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An even order symmetric B tensor is positive definite

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It is easily checkable if a given tensor is a B tensor, or a B_0 tensor or not. In this paper, we show that a symmetric B tensor can always be decomposed to the sum of a strictly diagonally dominated symmetric M tensor and several positive multiples of partially all one tensors, and a symmetric B_0 tensor can always be decomposed to the sum of a diagonally dominated symmetric M tensor and several positive multiples of partially all one tensors. When the order is even, this implies that the corresponding B tensor is positive definite, and the corresponding B_0 tensor is positive semi-definite. This gives a checkable sufficient condition for positive definite and semi-definite tensors. This approach is different from the approach in the literature for proving a symmetric B matrix is positive definite, as that matrix approach cannot be extended to the tensor case.

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M tensor
Partially all one tensor

1. Introduction

Denote $[n] := \{1, \cdots, n\}$. A real mth order n-dimensional tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ is a multi-array of real entries $a_{i_1 \cdots i_m}$, where $i_j \in [n]$ for $j \in [m]$. All the real mth order n-dimensional tensors form a linear space of dimension n^m. Denote this linear space by $T_{m,n}$. For $i \in [n]$, we call $a_{ii_2 \cdots i_m}$ for $i_j \in [n]$, $j = 2, \cdots, m$, the entries of $\mathcal{A}$ in the ith row, where $a_{i \cdots i}$ is the ith diagonal entry of $\mathcal{A}$, while the other entries are the off-diagonal entries of $\mathcal{A}$ in the ith row.

Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$. If the entries $a_{i_1 \cdots i_m}$ are invariant under any permutation of their indices, then $\mathcal{A}$ is called a **symmetric tensor**. All the real mth order n-dimensional symmetric tensors form a linear subspace of $T_{m,n}$. Denote this linear subspace by $S_{m,n}$. Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$ and $\mathbf{x} \in \Re^n$. Then $\mathcal{A}x^m$ is a homogeneous polynomial of degree m, defined by

$$\mathcal{A}x^m = \sum_{i_1, \cdots, i_m = 1}^{n} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}.$$

A tensor $\mathcal{A} \in T_{m,n}$ is called **positive semi-definite** if for any vector $\mathbf{x} \in \Re^n$, $\mathcal{A}\mathbf{x}^m \geq 0$, and is called **positive definite** if for any non-zero vector $\mathbf{x} \in \Re^n$, $\mathcal{A}\mathbf{x}^m > 0$. Clearly, if m is odd, there is no non-zero positive semi-definite tensors. Positive definiteness and semi-definiteness of real symmetric tensors and their corresponding homogeneous polynomials have applications in automatical control [1,5,12,27], polynomial problems [16, 24], magnetic resonance imaging [2,7,22,23] and spectral hypergraph theory [8–11,13,18, 20]. In [17], Qi introduced H-eigenvalues and Z-eigenvalues for real symmetric tensors, and showed that an even order real symmetric tensor is positive (semi-)definite if and only if all of its H-eigenvalues, or all of its Z-eigenvalues, are positive (non-negative). In matrix theory, it is well-known that a strictly diagonally dominated symmetric matrix is positive definite and a diagonally dominated symmetric matrix is positive semi-definite. Here, we may also easily show that an even order strictly diagonally dominated symmetric tensor is positive definite and an even order diagonally dominated symmetric tensor is positive semi-definite. We will show this in Section 2. Based upon this, we know that the Laplacian tensor in spectral hypergraph theory is positive semi-definite [9–11,18,23]. Song and Qi [25] showed that an even order Hilbert tensor is positive definite. This also extends the matrix result that a Hilbert matrix is positive definite. In matrix theory, a completely positive tensor is positive semi-definite, and a diagonally dominated symmetric non-negative tensor is completely positive. In [21], completely positive tensors were introduced. An even order completely positive tensor is also positive semi-definite. Then, it was shown in [21] that a strongly symmetric, hierarchically dominated non-negative tensor is completely positive. These are some checkable sufficient conditions for positive definite or semi-definite tensors in the literature.

In the matrix literature, there is another easily checkable sufficient condition for positive definite matrices. It is easy to check a given matrix is a B matrix or not [14,15]. A B matrix is a P matrix [14]. It is well-known that a symmetric matrix is a P matrix if and only it is positive definite [3, pp. 147, 153]. Thus, a symmetric B matrix is positive definite.

P matrices and B matrices were extended to P tensors and B tensors in [26]. It is easy to check a given tensor is a B tensor or not, while it is not easy to check a given tensor is a P tensor or not. It was proved there that a symmetric tensor is a P tensor if and only it is positive definite. However, it was not proved in [26] if an even order B tensor is a P tensor or not, or if an even order symmetric B tensor is positive definite or not. As pointed out in [26], an odd order identity tensor is a B tensor, but not a P tensor. Thus we know that an odd order B tensor may not be a P tensor.

The B tensor condition is not so strict compare with the strongly diagonal dominated tensor condition if the tensor is not sparse. A tensor in $T_{m,n}$ is strictly diagonally dominated tensor if every diagonal entry of that tensor is greater than the sum of the absolute values of all the off-diagonal entries in the same row. For each row, there are $n^{m-1} - 1$ such off-diagonal entries. Thus, this condition is quite strict when n and m are big and the tensor is not sparse. A tensor in $T_{m,n}$ is a B tensor if for every row of the tensor, the sum of all the entries in that row is positive, and each off-diagonal entry is less than the average value of the entries in the same row. An initial numerical experiment indicated that for $m = 4$ and $n = 2$, a symmetric B tensor is positive definite. Thus, it is possible that an even order symmetric B tensor is positive definite. If this is true, we will have an easily checkable, not very strict, sufficient condition for positive definite tensors.

However, the technique in [14] cannot be extended to the tensor case. It was proved in [14] that the determinant of every principal submatrix of a B matrix is positive. Thus, a B matrix is a P matrix. It was pointed out in [17] that the determinant of every principal sub-tensor of a symmetric positive definite tensor is positive, but this is only a necessary, not a sufficient condition for symmetric positive definite tensors. Hence, the technique in [14] cannot be extended to the tensor case.

In [26], P tensors were defined by extending an alternative definition for P matrices. But it is still unknown if an even order B tensor is a P tensor or not.

In this paper, we use a new technique to prove that an even order symmetric B tensor is positive definite. We show that a symmetric B tensor can always be decomposed to the sum of a strictly diagonally dominated symmetric M tensor and several positive multiples of partially all one tensors, and a symmetric B_0 tensor can always be decomposed to the sum of a diagonally dominated symmetric M tensor and several positive multiples of partially all one tensors. Even order partially all one tensors are positive semi-definite. As stated before, an even order diagonally dominated symmetric tensor is positive semi-definite, and an even order strictly diagonally dominated symmetric tensor is positive definite. Therefore, when the order is even, these imply that the corresponding symmetric B tensor is positive definite, and the corresponding symmetric B_0 tensor is positive semi-definite. Hence, this gives an easily checkable, not very strict, sufficient condition for positive definite and semi-definite tensors.

In the next section, we study diagonally dominated symmetric tensors. In Section 3, we define B, B_0 and partially all one tensors, and discuss their general properties. The main result is given in Section 4. We make some final remarks and raise some further questions in Section 5.

Throughout this paper, we assume that $m \geq 2$ and $n \geq 1$. We use small letters $x, u, v, \alpha, \cdots$, for scalers, small bold letters $\mathbf{x}, \mathbf{y}, \mathbf{u}, \cdots$, for vectors, capital letters $A, B, \cdots$, for matrices, calligraphic letters $\mathcal{A}, \mathcal{B}, \cdots$, for tensors. All the tensors discussed in this paper are real.

2. Diagonally dominated symmetric tensors

We define the generalized Kronecker symbol as

$$\delta_{i_1\cdots \text{ß}_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

The tensor $\mathcal{I} = (\delta_{i_1\cdots \text{ß}_m})$ is called the **identity tensor** of $T_{m,n}$.

Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$. If for $i \in [n]$,

$$a_{i\cdots i} \geq \sum \{ |a_{ii_2\cdots i_m}| : i_j \in [n], j = 2, \cdots, m, \delta_{ii_2\cdots i_m} = 0 \},$$

then $\mathcal{A}$ is called a **diagonally dominated tensor**. If for $i \in [n]$,

$$a_{i\cdots i} > \sum \{ |a_{ii_2\cdots i_m}| : i_j \in [n], j = 2, \cdots, m, \delta_{ii_2\cdots i_m} = 0 \},$$

then $\mathcal{A}$ is called a **strictly diagonally dominated tensor**.

Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$ and $\mathbf{x} \in C^n$. Define $\mathcal{A}\mathbf{x}^{m-1}$ as a vector in C^n with its ith component as

$$(\mathcal{A}\mathbf{x}^{m-1})_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}$$

for $i \in [n]$. For any vector $\mathbf{x} \in C^n$, define $\mathbf{x}^{[m-1]}$ as a vector in C^n with its ith component defined as x_i^{m-1} for $i \in [n]$. Let $\mathcal{A} \in T_{m,n}$. If there is a non-zero vector $\mathbf{x} \in C^n$ and a number $\lambda \in C$ such that

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}, \tag{1}$$

then λ is called an **eigenvalue** of $\mathcal{A}$ and $\mathbf{x}$ is called an **eigenvector** of $\mathcal{A}$, associated with λ. If the eigenvector $\mathbf{x}$ is real, then the eigenvector λ is also real. In this case, λ and

$\mathbf{x}$ are called an **H-eigenvalue** and an **H-eigenvector** of $\mathcal{A}$, respectively. The maximum modulus of the eigenvalues of $\mathcal{A}$ is called the **spectral radius** of $\mathcal{A}$, and denoted as $\rho(\mathcal{A})$. Eigenvalues and H-eigenvalues were first introduced in [17] for symmetric tensors. The following theorem is from [17, Theorem 5].

Theorem 1. *Suppose that $\mathcal{A} \in S_{m,n}$ and m is even. Then $\mathcal{A}$ always has H-eigenvalues. $\mathcal{A}$ is positive semi-definite if and only if all of its H-eigenvalues are non-negative. $\mathcal{A}$ is positive definite if and only if all of its H-eigenvalues are positive.*

The following theorem is from [17, Theorem 6]. Theorem 6 of [17] is restricted to symmetric tensors. But it is true for non-symmetric tensors, and the proof is the same.

Theorem 2. *Suppose that $\mathcal{A} \in T_{m,n}$. Then the eigenvalues λ of $\mathcal{A}$ satisfy the following constraints: for $i \in [n]$,*

$$|\lambda - a_{i\cdots i}| \le \sum \bigl\{ |a_{ii_2\cdots i_m}| : i_j \in [n], j = 2, \cdots, m, \delta_{ii_2\cdots i_m} = 0 \bigr\}.$$

We now have the following theorem.

Theorem 3. *Let $\mathcal{A} \in S_{m,n}$ and m be even. If $\mathcal{A}$ is diagonally dominated, then $\mathcal{A}$ is positive semi-definite. If $\mathcal{A}$ is strictly diagonally dominated, then $\mathcal{A}$ is positive definite.*

Proof. By Theorem 2 and the definition of diagonally dominated and strictly diagonally dominated tensors, all the H-eigenvalues of a diagonally dominated tensor, if exist, are non-negative, and all the H-eigenvalues of a strictly diagonally dominated tensor, if exist, are positive. The conclusions follow from Theorem 1 now. □

Let $\mathcal{A} \in T_{m,n}$. If all of the off-diagonal entries of $\mathcal{A}$ are non-positive, then $\mathcal{A}$ is called a **Z tensor**. If a Z tensor $\mathcal{A}$ can be written as $\mathcal{A} = c\mathcal{I} - \mathcal{B}$, such that $\mathcal{B}$ is a non-negative tensor and $c \ge \rho(\mathcal{B})$, then $\mathcal{A}$ is called an **M tensor** [29]. If $c > \rho(\mathcal{B})$, then $\mathcal{A}$ is called a **strong M tensor** [29]. It was proved in [29] that a diagonally dominated Z tensor is an M tensor, and a strictly diagonally dominated Z tensor is a strong M tensor. The properties of M and strong M tensors may be found in [4,6,29].

3. B, $\mathbf{B_0}$ and partially all one tensors

Let $\mathcal{B} = (b_{i_1\cdots i_m}) \in T_{m,n}$. We say that $\mathcal{B}$ is a **B tensor** if for all $i \in [n]$

$$\sum_{i_2,\cdots,i_m=1}^{n} b_{ii_2i_3\cdots i_m} > 0$$

and

$$\frac{1}{n^{m-1}}\left(\sum_{i_2,\cdots,i_m=1}^{n} b_{ii_2i_3\cdots i_m}\right) > b_{ij_2j_3\cdots j_m} \quad \text{for all } (j_2,j_3,\cdots,j_m) \neq (i,i,\cdots,i).$$

We say that $\mathcal{B}$ is a $\mathbf{B_0}$ **tensor** if for all $i \in [n]$

$$\sum_{i_2,\cdots,i_m=1}^{n} b_{ii_2i_3\cdots i_m} \geq 0$$

and

$$\frac{1}{n^{m-1}}\left(\sum_{i_2,\cdots,i_m=1}^{n} b_{ii_2i_3\cdots i_m}\right) \geq b_{ij_2j_3\cdots j_m} \quad \text{for all } (j_2,j_3,\cdots,j_m) \neq (i,i,\cdots,i).$$

This definition is a natural extension of the definition of B matrices [14,15,26]. It is easily checkable if a given tensor in $T_{m,n}$ is a B tensor, or a B_0 tensor or not. As discussed in the introduction, the definitions of B and B_0 tensors are not so strict, compared with the definitions of diagonally dominated and strictly diagonally dominated tensors, if the tensor is not sparse. We also can see that a Z tensor is diagonally dominated if and only if it is a B_0 tensor, and a Z tensor is strictly diagonally dominated if and only if it is a B tensor [26].

A tensor $\mathcal{C} \in T_{m,r}$ is called **a principal sub-tensor** of a tensor $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$ $(1 \leq r \leq n)$ if there is a set J that composed of r elements in $[n]$ such that

$$\mathcal{C} = (a_{i_1\cdots i_m}), \quad \text{for all } i_1, i_2, \cdots, i_m \in J.$$

This concept was first introduced and used in [17] for symmetric tensor. We denote by $\mathcal{A}_r^J$ the principal sub-tensor of a tensor $\mathcal{A} \in T_{m,n}$ such that the entries of $\mathcal{A}_r^J$ are indexed by $J \subset [n]$ with $|J| = r$ $(1 \leq r \leq n)$.

It was proved in [26] that all the principal sub-tensors of a B_0 tensor are B_0 tensors, and all the principal sub-tensors of a B tensor are B tensors.

Suppose that $\mathcal{A} \in S_{m,n}$ has a principal sub-tensor $\mathcal{A}_r^J$ with $J \subset [n]$ with $|J| = r$ $(1 \leq r \leq n)$ such that all the entries of $\mathcal{A}_r^J$ are one, and all the other entries of $\mathcal{A}$ are zero. Then $\mathcal{A}$ is called a **partially all one tensor**, and denoted by $\mathcal{E}^J$. If $J = [n]$, then we denote $\mathcal{E}^J$ simply by $\mathcal{E}$ and call it an **all one tensor**. An even order partially all one tensor is positive semi-definite. In fact, when m is even, if we denote by $\mathbf{x}_J$ the r-dimensional sub-vector of a vector $\mathbf{x} \in \Re^n$, with the components of $\mathbf{x}_J$ indexed by J, then for any $\mathbf{x} \in \Re^n$, we have

$$\mathcal{E}^J\mathbf{x}^m = \left(\sum\{x_j : j \in J\}\right)^m \geq 0.$$

4. Decomposition of B tensors

We now prove the main result of this paper.

Theorem 4. *Suppose that $\mathcal{B} = (b_{i_1 \cdots i_m}) \in S_{m,n}$ is a symmetric B_0 tensor. Then either $\mathcal{B}$ is a diagonally dominated symmetric M tensor itself, or we have*

$$\mathcal{B} = \mathcal{M} + \sum_{k=1}^{s} h_k \mathcal{E}^{J_k}, \tag{2}$$

where $\mathcal{M}$ is a diagonally dominated symmetric M tensor, s is a positive integer, $h_k > 0$ and $J_k \subset [n]$, for $k = 1, \cdots, s$, and $J_k \cap J_l = \emptyset$, for $k \neq l, k$ and $l = 1, \cdots, s$ when $s > 1$. If furthermore $\mathcal{B}$ is a B tensor, then either $\mathcal{B}$ is a strictly diagonally dominated symmetric M tensor itself, or we have (2) with $\mathcal{M}$ as a strictly diagonally dominated symmetric M tensor. An even order symmetric B_0 tensor is positive semi-definite. An even order symmetric B tensor is positive definite.

Proof. We now prove the first conclusion. Suppose that $\mathcal{B} = (b_{i_1 \cdots i_m}) \in S_{m,n}$ is a symmetric B_0 tensor. Define $\hat{J}(\mathcal{B}) \subset [n]$ as

$$\hat{J}(\mathcal{B}) = \big\{ i \in [n] : \text{there is at least one positive off-diagonal entry in the } i\text{th row of } \mathcal{B} \big\}.$$

If $\hat{J}(\mathcal{B})$ is an empty set, then $\mathcal{B}$ is a Z tensor, thus a diagonally dominated symmetric M tensor. The conclusion holds in this case. Assume that $\hat{J}(\mathcal{B})$ is not empty. Let $\mathcal{B}_1 = \mathcal{B}$. For each $i \in \hat{J}(\mathcal{B})$, let d_i be the value of the largest off-diagonal entry in the ith row of $\mathcal{B}_1$. Let

$$J_1 = \hat{J}(\mathcal{B}_1).$$

We see that $J_1 \neq \emptyset$. Let

$$h_1 = \min\{d_i : i \in J_1\}.$$

Then $h_1 > 0$.

Now consider $\mathcal{B}_2 = \mathcal{B}_1 - h_1 \mathcal{E}^{J_1}$. It is not difficult to see that $\mathcal{B}_2$ is still a symmetric B_0 tensor.

We now replace $\mathcal{B}_1$ by $\mathcal{B}_2$, and repeat this process. We see that

$$\hat{J}(\mathcal{B}_2) = \big\{ i \in [n] : \text{there is at least one positive off-diagonal entry in the } i\text{th row of } \mathcal{B}_2 \big\}$$

is a proper subset of $\hat{J}(\mathcal{B}_1)$. Repeat this process until $\hat{J}(B_{s+1}) = \emptyset$. Let $\mathcal{M} = B_{s+1}$. We see that (2) holds. Then we have

$$\hat{J}(\mathcal{B}_{k+1}) = \hat{J}(\mathcal{B}_k) \setminus J_k,$$

for $k \in [s]$. Thus, $J_k \cap J_l = \emptyset$, for $k \neq l, k$ and $l = 1, \cdots, s$ when $s > 1$. This proves the first conclusion.

Similarly, we may prove the second conclusion, i.e., if $\mathcal{B}$ is a B tensor, then either $\mathcal{B}$ itself is a strictly diagonally dominated symmetric M tensor, or in (2), $\mathcal{M}$ is a strictly diagonally dominated symmetric M tensor.

Suppose now $\mathcal{B}$ is a symmetric B_0 tensor and m is even. If $\mathcal{B}$ itself is a diagonally dominated symmetric M tensor, then it is positive semi-definite by Theorem 3. Otherwise, (2) holds with $s > 0$. Let $\mathbf{x} \in \Re^n$. Then by (2),

$$\mathcal{B}\mathbf{x}^m = \mathcal{M}\mathbf{x}^m + \sum_{k=1}^{s} h_k \mathcal{E}^{J_k}\mathbf{x}^m = \mathcal{M}\mathbf{x}^m + \sum_{k=1}^{s} h_k \|\mathbf{x}_{J_k}\|_m^m \geq \mathcal{M}\mathbf{x}^m \geq 0,$$

as by Theorem 3, a diagonally dominated symmetric M tensor is positive semi-definite. This proves the third conclusion.

The fourth conclusion can be proved similarly. $\square$

For non-symmetric B and B_0 tensors, some decomposition results may still be obtained. However, in this case, we cannot establish positive definiteness or semi-definiteness results as Theorems 1 and 3 cannot be applied to non-symmetric tensors.

By this theorem and Theorem 1, we have the following corollary.

Corollary 1. *All the H-eigenvalues of an even order symmetric B_0 tensor are non-negative. All the H-eigenvalues of an even order symmetric B tensor are positive.*

5. Final remarks and further questions

Theorem 4 gives an easily checkable sufficient condition for positive definite and semi-definite tensors. It is much more general compared with Theorem 3. The proof technique of Theorem 4 is totally different that in the B matrix literature [14,15]. It decomposes a symmetric B tensor as the sum of two kinds of somewhat basic tensors: strictly diagonally dominated symmetric M tensors and positive multiples of partially all one tensors.

Question 1. Can we apply this technique to give more general sufficient conditions for positive definite and semi-definite tensors?

In [26], it was proved that an even order symmetric tensor is positive definite if and only if it is a P tensor, and an even order symmetric tensor is positive semi-definite if and only if it is a P_0 tensor. Thus, an even order symmetric B tensor is a P tensor and an even order symmetric B_0 tensor is a P_0 tensor.

Question 2. Can we show that an even order non-symmetric B tensor is a P tensor and an even order non-symmetric B_0 tensor is a P_0 tensor? After the early draft of this paper at arXiv, Yuan and You [28] gave a counter example to answer this question.

In the literature, we know that several classes of tensors have the following two properties:

a). If the order is even, then they are positive semi-definite;
b). If the order is odd, then their H-eigenvalues, if exist, are non-negative.

This includes diagonally dominated tensors discussed in Section 2 of this paper, complete Hankel tensors and strong Hankel tensors [19], completely positive tensors [21] and P_0 tensors [26]. Some of them guarantee that H-eigenvalues exist even when the order is odd.

Question 3. Does an odd order symmetric B_0 tensor always have H-eigenvalues? If such H-eigenvalues exist, are they always non-negative?

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References

[1] N.K. Bose, A.R. Modaress, General procedure for multivariable polynomial positivity with control applications, IEEE Trans. Automat. Control AC21 (1976) 596–601.
[2] Y. Chen, Y. Dai, D. Han, W. Sun, Positive semidefinite generalized diffusion tensor imaging via quadratic semidefinite programming, SIAM J. Imaging Sci. 6 (2013) 1531–1552.
[3] R.W. Cottle, J.S. Pang, R.E. Stone, The Linear Complementarity Problem, Academic Press, Boston, 1992.
[4] W. Ding, L. Qi, Y. Wei, M-tensors and nonsingular M-tensors, Linear Algebra Appl. 439 (2013) 3264–3278.
[5] M.A. Hasan, A.A. Hasan, A procedure for the positive definiteness of forms of even-order, IEEE Trans. Automat. Control 41 (1996) 615–617.
[6] J. He, T.Z. Huang, Inequalities for M-tensors, J. Inequal. Appl. 2014 (2014) 114.
[7] S. Hu, Z. Huang, H. Ni, L. Qi, Positive definiteness of diffusion kurtosis imaging, Inverse Probl. Imaging 6 (2012) 57–75.
[8] S. Hu, L. Qi, Algebraic connectivity of an even uniform hypergraph, J. Comb. Optim. 24 (2012) 564–579.
[9] S. Hu, L. Qi, The eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors of a uniform hypergraph, Discrete Appl. Math. 169 (2014) 140–151.
[10] S. Hu, L. Qi, J. Shao, Cored hypergraphs, power hypergraphs and their Laplacian eigenvalues, Linear Algebra Appl. 439 (2013) 2980–2998.
[11] S. Hu, L. Qi, J. Xie, The largest Laplacian and signless Laplacian eigenvalues of a uniform hypergraph, arXiv:1304.1315, April 2013.
[12] E.I. Jury, M. Mansour, Positivity and nonnegativity conditions of a quartic equation and related problems, IEEE Trans. Automat. Control AC26 (1981) 444–451.
[13] G. Li, L. Qi, G. Yu, The Z-eigenvalues of a symmetric tensor and its application to spectral hypergraph theory, Numer. Linear Algebra Appl. 20 (2013) 1001–1029.

[14] J.M. Peña, A class of P-matrices with applications to the localization of the eigenvalues of a real matrix, SIAM J. Matrix Anal. Appl. 22 (2001) 1027–1037.
[15] J.M. Peña, On an alternative to Gerschgorin circles and ovals of Cassini, Numer. Math. 95 (2003) 337–345.
[16] B. Reznick, Some concrete aspects of Hilbert's 17th problem, in: Contemp. Math., vol. 253, American Mathematical Society, 2000, pp. 251–272.
[17] L. Qi, Eigenvalues of a real supersymmetric tensor, J. Symbolic Comput. 40 (2005) 1302–1324.
[18] L. Qi, H^+-eigenvalues of Laplacian and signless Laplacian tensors, Commun. Math. Sci. 12 (2014) 1045–1064.
[19] L. Qi, Hankel tensors: associated Hankel matrices and Vandermonde decomposition, Commun. Math. Sci. 12 (2014).
[20] L. Qi, J. Shao, Q. Wang, Regular uniform hypergraphs, s-cycles, s-paths and their largest Laplacian H-eigenvalues, Linear Algebra Appl. 443 (2014) 215–227.
[21] L. Qi, C. Xu, Y. Xu, Nonnegative tensor factorization, completely positive tensors and an Hierarchically elimination algorithm, arXiv:1305.5344v1, May 2013.
[22] L. Qi, G. Yu, E.X. Wu, Higher order positive semi-definite diffusion tensor imaging, SIAM J. Imaging Sci. 3 (2010) 416–433.
[23] L. Qi, G. Yu, Y. Xu, Nonnegative diffusion orientation distribution function, J. Math. Imaging Vision 45 (2013) 103–113.
[24] N.Z. Shor, Nondifferentiable Optimization and Polynomial Problems, Kluwer Academic Publishers, Boston, 1998.
[25] Y. Song, L. Qi, Infinite and finite dimensional Hilbert tensors, Linear Algebra Appl. 451 (2014) 1–14.
[26] Y. Song, L. Qi, An initial study on P, P_0, B and B_0 tensors, arXiv:1403.1118, March 2014.
[27] F. Wang, L. Qi, Comments on 'Explicit criterion for the positive definiteness of a general quartic form', IEEE Trans. Automat. Control 50 (2005) 416–418.
[28] P. Yuan, L. You, Some remarks on P, P_0, B and B_0 tensors, Manuscript, School of Mathematics, South China Normal University, May 2014.
[29] L. Zhang, L. Qi, G. Zhou, M-tensors and some applications, SIAM J. Matrix Anal. Appl. 35 (2014) 437–452.