Space Tensor Conic Programming*

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Abstract

Space tensors appear in physics and mechanics, and they are real physical entities. Mathematically, they are tensors in the three-dimensional Euclidean space. In the research of diffusion magnetic resonance imaging, convex optimization problems are formed where higher order positive semi-definite space tensors are involved. In this short paper, we investigate these problems from the viewpoint of conic linear programming (CLP). We characterize the dual cone of the positive semi-definite space tensor cone, and study the CLP formulation and the duality of the positive semi-definite space tensor conic programming problem.

Keywords: Space Tensor, Positive Semi-Definiteness, Cone, Dual Cone, Conic Linear Programming, Duality.

1. Introduction

Space tensors appear in physics and mechanics, and they are real physical entities. Mathematically, they are in the three-dimensional Euclidean space. Most early research work dealt with order-2 space tensors, that is, homogeneous quadratic polynomials in three variables. Recently, in the research of diffusion magnetic resonance imaging (MRI), higher order space tensors are typically involved [7, 8, 9, 10, 20, 30].

For example, the diffusion tensor imaging model (DTI) [5, 6] is now used widely in biological and clinical research. The diffusion tensor in DTI is a second order tensor of dimension three. However, DTI is known to have a limited capability in resolving multiple fiber orientations within one voxel. In order to describe the non-Gaussian diffusion process, one approach is to model the diffusivity function with higher order diffusion tensors (HODT) [20]. This model does not assume any a priori knowledge about the diffusivity profile and has potential to describe the non-Gaussian diffusion.

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Let $x = (x_1, x_2, x_3)^T$ denote the magnetic field gradient direction. Assume that we use an mth order diffusion tensor, where m is positive even integer. Then the diffusivity function can be expressed as

$$d(x) = \sum_{i=0}^{m} \sum_{j=0}^{m-i} d_{ij} x_1^i x_2^j x_3^{m-i-j}. \tag{1}$$

The coefficients of a diffusivity function x can be regarded as an mth order symmetric tensor [7, 9, 10, 20, 21]. Clearly, there are

$$n = \frac{1}{2}(m+1)(m+2)$$

terms or coefficients in (1) [20, 9]. Hence, each diffusivity function can also be regarded as a coefficient vector d in $\Re^n$, indexed by i, j, where $i = 0, \cdots, m; j = 0, \cdots, m-i$.

An intrinsic property of the diffusivity profile is the concept of *positive semi-definiteness* [8, 30]. In real applications, the diffusion tensor, either second or higher order, must be positive semi-definite to be physically meaningful. There are some approaches to preserve positive semi-definiteness of a diffusion tensor of second order or fourth order [3, 4, 9, 11]. None of them can work for arbitrary high order diffusion tensors. In [23], a comprehensive model, called PSDT (positive semi-definite tensor), was proposed to approximate the diffusivity function by a positive semi-definite diffusion tensor of either second or higher order.

We say that d is positive semi-definite if for all $x \in \Re^3$, $d(x) \geq 0$. Denote the set of all positive semi-definite diffusivity functions as $\mathcal{S} = \mathcal{S}(m)$. It was proved in [23] that $\mathcal{S}$ is a closed convex cone in $\Re^n$, together a PSDT minimization problem was proposed:

$$\min_{d \in \mathcal{S}} P(d) \equiv (d - \bar{d})^\top B (d - \bar{d}), \tag{2}$$

where $\bar{d}$ is the simple least squares solution of the problem, B is an $n \times n$ positive semi-definite matrix. An explicit form of (2) is

$$\min\{P(d) : \lambda_{\min}(d) \geq 0\}, \tag{3}$$

where $\lambda_{\min}(d)$ is the smallest Z-eigenvalue of d, defined by

$$\lambda_{\min}(d) = \min\{d(x) : x_1^2 + x_2^2 + x_3^2 = 1\}. \tag{4}$$

Problem (4) has only three variables.

In fact, the authors of [22, 23] presented a method to verify if an mth order symmetric space tensor is positive semi-definite or not. This involves to solve an equation of a one-dimensional polynomial of degree no more than $m^2 - m - 1$ [18] to find all stationary points of (4) and solve the minimum eigenvalue problem exhaustedly. Such a polynomial may be approximately solved to any given error bound in polynomial time of m or $n = \frac{1}{2}(m+1)(m+2)$. This forms a base for our discussion in this short paper.

Another application of space tensor is in fiber imaging as well. It was known that the diffusivity profile does not agree with the true synthetic fiber directions in regions of multiple fiber crossing. In 2004, Tuch [29] further introduced Q-balling imaging (QBI) to reconstruct the diffusion orientation distribution function (ODF) of the underlying fiber population of a biological tissue.

The ODF is a function on the unit sphere describing the probability averaged over the voxel that a particle will diffuse into any solid angle. As the water molecules in normal tissues tend to diffuse along fibers when contained in fiber bundles [6], the principal directions (maxima) of the ODF agree with the true synthetic fiber directions. Tuch [29] showed that the ODF can be estimated directly from the raw signal on a single sphere of q-space by the Funk-Radon transformation. This is the main idea of QBI.

The ODF, as a probability distribution function, should be nonnegative. In [24], a nonnegative ODF model was presented. The ODF values are strictly nonnegative in that model. Mathematically, that model is similar to (2) or (3), with an additional linear transformation resulted from the diffusivity function space to the q-space.

On the other hand, one of major developments in convex optimization for the last two decades was focusing on conic linear programming [15, 16, 27, 32]. Conic Linear programming (CLP) problems include linear programming (LP) problems, semi-definite programming (SDP) problems [1, 31, 33, 17] and second-order cone programming (SOCP) [2, 14]. It turned out that many space tensor optimization problems, such as (2), are convex optimization problem, and they can be converted to CLP problems. In this short paper, we study these problems from the viewpoint of CLP.

Our paper is organized as follows. In Section 2, we study the operations of symmetric space tensors and linear operations on symmetric space tensors. The structure of the positive semi-definite space tensor cone and its dual cone are discussed in Section 3. In Sections 4, we study the formulation and duality of space tensor linear and convex programming problems. Some final comments are made in Section 5.

Throughout this paper, let m be a positive even integer, and $n = \frac{1}{2}(m+1)(m+2)$.

2. Symmetric Space Tensors

Denote the set of all mth order real symmetric space tensors as $\mathcal{T} = \mathcal{T}(m)$. Let $A = (a_{i_1 \cdots i_m}) \in \mathcal{T}(m)$ and $B = (b_{i_1 \cdots i_m}) \in \mathcal{T}(m)$, define

$$A \bullet B = \sum_{i_1, \cdots, i_m = 1}^{3} a_{i_1 \cdots i_m} b_{i_1 \cdots i_m}$$

and

$$\|A\| = \sqrt{A \bullet A}.$$

It appears that the number of components of an mth order space tensor is 3^m. However, the number of independent components of a tensor in $\mathcal{T}$ is actually $n = \frac{1}{2}(m+1)(m+2)$. Thus $\mathcal{T}(m)$ is a linear space of dimension n – a polynomial in m.

In fact, we can define a linear transformation $\mathcal{L} : \mathcal{T}(m) \to \Re^n$. Let $A \in \mathcal{T}(m)$. Then $d = \mathcal{L}(A) \in \Re^n$ is defined as follows: For $k = 1, \cdots, n$, let $k = j + 1 + i(2m + 3 - i)/2$, where $i = 0, \cdots, m$; $j = 0, \cdots, m - i$. Then for $k = 1, \cdots, n$,

$$d_k = a_{i_1,\cdots,i_m} \sqrt{\binom{m}{i}\binom{m-i}{j}},$$

where i members of $\{i_1, \cdots, i_m\}$ are 1, j members of $\{i_1, \cdots, i_m\}$ are 2, and $m - i - j$ members of $\{i_1, \cdots, i_m\}$ are 3. Then we see that $\mathcal{L}$ establishes a one-to-one relation between $\mathcal{T}(m)$ and $\Re^n$. (Note that in the research of diffusion magnetic resonance imaging [7, 9, 10, 24], there is also a one-to-one relation between symmetric space tensors and spherical harmonics.) Furthermore, for any $A, B \in \mathcal{T}(m)$, we have

$$A \bullet B = \mathcal{L}(A) \bullet \mathcal{L}(B) = \mathcal{L}(A)^\top \mathcal{L}(B).$$

In particular, we have

$$\|A\| = \|\mathcal{L}(A)\|_2.$$

Example 2.1 *One can also represent $d = \mathcal{L}(A) \in \Re^n$ with double indices $i = 0, \cdots, m$ and $j = 0, \cdots, m - i$. For example, when $m = 4$, we have*

$$d = \begin{pmatrix} d_{00} & d_{01} & d_{02} & d_{03} & d_{04} \\ d_{10} & d_{11} & d_{12} & d_{13} & \\ d_{20} & d_{21} & d_{22} & & \\ d_{30} & d_{31} & & & \\ d_{40} & & & & \end{pmatrix}$$

For any $x \in \Re^3$, $x^m \equiv (x_{i_1} \cdots x_{i_m}) \in \mathcal{T}(m)$ is a rank-one space tensor:

$$\mathcal{L}(x^4) = \begin{pmatrix} (x_3)^4 & 2x_2(x_3)^3 & \sqrt{6}(x_2)^2(x_3)^2 & 2(x_2)^3x_3 & (x_2)^4 \\ 2x_1(x_3)^3 & \sqrt{12}x_1x_2(x_3)^2 & \sqrt{12}x_1(x_2)^2x_3 & 2x_1(x_2)^3 & \\ \sqrt{6}(x_1)^2(x_3)^2 & \sqrt{12}(x_1)^2x_2x_3 & \sqrt{6}(x_1)^2(x_2)^2 & & \\ 2(x_1)^3x_3 & 2(x_1)^3x_2 & & & \\ (x_1)^4 & & & & \end{pmatrix}.$$

We also have $d(x) = Ax^m \equiv A \bullet x^m$. Again, if $A \in \mathcal{T}(m)$ has the property that $Ax^m \geq 0$ for all $x \in \Re^3$, we say that A is *positive semi-definite*. If $A = (a_{i_1,\cdots,i_m}) \in \mathcal{T}(m)$ has the property that $Ax^m > 0$ for all $x \in \Re^3, x \neq 0$, then we say that A is *positive definite*.

A linear transformation $\mathbf{Q} : \mathcal{T}(\mathbf{m}) \to \mathcal{T}(\mathbf{m})$ is called symmetric if for any $A, B \in \mathcal{T}(m)$, $\mathbf{Q}A \bullet B = A \bullet \mathbf{Q}B$. If furthermore, for any $A \in \mathcal{T}(m)$, $A \bullet \mathbf{Q}A \geq 0$, then $\mathbf{Q}$ is called positive semi-definite. If for any $A \in \mathcal{T}(m), A \neq 0$, $A \bullet \mathbf{Q}A > 0$, then $\mathbf{Q}$ is called positive definite.

For any linear transformation $\mathbf{Q} : \mathcal{T}(m) \to \mathcal{T}(m)$, there is an $n \times n$ matrix Q such that for any $A \in \mathcal{T}(m)$

$$Q\mathcal{L}(A) = \mathcal{L}(\mathbf{Q}A).$$

It is not difficult to see that $\mathbf{Q}$ is symmetric / positive semi-definite / positive definite if and only if Q is symmetric / positive semi-definite / positive definite.

If $\mathbf{Q}$ is positive semi-definite linear transformation from $\mathcal{T}(m)$ to $\mathcal{T}(m)$, then there is another positive semi-definite linear transformation $\mathbf{P}$ from $\mathcal{T}(m)$ to $\mathcal{T}(m)$ such that $\mathbf{P}^2 = \mathbf{Q}$.

For $k = 1, \cdots, n$, let $k = j + 1 + i(2m + 3 - i)/2$, where $i = 0, \cdots, m - i, i = 0, \cdots, m$. Let $E_k = (e_{i_1 \cdots i_m})$ be defined as $e_{i_1 \cdots i_m} = 0$ unless there are exactly i members of $\{i_1, \cdots, i_m\}$ are 1, and exactly j members of $\{i_1, \cdots, i_m\}$ are 2. In the latter case, $e_{i_1, \cdots, i_m} = 1$. Then $E_1, \cdots, E_n$ form a base of $\mathcal{T}(m)$.

Let $\mathbf{P}$ be a linear transformation from $\mathcal{T}(m)$ to $\mathcal{T}(m)$. Then we call $P_k = \mathbf{P}E_k$ the kth row space tensor of $\mathbf{P}$. For any $X \in \mathcal{T}(m)$, we have

$$\mathcal{L}(\mathbf{P}X)_k = P_k \bullet X. \tag{5}$$

3. Semi-Definite Space Tensor Cones

Let $\mathcal{K} \subset \mathcal{T}(m)$ be a cone. Then, the dual cone of $\mathcal{K}$ is defined by

$$\mathcal{K}^* = \{A \in \mathcal{T}(m) : A \bullet B \geq 0, \ \forall \ B \in \mathcal{K}\}.$$

Denote the set of all mth order real positive semi-definite symmetric space tensors by $\mathcal{S}(m)$. Then $\mathcal{S}(m)$ must be a closed convex cone. Furthermore, the cone has an interior.

Theorem 1 *A space tensor $A \in \mathcal{T}(m)$ is an interior point of $\mathcal{S}(m)$ if and only if it is positive definite.*

Proof. First, suppose that $A \in \mathcal{S}(m)$ is not positive definite. Then, there is $x \in \Re^3, x \neq 0$, such that $Ax^m = 0$. Then

$$(A - \epsilon x^m)\, x^m = -\epsilon \left(x^\top x\right)^m < 0,$$

for any $\epsilon > 0$. This implies that A is not an interior point of $\mathcal{S}(m)$.

On the other hand, suppose that $A \in \mathcal{S}(m)$ is not an interior point of $\mathcal{S}(m)$. Then there is a sequence $\{(B_k, \epsilon_k) \in \mathcal{T}(m) \times \Re : \|B_k\| = 1, \epsilon_k > 0, \text{ for } k = 1, 2, \cdots\}$ such that $A + \epsilon_k B_k \notin \mathcal{S}(m)$ for $k = 1, 2, \cdots$, and $\lim_{k \to \infty} \epsilon_k = 0$. Then there are $x^{(k)}$ with $\|x^{(k)}\| = 1$ for $k = 1, 2, \cdots$, such that

$$(A + \epsilon_k B_k) \left(x^{(k)}\right)^m \leq 0.$$

Let x be a limiting point of $\{x^{(k)}\}$. Then $\|x\| = 1$ and $Ax^m \leq 0$. This shows that A is not positive definite and completes the proof. $\square$

Example 3.1 *The following tensor is an interior point of* $\mathcal{S}(4)$

$$d = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & & \\ 0 & 0 & & & \\ 1 & & & & \end{pmatrix}$$

Let $\mathcal{U}(m) = \{x^m : x \in \Re^3\}$. Then $\mathcal{U}(m)$ is the cone of real symmetric positive semi-definite so-called *rank-one* space tensors. We have the following theorem.

Theorem 2 *We have* $\mathcal{U}(m) \subset \mathcal{S}(m)$ *and* $\mathcal{S}(m) = \mathcal{U}(m)^*$. *Furthermore, for any* $x \in \Re^3$, x^m *is not an interior point of* $\mathcal{S}(m)$. *Hence, the interior of* $\mathcal{U}(m)$ *is empty. For any* $x \in \Re^3$, *we also have* $\|x^m\| = \|x\|_2^m$.

Proof. For any $x, y \in \Re^3$,

$$x^m \bullet y^m = \left(x^\top y\right)^m \geq 0.$$

This implies that x^m is positive semi-definite. Hence, $\mathcal{U}(m) \subset \mathcal{S}(m)$. By definition, we have $\mathcal{S}(m) = \mathcal{U}(m)^*$. For any $x \in \Re^3$, let $y \in \Re^3, y \neq 0$ and $x^\top y = 0$. Then $x^m \bullet y^m = 0$. This implies that x^m is not positive definite and thus not an interior point of $\mathcal{S}(m)$ by Theorem 1. For any $x \in \Re^3$,

$$\|x^m\| = \sqrt{x^m \bullet x^m} = \sqrt{\left(x^\top x\right)^m} = \|x\|_2^m.$$

This completes the proof. $\square$

Theorem 3 *Let* l *be a positive integer and*

$$\mathcal{V}(m, l) = \left\{ \sum_{i=1}^{l} \left(x^{(i)}\right)^m : x^{(i)} \in \Re^3, \text{ for } i = 1, \cdots, l \right\}.$$

Then $\mathcal{V}(m, l) \subset \mathcal{S}(m)$ *and it is a closed cone.*

Proof. We see that members of $\mathcal{V}(m, l)$ are convex combinations of members of $\mathcal{U}(m)$. As $\mathcal{U}(m) \subset \mathcal{S}(m)$ and $\mathcal{S}(m)$ is convex, we have $\mathcal{V}(m, l) \subset \mathcal{S}(m)$. We now prove that $\mathcal{V}(m, l)$ is closed. Let A be a limiting point of $\mathcal{V}(m, l)$. Then there is a sequence $\{A_j : j = 1, 2, \cdots\} \subset V(m, l)$ such that $\lim_{j \to \infty} A_j = A$. Then $\{\|A_j\| : j = 1, 2, \cdots\}$ is bounded, and there are $x^{(ij)} \in \Re^3$ for $i = 1, \cdots, l, j = 1, 2, \cdots$, such that

$$A_j = \sum_{i=1}^{l} \left(x^{(ij)}\right)^m,$$

for $j = 1, 2, \cdots$. Then

$$\|A_j\|^2 = \sum_{i=1}^{l} \|x^{(ij)}\|^{2m} + \sum_{i \neq k} \left[\left(x^{(ij)}\right)^\top \left(x^{(kj)}\right) \right]^m .$$

As every term in the above sum is nonnegative and $\{\|A_j\|\}$ is bounded, $\{x^{(ij)} : j = 1, 2, \cdots\}$ is bounded for $i = 1, \cdots, l$. Without loss of generality, we may assume that there are $x^{(i)}$ such that $\lim_{j \to \infty} x^{(ij)} = x^{(i)}$ for $i = 1, \cdots, l$. Then

$$A = \lim_{j \to \infty} A_j = \sum_{i=1}^{l} \left(x^{(i)}\right)^m \in \mathcal{V}(m, l).$$

This shows that $\mathcal{V}(m, l)$ is closed. The proof is completed. $\square$

Remark 1. The proof of this theorem was suggested by Xinzhen Zhang. This result is nontrivial actually. According to [13], for $l \geq 2$ and $m \geq 3$, the rank-l set

$$\left\{ \sum_{i=1}^{l} \alpha_i \left(x^{(i)}\right)^m : x^{(i)} \in \Re^3, \alpha_i \in \Re, \text{ for } i = 1, \cdots, l \right\}$$

is not closed.

Theorem 4 *Let*

$$\mathcal{V}(m) = \mathcal{V}(m, n).$$

Then $\mathcal{V}(m) \subset \mathcal{S}(m)$ and is a closed convex cone. Furthermore, we have

$$\mathcal{S}(m)^* = \mathcal{V}(m) = \mathrm{cl}\left(\mathrm{conv}\left(\mathcal{U}(m)\right)\right)$$

and

$$\mathcal{S}(m)^{**} = \mathcal{S}(m).$$

Proof. Since $\mathcal{S}(m) = \mathcal{U}(m)^*$, by convex analysis [26], we have

$$\mathcal{S}(m)^* = \mathcal{U}(m)^{**} = \mathrm{cl}\left(\mathrm{conv}\left(\mathcal{U}(m)\right)\right).$$

As the dimension of $\mathcal{T}(m)$ is n, by the Carathéodory's theorem [26], we have

$$\mathcal{V}(m) = \mathrm{conv}\left(\mathcal{U}(m)\right).$$

By Theorem 3, $\mathcal{V}(m)$ is closed and $\mathcal{V}(m) \subset \mathcal{S}(m)$. Hence,

$$\mathcal{V}(m) = \mathrm{cl}\left(\mathrm{conv}\left(\mathcal{U}(m)\right)\right).$$

Finally, as $\mathcal{S}(m)$ is a closed convex cone, by convex analysis [26],

$$\mathcal{S}(m)^{**} = \mathcal{S}(m).$$

This completes the proof. $\square$

Remark 2. If $m = 2$, then the space tensor cone is self-dual, that is, $\mathcal{S}(2)^* = \mathcal{V}(2) = \mathcal{S}(2)$, which can be represents by the 3×3 positive semidefinite (PSD) matrix cone with 6 independent components. If $m = 4$, the space tensor cone is also self-dual or $\mathcal{S}(4)^* = \mathcal{V}(4) = \mathcal{S}(4)$, which can be represented by a sum-of-square (SOS) cone. This is due to Hilbert who showed in 1888 (see [28]) that if $f(x)$ is a form of three variables and has degree 4, then $f(x)$ is a sum of squares; also see proof in [25]. Note that SOS cone has a finite representation of PSD cones; see [12, 19]. However, this is not true for $m > 4$ in general. Hilbert also showed ([28]) that, when $m > 4$, there is a nonnegative homogeneous polynomial $f(x)$ of even degree m, which cannot be a sum of squares of real polynomials. Thus this implies that $A \in \mathcal{S}(m)$ but $A \notin \mathcal{V}(m)$, i.e., $\mathcal{S}(m) \neq \mathcal{V}(m) \equiv \mathcal{S}(m)^*$.

Clearly, $\mathcal{S}(m)$ and $\mathcal{V}(m) \equiv \mathcal{S}(m)^*$ are two most important closed convex cones in the tensor space $\mathcal{T}(m)$. They form the base of the positive semi-definite tensor linear programming (SDTLP) problem, which we will discuss in the next sections.

4. The Space Tensor Conic Programming Problem

Denote $\mathcal{S}(m)$ and $\mathcal{S}(m)^* = \mathcal{V}(m)$ as $\mathcal{S}$ and $\mathcal{S}^*$ in this section. A closed nonempty convex cone K is called a pointed cone if $K \cap (-K) = \{0\}$ [15, 16]. Clearly, $\mathcal{S}$ and $\mathcal{S}^*$ are pointed cones.

For any pointed cone K, define the relation $\succeq_K$ by

$$u \succeq_K v \text{ if and only if } u - v \in K.$$

If $\mathrm{int}(K)$, the interior of K, is nonempty, further define the relation $\succ_K$ by

$$u \succ_K v \text{ if and only if } u - v \in \mathrm{int}(K).$$

Then we may define the following space tensor conic linear programming (STLP) problem as

$$\begin{array}{rl} w_p^* := \inf\limits_X & C \bullet X \\ \text{subject to} & A_i \bullet X = b_i, \text{ for } i = 1, \cdots, p, \\ & X \succeq_{\mathcal{S}} 0, \end{array} \tag{6}$$

where $C, A_1, \cdots, A_p \in \mathcal{T}(m)$, and $b_1, \cdots, b_p \in \Re$. Thus, STLP (6) is a convex optimization problem. According to the theory of conic linear programming (CLP) [15, 16, 32], there is a dual problem, associated with (6), with the following form:

$$\begin{array}{rl} w_d^* := \sup\limits_{y,S} & b^\top y \\ \text{subject to} & \sum_{i=1}^p y_i A_i + S = C, \\ & y \in \Re^p, \ S \succeq_{\mathcal{S}^*} 0. \end{array} \tag{7}$$

By the CLP weak duality theorem [15, 16], we have the following theorem.

Theorem 5 *Let $\bar{X} \in \mathcal{S}$ be feasible for (6) and $(\bar{y}, \bar{S}) \in \Re^p \times \mathcal{S}^*$ be feasible for (7). Then $b^\top \bar{y} \leq C \bullet \bar{X}$.*

Corollary 1 *Given a primal feasible solution X^* and a dual feasible solution (y^*, S^*), they are optimal for (6) and (7), respectively, if $C \bullet X^* = b^\top y^*$.*

According to the CLP strong duality theorem, we have the following three theorems.

Theorem 6 *Suppose that (6) is bounded below and strictly feasible, i.e., there exists a primal feasible solution $\bar{X} \succ_{\mathcal{S}} 0$. Then, we have $w_p^* = w_d^*$, i.e., there exists a dual feasible solution (y^*, S^*) such that $b^\top y^* = w_p^* = w_d^*$, i.e., the common optimal value is attained by some dual feasible solution.*

According to Theorem 1, $\bar{X} \succ_{\mathcal{S}} 0$ if and only if $\bar{X}$ is positive definite.

Theorem 7 *Suppose that (7) is bounded above and strictly feasible, i.e., there exists a dual feasible solution $(\bar{y}, \bar{S})$ such that $\bar{S} \succ_{\mathcal{S}^*} 0$. Then, we have $w_p^* = w_d^*$, i.e., there exists a primal feasible solution X^* such that $C \bullet X^* = w_p^* = w_d^*$, i.e., the common optimal value is attained by some primal feasible solution.*

One open question is how to identify $\bar{S}$ such that $\bar{S} \succ_{\mathcal{S}^*} 0$.

Theorem 8 *Suppose either (6) or (7) is bounded and strictly feasible. Then, given a primal feasible solution X^* and a dual feasible solution (y^*, S^*), they are optimal for (6) and (7), respectively, if and only if the complementary slackness holds, i.e., $X^* \bullet S^* = 0$.*

In some cases, space tensor (convex) nonlinear programming can be rewritten as space tensor conic linear programming. Consider the space tensor convex quadratic programming (STQP) problem:

$$\begin{array}{ll} \inf\limits_{X} & (X - \bar{X}) \bullet \mathbf{Q}(X - \bar{X}) \\ \text{subject to} & A_i \bullet X = b_i, \text{ for } i = 1, \cdots, p, \\ & X \succeq_{\mathcal{S}} 0, \end{array} \tag{8}$$

where $C, A_1, \cdots, A_p \in \mathcal{T}(m)$, $b_1, \cdots, b_p \in \Re$, $\bar{X} \in \mathcal{T}(m)$, and $\mathbf{Q} : \mathcal{T}(m) \to \mathcal{T}(m)$ is a positive semi-definite symmetric linear transformation. Clearly, STQP (8) is also a convex optimization problem. The PSDT problem (2) discussed earlier is a special case of (8) with $p = 0$.

Problem (8) is equivalent to the following problem:

$$\begin{array}{ll} \inf\limits_{X} & \|\mathbf{P}(X - \bar{X})\| \\ \text{subject to} & A_i \bullet X = b_i, \text{ for } i = 1, \cdots, p, \\ & X \succeq_{\mathcal{S}} 0, \end{array} \tag{9}$$

where $\mathbf{P} : \mathcal{T}(m) \to \mathcal{T}(m)$ is also a positive semi-definite symmetric linear transformation, and $\mathbf{P}^2 = \mathbf{Q}$. Let $d = \mathcal{L}(\mathbf{P}\bar{X})$, and there $P_1, \cdots, P_n$ are row space tensors of $\mathbf{P}$ as defined in Section 3. Then (9) is equivalent to

$$\begin{array}{rl} v_p^* = \inf\limits_{X,u,t} & t \\ \text{subject to} & A_i \bullet X = b_i, \text{ for } i = 1, \cdots, p, \\ & P_i \bullet X - u_i = d_i, \text{ for } i = 1, \cdots, n, \\ & \|u\|_2 \leq t, \ X \succeq_{\mathcal{S}} 0, \end{array} \tag{10}$$

where $u, d \in \Re^n$ and $t \in \Re$. We may further write (10) as

$$\begin{array}{rl} v_p^* = \inf\limits_{X,u,t} & t \\ \text{subject to} & A_i \bullet X = b_i, \text{ for } i = 1, \cdots, p, \\ & P_i \bullet X - u_i = d_i, \text{ for } i = 1, \cdots, n, \\ & \binom{t}{u} \in L, \ X \succeq_{\mathcal{S}} 0, \end{array} \tag{11}$$

where

$$L = \left\{ \begin{pmatrix} t \\ u \end{pmatrix} \in \Re^{n+1} : \|u\|_2 \leq t \right\}$$

is the second order cone, which is also a pointed cone [16, 15].

Problem (11) is actually a conic linear programming problem. The dual problem, associated with (11), has the following form:

$$\begin{array}{rl} v_d^* := \sup\limits_{y,w,S} & b^\top y + d^\top w \\ \text{subject to} & \sum_{i=1}^p y_i A_i + \sum_{i=1}^n w_i P_i + S = 0, \\ & \binom{1}{w} \succeq_L 0, \ S \succeq_{\mathcal{S}^*} 0, \end{array} \tag{12}$$

where $y \in \Re^p$ and $w \in \Re^n$. As (6) and (7), (11) and (12) have their duality results. We do not go to details.

5. Conclusions and Final Remarks

As we mentioned before, space tensors naturally appear in physics and mechanics. They describe the diffusion properties or their probability functions of particles in liquid. The diffusion properties and their probability functions are antipodally symmetric and non-negative. Hence, these space tensors are positive semi-definite symmetric tensors. The diffusion properties and their probability functions are spherically homogeneous. In the multiple fiber region, the diffusion can be highly nonlinear. Then high order positive semi-definite symmetric tensors are involved. These are application bases of space tensor conic programming.

We have formulated space tensor programming problems as CLP problems in Section 4. This gives them not only nice duality structure, but also efficient solution possibilities.

We may use the primal potential reduction algorithm [32], which is a polynomial-time algorithm. It will be one of our next research topics to establish practical CLP methods to solve these space tensor conic programming problems.

There are also many other open questions and we list a few of them for further research.

1. For any $x \in \Re^3$, is rank-one space tensor x^m an extreme point of $\mathcal{S}(m)$?
2. For $m = 4$, what is the dimension of a PSD cone to represent the space tensor SOS cone $\mathcal{S}(4)$?
3. If $m = 2$, then $\mathcal{V}(m) = \mathcal{V}(m, 3)$. When $m > 2$, what is the minimum value of l such that $\mathcal{V}(m) = \mathcal{V}(m, l)$?
4. Suppose that $m > 4$. It is known $\mathcal{V}(m) \neq \mathcal{S}(m)$. Then, does $\mathcal{V}(m)$ have a nonempty interior? If the interior point set of $\mathcal{V}(m)$ is not empty, how can one identify a point is an interior point of $\mathcal{V}(m)$? The answer to this question is useful for the duality theory of the space tensor programming problem.
5. What is the efficient barrier function of the space tensor cone $\mathcal{S}(m)$ or $\mathcal{V}(m)$?

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