\left(\mathcal{A}x^{p-1}y^q\right)_i = \sum_{i_2,\ldots,i_p=1}^m \sum_{j_1,\ldots,j_q=1}^n a_{ii_2\cdots i_p j_i \cdots j_q} x_{i_2} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}, \quad i = 1, 2, \ldots, m.$$

Similarly, let $\mathcal{A}x^p y^{q-1}$ be a vector in $\Re^n$ such that

$$\left(\mathcal{A}x^p y^{q-1}\right)_j = \sum_{i_1,\ldots,i_p=1}^m \sum_{j_2,\ldots,j_q=1}^n a_{i_1\cdots i_p j j_2 \cdots j_q} x_{i_1} \cdots x_{i_p} y_{j_2} \cdots y_{j_q}, \quad j = 1, 2, \ldots, n.$$

Throughout this paper, we let $M = p + q$ and $N = m + n$. For a vector $x = (x_1, \ldots, x_m)^\top$, and an integer M, we denote $x^{[M-1]} = (x_1^{M-1}, \ldots, x_m^{M-1})^\top$. Similarly we have $y^{[M-1]}$ for $y \in \mathcal{C}^n$. Consider

$$\begin{cases} \mathcal{A}x^{p-1}y^q = \lambda x^{[M-1]}, \\ \mathcal{A}x^p y^{q-1} = \lambda y^{[M-1]}. \end{cases} \tag{3}$$

If $\lambda \in \mathcal{C}$, $x \in \mathcal{C}^m \setminus \{0\}$ and $y \in \mathcal{C}^n \setminus \{0\}$ are solutions of (3), then we say that λ is a **singular value** of $\mathcal{A}$, x and y are a **left** and a **right eigenvectors** of $\mathcal{A}$, associated with the singular value λ. If $\lambda \in \Re$, $x \in \Re^m$ and $y \in \Re^n$ are solutions of (3), then we say that λ is an **H-singular value** of $\mathcal{A}$, x and y are a **left** and a **right H-eigenvectors** of $\mathcal{A}$, associated with the H-singular value λ. If a singular value is not an H-singular value, we call it an **N-singular value** of $\mathcal{A}$. If $p = q = 1$, then this is just the usual definition of singular values for a rectangular matrix. Hence, this definition extends the classical concept of singular values of rectangular matrices to higher order rectangular tensors. Here, we use the words "singular value", "H-singular value", "N-singular value" parallel to the usage of "eigenvalue", "H-eigenvalue" and "N-singular value" for symmetric tensors [16]. When M is even, our definition is the same as in [12]. When M is odd, our definition is slightly different from that in [12], but parallel to the definition of eigenvalues of square matrices [3].

Note that when $p > 1$, $\lambda = 0$, $x = 0$ and any nonzero y form a solution of (3). Similarly, when $q > 1$, $\lambda = 0$, $y = 0$ and any nonzero x form a solution of (3). In these cases, we say that 0 is a **trivial value** of $\mathcal{A}$.

Let vector

$$z = \begin{pmatrix} x \\ y \end{pmatrix}.$$

According to algebraic geometry [5,8], the resultant of a homogeneous polynomial system is an irreducible polynomial of the coefficients of the homogeneous polynomial system such that the polynomial vanishes if and only if the homogeneous polynomial system has nontrivial solutions. System (3) can be regarded as a homogeneous polynomial system of $z \in \Re^N$, with λ as a parameter. Then the resultant of (3) is a one-dimensional polynomial of λ. Denote it as $\phi(\lambda)$, and call it the **characteristic polynomial** of $\mathcal{A}$.

Theorem 1. *Suppose that $\mathcal{A}$ is a real (p,q)th order $(m \times n)$-dimensional rectangular tensor. We have the following conclusions on singular values of $\mathcal{A}$:*

(a) *If x and y are a left and a right eigenvectors of $\mathcal{A}$, associated with a singular value λ of $\mathcal{A}$, then*

$$f(x, y) = \lambda \sum_{i=1}^{p} x_i^M = \lambda \sum_{j=1}^{q} y_j^M. \tag{4}$$

(b) *When both p and q are odd, if λ is a singular value of $\mathcal{A}$, then $-\lambda$ is also a singular value of $\mathcal{A}$.*
(c) *Any singular value of $\mathcal{A}$ is a root of the characteristic polynomial ϕ. Any nonzero root of ϕ is a singular value of $\mathcal{A}$.*
(d) *The number of singular values is at most $N(M-1)^{N-1}$.*

Proof. By the first equation of (3), we have

$$f(x, y) = \mathcal{A}x^p y^q = \lambda\big(x^{[M-1]}\big)^\top x = \lambda \sum_{i=1}^{p} x_i^M.$$

By the second equation of (3), we have

$$f(x, y) = \mathcal{A}x^p y^q = \lambda\big(y^{[M-1]}\big)^\top y = \lambda \sum_{j=1}^{q} y_j^M.$$

We thus have conclusion (a).

Suppose that both p and q are odd. If λ is a singular value of $\mathcal{A}$ with x and y as a left and a right eigenvectors. Then $-\lambda$ is a singular value of $\mathcal{A}$ with $-x$ and y as a left and a right eigenvectors. This proves conclusion (b).

According to the definition of the resultant [5,8], (3) has a nonzero solution (x, y) if and only if $\phi(\lambda) = 0$. If $x \neq 0$ and $y \neq 0$, then λ is a singular value of $\mathcal{A}$. Otherwise, $\lambda = 0$ is a trivial value of $\mathcal{A}$. The conclusion (c) follows.

By the knowledge of resultants [5,8], the degree of ϕ is at most $N(M-1)^{N-1}$. Hence, by (c), the conclusion (d) follows. □

3. Singular values of a real partially symmetric rectangular tensor

Suppose that $\mathcal{A} = (a_{i_1 \cdots i_p j_i \cdots j_q})$ is a real (p,q)th order $(m \times n)$-dimensional rectangular tensor. We say that $\mathcal{A}$ is a real partially symmetric rectangular tensor, if $a_{i_1 \cdots i_p j_i \cdots j_q}$ is invariant under any permutation of indices among $i_1, \ldots, i_p$, and any permutation of indices among $j_1, \ldots, j_q$, i.e.,

$$a_{\pi(i_1 \cdots i_p)\sigma(j_1 \cdots j_q)} = a_{i_1 \cdots i_p j_i \cdots j_q}, \quad \pi \in S_p,\ \sigma \in S_q,$$

where S_r is the permutation group of r indices.

When $p = q = 1$, such a tensor $\mathcal{A}$ is simply an $m \times n$ rectangular matrix. Hence, we call such a tensor a **partially symmetric rectangular tensor**. When $p = q = 2$ and $m = n = 2$ or 3, the elasticity tensor is such a tensor [10,11,19,22,24,25]. When $p = q = 2$, such a partially symmetric rectangular tensor is useful for the entanglement problem in quantum physics [6,7,13,23,26].

When both p and q are even, if $f(x, y) > 0$ for all $x \in \Re^m$, $x \neq 0$, $y \in \Re^n$, $y \neq 0$, then we say that $\mathcal{A}$ is positive definite. When $\mathcal{A}$ is the elasticity tensor, the strong ellipticity condition holds if and only if $\mathcal{A}$ is positive definite [19]. Since the strong ellipticity condition plays an important role in nonlinear elasticity and materials, positive definiteness of such a partially symmetric tensor has a sound application background.

We now have the following theorem on H-singular values of $\mathcal{A}$.

Theorem 2. *Suppose that $\mathcal{A}$ is a real (p, q)th order $(m \times n)$-dimensional partially symmetric rectangular tensor. We have the following conclusions on H-singular values of $\mathcal{A}$:*

(a) *If M is even, then H-singular values always exist.*
(b) *When both p and q are even, $\mathcal{A}$ is positive definite if and only if all of its H-singular values are positive.*

Proof. Consider the optimization problem

$$\min\left\{ f(x, y)\colon \sum_{i=1}^{p} x_i^M = 1,\ \sum_{j=1}^{q} y_j^M = 1 \right\}. \tag{5}$$

The objective function of (5) is continuous. When M is even, the feasible set of (5) is compact. Hence, when M is even, (5) has at least one maximizer and one minimizer. Since the constraints of (5) satisfy the linear independence constraint qualification, this minimizer or maximizer satisfies the following optimality conditions of (5):

$$\begin{cases} \mathcal{A}x^{p-1}y^q = \lambda x^{[M-1]}, \\ \mathcal{A}x^p y^{q-1} = \mu y^{[M-1]}, \\ \sum_{i=1}^{p} x_i^M = 1, \\ \sum_{j=1}^{q} y_j^M = 1, \end{cases} \tag{6}$$

where $\frac{M}{p}\lambda$ and $\frac{M}{q}\mu$ are the optimal Lagrangian multipliers. By the first and the third equations of (14), we have

$$f(x, y) = \mathcal{A}x^p y^q = \lambda \big(x^{[M-1]}\big)^\top x = \lambda \sum_{i=1}^{p} x_i^M = \lambda.$$

By the second and the fourth equations of (14), we have

$$f(x, y) = \mathcal{A}x^p y^q = \lambda \big(y^{[M-1]}\big)^\top y = \lambda \sum_{j=1}^{q} y_j^M = \mu.$$

Hence $\lambda = \mu$. i.e., λ, x and y satisfy (3). This proves (a).

When m and n are even, $\mathcal{A}$ is positive definite if and only if the optimal objective function value of (5) is positive. Suppose that all the H-singular values of $\mathcal{A}$ are positive. By the proof for (a), the optimal solution (x^*, y^*) and λ_*, where $\frac{M}{p}\lambda_*$ and $\frac{M}{q}\lambda_*$ are optimal Lagrangian multipliers of (5), satisfy (3) and $\lambda_* = f(x^*, y^*)$. This shows that λ_* is an H-singular value of $\mathcal{A}$. Then $\lambda_* > 0$. Hence the optimal objective function value of (5), $f(x^*, y^*) = \lambda_* > 0$. Hence, $\mathcal{A}$ is positive definite. This proves the "if" part of (b).

On the other hand, suppose that $\mathcal{A}$ is positive definite. Let λ be an H-singular value of $\mathcal{A}$, x and y be a left and a right eigenvectors of $\mathcal{A}$, associated with λ. The $f(x, y) > 0$. Since $x \neq 0$ and $y \neq 0$, $\sum_{i=1}^{p} x_i^M > 0$ and $\sum_{j=1}^{q} y_j^M > 0$. By Theorem 1(a), we have $\lambda > 0$. This proves the "only if" part of (b). The proof of the theorem is completed. □

Theorem 3. *Suppose that $\mathcal{A}$ is a real (p, p)th order $(m \times n)$-dimensional partially symmetric rectangular tensor. Then the sum of all the singular values of $\mathcal{A}$ is zero. If further more p is even and $\mathcal{A}$ is positive definite, then the sum of all the N-singular values of $\mathcal{A}$ is a negative number.*

Proof. Consider

$$f(z) \equiv f(x, y) = \mathcal{A}x^p y^p.$$

It is a homogeneous polynomial of z with degree $M = 2p$. There exists a unique Mth order N-dimensional symmetric square tensor $\mathcal{B} = (b_{i_1 \cdots i_N})$ such that

$$f(z) \equiv \mathcal{B}z^M.$$

It is not difficult to see that the left-hand side of (3) is $\frac{1}{p}\nabla f(z) = 2\mathcal{B}z^{M-1}$, where $\nabla f(z) = (\frac{\partial f(z)}{\partial z_1}, \ldots, \frac{\partial f(z)}{\partial z_N})^\top$. On the other side, the right-hand side of (3) is $\lambda z^{[M-1]}$. By [16], if $z \neq 0$ and λ are solutions of

$$\mathcal{B}z^{m-1} = \lambda z^{[M-1]},$$

then λ is an eigenvalue of $\mathcal{B}$. Then, in this case, by Theorem 1 of [16], the sum of all the eigenvalues of $\mathcal{B}$ is $(M-1)^{N-1}\operatorname{tr}(\mathcal{B})$. By the definition of $\mathcal{B}$, $b_{i,\ldots,i} = 0$ for $i = 1, \ldots, N$, i.e., $\operatorname{tr}(\mathcal{B}) = 0$. Since the collection of all the eigenvalues of $\mathcal{B}$ is the collection of all the singular and trivial values of $\mathcal{A}$, and trivial values are zeros, we have the first conclusion. The second conclusion follows from the first conclusion and Theorem 2(b). □

The last conclusion of this theorem is very different from the matrix case, as discussed in the introduction. It is not sure if this conclusion also holds in the case $p \neq q$.

We may prove a Gerschgorin-type theorem for singular values of regular tensors as for eigenvalues of symmetric tensors in (3). When $m = n = 2$, a direct method for finding singular values can be established like in [19]. This is useful for checking strong ellipticity condition in plane [9–11,19,22,24,25].

4. The Perron–Frobenius theorem

In this section we extend the Perron–Frobenius theorem for eigenvalues of nonnegative square tensors in [2] to singular values of nonnegative rectangular tensors. The crucial point is to define the irreducibility of rectangular tensors. The argument used in the following proof is parallel to that in [2]. We proceed the proof for completeness.

Let $P_k = \{x \in \Re^k\colon\ x_i \geqslant 0,\ i = 1, 2, \ldots, k\}$ and $int(P_k) = \{x \in \Re^k\colon\ x_i > 0,\ i = 1, 2, \ldots, k\}$. A vector $x \in \Re^k$ is called nonnegative if $x \in P_k$ and it is called strongly positive if $x \in int(P_k)$. In this section, we denote the zero vector in $\Re^k$ by θ.

Let $\mathcal{A} = (a_{i_1\cdots i_p j_i \cdots j_q})$ be a (p,q)th order $(m \times n)$-dimensional nonnegative rectangular tensor, where $p, q \geqslant 1$. Denote $\{e_i\}_1^m$ and $\{f_i\}_1^n$ the basis of $\Re^m$ and $\Re^n$, respectively, and let $e_i^p = e_i \otimes \cdots \otimes e_i$ (p times) and $f_j^q = f_j \otimes \cdots \otimes f_j$ (q times), where $\otimes$ is the notation of tensor product of vectors.

For any $j = 1, 2, \ldots, n$, let $\mathcal{A}(\cdot, f_j^q) = (a_{i_1,\ldots,i_p,j,\ldots,j})$ be a pth order m-dimensional square tensor.

For any $i = 1, 2, \ldots, m$, let $\mathcal{A}(e_i^p, \cdot) = (a_{i,\ldots,i,j_1,\ldots,j_q})$ be a qth order n-dimensional square tensor.

Definition 1. A nonnegative rectangular tensor $\mathcal{A}$ is called irreducible if all the square tensors $\mathcal{A}(\cdot, f_j^q)$, $j = 1, \ldots, n$, and $\mathcal{A}(e_i^p, \cdot)$, $i = 1, \ldots, m$, are irreducible in the sense of [2].

Lemma 1. *If $\mathcal{A}$ is irreducible, then all the tensors $\mathcal{A}(\cdot, f_j^q)$, $j = 1, \ldots, n$, and $\mathcal{A}(e_i^p, \cdot)$, $i = 1, \ldots, m$, do not have eigenvalue* 0.

Proof. Suppose that it is not true. Then, either there exists j_0 such that $\mathcal{A}(\cdot, f_{j_0}^q)$ has eigenvalue 0 or there exists i_0 such that $\mathcal{A}(e_{i_0}^p, \cdot)$ has eigenvalue 0. Say $\mathcal{A}(\cdot, f_{j_0}^q)$ has eigenvalue 0, i.e, $\exists x_0 \neq \theta$, such that $\mathcal{A}(x_0^{p-1}, f_{j_0}^q) = \theta$. If $(x_0)_i > 0$, $\forall i$, then $\mathcal{A}(\cdot, f_{j_0}^q) = 0$, and then it is reducible, a contradiction. Otherwise, there exist a nonempty index set I and $\delta > 0$, such that $(x_0)_i = 0$, $\forall i \in I$, and $(x_0)_i \geqslant \delta$, $\forall i \notin I$. We have

$$\delta^{p-1} \sum_{i_2,\ldots,i_p \notin I} a_{i,i_2,\ldots,i_p,j_0,\ldots,j_0} \leqslant \sum_{1 \leqslant i_2,\ldots,i_p \leqslant m} a_{i,i_2,\ldots,i_p,j_0,\ldots,j_0}(x_0)_{i_2}\cdots(x_0)_{i_p} = \left(\mathcal{A}x_0^{p-1}, f_{j_0}\right)_i = 0, \quad \forall i.$$

It implies

$$a_{i_1,i_2,\ldots,i_p,j_0,\ldots,j_0} = 0, \quad \forall i_1 \in I,\ \forall i_2, \ldots, i_p \notin I.$$

Then $\mathcal{A}(\cdot, f_{j_0}^q)$ is reducible. This is a contradiction.

Similarly, we prove that $\mathcal{A}(e_{i_0}^p, \cdot)$ cannot have eigenvalue 0. □

Lemma 2. *If $\mathcal{A}$ is irreducible, then for any $(x, y) \in (P_m \setminus \{\theta\}) \times (P_n \setminus \{\theta\})$, $\mathcal{A}x^{p-1}y^q \neq \theta$ and $\mathcal{A}x^p y^{q-1} \neq \theta$.*

Proof. Suppose $\mathcal{A}x^{p-1}y^q = \theta$, i.e., $(\mathcal{A}x^{p-1}y^q)_i = 0$, $\forall i$. Since $y \neq \theta$, $\exists j_0$ and $\delta > 0$ such that $y \geqslant \delta f_{j_0}$, we have

$$0 = \left(\mathcal{A}x^{p-1}y^q\right)_i \geqslant \delta^q\left(\mathcal{A}\left(x^{p-1}, f_{j_0}^q\right)\right)_i \geqslant 0, \quad \forall i.$$

Namely,

$$\mathcal{A}\left(x^{p-1}, f_{j_0}^q\right) = \theta.$$

This means that x is an eigenvector of $\mathcal{A}(\cdot, f_{j_0}^q)$ with eigenvalue 0. According to Lemma 1, this is a contradiction. Similarly we prove $\mathcal{A}x^p y^{q-1} \neq \theta$. □

The following lemma is a version of Lemma 4.3 in [2].

Lemma 3. *Let $\mathcal{A}$ be nonnegative and irreducible, and let $(\lambda, (x, y)) \in R_+ \times (int(P_m) \times int(P_n))$ be a solution of (3). If $(\mu, (u, v)) \in R_+ \times ((P_m \setminus \theta) \times (P_n \setminus \theta))$ satisfies*

$$\mathcal{A}u^{p-1}v^q \geqslant \ (or \ \leqslant) \ \mu u^{M-1}, \quad and \quad \mathcal{A}u^p v^{q-1} \geqslant \ (or \ \leqslant) \ \mu v^{M-1}$$

then $\mu \leqslant$ (or $\geqslant$ resp.) λ.

Proof. Define $t_0 = max\{s \geqslant 0 \mid x - su \in P_m,\ \ y - sv \in P_n\}$. Since $(x, y) \in int(P_m) \times int(P_n)$, $t_0 > 0$. Also, we have

$$\begin{cases} x - tu \geqslant 0, \\ y - tv \geqslant 0, \end{cases} \tag{7}$$

if and only if $t \in [0, t_0]$. Thus

$$\begin{cases} \lambda x^{[M-1]} = \mathcal{A}x^{p-1}y^q \geqslant t_0^{M-1}\mathcal{A}u^{p-1}v^q \geqslant t_0^{M-1}\mu u^{M-1}, \\ \lambda y^{[M-1]} = \mathcal{A}x^p y^{q-1} \geqslant t_0^{M-1}\mathcal{A}u^p v^{q-1} \geqslant t_0^{M-1}\mu v^{M-1}, \end{cases} \tag{8}$$

i.e.,

$$\begin{cases} x \geqslant t_0\left(\dfrac{\mu}{\lambda}\right)^{\frac{1}{M-1}} u, \\[2ex] y \geqslant t_0\left(\dfrac{\mu}{\lambda}\right)^{\frac{1}{M-1}} v. \end{cases} \tag{9}$$

This implies $\mu \leqslant \lambda$. $\square$

Theorem 4. *Assume that the nonnegative tensor $\mathcal{A}$ is irreducible, then there exists a solution $(\lambda_0, (x_0, y_0))$ of system (1), satisfying $\lambda_0 > 0$ and $(x_0, y_0) \in int(P_m) \times int(P_n)$.*

Moreover, if λ is a singular value with strongly positive left and right eigenvectors, then $\lambda = \lambda_0$. The strongly positive left and right eigenvectors are unique up to a multiplicative constant.

Proof. Denote $D_k = \{z = (z_1, \ldots, z_k) \in P_k \mid \sum_{i=1}^k z_i = 1\}$. Provided by Lemma 2, the map F on $D_m \times D_n$ into itself:

$$F(\xi, \eta) = \left(\frac{(\mathcal{A}\xi^{p-1}\eta^q)_i^{\frac{1}{M-1}}}{\sum_{i=1}^m (\mathcal{A}\xi^{p-1}\eta^q)_i^{\frac{1}{M-1}}}, \frac{(\mathcal{A}\xi^p\eta^{q-1})_j^{\frac{1}{M-1}}}{\sum_{j=1}^n (\mathcal{A}\xi^p\eta^{q-1})_j^{\frac{1}{M-1}}}\right)$$

is well defined.

According to the Brouwer Fixed Point Theorem, there exists $(\xi_0, \eta_0) \in D_m \times D_n$ such that

$$\begin{cases} \mathcal{A}\xi_0^{p-1}\eta_0^q = \mu_0\xi_0^{[M-1]}, \\ \mathcal{A}\xi_0^p\eta_0^{q-1} = \nu_0\eta_0^{[M-1]}, \end{cases} \tag{10}$$

where

$$\begin{cases} \mu_0 = \left(\displaystyle\sum_{i=1}^m \left(\mathcal{A}\xi_0^{p-1}\eta_0^q\right)_i^{\frac{1}{M-1}}\right)^{M-1}, \\[3ex] \nu_0 = \left(\displaystyle\sum_{j=1}^n \left(\mathcal{A}\xi_0^p\eta_0^{q-1}\right)_j^{\frac{1}{M-1}}\right)^{M-1}. \end{cases} \tag{11}$$

Define $t = (\frac{\nu_0}{\mu_0})^{\frac{1}{M}}$, $x_0 = \xi_0$, $y_0 = t\eta_0$ and $\lambda_0 = (\mu_0^p\nu_0^q)^{\frac{1}{M}}$. Then, $(\lambda_0, (x_0, y_0))$ is a solution of (3).

Now we want to show: $(x_0, y_0) \in int(P_m) \times int(P_n)$. Suppose not, then either there exists a proper nonempty index subset $I \subset \{1, \ldots, m\}$ and a nonempty subset $J \subset \{1, \ldots, n\}$, such that $(x_0)_i = 0$, $\forall i \notin I$, $(x_0)_i \geqslant \delta > 0$, $\forall i \in I$ and $(y_0)_j \geqslant \delta > 0$, $\forall j \in J$, or a proper nonempty index subset $J \subset \{1, \ldots, n\}$ and a nonempty subset $I \subset \{1, \ldots, m\}$, such that $(y_0)_j = 0$, $\forall j \notin J$, $(y_0)_j \geqslant \delta > 0$, $\forall j \in J$, and $(x_0)_i \geqslant \delta$, $\forall i \in I$.

Since $\mathcal{A}(\cdot, f_j^q)$, $\forall j \in J$ are all irreducible, if I is proper, $\forall i \notin I$, $j \in J$ we have

$$\delta^{M-1}\sum_{i_2,\dots,i_p\in I} a_{i,i_2,\dots,i_p,j,j,\dots,j} \leqslant \delta^q \sum_{1\leqslant i_2\cdots i_p\leqslant m,\,1\leqslant j\leqslant n} a_{i_1,i_2,\dots,i_p,j,\dots,j}(x_0)_{i_2}\cdots(x_0)_{i_p}$$
$$\leqslant \sum_{1\leqslant i_2,\dots,i_p\leqslant m,\,1\leqslant j_1,\dots,j_q\leqslant n} a_{i,i_2,\dots,i_p,j_1,\dots,j_q}(x_0)_{i_2}\cdots(x_0)_{i_p}(y_0)_{j_1}\cdots(y_0)_{j_q}$$
$$= \left(\mathcal{A}x_0^{p-1}y_0^q\right)_i = 0.$$

This contradicts the irreducibility of $\mathcal{A}(\cdot, f_j^q)$, $\forall j \in J$. Therefore I is not proper. Similarly, we prove that J is not proper. This implies that $x_0 \in int(P_m)$ and $y_0 \in int(P_n)$.

The uniqueness of the positive singular value with strongly positive left and right eigenvectors now follows from Lemma 3 directly. The uniqueness up to a multiplicative constant of the strongly positive left and right eigenvectors is proved in the same way as in [2]. □

The following minimax characterization of the unique positive eigenvalue with strongly positive eigenvectors for nonnegative tensors in [2] is also extended to the unique singular value with strongly positive left and right eigenvectors for nonnegative rectangular tensors.

Theorem 5. *Assume that $\mathcal{A}$ is an irreducible nonnegative rectangular tensor of order (p,q) and dimension $m\times n$, then*

$$\min_{(x,y)\in(P_m\setminus\{\theta\})\times(P_n\setminus\{\theta\})}\max_{i,j}\left(\frac{(\mathcal{A}x^{p-1}y^q)_i}{x_i^{M-1}}, \frac{(\mathcal{A}x^p y^{q-1})_j}{y_j^{M-1}}\right) = \lambda_0$$
$$= \max_{(x,y)\in(P_m\setminus\{\theta\})\times(P_n\setminus\{\theta\})}\min_{i,j}\left(\frac{(\mathcal{A}x^{p-1}y^q)_i}{x_i^{M-1}}, \frac{(\mathcal{A}x^p y^{q-1})_j}{y_j^{M-1}}\right),$$

where λ_0 is the unique positive singular value corresponding to strongly positive left and right eigenvectors.

Proof. On $(P_m\setminus\{\theta\})\times(P_n\setminus\{\theta\})$ we define the function:

$$\mu_*(x,y) = \min_{i,j}\left(\frac{(\mathcal{A}x^{p-1}y^q)_i}{x_i^{M-1}}, \frac{(\mathcal{A}x^p y^{q-1})_j}{y_j^{M-1}}\right).$$

Since it is a positively 0-homogeneous function, it can be restricted on $(D_m)\times(D_n)$. Let

$$r_* := \mu_*(x_*, y_*) = \max_{x\in\Delta_m, y\in D_n}\mu_*(x,y) = \max_{(x,y)\in(P_m\setminus\{\theta\})\times(P_n\setminus\{\theta\})}\mu_*(x,y). \tag{12}$$

Let $(\lambda_0,(x_0,y_0)) \in \mathbb{R}_+\times(int(P_m)\times int(P_n))$ be the solution of (3). On one hand we have

$$\lambda_0 = \mu_*(x_0,y_0) \leqslant \mu_*(x_*,y_*),$$

i.e.,

$$\lambda_0 \leqslant r_*. \tag{13}$$

On the other hand, by the definition of $\mu_*(x,y)$, we have

$$r_* = \mu_*(x_*,y_*) = \min_{i,j}\left(\frac{(\mathcal{A}x_*^{p-1}y_*^q)_i}{(x_*)_i^{M-1}}, \frac{(\mathcal{A}x_*^p y_*^{q-1})_j}{(y_*)_j^{M-1}}\right).$$

This means

$$\begin{cases} \mathcal{A}x_*^{p-1}y_*^q \geqslant r_* x_*^{[M-1]}, \\ \mathcal{A}x_*^p y_*^{q-1} \geqslant r_* y_*^{[M-1]}. \end{cases} \tag{14}$$

According to Lemma 3, we have $r_* \leqslant \lambda_0$, and then

$$\lambda_0 = r_*.$$

Similarly, we prove the other equality. □

As a consequence, we have

Theorem 6. *Assume that $\mathcal{A}$ is an irreducible nonnegative rectangular tensor, and λ_0 is the positive singular value with strongly positive left and right eigenvectors. Then $|\lambda| \leqslant \lambda_0$ for all singular values λ of $\mathcal{A}$.*

For more discussion on this topic, see [4].

5. An algorithm

In this section, based on Theorems 5 and 6, we give an algorithm for calculating the largest singular value of an irreducible nonnegative rectangular tensor $\mathcal{A}$. This algorithm is parallel to the one in [14] for finding the largest eigenvalue of an irreducible nonnegative square tensor. We first give some results which will be used later.

For any two vectors $x \in \Re^k$ and $y \in \Re^k$, $x \geqslant y$ and $x > y$ mean that $x - y \in P_k$ and $x - y \in int(P_k)$, respectively. Here, P_k and $int(P_k)$ are defined in Section 4. By a direct computation, we obtain the following lemma.

Lemma 4. *Suppose that $\mathcal{A}$ is a nonnegative (p,q)th order $(m \times n)$-dimensional rectangular tensor, $x \in \Re^m$, $\bar{x} \in \Re^m$, $y \in \Re^n$, and $\bar{y} \in \Re^n$ are four nonnegative column vectors, and t is a positive number. Then, we have*

(1) *If $x \geqslant \bar{x} \geqslant 0$ and $y \geqslant \bar{y} \geqslant 0$, then $\mathcal{A}x^{p-1}y^q \geqslant \mathcal{A}\bar{x}^{p-1}\bar{y}^q$ and $\mathcal{A}x^p y^{q-1} \geqslant \mathcal{A}\bar{x}^p \bar{y}^{q-1}$.*
(2) *$\mathcal{A}(tx)^{p-1}(ty)^q = t^{M-1}\mathcal{A}x^{p-1}y^q$ and $\mathcal{A}(tx)^p(ty)^{q-1} = t^{M-1}\mathcal{A}x^p y^{q-1}$.*

Lemma 5. *Suppose that a nonnegative (p,q)th order $(m \times n)$-dimensional rectangular tensor $\mathcal{A}$ is irreducible. Then, for any two strongly positive vectors $x > 0$, $x \in \Re^m$ and $y > 0$, $y \in \Re^n$, $\mathcal{A}x^{p-1}y^q$ and $\mathcal{A}x^p y^{q-1}$ are strongly positive vectors, i.e.,*

$$\mathcal{A}x^{p-1}y^q > 0, \qquad \mathcal{A}x^p y^{q-1} > 0.$$

Proof. Clearly, $\mathcal{A}x^{p-1}y^q \geqslant 0$. Suppose $(\mathcal{A}x^{p-1}y^q)_i = 0$, for some i. Since $y > 0$, there exist j_0 and $\delta > 0$ such that $y \geqslant \delta f_{j_0}$. So we have

$$0 = \left(\mathcal{A}x^{p-1}y^q\right)_i \geqslant \delta^q\left(\mathcal{A}\left(x^{p-1}, f_{j_0}^q\right)\right)_i \geqslant 0.$$

Hence,

$$\left(\mathcal{A}\left(x^{p-1}, f_{j_0}^q\right)\right)_i = 0. \tag{15}$$

Since $x > 0$ and $\mathcal{A}(\cdot, f_{j_0}^q)$ is irreducible, by Lemma 2.2 [2], we have $\mathcal{A}(x^{p-1}, f_{j_0}^q) > 0$. This contradicts with (15). Therefore, $\mathcal{A}x^{p-1}y^q > 0$.

Similarly, we can prove $\mathcal{A}x^p y^{q-1} > 0$. □

Now we give the main result of this section. Based on this result, we will obtain an iterative method to calculate a lower bound and an upper bound of the largest singular value of an irreducible nonnegative rectangular tensor $\mathcal{A}$.

Theorem 7. *Suppose that a nonnegative (p,q)th order $(m \times n)$-dimensional rectangular tensor $\mathcal{A}$ is irreducible. Let $x^{(0)} \in \Re^m$ and $y^{(0)} \in \Re^n$ are two arbitrary strongly positive vectors. Let $\xi^{(0)} = \mathcal{A}(x^{(0)})^{p-1}(y^{(0)})^q$ and $\eta^{(0)} = \mathcal{A}(x^{(0)})^p(y^{(0)})^{q-1}$. Define*

$$x^{(1)} = \frac{(\xi^{(0)})^{[\frac{1}{M-1}]}}{\|(\xi^{(0)}, \eta^{(0)})^{[\frac{1}{M-1}]}\|}, \qquad y^{(1)} = \frac{(\eta^{(0)})^{[\frac{1}{M-1}]}}{\|(\xi^{(0)}, \eta^{(0)})^{[\frac{1}{M-1}]}\|},$$

$$\xi^{(1)} = \mathcal{A}\left(x^{(1)}\right)^{p-1}\left(y^{(1)}\right)^q, \qquad \eta^{(1)} = \mathcal{A}\left(x^{(1)}\right)^p\left(y^{(1)}\right)^{q-1},$$

$$\vdots$$

$$x^{(k+1)} = \frac{(\xi^{(k)})^{[\frac{1}{M-1}]}}{\|(\xi^{(k)}, \eta^{(k)})^{[\frac{1}{M-1}]}\|}, \qquad y^{(k+1)} = \frac{(\eta^{(k)})^{[\frac{1}{M-1}]}}{\|(\xi^{(k)}, \eta^{(k)})^{[\frac{1}{M-1}]}\|},$$

$$\xi^{(k+1)} = \mathcal{A}\left(x^{(k+1)}\right)^{p-1}\left(y^{(k+1)}\right)^q, \qquad \eta^{(k+1)} = \mathcal{A}\left(x^{(k+1)}\right)^p\left(y^{(k+1)}\right)^{q-1}, \quad k \geqslant 1,$$

$$\vdots$$

and let

$$\underline{\lambda}_k = \min_{x_i^{(k)} > 0,\, y_j^{(k)} > 0} \left\{ \frac{\xi_i^{(k)}}{(x_i^{(k)})^{M-1}}, \frac{\eta_j^{(k)}}{(y_j^{(k)})^{M-1}} \right\},$$

$$\bar{\lambda}_k = \max_{x_i^{(k)} > 0,\, y_j^{(k)} > 0} \left\{ \frac{\xi_i^{(k)}}{(x_i^{(k)})^{M-1}}, \frac{\eta_j^{(k)}}{(y_j^{(k)})^{M-1}} \right\}, \quad k = 1, 2, \ldots.$$

Assume that λ_0 is the unique positive singular value of $\mathcal{A}$. Then,

$$\underline{\lambda}_1 \leqslant \underline{\lambda}_2 \leqslant \cdots \leqslant \lambda_0 \leqslant \cdots \leqslant \bar{\lambda}_2 \leqslant \bar{\lambda}_1.$$

Proof. Clearly, by Theorem 5, for $k = 1, 2, \ldots,$

$$\underline{\lambda}_k \leqslant \lambda_0 \leqslant \bar{\lambda}_k.$$

We now prove for any $k \geqslant 1$,

$$\underline{\lambda}_k \leqslant \underline{\lambda}_{k+1} \quad \text{and} \quad \bar{\lambda}_{k+1} \leqslant \bar{\lambda}_k.$$

For each $k = 1, 2, \ldots,$ by the definition of $\underline{\lambda}_k$ and Lemma 5, we have

$$\xi^{(k)} \geqslant \underline{\lambda}_k \big(x^{(k)}\big)^{[M-1]} > 0, \qquad \eta^{(k)} \geqslant \underline{\lambda}_k \big(y^{(k)}\big)^{[M-1]} > 0.$$

Then,

$$\big(\xi^{(k)}\big)^{[\frac{1}{M-1}]} \geqslant (\underline{\lambda}_k)^{\frac{1}{M-1}} x^{(k)} > 0, \qquad \big(\eta^{(k)}\big)^{[\frac{1}{M-1}]} \geqslant (\underline{\lambda}_k)^{\frac{1}{M-1}} y^{(k)} > 0.$$

So,

$$x^{(k+1)} = \frac{(\xi^{(k)})^{[\frac{1}{M-1}]}}{\|(\xi^{(k)}, \eta^{(k)})^{[\frac{1}{M-1}]}\|} \geqslant \frac{(\underline{\lambda}_k)^{\frac{1}{M-1}} x^{(k)}}{\|(\xi^{(k)}, \eta^{(k)})^{[\frac{1}{M-1}]}\|} > 0,$$

$$y^{(k+1)} = \frac{(\eta^{(k)})^{[\frac{1}{M-1}]}}{\|(\xi^{(k)}, \eta^{(k)})^{[\frac{1}{M-1}]}\|} \geqslant \frac{(\underline{\lambda}_k)^{\frac{1}{M-1}} y^{(k)}}{\|(\xi^{(k)}, \eta^{(k)})^{[\frac{1}{M-1}]}\|} > 0.$$

Hence, by Lemma 4, we get

$$\begin{aligned}\mathcal{A}\big(x^{(k+1)}\big)^{p-1}\big(y^{(k+1)}\big)^q &\geqslant \frac{\underline{\lambda}_k \mathcal{A}(x^{(k)})^{p-1}(y^{(k)})^q}{\|(\xi^{(k)}, \eta^{(k)})^{[\frac{1}{M-1}]}\|^{M-1}} \\ &= \frac{\underline{\lambda}_k \xi^{(k)}}{\|(\xi^{(k)}, \eta^{(k)})^{[\frac{1}{M-1}]}\|^{M-1}} \\ &= \underline{\lambda}_k \big(x^{(k+1)}\big)^{[M-1]}\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}\big(x^{(k+1)}\big)^{p}\big(y^{(k+1)}\big)^{q-1} &\geqslant \frac{\underline{\lambda}_k \mathcal{A}(x^{(k)})^{p}(y^{(k)})^{q-1}}{\|(\xi^{(k)}, \eta^{(k)})^{[\frac{1}{M-1}]}\|^{M-1}} \\ &= \frac{\underline{\lambda}_k \eta^{(k)}}{\|(\xi^{(k)}, \eta^{(k)})^{[\frac{1}{M-1}]}\|^{M-1}} \\ &= \underline{\lambda}_k \big(y^{(k+1)}\big)^{[M-1]},\end{aligned}$$

which means for each $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,$

$$\underline{\lambda}_k \leqslant \frac{(\mathcal{A}(x^{(k+1)})^{p-1}(y^{(k+1)})^q)_i}{(x_i^{(k+1)})^{M-1}}, \qquad \underline{\lambda}_k \leqslant \frac{(\mathcal{A}(x^{(k+1)})^{p}(y^{(k+1)})^{q-1})_j}{(y_j^{(k+1)})^{M-1}}.$$

Therefore, we obtain

$$\underline{\lambda}_k \leqslant \underline{\lambda}_{k+1}.$$

Similarly, we can prove that

$$\bar{\lambda}_{k+1} \leqslant \bar{\lambda}_k.$$

This completes our proof. □

Based on Theorem 7, we state our algorithm as follows:

Algorithm 1.

Step 0. *Choose* $x^{(0)} > 0$, $x^{(0)} \in \Re^m$ *and* $y^{(0)} > 0$, $y^{(0)} \in \Re^n$. *Let* $\xi^{(0)} = \mathcal{A}(x^{(0)})^{p-1}(y^{(0)})^q$ *and* $\eta^{(0)} = \mathcal{A}(x^{(0)})^p(y^{(0)})^{q-1}$. *Set* $k := 0$.

Table 1
Numerical results of Algorithm 1 for randomly generated tensors.

(p,q)	(m,n)	Ite	$\bar{\lambda}-\underline{\lambda}$	λ	NormX	NormY
(2, 2)	(2, 2)	13	9.89e−008	44.64	6.36e−009	6.68e−009
(2, 2)	(2, 4)	19	6.42e−008	109.86	3.11e−009	2.20e−009
(2, 2)	(2, 6)	20	6.78e−008	215.38	2.70e−009	1.24e−009
(2, 2)	(2, 8)	21	4.38e−008	331.93	1.54e−009	6.40e−010
(2, 2)	(2, 10)	21	6.61e−008	430.71	2.09e−009	7.03e−010
(2, 3)	(2, 2)	11	3.34e−008	82.23	1.07e−009	1.10e−009
(2, 3)	(2, 4)	16	7.12e−008	441.00	1.49e−009	8.68e−010
(2, 3)	(2, 6)	17	7.12e−008	1116.28	1.04e−009	4.66e−010
(2, 3)	(2, 8)	18	4.48e−008	2220.24	5.19e−010	1.78e−010
(2, 3)	(2, 10)	18	9.01e−008	3849.86	8.59e−010	2.45e−010
(3, 3)	(2, 2)	10	4.65e−008	180.79	7.66e−010	7.90e−010
(3, 3)	(2, 4)	15	2.09e−008	924.37	1.73e−010	1.04e−010
(3, 3)	(2, 6)	15	8.97e−008	2491.53	4.92e−010	1.98e−010
(3, 3)	(2, 8)	16	4.70e−008	5171.52	1.79e−010	5.89e−011
(3, 3)	(2, 10)	16	9.41e−008	8918.69	2.76e−010	7.30e−011

Step 1. *Compute*

$$x^{(k+1)} = \frac{(\xi^{(k)})^{[\frac{1}{M-1}]}}{\|(\xi^{(k)}, \eta^{(k)})^{[\frac{1}{M-1}]}\|},$$

$$y^{(k+1)} = \frac{(\eta^{(k)})^{[\frac{1}{M-1}]}}{\|(\xi^{(k)}, \eta^{(k)})^{[\frac{1}{M-1}]}\|},$$

$$\xi^{(k+1)} = \mathcal{A}\big(x^{(k+1)}\big)^{p-1}\big(y^{(k+1)}\big)^{q},$$

$$\eta^{(k+1)} = \mathcal{A}\big(x^{(k+1)}\big)^{p}\big(y^{(k+1)}\big)^{q-1}.$$

Let

$$\underline{\lambda}_{k+1} = \min_{x_i^{(k+1)}>0,\, y_j^{(k+1)}>0} \left\{ \frac{\xi_i^{(k+1)}}{(x_i^{(k+1)})^{M-1}}, \quad \frac{\eta_j^{(k+1)}}{(y_j^{(k+1)})^{M-1}} \right\},$$

$$\bar{\lambda}_{k+1} = \max_{x_i^{(k+1)}>0,\, y_j^{(k+1)}>0} \left\{ \frac{\xi_i^{(k+1)}}{(x_i^{(k+1)})^{M-1}}, \frac{\eta_j^{(k+1)}}{(y_j^{(k+1)})^{M-1}} \right\}.$$

Step 2. *If $\bar{\lambda}_{k+1} = \underline{\lambda}_{k+1}$, stop. Otherwise, replace k by $k+1$ and go to Step* 1.

For Algorithm 1, by Theorem 7, we obtain the following result.

Theorem 8. *Suppose that a nonnegative (p,q)th order $(m \times n)$-dimensional rectangular tensor $\mathcal{A}$ is irreducible. Assume that λ_0 is the unique positive singular value of $\mathcal{A}$. Then, Algorithm* 1 *produces the value of λ_0 in a finite number of steps, or generates two convergent sequences $\{\underline{\lambda}_k\}$ and $\{\bar{\lambda}_k\}$. Furthermore, let $\underline{\lambda} = \lim_{k\to+\infty} \underline{\lambda}_k$ and $\bar{\lambda} = \lim_{k\to+\infty} \bar{\lambda}_k$. Then, $\underline{\lambda}$ and $\bar{\lambda}$ are a lower bound and an upper bound of λ_0, respectively. If $\underline{\lambda} = \bar{\lambda}$, then $\lambda_0 = \underline{\lambda} = \bar{\lambda}$.*

Since $\{\underline{\lambda}_k\}$ is a monotonic increasing sequence and has an upper bound, the limit exists. $\{\bar{\lambda}_k\}$ is monotonic decreasing sequence and has a lower bound, so the limit exists. Since $\{x^{(k)}\}$ and $\{y^{(k)}\}$ are two bounded sequences, $\{x^{(k)}\}$ and $\{y^{(k)}\}$ have convergent subsequences which converge to a vector x and a vector y, respectively.

In the following, in order to show the viability of Algorithm 1, we used Matlab 7.1 to test it some randomly generated rectangular tensors. For these randomly generated tensors, the value of each entry is between 0 and 10. Throughout the computational experiments, $x^{(0)} = [1,1,\ldots,1]^T \in \Re^m$ and $y^{(0)} = [1,1,\ldots,1]^T \in \Re^n$. We terminated our iteration when $\bar{\lambda}_k - \underline{\lambda}_k \leqslant 10^{-7}$.

Our numerical results are reported in Table 1. In this table, (p,q) and (m,n) specify the order and the dimension of the randomly generated tensor, respectively. **Ite** denotes the number of iterations, $\bar{\lambda} - \underline{\lambda}$ and λ denote the values of $\bar{\lambda}_k - \underline{\lambda}_k$ and $0.5(\bar{\lambda}_k + \underline{\lambda}_k)$ at the final iteration, respectively. **NormX** and **NormY** denote the values of $\|\mathcal{A}(x^{(k)})^{p-1}(y^{(k)})^{q} - \lambda_k (x^{(k)})^{[M-1]}\|_\infty$ and $\|\mathcal{A}(x^{(k)})^{p}(y^{(k)})^{q-1} - \lambda_k (y^{(k)})^{[M-1]}\|_\infty$ at the final iteration, respectively. The results reported in Table 1 show that the proposed algorithm is promising. The algorithm is able to produce the largest singular values for all these randomly generated tensors.

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