E-Determinants of Tensors

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Abstract

We generalize the concept of the *symmetric hyperdeterminants* for symmetric tensors to the *E-determinants* for general tensors. We show that the E-determinant inherits many properties of the determinant of a matrix. These properties include: solvability of polynomial systems, the E-determinant of the composition of tensors, product formula for the E-determinant of a block tensor, Hadamard's inequality, Geršgrin's inequality and Minikowski's inequality. As a simple application, we show that if the leading coefficient tensor of a polynomial system is a triangular tensor with nonzero diagonal elements, then the system definitely has a solution. We investigate the characteristic polynomial of a tensor through the E-determinant. Explicit formulae for the coefficients of the characteristic polynomial are given when the dimension is two.

Key words: Tensor, eigenvalue, determinant, characteristic polynomial

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1 Introduction

Eigenvalues of tensors, proposed by Qi [26] and Lim [20] independently in 2005, have attracted much attention in the literature and found various applications in science and engineering, see [2, 3, 6, 7, 8, 17, 19, 21, 25, 27, 28, 29, 30, 31, 32, 36, 37] and references therein. The concept of *symmetric hyperdeterminant* was introduced by Qi [26] to investigate the eigenvalues of a symmetric tensor. Let $\mathcal{T} = (t_{i_1 \ldots i_m})$ be an m-th order n-dimensional tensor, $\mathbf{x} = (x_i) \in \mathbb{C}^n$ (the n-dimensional complex space) and $\mathcal{T}\mathbf{x}^{m-1}$ be an n-dimensional vector with its i-th element being $\sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} t_{ii_2 \ldots i_m} x_{i_2} \cdots x_{i_m}$. Then, when $\mathcal{T}$ is symmetric, its symmetric hyperdeterminant is the resultant $\mathrm{Res}(\mathcal{T}\mathbf{x}^{m-1})$ (for the definition of the resultant, see the next section). The symmetric hyperdeterminant of a symmetric tensor is equal to the product of all of the eigenvalues of that tensor [26]. Recently, Li, Qi and Zhang [19] proved that the constant term of the E-characteristic polynomial of tensor $\mathcal{T}$ (not necessarily symmetric) is a power of the resultant $\mathrm{Res}(\mathcal{T}\mathbf{x}^{m-1})$. They further found that the resultant $\mathrm{Res}(\mathcal{T}\mathbf{x}^{m-1})$ is an invariant of $\mathcal{T}$ under the orthogonal linear transformation group. Li, Qi and Zhang [19] pointed out that the resultant $\mathrm{Res}(\mathcal{T}\mathbf{x}^{m-1})$ deserves further study, since it has close relation to the eigenvalue theory of tensors. In this paper, we study $\mathrm{Res}(\mathcal{T}\mathbf{x}^{m-1})$ systematically. Note that $\mathrm{Res}(\mathcal{T}\mathbf{x}^{m-1})$ is different from the hyperdeterminant investigated in [1, 4, 5, 10, 12, 13, 14, 22, 23, 24, 33, 35]. We now give the following definition.

Definition 1.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ (the space of m-th order n-dimensional tensors). Then its* E-determinant*, denoted by Edet$(\mathcal{T})$, is defined as the resultant of the polynomial system $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$.*

Here we use the prefix "E" to highlight the relation of $\mathrm{Edet}(\mathcal{T})$ with the eigenvalue theory of tensors.

The rest of this paper is organized as follows.

In the next section, we present some basic properties of the E-determinant. Then, in Section 3, we show that the solvability of a polynomial system is characterized by the E-determinant of the leading coefficient tensor of the polynomial system.

A tensor $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ induces a homogenous polynomial map $\hat{\mathcal{T}} : \mathbb{C}^n \to \mathbb{C}^n$ as $\mathbf{x} \mapsto \mathcal{T}\mathbf{x}^{m-1}$. Let $\mathcal{U} \in \mathbb{T}(\mathbb{C}^n, p)$ and $\mathcal{V} \in \mathbb{T}(\mathbb{C}^n, q)$ for $p, q \geq 2$, and homogenous polynomial maps $\hat{\mathcal{U}}$ and $\hat{\mathcal{V}}$ be induced by $\mathcal{U}$ and $\mathcal{V}$ respectively. In Section 4, we show that the composition of $\hat{\mathcal{U}}$ and $\hat{\mathcal{V}}$ is another homogenous polynomial map $\hat{\mathcal{W}}$ induced by a tensor $\mathcal{W} \in \mathbb{T}(\mathbb{C}^n, (p-1)(q-1)+1)$. We show that $\mathrm{Edet}(\mathcal{W}) = 0$ if and only if $\mathrm{Edet}(\mathcal{U})\mathrm{Edet}(\mathcal{V}) = 0$.

We conjecture that

$$\mathrm{Edet}(\mathcal{W}) = (\mathrm{Edet}(\mathcal{U}))^{(q-1)^{n-1}} (\mathrm{Edet}(\mathcal{V}))^{(p-1)^n}$$

and prove that this conjecture is true when $\min\{p, q\} = 2$.

Block tensors are discussed in Section 5. We give an expression of the E-determinant of a tensor which has an "upper triangular structure", based on the E-determinants of its two diagonal sub-tensors.

As a simple application of the E-determinant theory, in Section 6, we show that if the leading coefficient tensor of a polynomial system is a triangular tensor with nonzero diagonal elements, then the system definitely has a solution.

Based upon a result of Morozov and Shakirov [23], in Section 7, we give a trace formula for the E-determinant. This formula involves some differential operators. Using this formula, we will establish an explicit formula for the E-determinant when the dimension is two. As this needs to use some results in Section 8, we will do this in Section 9.

The E-determinant contributes to the characteristic polynomial theory of tensors. In Section 8, we analyze various related properties of the characteristic polynomial and the E-determinant. Especially, a trace formula for the characteristic polynomial is presented, which has potential applications in various areas, such as scientific computing and geometrical analysis of eigenvalues. We also generalize the eigenvalue representation for the determinant of a matrix to the E-determinant of a tensor. Under an assumption (Assumption 8.1), we transform the positive semidefiniteness problem of an even order tensor to a computable condition (see Proposition 8.1).

In Section 9, we give explicit formulae for the E-determinant and the characteristic polynomial when the dimension is two.

We generalize some inequalities of the determinant for a matrix to the E-determinant for a tensor in Section 10. Among them, we present generalizations of Hadamard's inequality, Geršgrin's inequality and Minikowski's inequality. These inequalities give estimations for the E-determinant in terms of the entries of the underlying tensor.

Some final remarks are given in Section 11.

The following is the notation that is used in the sequel. Scalars are written as lowercase letters ($\lambda, a, \ldots$); vectors are written as bold lowercase letters ($\mathbf{x} = (x_i), \ldots$); matrices are written as italic capitals ($A = (a_{ij}), \ldots$); tensors are written as calligraphic letters ($\mathcal{T} = (t_{i_1 \ldots i_m}), \ldots$); and, sets are written as blackboard bold letters ($\mathbb{T}, \mathbb{S}, \ldots$).

Given a ring $\mathbb{K}$ (hereafter, we mean a commutative ring with 1 [18, Pages 83-84], e.g., $\mathbb{C}$), we denote by $\mathbb{K}[\mathbb{E}]$ the polynomial ring consists of polynomials in the set of indeterminate $\mathbb{E}$ with coefficients in $\mathbb{K}$. Especially, we denote by $\mathbb{K}[\mathcal{T}]$ the polynomial ring consists of polynomials in indeterminate $\{t_{i_1\dots i_m}\}$ with coefficients in $\mathbb{K}$, and similarly for $\mathbb{K}[\lambda]$, $\mathbb{K}[A]$, $\mathbb{K}[\lambda, \mathcal{T}]$, etc.

For a matrix A, A^T denotes its transpose and $\mathrm{Tr}(A)$ denotes its trace. We denote by $\mathbb{N}_+$ the set of all positive integers and $\mathbf{e}_i$ the i-th identity vector, i.e., the i-th column vector of the identity matrix E. Throughout this paper, unless stated otherwise, integers $m, n \geq 2$ and tensors refer to m-th order n-dimensional tensors with entries in $\mathbb{C}$.

2 Basic Properties of the E-Determinant

Let $\mathcal{E}$ be the identity tensor of appropriate order and dimension, e.g., $e_{i_1\dots i_m} = 1$ if and only if $i_1 = \cdots = i_m \in \{1, \dots, n\}$, and zero otherwise. The following definitions were introduced by Qi [26].

Definition 2.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. For some $\lambda \in \mathbb{C}$, if system $(\lambda\mathcal{E} - \mathcal{T})\,\mathbf{x}^{m-1} = \mathbf{0}$ has a solution $\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, then λ is called an eigenvalue of tensor $\mathcal{T}$ and $\mathbf{x}$ an eigenvector of $\mathcal{T}$ associated with λ.*

We denote by $\sigma(\mathcal{T})$ the set of all eigenvalues of tensor $\mathcal{T}$.

Definition 2.2 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. The E-determinant of $\lambda\mathcal{E} - \mathcal{T}$ which is a polynomial in $(\mathbb{C}[\mathcal{T}])\,[\lambda]$, denoted by $\psi(\lambda)$, is called the* characteristic polynomial *of tensor $\mathcal{T}$.*

If λ is a root of $\psi(\lambda)$ of multiplicity s, then we call s the *algebraic multiplicity* of eigenvalue λ. Denote by $\mathbb{V}(f)$ the variety of the principal ideal $\langle f \rangle$ generated by f [10, 11, 18]. Then, we have the following result.

Theorem 2.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. Then $\psi \in \mathbb{C}[\lambda, \mathcal{T}]$ is homogenous of degree $n(m-1)^{n-1}$ and*

$$\mathbb{V}(\psi(\lambda)) = \sigma(\mathcal{T}). \tag{1}$$

When $\mathcal{T}$ is symmetric, Qi proved (1) in [26, Theorem 1(a)].

For $f \in \mathbb{K}[\mathbf{x}]$, we denote by $\deg(f)$ the degree of f. If every monomial in f has degree $\deg(f)$, then f is called homogenous of degree $\deg(f)$. Given a system of polynomials $h := \{h_1, \dots, h_n\}$ with $h_i \in \mathbb{C}[\mathbf{x}]$ being homogenous of degree $r_i \in \mathbb{N}_+$. The *resultant* of the polynomial system h, denoted by $\mathrm{Res}_{r_1,\dots,r_n}(h)$ or simply $\mathrm{Res}(h)$, is defined as an irreducible polynomial in the coefficients of h such that $\mathrm{Res}(h) = 0$ if and only if $h = \mathbf{0}$ has a solution in $\mathbb{C}^n \setminus \{\mathbf{0}\}$. Furthermore, $\mathrm{Res}(h)$ is homogenous of degree $\Pi_{i \neq j} r_i$ in the coefficients of h_j for every $j \in \{1, \dots, n\}$. So, it is homogeneous of total degree $\sum_{i=1}^n \Pi_{j \neq i}^n r_j$ [13, Proposition 13.1.1], see also [23, Page 713]. These, together with Definition 1.1 and [10, Theorem 3.2.3(b)], immediately imply the following proposition.

Proposition 2.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. Then,*

(i) For every $i \in \{1, \dots, n\}$, let $\mathbb{K}_i := \mathbb{C}[\{t_{ji_2\dots i_m} \mid j, i_2, \dots, i_m = 1, \dots, n,\ j \neq i\}]$. Then Edet($\mathcal{T}$) $\in \mathbb{K}_i[\{t_{ii_2\dots i_m} \mid i_2, \dots, i_m = 1, \dots, n\}]$ is homogenous of degree $(m-1)^{n-1}$.

(ii) Edet($\mathcal{T}$) $\in \mathbb{C}[\mathcal{T}]$ is irreducible and homogeneous of degree $n(m-1)^{n-1}$.

(iii) Edet($\mathcal{E}$) $= 1$.

By Proposition 2.1, we have the following corollary.

Corollary 2.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. If for some i, $t_{ii_2\dots i_m} = 0$ for all $i_2, \dots, i_m \in \{1, \dots, n\}$, then Edet($\mathcal{T}$) $= 0$. In particular, the E-determinant of the zero tensor is zero.*

Proof. Let $\mathbb{K}_i := \mathbb{C}[\{t_{ji_2\dots i_m} \mid j, i_2, \dots, i_m = 1, \dots, n,\ j \neq i\}]$. Then by Proposition 2.1 (i) $\mathrm{Edet}(\mathcal{T})$ is a homogenous polynomial in the variable set $\{t_{ii_2\dots i_m} \mid i_2, \dots, i_m = 1, \dots, n\}$ with coefficients in the ring $\mathbb{K}_i$. As $t_{ii_2\dots i_m} = 0$ for all $i_2, \dots, i_m \in \{1, \dots, n\}$ by the assumption, $\mathrm{Edet}(\mathcal{T}) = 0$ as desired. □

By Proposition 2.1 (ii), we have another corollary as follows.

Corollary 2.2 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ and $\alpha \in \mathbb{C}$. Then*

$$Edet(\alpha\mathcal{T}) = (\alpha)^{n(m-1)^{n-1}}\, Edet(\mathcal{T}).$$

3 Solvability of Polynomial Equations

Let matrix $A \in \mathbb{T}(\mathbb{C}^n, 2)$, we know that

- $\mathrm{Det}(A) = 0$ if and only if $A\mathbf{x} = \mathbf{0}$ has a solution in $\mathbb{C}^n \setminus \{\mathbf{0}\}$; and,
- $\mathrm{Det}(A) \neq 0$ if and only if $A\mathbf{x} = \mathbf{b}$ has a unique solution in $\mathbb{C}^n$ for every $\mathbf{b} \in \mathbb{C}^n$.

We generalize such a result to the E-determinant and polynomial system in this section.

Theorem 3.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. Then,*

(i) Edet($\mathcal{T}$) $= 0$ if and only if $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$ has a solution in $\mathbb{C}^n \setminus \{\mathbf{0}\}$.

(ii) Edet($\mathcal{T}$) $\neq 0$ only if $\mathcal{T}\mathbf{x}^{m-1} = \mathcal{B}^{m-1}\mathbf{x}^{m-2} + \cdots + \mathcal{B}^3\mathbf{x}^2 + A\mathbf{x} + \mathbf{b}$ has a solution in $\mathbb{C}^n$ for every $\mathbf{b} \in \mathbb{C}^n$, $A \in \mathbb{T}(\mathbb{C}^n, 2)$, and $\mathcal{B}^j \in \mathbb{T}(\mathbb{C}^n, j)$ for $j = 3, \ldots, m-1$.

Proof. (i) It follows from Definition 1.1 immediately.

(ii) Suppose that $\mathrm{Edet}(\mathcal{T}) \neq 0$. For any $\mathbf{b} \in \mathbb{C}^n$, $A \in \mathbb{T}(\mathbb{C}^n, 2)$, and $\mathcal{B}^j \in \mathbb{T}(\mathbb{C}^n, j)$ for $j = 3, \ldots, m-1$, we define tensor $\mathcal{U} \in \mathbb{T}(\mathbb{C}^{n+1}, m)$ as follows:

$$u_{i_1 i_2 \ldots i_m} := \begin{cases} t_{i_1 i_2 \ldots i_m} & \forall i_j \in \{1, \ldots, n\},\ j \in \{1, \ldots, m\}, \\ -b_{i_1} & \forall i_1 \in \{1, \ldots, n\} \text{ and } i_2 = \cdots = i_m = n+1, \\ -a_{i_1 i_2} & \forall i_1, i_2 \in \{1, \ldots, n\} \text{ and } i_3 = \cdots = i_m = n+1, \\ -b^k_{i_1 \ldots i_k} & \forall i_1, \ldots, i_k \in \{1, \ldots, n\} \text{ and } i_{k+1} = \cdots = i_m = n+1, \\ & \quad \forall k = 3, \ldots, m-1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

By Corollary 2.1, we have that $\mathrm{Edet}(\mathcal{U}) = 0$. Hence, by (i), there exists $\mathbf{y} := (\mathbf{x}^T, \alpha)^T \in \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ such that $\mathcal{U}\mathbf{y}^{m-1} = \mathbf{0}$. Consequently, by (2) and the first n equations in $\mathcal{U}\mathbf{y}^{m-1} = \mathbf{0}$, we know that

$$\mathcal{T}\mathbf{x}^{m-1} - \alpha\mathcal{B}^{m-1}\mathbf{x}^{m-2} - \cdots - \alpha^{m-3}\mathcal{B}^3\mathbf{x}^2 - \alpha^{m-2}A\mathbf{x} - \alpha^{m-1}\mathbf{b} = \mathbf{0}. \tag{3}$$

Furthermore, we claim that $\alpha \neq 0$. Otherwise, from (3), $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$ which means $\mathrm{Edet}(\mathcal{T}) = 0$ by (i). It is a contradiction. Hence, from (3) we know that $\frac{\mathbf{x}}{\alpha}$ is a solution to

$$\mathcal{T}\mathbf{x}^{m-1} = \mathcal{B}^{m-1}\mathbf{x}^{m-2} + \cdots + \mathcal{B}^3\mathbf{x}^2 + A\mathbf{x} + \mathbf{b}.$$

The proof is complete. □

So, like the determinants of linear equations, the E-determinants are criterions for the solvability of non-linear polynomial equations. It is interesting to investigate whether $\mathcal{T}\mathbf{x}^{m-1} = \mathcal{B}^{m-1}\mathbf{x}^{m-2} + \cdots + \mathcal{B}^3\mathbf{x}^2 + A\mathbf{x} + \mathbf{b}$ has only finitely many solutions whenever $\mathrm{Edet}(\mathcal{T}) \neq 0$.

4 Composition of Homogenous Polynomial Maps

Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. Then it induces a homogenous polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ defined as $\mathbf{x} \mapsto \mathcal{T}\mathbf{x}^{m-1}$. We denote this homogenous polynomial map as $\hat{\mathcal{T}}$. Let $\mathcal{U} \in \mathbb{T}(\mathbb{C}^n, p)$ and $\mathcal{V} \in \mathbb{T}(\mathbb{C}^n, q)$ with $p, q \geq 2$. Then, the composition map denoted as $\hat{\mathcal{U}} \circ \hat{\mathcal{V}}$ in the usual sense is well-defined. Actually, $\hat{\mathcal{U}} \circ \hat{\mathcal{V}} : \mathbb{C}^n \to \mathbb{C}^n$ and $\left(\hat{\mathcal{U}} \circ \hat{\mathcal{V}}\right)(\mathbf{x}) := \hat{\mathcal{U}}\left(\hat{\mathcal{V}}(\mathbf{x})\right) = \mathcal{U}\left(\mathcal{V}\mathbf{x}^{q-1}\right)^{p-1}$. Then, it is easy to see that

$$\left[\left(\hat{\mathcal{U}} \circ \hat{\mathcal{V}}\right)(\mathbf{x})\right]_i := \sum_{i_2,\dots,i_p=1}^{n} u_{ii_2\dots i_p}\left(\mathcal{V}\mathbf{x}^{q-1}\right)_{i_2}\cdots\left(\mathcal{V}\mathbf{x}^{q-1}\right)_{i_p}, \quad \forall \mathbf{x} \in \mathbb{C}^n,\ \forall i = 1, \dots, n. \tag{4}$$

By the definition of $\hat{\mathcal{T}}$ for tensor $\mathcal{T}$, we can see that there exists a tensor $\mathcal{W}$ such that $\hat{\mathcal{W}} = \hat{\mathcal{U}} \circ \hat{\mathcal{V}}$. Furthermore, it is easy to get the following result.

Proposition 4.1 *Let $\mathcal{U} \in \mathbb{T}(\mathbb{C}^n, p)$ and $\mathcal{V} \in \mathbb{T}(\mathbb{C}^n, q)$ with $p, q \geq 2$. Then, with $\mathcal{W} \in \mathbb{T}(\mathbb{C}^n, 1 + (p-1)(q-1))$ defined as*

$$w_{ij_{(i_2,2)}\cdots j_{(i_2,q)}\cdots j_{(i_p,2)}\cdots j_{(i_p,q)}} := \sum_{i_2,\dots,i_p=1}^{n} u_{ii_2\dots i_p} v_{i_2 j_{(i_2,2)}\cdots j_{(i_2,q)}} \cdots v_{i_p j_{(i_p,2)}\cdots j_{(i_p,q)}}, \tag{5}$$

we have $\hat{\mathcal{W}} = \hat{\mathcal{U}} \circ \hat{\mathcal{V}}$. So, it is reasonable to define the composition of $\mathcal{U}$ and $\mathcal{V}$ as $\mathcal{U} \circ \mathcal{V} := \mathcal{W}$.

Note that when $p = q = 2$, i.e., both $\mathcal{U}$ and $\mathcal{V}$ are matrices. Then the composition $\mathcal{U} \circ \mathcal{V}$ reduces to the usual multiplication of matrices, and moreover it is uniquely defined. Nonetheless, when $m > 2$, there exist many $\mathcal{W}$'s satisfying $\hat{\mathcal{W}} = \hat{\mathcal{U}} \circ \hat{\mathcal{V}}$. Hence, we need (5) to make the composition of $\mathcal{U}$ and $\mathcal{V}$ uniquely defined.

Proposition 4.2 *Let $\mathcal{U} \in \mathbb{T}(\mathbb{C}^n, p)$ and $\mathcal{V} \in \mathbb{T}(\mathbb{C}^n, q)$ with $p, q \geq 2$. Then, we have*

$$\mathit{Edet}(\mathcal{U} \circ \mathcal{V}) = 0 \quad \Longleftrightarrow \quad \mathit{Edet}(\mathcal{U})\mathit{Edet}(\mathcal{V}) = 0.$$

Proof. The proof for "$\Leftarrow$": Suppose that $\mathrm{Edet}(\mathcal{V}) = 0$, by Theorem 3.1 (i), there exists $\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ such that $\mathcal{V}\mathbf{x}^{q-1} = \mathbf{0}$. From (4), it is easy to see that $(\mathcal{U} \circ \mathcal{V})\,\mathbf{x}^{(p-1)(q-1)} = \mathbf{0}$. By Theorem 3.1 (i) again, we know that $\mathrm{Edet}(\mathcal{U} \circ \mathcal{V}) = 0$.

Now, if $\mathrm{Edet}(\mathcal{V}) \neq 0$, then $\mathrm{Edet}(\mathcal{U}) = 0$ by the hypothesis, which implies that there exists $\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ such that $\mathcal{U}\mathbf{x}^{p-1} = \mathbf{0}$. Since $\mathrm{Edet}(\mathcal{V}) \neq 0$, by Theorem 3.1 (ii), we know that for $\mathbf{x}$ there exists $\mathbf{y} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ such that $\mathcal{V}\mathbf{y}^{q-1} = \mathbf{x}$. Consequently, by (4), we get that $(\mathcal{U} \circ \mathcal{V})\,\mathbf{y}^{(p-1)(q-1)} = \mathbf{0}$. By Theorem 3.1 (i), $\mathrm{Edet}(\mathcal{U} \circ \mathcal{V}) = 0$.

The proof for " $\Rightarrow$ ": Suppose that $\mathrm{Edet}(\mathcal{U} \circ \mathcal{V}) = 0$. By Theorem 3.1 (i), there exists $\mathbf{y} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ such that $(\mathcal{U} \circ \mathcal{V})\,\mathbf{y}^{(p-1)(q-1)} = \mathbf{0}$. If $\mathrm{Edet}(\mathcal{V}) = 0$, then we are done. If $\mathrm{Edet}(\mathcal{V}) \neq 0$, then $\mathbf{x} := \mathcal{V}\mathbf{y}^{q-1} \neq \mathbf{0}$. Consequently, it holds that $\mathcal{U}\mathbf{x}^{p-1} = (\mathcal{U} \circ \mathcal{V})\,\mathbf{y}^{(p-1)(q-1)} = \mathbf{0}$. By Theorem 3.1 (i) again, we know that $\mathrm{Edet}(\mathcal{U}) = 0$. Therefore, $\mathrm{Edet}(\mathcal{U})\mathrm{Edet}(\mathcal{V}) = 0$. $\square$

The following corollary is a direct consequence of [26, Theorem 1(b)] and Proposition 4.2.

Corollary 4.1 *Let $\mathcal{U} \in \mathbb{T}(\mathbb{C}^n, p)$ and $\mathcal{V} \in \mathbb{T}(\mathbb{C}^n, q)$ with $p, q \geq 2$. Then, $\mathcal{U} \circ \mathcal{V}$ has zero as its eigenvalue if and only if one of $\mathcal{U}$ and $\mathcal{V}$ has zero as its eigenvalue.*

We have the following conjecture.

Conjecture 4.1 *Let $\mathcal{U} \in \mathbb{T}(\mathbb{C}^n, p)$ and $\mathcal{V} \in \mathbb{T}(\mathbb{C}^n, q)$ with $p, q \geq 2$. Then, we have*

$$Edet(\mathcal{U} \circ \mathcal{V}) = (Edet(\mathcal{U}))^{(q-1)^{n-1}} (Edet(\mathcal{V}))^{(p-1)^n}.$$

When $p = q = 2$, this conjecture is true as it reduces to the Cauchy-Binet formula for matrices, i.e, for $A, B \in \mathbb{T}(\mathbb{C}^n, 2)$, it holds that $\mathrm{Det}(AB) = \mathrm{Det}(A)\mathrm{Det}(B)$ [15, Page 22]. In the remainder of this section, we show that this conjecture is true when $\min\{p, q\} = 2$.

Given a set $\mathbb{E} \subseteq \mathbb{C}^n$, we denote by $\mathbb{I}(\mathbb{E}) \subseteq \mathbb{C}[\mathbf{x}]$ the ideal of polynomials in $\mathbb{C}[\mathbf{x}]$ which vanishes identically on $\mathbb{E}$. Given a set of polynomials $\mathbb{F} := \{f_1, \ldots, f_s : f_i \in \mathbb{C}[\mathbf{x}]\}$, we denote by $\mathbb{V}(\mathbb{F}) \subseteq \mathbb{C}^n$ the variety of $\mathbb{F}$, i.e., the set of the common roots of polynomials in $\mathbb{F}$ [10, 18].

The following proposition follows from (5), Proposition 4.2 and [10, Theorem 3.3.5(a)].

Proposition 4.3 *Conjecture 4.1 is true if $p = 2$, i.e., if $G \in \mathbb{T}(\mathbb{C}^n, 2)$ and $\mathcal{V} \in \mathbb{T}(\mathbb{C}^n, q)$ for $q \geq 2$, then,*

$$Edet(G \circ \mathcal{V}) = (Det(G))^{(q-1)^{n-1}} Edet(\mathcal{V}).$$

Now, we prove the following result.

Proposition 4.4 *Conjecture 4.1 is true if $q = 2$, i.e., if $\mathcal{U} \in \mathbb{T}(\mathbb{C}^n, p)$ and $G \in \mathbb{T}(\mathbb{C}^n, 2)$ for $p \geq 2$, then*

$$Edet(\mathcal{U} \circ G) = Edet(\mathcal{U}) \left(Det(G)\right)^{(p-1)^n}.$$

Proof. If G is singular, then the conclusion follows from Proposition 4.2. We now assume that G is nonsingular. By the fact that $(\mathcal{U} \circ G)\, \mathbf{x}^{p-1} = \mathcal{U}\, (G\mathbf{x})^{p-1}$ for any $\mathbf{x} \in \mathbb{C}^n$ and $G \in \mathbb{T}(\mathbb{C}^n, 2)$, we see that

$$\mathbb{I}\left(\mathbb{V}(\mathrm{Edet}(\mathcal{U} \circ G))\right) = \mathbb{I}\left(\mathbb{V}(\mathrm{Edet}(\mathcal{U}))\right)$$

for any nonsingular $G \in \mathbb{T}(\mathbb{C}^n, 2)$. Hence, $\mathrm{Edet}(\mathcal{U} \circ G) \in \mathbb{I}\left(\mathbb{V}(\mathrm{Edet}(\mathcal{U} \circ G))\right) = \mathbb{I}\left(\mathbb{V}(\mathrm{Edet}(\mathcal{U}))\right)$. Since $\mathrm{Edet}(\mathcal{U}) \in \mathbb{C}[\mathcal{U}]$ is irreducible, by Hilbert's Nullstellensatz [11, Theorem 4.2], we have that

$$\mathrm{Edet}(\mathcal{U} \circ G) = p(\mathcal{U}, G)\mathrm{Edet}(\mathcal{U}) \tag{6}$$

for some $p(\mathcal{U}, G) \in \mathbb{C}[\mathcal{U}, G]$. Let $\mathcal{R} := \mathcal{U} \circ G$. By Proposition 2.1 (ii), $\mathrm{Edet}(\mathcal{R}) = \mathrm{Edet}(\mathcal{U} \circ G) \in \mathbb{C}[\mathcal{R}]$ is homogenous of degree $n(p-1)^{n-1}$ in variables $r_{i_1 \ldots i_m}$'s. Hence, by (5), it is homogenous of degree $n(p-1)^{n-1}$ in variables $u_{i_1 \ldots i_m}$'s and homogenous of degree $n(p-1)^n$ in variables g_{ij}'s. Since $\mathrm{Edet}(\mathcal{U}) \in \mathbb{C}[\mathcal{U}]$ is homogenous of degree $n(p-1)^{n-1}$ by Proposition 2.1 (ii) again, $p(\mathcal{U}, G)$ is independent of $\mathcal{U}$ and homogeneous of degree $n(p-1)^n$ in variables g_{ij}'s. Let $\mathcal{U} = \mathcal{E}$, it holds that

$$(\mathcal{E} \circ G)\, \mathbf{x}^{p-1} = \mathcal{E}\, (G\mathbf{x})^{p-1} = \begin{pmatrix} (\sum_{j=1}^n g_{1j} x_j)^{p-1} \\ \vdots \\ (\sum_{j=1}^n g_{nj} x_j)^{p-1} \end{pmatrix}.$$

Consequently, by [10, Theorem 3.3.2(b)], we have

$$\mathrm{Edet}(\mathcal{E} \circ G) = \left[\mathrm{Res} \begin{pmatrix} (\sum_{j=1}^n g_{1j} x_j) \\ \vdots \\ (\sum_{j=1}^n g_{nj} x_j) \end{pmatrix} \right]^{(p-1)^n} = \left(\mathrm{Det}(G)\right)^{(p-1)^n}.$$

Therefore, by Proposition 2.1 (iii) and (6), we conclude that $p(\mathcal{U}, G) = \left(\mathrm{Det}(G)\right)^{(p-1)^n}$ and complete the proof. □

The following is a direct corollary of Propositions 4.3 and 4.4.

Corollary 4.2 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ and $G \in \mathbb{T}(\mathbb{C}^n, 2)$. Then,*

$$Edet(G \circ \mathcal{T}) \left(Det(G)\right)^{(m-2)(m-1)^{n-1}} = Edet(\mathcal{T} \circ G).$$

5 Block Tensors

In the context of matrices, if a square matrix A can be partitioned as

$$A = \begin{pmatrix} B & \sharp \\ 0 & C \end{pmatrix}$$

with square sub-matrices B and C, then $\mathrm{Det}(A) = \mathrm{Det}(B)\mathrm{Det}(C)$. We now generalize this property to tensors. The following definition is straightforward.

Definition 5.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ and $1 \le k \le n$. Tensor $\mathcal{U} \in \mathbb{T}(\mathbb{C}^k, m)$ is called a* sub-tensor *of $\mathcal{T}$ associated to the index set $\{j_1, \ldots, j_k\} \subseteq \{1, \ldots, n\}$ if and only if $u_{i_1 \ldots i_m} = t_{j_{i_1} \cdots j_{i_m}}$ for all $i_1, \ldots, i_m \in \{1, \ldots, k\}$.*

Theorem 5.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. Suppose that there exists an integer k satisfying $1 \le k \le n-1$ and $t_{ii_2 \ldots i_m} = 0$ for every $i \in \{k+1, \ldots, n\}$ and all indices $i_2, \ldots, i_m$ such that $\{i_2, \ldots, i_m\} \cap \{1, \ldots, k\} \neq \emptyset$. Denote by $\mathcal{U} \in \mathbb{T}(\mathbb{C}^k, m)$ and $\mathcal{V} \in \mathbb{T}(\mathbb{C}^{n-k}, m)$ the* sub-tensors *of $\mathcal{T}$ associated to $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$, respectively. Then, it holds that*

$$Edet(\mathcal{T}) = [Edet(\mathcal{U})]^{(m-1)^{n-k}} [Edet(\mathcal{V})]^{(m-1)^k}. \tag{7}$$

Proof. We first show that

$$\mathrm{Edet}(\mathcal{T}) = 0 \iff \mathrm{Edet}(\mathcal{U})\mathrm{Edet}(\mathcal{V}) = 0. \tag{8}$$

Suppose that $\mathrm{Edet}(\mathcal{T}) = 0$. Then there exists $\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ such that $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$. Denote by $\mathbf{u} \in \mathbb{C}^k$ the vector consists of $x_1, \ldots, x_k$, and $\mathbf{v} \in \mathbb{C}^{n-k}$ the vector consists of $x_{k+1}, \ldots, x_n$. If $\mathbf{v} \neq \mathbf{0}$, then $\mathrm{Edet}(\mathcal{V}) = 0$, and otherwise $\mathrm{Edet}(\mathcal{U}) = 0$. Hence, we have

$$\mathrm{Edet}(\mathcal{T}) = 0 \implies \mathrm{Edet}(\mathcal{U})\mathrm{Edet}(\mathcal{V}) = 0.$$

Conversely, suppose that $\mathrm{Edet}(\mathcal{U})\mathrm{Edet}(\mathcal{V}) = 0$. If $\mathrm{Edet}(\mathcal{U}) = 0$, then there exists $\mathbf{u} \in \mathbb{C}^k \setminus \{\mathbf{0}\}$ such that $\mathcal{U}\mathbf{u}^{m-1} = \mathbf{0}$. Denote $\mathbf{x} := (\mathbf{u}^T, \mathbf{0})^T \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, then $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$, which implies $\mathrm{Edet}(\mathcal{T}) = 0$ by Theorem 3.1 (i). If $\mathrm{Edet}(\mathcal{U}) \neq 0$, then $\mathrm{Edet}(\mathcal{V}) = 0$, which implies that there exists $\mathbf{v} \in \mathbb{C}^{n-k} \setminus \{\mathbf{0}\}$ such that $\mathcal{V}\mathbf{v}^{m-1} = \mathbf{0}$. Now, by vector $\mathbf{v}$ and tensor $\mathcal{T}$, we construct vector $\mathbf{b} \in \mathbb{C}^k$ as

$$b_i := \sum_{j_2, \ldots, j_m = k+1}^{n} t_{ij_2 \ldots j_m} v_{j_2 - k} \cdots v_{j_m - k}, \ \forall i \in \{1, \ldots, k\}; \tag{9}$$

matrix $A \in \mathbb{T}(\mathbb{C}^k, 2)$ as

$$a_{ij} := \sum_{(q_2,\dots,q_m)\in\mathbb{D}} t_{iq_2\dots q_m} \prod_{q_w>k} v_{q_w-k},\ \forall i,j \in \{1,\dots,k\} \tag{10}$$

with $\mathbb{D} := \{(q_2,\dots,q_m) \mid q_p = j,\ \text{for some } p = 2,\dots,m,\ \text{and } q_l = k+1,\dots,n, l \neq p\}$; and, tensors $\mathcal{B}^s \in \mathbb{T}(\mathbb{C}^k, s)$ for $s = 3,\dots,m-1$ as

$$b^s_{ij_2\dots j_s} := \sum_{(q_2,\dots,q_m)\in\mathbb{D}^s} t_{iq_2\dots q_m} \prod_{q_w>k} v_{q_w-k},\ \forall i, j_2,\dots,j_m \in \{1,\dots,k\} \tag{11}$$

with

$$\begin{aligned}\mathbb{D}^s \ := \ & \{(q_2,\dots,q_m) \mid \{q_{t_2},\dots,q_{t_s}\} = \{j_2,\dots,j_s\} \text{ for some pairwise different } t_2,\dots,t_s \\ & \text{in } \{2,\dots,m\},\ \text{and } q_l = k+1,\dots,n, l \notin \{t_2,\dots,t_s\}\}.\end{aligned}$$

Since $\mathrm{Edet}(\mathcal{U}) \neq 0$, by Theorem 3.1 (ii),

$$\mathcal{U}\mathbf{u}^{m-1} + \mathcal{B}^{m-1}\mathbf{u}^{m-2} + \cdots + \mathcal{B}^3\mathbf{u}^2 + A\mathbf{u} + \mathbf{b} = \mathbf{0}$$

has a solution $\mathbf{u} \in \mathbb{C}^k$. Let $\mathbf{x} := (\mathbf{u}^T, \mathbf{v}^T)^T \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ as $\mathbf{v} \in \mathbb{C}^{n-k} \setminus \{\mathbf{0}\}$. By (9), (10) and (11), we have that

$$\left(\mathcal{T}\mathbf{x}^{m-1}\right)_i = \left(\mathcal{U}\mathbf{u}^{m-1} + \mathcal{B}^{m-1}\mathbf{u}^{m-2} + \cdots + \mathcal{B}^3\mathbf{u}^2 + A\mathbf{u} + \mathbf{b}\right)_i = 0,\ \forall i = 1,\dots,k.$$

Furthermore,

$$\left(\mathcal{T}\mathbf{x}^{m-1}\right)_i = \left(\mathcal{V}\mathbf{v}^{m-1}\right)_i = 0,\ \forall i = k+1,\dots,n.$$

Consequently, $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$ which implies $\mathrm{Edet}(\mathcal{T}) = 0$ by Theorem 3.1 (i).

Hence, we proved (8). In the following, we show that (7) holds. Since $\mathrm{Edet}(\mathcal{U}), \mathrm{Edet}(\mathcal{V}) \in \mathbb{C}[\mathcal{T}]$, by (8), we have

$$\mathbb{V}(\mathrm{Edet}(\mathcal{U})\mathrm{Edet}(\mathcal{V})) = \mathbb{V}(\mathrm{Edet}(\mathcal{T})),$$

which implies that

$$\mathbb{I}(\mathbb{V}(\mathrm{Edet}(\mathcal{T}))) = \mathbb{I}(\mathbb{V}(\mathrm{Edet}(\mathcal{U})\mathrm{Edet}(\mathcal{V}))).$$

By Proposition 2.1 (ii), both $\mathrm{Edet}(\mathcal{U}) \in \mathbb{C}[\mathcal{U}]$ and $\mathrm{Edet}(\mathcal{V}) \in \mathbb{C}[\mathcal{V}]$ are irreducible. These, together with $\mathrm{Edet}(\mathcal{T}) \in \mathbb{I}(\mathbb{V}(\mathrm{Edet}(\mathcal{T}))) = \mathbb{I}(\mathbb{V}(\mathrm{Edet}(\mathcal{U})\mathrm{Edet}(\mathcal{V}))) = \langle \mathrm{Edet}(\mathcal{U})\mathrm{Edet}(\mathcal{V})\rangle$ and Hilbert's Nullstellensatz [11, Theorem 4.2], imply

$$\mathrm{Edet}(\mathcal{T}) = p(\mathcal{T})\mathrm{Edet}(\mathcal{U})\mathrm{Edet}(\mathcal{V}) \tag{12}$$

for some $p(\mathcal{T}) \in \mathbb{C}[\mathcal{T}]$. Now, suppose that

$$\mathrm{Edet}(\mathcal{T}) = q(\mathcal{T}) \left(\mathrm{Edet}(\mathcal{U})\right)^{r_1} \left(\mathrm{Edet}(\mathcal{V})\right)^{r_2} \tag{13}$$

for the maximal $r_1, r_2 \in \mathbb{N}_+$ and $q(\mathcal{T}) \in \mathbb{C}[\mathcal{T}]$. Suppose that $q(\mathcal{T}) = \prod_{i=1}^{k} (q_i(\mathcal{T}))^{k_i}$ is the decomposition of q into irreducible components. Then, by (13), we have $\mathrm{Edet}(\mathcal{U})\mathrm{Edet}(\mathcal{V}) \in \mathbb{I}(\mathbb{V}(\mathrm{Edet}(\mathcal{U})\mathrm{Edet}(\mathcal{V}))) = \mathbb{I}(\mathbb{V}(\mathrm{Edet}(\mathcal{T}))) = \langle \prod_{i=1}^{k} q_i(\mathcal{T})\mathrm{Edet}(\mathcal{U})\mathrm{Edet}(\mathcal{V})\rangle$. This, together with Hilbert's Nullstellensatz [11, Theorem 4.2], implies

$$\left(\mathrm{Edet}(\mathcal{U})\mathrm{Edet}(\mathcal{V})\right)^r = w(\mathcal{T}) \prod_{i=1}^{k} q_i(\mathcal{T})\mathrm{Edet}(\mathcal{U})\mathrm{Edet}(\mathcal{V}) \tag{14}$$

for some $r \in \mathbb{N}_+$ and $w(\mathcal{T}) \in \mathbb{C}[\mathcal{T}]$.

Now, we claim that $q(\mathcal{T}) = 1$. Since otherwise, every q_i is co-prime with both $\mathrm{Edet}(\mathcal{U})$ and $\mathrm{Edet}(\mathcal{V})$ as r_1, r_2 being maximal. While, this contradicts the equality (14). Consequently, by (13),

$$\mathrm{Edet}(\mathcal{T}) = \left(\mathrm{Edet}(\mathcal{U})\right)^{r_1} \left(\mathrm{Edet}(\mathcal{V})\right)^{r_2}.$$

Comparing the degrees of the both sides with Proposition 2.1 (ii), we get (7). The proof is complete. □

6 A Simple Application: Triangular Tensors

Let $\mathcal{T} = (t_{i_1 \cdots i_m}) \in \mathbb{T}(\mathbb{C}^n, m)$. Suppose that $t_{i_1 \cdots i_m} \equiv 0$ if $\min\{i_2, \cdots, i_m\} < i_1$. Then $\mathcal{T}$ is called an upper triangular tensor. Suppose that $t_{i_1 \cdots i_m} \equiv 0$ if $\max\{i_2, \cdots, i_m\} > i_1$. Then $\mathcal{T}$ is called a lower triangular tensor. If $\mathcal{T}$ is either upper or lower triangular, then $\mathcal{T}$ is called a triangular tensor. In particular, a diagonal tensor is a triangular tensor.

By Theorem 5.1, we have the following proposition.

Proposition 6.1 *Suppose that $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ is a triangular tensor. Then*

$$Edet(\mathcal{T}) = \prod_{i=1}^{n} (t_{i\ldots i})^{(m-1)^{n-1}}.$$

By Definition 2.2 and the above proposition, we have the following corollary.

Corollary 6.1 *Suppose that $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ is a triangular tensor. Then*

$$\sigma(\mathcal{T}) = \{t_{i\ldots i} \mid i = 1, \ldots, n\},$$

and the algebraic multiplicity of $t_{i\ldots i}$ is $(m-1)^{n-1}$ for all $i = 1, \ldots, n$.

With Theorem 3.1, we have the following simple application of the E-determinant theory.

Theorem 6.1 *Suppose that $\mathcal{T}$ is a triangular tensor with nonzero diagonal elements. Then $\mathcal{T}\mathbf{x}^{m-1} = \mathcal{B}^{m-1}\mathbf{x}^{m-2} + \cdots + \mathcal{B}^3\mathbf{x}^2 + A\mathbf{x} + \mathbf{b}$ has a solution in $\mathbb{C}^n$ for every $\mathbf{b} \in \mathbb{C}^n$, $A \in \mathbb{T}(\mathbb{C}^n, 2)$, and $\mathcal{B}^j \in \mathbb{T}(\mathbb{C}^n, j)$ for $j = 3, \ldots, m-1$.*

We can show that the composition of two upper (lower) triangular tensors is still an upper (lower) triangular tensor, and Conjecture 4.1 is true for two upper (lower) triangular tensors. We do not go to the details.

7 A Trace Formula of the E-Determinant

For the determinant, one generalization of Newton's identities:

$$\operatorname{Det}(E + A) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(- \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \operatorname{Tr}(A^j) \right)^k$$

is of irreplaceable importance in the theory of the determinant. Very recently, Morozov and Shakirov [23] generalized it to the context of the resultant of a homogenous polynomial system. In this section, we present a trace formula for the E-determinant based on the result of Morozov and Shakirov [23].

Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. Define the following differential operators:

$$\hat{g}_i := \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} t_{ii_2\ldots i_m} \frac{\partial}{\partial a_{ii_2}} \cdots \frac{\partial}{\partial a_{ii_m}}, \ \forall i = 1, \ldots, n, \tag{15}$$

where A is an auxiliary $n \times n$ variable matrix consists of elements a_{ij}'s. It is clear that for every i, $\hat{g}_i$ is a differential operator which belongs to the operator algebra $\mathbb{C}[\partial A]$, here ∂A is the $n \times n$ matrix with elements $\frac{\partial}{\partial a_{ij}}$'s. In order to make the operators in (15) convenient to use and the resulting formulae tidy, we reformulate $\hat{g}_i$ in the following way:

$$\hat{g}_i =: \sum_{1 \le i_2 \le i_3 \le \cdots \le i_m \le n} w_{ii_2\ldots i_m} \frac{\partial}{\partial a_{ii_2}} \cdots \frac{\partial}{\partial a_{ii_m}}, \ \forall i = 1, \ldots, n. \tag{16}$$

While, the corresponding g_i is defined as

$$g_i(\mathbf{x}) := \sum_{1\le i_2\le i_3\le\cdots\le i_m\le n} w_{ii_2\ldots i_m} x_{i_2}\cdots x_{i_m},\ \forall i=1,\ldots,n. \tag{17}$$

Then, by direct computation we have

$$(\hat{g}_i)^s \left(\sum_{j=1}^n a_{ij}x_j\right)^t \Bigg|_{A=0} = \begin{cases} (s(m-1))!\,[g_i(\mathbf{x})]^s & \text{if } t=s(m-1), \\ 0 & \text{otherwise.} \end{cases}$$

We now have the following proposition.

Proposition 7.1 *Let $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m)$. Then,*

$$\log Edet(\mathcal{E}-\mathcal{T}) = \prod_{i=1}^n \left[\sum_{k=0}^\infty \frac{m-1}{((m-1)k)!}(\hat{g}_i)^k\right] \log Det(E-A)\Bigg|_{A=0}.$$

So,

$$\begin{aligned} Edet(\mathcal{E}-\mathcal{T}) &= \exp\left(\prod_{i=1}^n \left[\sum_{k=0}^\infty \frac{m-1}{((m-1)k)!}(\hat{g}_i)^k\right] \log Det(E-A)\Bigg|_{A=0}\right) \\ &= \exp\left(\sum_{k_1=0}^\infty\cdots\sum_{k_n=0}^\infty -Tr_{k_1,\ldots,k_n}(\mathcal{T})\right) \end{aligned} \tag{18}$$

with the graded components defined as

$$Tr_{k_1,\ldots,k_n}(\mathcal{T}) := (m-1)^n \prod_{i=1}^n \left[\frac{(\hat{g}_i)^{k_i}}{((m-1)k_i)!}\right] \frac{Tr(A^{(m-1)(\sum_{i=1}^n k_i)})}{(m-1)(\sum_{i=1}^n k_i)} \tag{19}$$

and $Tr_{0,\ldots,0}(\mathcal{T}) := 0$.

Proof. The results follow from [23, Sections 4-7]. We omit the details. □

Note that we can derive the expansion of the right hand side of (18) by using multi-Schur polynomials in terms of $\mathrm{Tr}_{k_1,\ldots,k_n}(\mathcal{T})$'s. Then, a trace formula for $\mathrm{Edet}(\mathcal{T})$ can be derived.

Motivated by [9, 23], we define the *d-th trace* of tensor $\mathcal{T}$ as

$$\begin{aligned} \mathrm{Tr}_d(\mathcal{T}) &:= (m-1)^{n-1}\left[\sum_{\sum_{i=1}^n k_i=d}\prod_{i=1}^n \frac{(\hat{g}_i)^{k_i}}{((m-1)k_i)!}\right]\mathrm{Tr}(A^{(m-1)d}) \\ &= d\sum_{\sum_{i=1}^n k_i=d}\mathrm{Tr}_{k_1,\ldots,k_n}(\mathcal{T}). \end{aligned} \tag{20}$$

Then, it follows from (18), (19) and (20) that

$$\mathrm{Edet}(\mathcal{E}-\mathcal{T})=\exp\left(\sum_{d=0}^{\infty}-\frac{\mathrm{Tr}_d(\mathcal{T})}{d}\right). \tag{21}$$

We remark that (21) is a generalization of the well known identity

$$\log\mathrm{Det}(E-A)=\mathrm{Tr}\left(\log(E-A)\right)$$

for $A\in\mathbb{T}(\mathbb{C}^n,2)$, i.e., a square matrix, by noticing that $\log(E-A)=-\sum_{k=1}^{\infty}\frac{A^k}{k}$. In order to derive the expansion of the right hand side in (21), we need Schur polynomials which are defined as:

$$p_0(t_0)=1,\ \text{and}\ p_k(t_1,\dots,t_k):=\sum_{i=1}^{k}\sum_{d_j>0,\sum_{j=1}^{i}d_j=k}\frac{\Pi_{j=1}^{i}t_{d_j}}{i!},\quad\forall k\geq 1, \tag{22}$$

where $\{t_0,t_1,\dots\}$ are variables. Let $t_0=0$, we obtain the following expansion:

$$\exp\left(\sum_{k=0}^{\infty}t_k\alpha^k\right)=1+\sum_{k=1}^{\infty}p_k(t_1,\dots,t_k)\alpha^k.$$

This, together with (21), implies

$$\begin{aligned}\mathrm{Edet}(\mathcal{T}) &= \mathrm{Edet}(\mathcal{E}-(\mathcal{E}-\mathcal{T}))\\ &= 1+\sum_{k=1}^{\infty}p_k\left(-\frac{\mathrm{Tr}_1(\mathcal{E}-\mathcal{T})}{1},\dots,-\frac{\mathrm{Tr}_k(\mathcal{E}-\mathcal{T})}{k}\right).\end{aligned} \tag{23}$$

Now, we improve (23) to give an expression of $\mathrm{Edet}(\mathcal{T})$ with only finitely many terms. To this end, we need the following proposition.

Proposition 7.2 *Let $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m)$. Then, the followings hold:*

(i) for every $d\in\mathbb{N}_+$, $Tr_d(\mathcal{T})\in\mathbb{C}[\mathcal{T}]$ is homogenous of degree d;

(ii) for every $k\in\mathbb{N}_+$, $p_k\left(-\frac{Tr_1(\mathcal{T})}{1},\dots,-\frac{Tr_k(\mathcal{T})}{k}\right)\in\mathbb{C}[\mathcal{T}]$ is homogenous of degree k; and,

(iii) for any integer $k>n(m-1)^{n-1}$, $p_k\left(-\frac{Tr_1(\mathcal{T})}{1},\dots,-\frac{Tr_k(\mathcal{T})}{k}\right)\in\mathbb{C}[\mathcal{T}]$ is zero.

Proof. (i) By the formulae of $\hat{g}_i$'s as in (15), it is easy to see that

$$\sum_{\sum_{i=1}^{n}k_i=d}\prod_{i=1}^{n}\frac{(\hat{g}_i)^{k_i}}{((m-1)k_i)!}\in\mathbb{C}[\mathcal{T},\partial A]$$

is homogeneous, and more explicitly, homogenous of degree d in the variable $\mathcal{T}$ and homogeneous of degree $(m-1)d$ in the variable ∂A. It is also known that

$$\mathrm{Tr}(A^k) = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1} \in \mathbb{C}[A] \tag{24}$$

is homogeneous of degree k. These, together with (20), implies that $\mathrm{Tr}_d(\mathcal{T}) \in \mathbb{C}[\mathcal{T}]$ is homogenous of degree d as desired.

(ii) It follows from (i) and the definitions of Schur polynomials as in (22) directly.

(iii) From Proposition 2.1 (ii), it is clear that $\mathrm{Edet}(\mathcal{E}-\mathcal{T})$ is an irreducible polynomial which is homogenous of degree $n(m-1)^{n-1}$ in the entries of $\mathcal{B} := \mathcal{E}-\mathcal{T}$. Since the entries of $\mathcal{B}$ consist of 1 and the entries of tensor $\mathcal{T}$, the highest degree of $\mathrm{Edet}(\mathcal{E}-\mathcal{T})$ viewed as a polynomial in $\mathbb{C}[\mathcal{T}]$ is not greater than $n(m-1)^{n-1}$. This, together with (ii) which asserts that $p_k(-\frac{\mathrm{Tr}_1(\mathcal{T})}{1}, \ldots, -\frac{\mathrm{Tr}_k(\mathcal{T})}{k}) \in \mathbb{C}[\mathcal{T}]$ is homogenous of degree k, implies the result (iii).

The proof is complete. □

By Proposition 7.2 and (23), we immediately get

$$\begin{aligned}
\mathrm{Edet}(\mathcal{T}) &= \mathrm{Edet}(\mathcal{E}-(\mathcal{E}-\mathcal{T})) \\
&= 1 + \sum_{k=1}^{n(m-1)^{n-1}} p_k\left(-\frac{\mathrm{Tr}_1(\mathcal{E}-\mathcal{T})}{1}, \ldots, -\frac{\mathrm{Tr}_k(\mathcal{E}-\mathcal{T})}{k}\right).
\end{aligned} \tag{25}$$

This is a trace formula for the E-determinant. It involves the differential operators $\hat{g}_i$'s. In Section 9, we will give an explicit formula when $n = 2$.

8 The Characteristic Polynomial

By Definition 2.2, for any $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$, its characteristic polynomial is $\psi(\lambda) = \mathrm{Edet}(\lambda\mathcal{E}-\mathcal{T})$. The characteristic polynomial of a tensor was proposed by Qi in [26], and investigated by Cooper and Dutle in [9] very recently for spectral hypergraph theory. Following up Qi [26], Morozov and Shakirov [23] and Cooper and Dutle [9], we discuss some properties of the characteristic polynomial of a tensor related to the E-determinant.

Theorem 8.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. Then*

$$\psi(\lambda) = Edet(\lambda\mathcal{E}-\mathcal{T})$$

$$
\begin{aligned}
&= \lambda^{n(m-1)^{n-1}} + \sum_{k=1}^{n(m-1)^{n-1}} \lambda^{n(m-1)^{n-1}-k} p_k\left(-\frac{Tr_1(\mathcal{T})}{1}, \ldots, -\frac{Tr_k(\mathcal{T})}{k}\right) \\
&= \Pi_{\lambda_i\in\sigma(\mathcal{T})}(\lambda-\lambda_i)^{m_i},
\end{aligned}
$$

where m_i is the algebraic multiplicity of eigenvalue λ_i.

Proof. The first equality follows from Definition 2.2, and the last one from Theorem 2.1.

By Proposition 7.2 and (25), we can get

$$
\psi(1) = \mathrm{Edet}(\mathcal{E}-\mathcal{T}) \;=\; 1 + \sum_{k=1}^{n(m-1)^{n-1}} p_k\left(-\frac{\mathrm{Tr}_1(\mathcal{T})}{1}, \ldots, -\frac{\mathrm{Tr}_k(\mathcal{T})}{k}\right). \tag{26}
$$

Consequently, when $\lambda \neq 0$,

$$
\begin{aligned}
\psi(\lambda) &= \mathrm{Edet}(\lambda\mathcal{E}-\mathcal{T}) \\
&= \lambda^{n(m-1)^{n-1}}\mathrm{Edet}(\mathcal{E}-\frac{\mathcal{T}}{\lambda}) \\
&= \lambda^{n(m-1)^{n-1}}\left[1+\sum_{k=1}^{n(m-1)^{n-1}} p_k\left(-\frac{\mathrm{Tr}_1(\frac{\mathcal{T}}{\lambda})}{1}, \ldots, -\frac{\mathrm{Tr}_k(\frac{\mathcal{T}}{\lambda})}{k}\right)\right] \\
&= \lambda^{n(m-1)^{n-1}}\left[1+\sum_{k=1}^{n(m-1)^{n-1}} \frac{1}{\lambda^k}p_k\left(-\frac{\mathrm{Tr}_1(\mathcal{T})}{1}, \ldots, -\frac{\mathrm{Tr}_k(\mathcal{T})}{k}\right)\right] \\
&= \lambda^{n(m-1)^{n-1}} + \sum_{k=1}^{n(m-1)^{n-1}} \lambda^{n(m-1)^{n-1}-k} p_k\left(-\frac{\mathrm{Tr}_1(\mathcal{T})}{1}, \ldots, -\frac{\mathrm{Tr}_k(\mathcal{T})}{k}\right).
\end{aligned}
$$

Here the second equality comes from Corollary 2.2; the third from (26); and, the fourth from Proposition 7.2. Hence, the result follows from the fact that the field $\mathbb{C}$ is of characteristic zero. The proof is complete. □

Theorem 8.1 gives a trace formula for the characteristic polynomial of tensor $\mathcal{T}$ as well as an eigenvalue representation for it.

For the sequel analysis, we present the following hypothesis.

Assumption 8.1 *Let $\mathcal{T} \in \mathbb{S}(\mathbb{R}^n, m)$ (the subspace of $\mathbb{T}(\mathbb{R}^n, m)$ consists of symmetric tensors). Suppose that for every negative eigenvalue λ of $\mathcal{T}$, it possesses a real eigenvector associated to λ.*

When $m = 2$, Assumption 8.1 holds. Note that the tensors in [26, Examples 1 and 2] satisfy Assumption 8.1.

A tensor $\mathcal{T} \in \mathbb{S}(\mathbb{R}^n, m)$ is called positive semidefinite if and only if $\mathbf{x}^T(\mathcal{T}\mathbf{x}^{m-1}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Obviously, m being even is necessary for positive semidefinite tensors.

Lemma 8.1 *Let $\mathcal{T} \in \mathbb{S}(\mathbb{R}^n, m)$ and m be even. Suppose that Assumption 8.1 holds. Then, $\mathcal{T}$ is positive semidefinite if and only if all the real eigenvalues of $\mathcal{T}$ are nonnegative.*

Proof. If all the real eigenvalues of $\mathcal{T}$ are nonnegative, then $\mathcal{T}$ is positive semidefinite by [26, Theorem 5].

If $\mathcal{T}$ is positive semidefinite and it has a negative eigenvalue, then $\mathcal{T}$ has a real eigenvector associated to this eigenvalue by the hypothesis. As $\mathcal{T}$ is symmetric, this eigenvector violates the definition of positive semidefiniteness by [26, Theorem 3]. Consequently, if $\mathcal{T}$ is positive semidefinite, then all its real eigenvalues are nonnegative. □

The following classical result on real roots of a polynomial is Déscartes's Rule of Signs [34, Theorem 1.5].

Lemma 8.2 *The number of positive real roots of a polynomial is at most the number of sign changes in its coefficients.*

Let $\operatorname{sgn}(\cdot)$ be the sign function for scalars, i.e., $\operatorname{sgn}(\alpha) = 1$ if $\alpha > 0$, $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(\alpha) = -1$ if $\alpha < 0$.

Corollary 8.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{R}^n, m)$, and*

$$\psi(\lambda) \quad = \quad \lambda^{n(m-1)^{n-1}} + \sum_{k=1}^{n(m-1)^{n-1}} \lambda^{n(m-1)^{n-1}-k} p_k\left(-\frac{Tr_1(\mathcal{T})}{1}, \dots, -\frac{Tr_k(\mathcal{T})}{k}\right).$$

Suppose that

$$sgn\left(p_k\left(-\frac{Tr_1(\mathcal{T})}{1}, \dots, -\frac{Tr_k(\mathcal{T})}{k}\right)\right) = (-1)^k$$

for all $p_k\left(-\frac{Tr_1(\mathcal{T})}{1}, \dots, -\frac{Tr_k(\mathcal{T})}{k}\right) \neq 0$ with $1 \leq k \leq n(m-1)^{n-1}$. Then, all the real roots of ψ are nonnegative.

Proof. Suppose that m is even and n is odd. Then, $n(m-1)^{n-1}$ is odd. Consequently,

$$\phi(\lambda) := \psi(-\lambda) = -\lambda^{n(m-1)^{n-1}} + \sum_{k=1}^{n(m-1)^{n-1}} (-1)^{k+1}\lambda^{n(m-1)^{n-1}-k} p_k\left(-\frac{\mathrm{Tr}_1(\mathcal{T})}{1}, \dots, -\frac{\mathrm{Tr}_k(\mathcal{T})}{k}\right).$$

Then, by Lemma 8.2, ϕ defined as above has no positive real root, since the sign of the coefficient of ϕ is negative when it is nonzero. Hence, ψ has no negative real root.

The proofs for the other cases for m and n are similar. Consequently, the result follows. The proof is complete. $\square$

Proposition 8.1 *Let m be even, $\mathcal{T} \in \mathbb{S}(\mathbb{R}^n, m)$, and*

$$\psi(\lambda) = \lambda^{n(m-1)^{n-1}} + \sum_{k=1}^{n(m-1)^{n-1}} \lambda^{n(m-1)^{n-1}-k} p_k \left(-\frac{Tr_1(\mathcal{T})}{1}, \ldots, -\frac{Tr_k(\mathcal{T})}{k} \right).$$

Suppose that Assumption 8.1 holds. Then, $\mathcal{T}$ is positive semidefinite if

$$sgn \left(p_k \left(-\frac{Tr_1(\mathcal{T})}{1}, \ldots, -\frac{Tr_k(\mathcal{T})}{k} \right) \right) = (-1)^k \tag{27}$$

for all $p_k \left(-\frac{Tr_1(\mathcal{T})}{1}, \ldots, -\frac{Tr_k(\mathcal{T})}{k} \right) \neq 0$ with $1 \leq k \leq n(m-1)^{n-1}$. Furthermore, when n is even and all the complex eigenvalues of $\mathcal{T}$ have nonnegative real parts, then $\mathcal{T}$ is positive semidefinite if and only if (27) holds for all $p_k \left(-\frac{Tr_1(\mathcal{T})}{1}, \ldots, -\frac{Tr_k(\mathcal{T})}{k} \right) \neq 0$ with $1 \leq k \leq n(m-1)^{n-1}$.

Proof. The first result follows from Lemma 8.1 and Corollary 8.1 directly.

Now, we prove the second result. By Lemma 8.1, all the real roots of polynomial ψ are nonnegative. Consequently, all the real roots of polynomial $\phi(\lambda) := \psi(-\lambda)$ are nonpositive. Suppose that ϕ has negative roots $\{-\alpha_1, \ldots, -\alpha_s\}$ with the corresponding multiplicity set $\{m_1, \ldots, m_s\}$; zero root with multiplicity k; and, complex root pairs $\{(\mu_1, \bar{\mu}_1), \ldots, (\mu_t, \bar{\mu}_t)\}$ with the corresponding multiplicity set $\{r_1, \ldots, r_s\}$. Consequently,

$$\phi(\lambda) = \lambda^k \Pi_{i=1}^s (\lambda + \alpha_i)^{m_i} \Pi_{i=1}^t (\lambda^2 + (\mu_i + \bar{\mu}_i)\lambda + |\mu_i|^2)^{r_i}.$$

Hence, all the coefficients of ϕ are nonnegative by the assumption that all the complex eigenvalues of $\mathcal{T}$ have nonnegative real parts. Moreover, $n(m-1)^{n-1}$ is even, since n is even. These, together with the definition of ϕ, imply (27) immediately.

The proof is complete. $\square$

Remark 8.1 Proposition 8.1 is meaningful: we do not need to compute all the real eigenvalues of $\mathcal{T}$ associated with real eigenvectors, even the smallest, as [26, Theorem 5]. All we need is to check the condition (27), when the additional hypotheses in Proposition 8.1 are satisfied.

Here are some properties concerning the coefficients of $\psi(\lambda)$.

Proposition 8.2 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. Then,*

(i) $p_1(-Tr_1(\mathcal{T})) = -Tr_1(\mathcal{T}) = -(m-1)^{n-1}\sum_{i=1}^n t_{ii\ldots i}$*; and,*

(ii) $p_{n(m-1)^{n-1}}\left(-\frac{Tr_1(\mathcal{T})}{1}, \ldots, -\frac{Tr_{n(m-1)^{n-1}}(\mathcal{T})}{n(m-1)^{n-1}}\right) = (-1)^{n(m-1)^{n-1}} Edet(\mathcal{T})$*.*

Proof. (i) By (22), we know that $p_1(-\mathrm{Tr}_1(\mathcal{T})) = -\mathrm{Tr}_1(\mathcal{T})$. Furthermore, by (20), it is easy to see that

$$\begin{aligned}
\mathrm{Tr}_1(\mathcal{T}) &= (m-1)^{n-1}\sum_{i=1}^n \frac{\hat{g}_i}{(m-1)!}\mathrm{Tr}(A^{m-1}) \\
&= \frac{(m-1)^{n-1}}{(m-1)!}\sum_{i=1}^n \left[\sum_{i_2=1}^n \cdots \sum_{i_m=1}^n t_{ii_2\ldots i_m}\frac{\partial}{\partial a_{ii_2}} \cdots \frac{\partial}{\partial a_{ii_m}}\right]\mathrm{Tr}(A^{m-1}) \\
&= \frac{(m-1)^{n-1}}{(m-1)!}\sum_{i=1}^n \left[\sum_{i_2=1}^n \cdots \sum_{i_m=1}^n t_{ii_2\ldots i_m}\frac{\partial}{\partial a_{ii_2}} \cdots \frac{\partial}{\partial a_{ii_m}}\right] \\
&\quad \cdot\left(\sum_{i_1=1}^n \cdots \sum_{i_{m-1}=1}^n a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{m-2}i_{m-1}}a_{i_{m-1}i_1}\right) \\
&= \frac{(m-1)^{n-1}}{(m-1)!}\sum_{i=1}^n \left[t_{ii\ldots i}\frac{\partial}{\partial a_{ii}} \cdots \frac{\partial}{\partial a_{ii}}(a_{ii})^{m-1}\right] \\
&= (m-1)^{n-1}\sum_{i=1}^n t_{ii\ldots i}.
\end{aligned}$$

Here, the fourth equality follows from the fact that: (a) the differential operator in the right hand side of the third equality contains only items $\frac{\partial}{\partial a_{i\star}}$'s for $\star \in \{1, \ldots, n\}$ and the total degree is $m-1$; and, (b) only terms in $\mathrm{Tr}(A^{m-1})$ that contain the same $\frac{\partial}{\partial a_{i\star}}$'s of total degree $m-1$ can contribute to the result and this case occurs only when every $\star = i$ by (24). Consequently, the result (i) follows.

(ii) By Theorem 8.1, it is clear that

$$\psi(0) = \mathrm{Edet}(-\mathcal{T}) = p_{n(m-1)^{n-1}}\left(-\frac{\mathrm{Tr}_1(\mathcal{T})}{1}, \ldots, -\frac{\mathrm{Tr}_{n(m-1)^{n-1}}(\mathcal{T})}{n(m-1)^{n-1}}\right).$$

Moreover, $\mathrm{Edet}(-\mathcal{T}) \in \mathbb{C}[\mathcal{T}]$ is homogenous of degree $n(m-1)^{n-1}$ by Proposition 2.1 (ii), which implies $\mathrm{Edet}(-\mathcal{T}) = (-1)^{n(m-1)^{n-1}}\mathrm{Edet}(\mathcal{T})$. Consequently, the result follows. □

Corollary 8.2 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. Then,* *(i)* $\sum_{\lambda_i \in \sigma(\mathcal{T})} m_i \lambda_i = (m-1)^{n-1} \sum_{i=1}^n t_{ii\ldots i} = Tr_1(\mathcal{T})$, *and*

(ii) $\Pi_{\lambda_i \in \sigma(\mathcal{T})} \lambda_i^{m_i} = Edet(\mathcal{T})$.

Here m_i is the algebraic multiplicity of eigenvalue λ_i.

Proof. The results follow from the eigenvalue representation of $\psi(\lambda)$ in Theorem 8.1 and the coefficients of $\psi(\lambda)$ in Proposition 8.2 immediately. $\square$

Remark 8.2 In [26], Qi proved the results in Corollary 8.2 for $\mathcal{T} \in \mathbb{S}(\mathbb{R}^n, m)$. By Theorem 3.1 and Corollary 8.2, we see that the solvability of homogeneous polynomial equations is characterized by the zero eigenvalue of the underlying tensor.

9 Explicit Formulae When $n = 2$

We discuss more on the characteristic polynomial $\psi(\lambda)$ and the E-determinant $\mathrm{Edet}(\mathcal{T})$ of a tensor $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ in this section. Note that the trace formulae of both the characteristic polynomial and the E-determinant depend on the d-th traces of the underlying tensor for all $d = 1, \ldots, n(m-1)^{n-1}$. Nevertheless, it is very complicated [9, 23]. So, we give preliminary results on the computation of the d-th traces of a tensor. In particular, we give explicit formulae of $\mathrm{Tr}_2(\mathcal{T})$ of the tensor $\mathcal{T}$ for any order and dimension, and the characteristic polynomial $\psi(\lambda)$ and the E-determinant $\mathrm{Edet}(\mathcal{T})$ of the tensor $\mathcal{T}$ when $n = 2$.

The following lemma is a generalization of Proposition 8.2 (i).

Lemma 9.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. We have*

$$\frac{(\hat{g}_i)^k}{((m-1)k)!} Tr(A^{(m-1)k}) = t_{ii\ldots i}^k \tag{28}$$

for all $k \geq 0$ and $i \in \{1, \ldots, n\}$. So,

$$\sum_{i=1}^n \frac{(\hat{g}_i)^k}{((m-1)k)!} Tr(A^{(m-1)k}) = \sum_{i=1}^n t_{ii\ldots i}^k. \tag{29}$$

Proof. By (15) and (24), similar to the proof of Proposition 8.1 (i), we have

$$(\hat{g}_i)^k \mathrm{Tr}(A^{(m-1)k}) = \left[\sum_{i_2=1}^n \cdots \sum_{i_m=1}^n t_{ii_2\ldots i_m} \frac{\partial}{\partial a_{ii_2}} \cdots \frac{\partial}{\partial a_{ii_m}} \right]^k \mathrm{Tr}(A^{(m-1)k})$$

$$
\begin{aligned}
&= \left[\sum_{i_2=1}^{n}\cdots\sum_{i_m=1}^{n} t_{ii_2\ldots i_m}\frac{\partial}{\partial a_{ii_2}}\cdots\frac{\partial}{\partial a_{ii_m}}\right]^k \\
&\quad \cdot\left(\sum_{i_1=1}^{n}\cdots\sum_{i_{(m-1)k}=1}^{n} a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{(m-1)k-1}i_{(m-1)k}}a_{i_{(m-1)k}i_1}\right) \\
&= \left[t_{ii\ldots i}\frac{\partial}{\partial a_{ii}}\cdots\frac{\partial}{\partial a_{ii}}\right]^k (a_{ii})^{(m-1)k} \\
&= ((m-1)k)!t_{ii\ldots i}^k,
\end{aligned}
$$

which implies (28), and hence (29). □

Before further analysis, we need the following combinatorial result.

Lemma 9.2 *Let $i \neq j$, $k \geq 1$, $h \geq 1$ and $s \in \{1, \ldots, \min\{h,k\}(m-1)\}$ be arbitrary but fixed. Then, the number of term $(a_{ii})^{k(m-1)-s}(a_{ij})^s(a_{ji})^s(a_{jj})^{h(m-1)-s}$ in the expansion of $Tr(A^{(k+h)(m-1)})$ is*

$$
\binom{k(m-1)}{s}\binom{h(m-1)-1}{s-1}+\binom{h(m-1)}{s}\binom{k(m-1)-1}{s-1}.
$$

Proof. For the convenience of the sequel analysis, we define a *packaged element* of i as an ordered collection of a_{ij}, a_{jj}'s and a_{ji} with the form:

$$
a_{ij}\underbrace{a_{jj}\cdots a_{jj}}_{p}a_{ji}.
$$

The number p of a_{jj}'s in a packaged element of i can vary from 0 to the maximal number. A packaged element of j can be defined similarly.

Note that any term in

$$
\mathrm{Tr}(A^{(k+h)(m-1)}) = \sum_{i_1=1}^{n}\cdots\sum_{i_{(k+h)(m-1)}=1}^{n} a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{(k+h)(m-1)-1}i_{(k+h)(m-1)}}a_{i_{(k+h)(m-1)}i_1}
$$

which results in $(a_{ii})^{k(m-1)-s}(a_{ij})^s(a_{ji})^s(a_{jj})^{h(m-1)-s}$ has and only has either the packaged elements of i or the packaged elements of j if we count from the left most in the expression, and is totally determined by the numbers of a_{jj}'s in the packaged elements and the positions of the packaged elements in the expression

$$
a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{(k+h)(m-1)-1}i_{(k+h)(m-1)}}a_{i_{(k+h)(m-1)}i_1}. \tag{30}
$$

So, the number of term $(a_{ii})^{k(m-1)-s}(a_{ij})^s(a_{ji})^s(a_{jj})^{h(m-1)-s}$ in the expansion

$$
\mathrm{Tr}(A^{(k+h)(m-1)}) = \sum_{i_1=1}^{n}\cdots\sum_{i_{(k+h)(m-1)}=1}^{n} a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{(k+h)(m-1)-1}i_{(k+h)(m-1)}}a_{i_{(k+h)(m-1)}i_1}
$$

is totally determined by the number of cases how the packaged elements are arranged multiplying the number of cases of the positions of the packaged elements in the expression (30).

In the following, we consider only the situation of packaged elements of i. The other situation is similar. Note that there are altogether s packaged elements of i in every expression (30) which results in $(a_{ii})^{k(m-1)-s}(a_{ij})^s(a_{ji})^s(a_{jj})^{h(m-1)-s}$.

Firstly, note that there are $h(m-1)-s$ a_{jj}'s in $(a_{ii})^{k(m-1)-s}(a_{ij})^s(a_{ji})^s(a_{jj})^{h(m-1)-s}$. Then we have

$$\binom{h(m-1)-s+(s-1)}{s-1} = \binom{h(m-1)-1}{s-1}$$

different cases of s packaged elements of i with every case results in $(a_{ij})^s(a_{ji})^s(a_{jj})^{h(m-1)-s}$. Secondly, for an arbitrary but fixed case of s packaged elements of i in the first step, we get $k(m-1)$ "mixed" elements consist of the s packaged elements and the rest $k(m-1)-s$ a_{ii}'s. Consequently, there are exactly

$$\binom{k(m-1)}{s}$$

cases of the expression (30), under which the expression (30) results in the term $(a_{ii})^{k(m-1)-s}(a_{ij})^s(a_{ji})^s(a_{jj})^{h(m-1)-s}$.

Consequently, the number of term $(a_{ii})^{k(m-1)-s}(a_{ij})^s(a_{ji})^s(a_{jj})^{k(m-1)-s}$ in the expansion $\mathrm{Tr}(A^{(k+h)(m-1)})$ is

$$\binom{k(m-1)}{s}\binom{h(m-1)-1}{s-1}$$

in the situation of packaged elements of i.

By the symmetry of i and j, we can prove similarly that, in the situation of packaged elements of j, the number of term $(a_{ii})^{k(m-1)-s}(a_{ij})^s(a_{ji})^s(a_{jj})^{k(m-1)-s}$ in the expansion $\mathrm{Tr}(A^{(k+h)(m-1)})$ is

$$\binom{h(m-1)}{s}\binom{k(m-1)-1}{s-1}.$$

Hence, we have that the number of term $(a_{ii})^{k(m-1)-s}(a_{ij})^s(a_{ji})^s(a_{jj})^{k(m-1)-s}$ in the expansion $\mathrm{Tr}(A^{(k+h)(m-1)})$ is altogether

$$\binom{k(m-1)}{s}\binom{h(m-1)-1}{s-1}+\binom{h(m-1)}{s}\binom{k(m-1)-1}{s-1}.$$

The proof is complete. $\square$

In the sequel, let $\hat{g}$ and g be defined as in (16)and (17), respectively.

Lemma 9.3 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. For arbitrary $i < j$, and $h, k \geq 1$, we have*

$$\begin{aligned}
&\frac{(\hat{g}_i)^h(\hat{g}_j)^k}{(h(m-1))!(k(m-1))!} Tr(A^{(h+k)(m-1)}) \\
=\ & (\frac{h+k}{hk(m-1)}) \sum_{s=1}^{\min\{h,k\}(m-1)} \sum_{(a_1,\ldots,a_h)\in\mathbb{D}^s,\ (b_1,\ldots,b_k)\in\mathbb{E}^s} s \prod_{p=1}^{h}\prod_{q=1}^{k} w_{ii\ldots i\underbrace{j\ldots j}_{a_p}} w_{j\underbrace{i\ldots i}_{b_q}j\ldots j},
\end{aligned} \tag{31}$$

with $\mathbb{D}^s := \{(a_1, \ldots, a_h) \mid a_1 + \cdots + a_h = s,\ 0 \leq a_p \leq m-1\ \forall p = 1, \ldots, h\}$ and $\mathbb{E}^s := \{(b_1, \ldots, b_k) \mid b_1 + \cdots + b_k = s,\ 0 \leq b_q \leq m-1\ \forall q = 1, \ldots, k\}$.

Proof. Let $w := \min\{h, k\}(m-1)$, $\mathbb{D}^s := \{(a_1, \ldots, a_h) \mid a_1 + \cdots + a_h = s,\ 0 \leq a_p \leq m-1\ \forall p = 1, \ldots, h\}$ and $\mathbb{E}^s := \{(b_1, \ldots, b_k) \mid b_1+\cdots+b_k = s,\ 0 \leq b_q \leq m-1\ \forall q = 1, \ldots, k\}$ for all $s = 1, \ldots, w$. By (16) and (24), we have

$$\begin{aligned}
&(\hat{g}_i)^h(\hat{g}_j)^k \mathrm{Tr}(A^{(h+k)(m-1)}) \\
=\ & \left[\sum_{i_2\leq\cdots\leq i_m} w_{ii_2\ldots i_m} \frac{\partial}{\partial a_{ii_2}} \cdots \frac{\partial}{\partial a_{ii_m}}\right]^h \left[\sum_{j_2\leq\cdots\leq j_m} w_{jj_2\ldots j_m} \frac{\partial}{\partial a_{jj_2}} \cdots \frac{\partial}{\partial a_{jj_m}}\right]^k \mathrm{Tr}(A^{(h+k)(m-1)}) \\
=\ & \left[\sum_{i_2\leq\cdots\leq i_m} w_{ii_2\ldots i_m} \frac{\partial}{\partial a_{ii_2}} \cdots \frac{\partial}{\partial a_{ii_m}}\right]^h \left[\sum_{j_2\leq\cdots\leq j_m} w_{jj_2\ldots j_m} \frac{\partial}{\partial a_{jj_2}} \cdots \frac{\partial}{\partial a_{jj_m}}\right]^k \\
&\cdot \left(\sum_{i_1=1}^{n} \cdots \sum_{i_{(h+k)(m-1)}=1}^{n} a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{(h+k)(m-1)-1}i_{(h+k)(m-1)}} a_{i_{(h+k)(m-1)}i_1}\right) \\
=\ & \left(\sum_{s=1}^{w} \sum_{(a_1,\ldots,a_h)\in\mathbb{D}^s,\ (b_1,\ldots,b_k)\in\mathbb{E}^s} \prod_{p=1}^{h}\prod_{q=1}^{k} w_{ii\ldots i\underbrace{j\ldots j}_{a_p}} w_{j\underbrace{i\ldots i}_{b_q}j\ldots j}\right. \\
&\left. \cdot \left(\frac{\partial}{\partial a_{ii}}\right)^{h(m-1)-s} \left(\frac{\partial}{\partial a_{ij}}\right)^{s} \left(\frac{\partial}{\partial a_{ji}}\right)^{s} \left(\frac{\partial}{\partial a_{jj}}\right)^{k(m-1)-s}\right) \\
&\cdot \left\{\sum_{s=1}^{w} \left[\begin{pmatrix} k(m-1) \\ s \end{pmatrix} \begin{pmatrix} h(m-1)-1 \\ s-1 \end{pmatrix}\right.\right. \\
&\left.\left. + \begin{pmatrix} h(m-1) \\ s \end{pmatrix} \begin{pmatrix} k(m-1)-1 \\ s-1 \end{pmatrix}\right] \cdot (a_{ii})^{h(m-1)-s}(a_{ij})^s(a_{ji})^s(a_{jj})^{k(m-1)-s}\right\} \\
=\ & \sum_{s=1}^{w} \sum_{(a_1,\ldots,a_h)\in\mathbb{D}^s,\ (b_1,\ldots,b_k)\in\mathbb{E}^s} \prod_{p=1}^{h}\prod_{q=1}^{k} w_{ii\ldots i\underbrace{j\ldots j}_{a_p}} w_{j\underbrace{i\ldots i}_{b_q}j\ldots j} \\
&\cdot s\left((k(m-1))!(h(m-1)-1)! + (h(m-1))!(k(m-1)-1)!\right).
\end{aligned}$$

Here, the third equality follows from Lemma 9.2. Consequently, (31) follows. □

Especially, the following is a direct corollary of Lemma 9.3.

Corollary 9.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. For arbitrary $i < j$, we have*

$$\frac{\hat{g}_i\hat{g}_j}{[(m-1)!]^2} Tr(A^{2(m-1)}) = \sum_{s=1}^{m-1} \binom{2s}{m-1} w_{ii\dots i\underbrace{j\dots j}_{s}} w_{ji\dots i\underbrace{j\dots j}_{m-1-s}}.$$

Given an index set $\mathbb{L} := \{k_1, \dots, k_l\}$ with k_s taking value in $\{1, \dots, n\}$, denote by $\mathbb{H}_i(\mathbb{L})$ the set of indices in $\mathbb{L}$ taking value i. We denote by $|\mathbb{E}|$ the cardinality of a set $\mathbb{E}$.

Proposition 9.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. We have*

$$\begin{aligned} Tr_2(\mathcal{T}) &= (m-1)^{n-1}\left[\sum_{i=1}^n \frac{(\hat{g}_i)^2}{(2(m-1))!} + \sum_{i<j} \frac{\hat{g}_i\hat{g}_j}{[(m-1)!]^2}\right] Tr(A^{2(m-1)}) \\ &= (m-1)^{n-1}\left[\sum_{i=1}^n t_{ii\dots i}^2 + \sum_{i<j}\sum_{s=1}^{m-1} \binom{2s}{m-1}\right. \\ &\quad \cdot \left(\sum_{|\mathbb{H}_i(\{i_2,\dots,i_m\})|=m-1-s,\ |\mathbb{H}_j(\{i_2,\dots,i_m\})|=s} t_{ii_2\dots i_m}\right) \\ &\quad \cdot \left.\left(\sum_{|\mathbb{H}_i(\{j_2,\dots,j_m\})|=s,\ |\mathbb{H}_j(\{j_2,\dots,j_m\})|=m-1-s} t_{jj_2\dots j_m}\right)\right]. \end{aligned}$$

Proof. The result follows from Lemma 9.1, Corollary 9.1, (15) and (16) immediately. □

When $n = 2$, we can get the coefficients of the characteristic polynomial $\psi(\lambda)$ explicitly in terms of the entries of the underlying tensor by using Theorem 8.1, Lemmas 9.1 and 9.3. It is an alternative to Sylvester's formula [35]. In the following theorem, w, $\mathbb{D}^s$'s and $\mathbb{E}^s$'s are defined as in those in Lemma 9.3.

Theorem 9.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^2, m)$. We have*

$$\psi(\lambda) = \lambda^{2(m-1)} + \sum_{k=1}^{2(m-1)} \lambda^{2(m-1)-k} \sum_{i=1}^{k} \frac{1}{i!} \sum_{d_j>0, \sum_{j=1}^i d_j = k} \prod_{j=1}^{i} \frac{-Tr_{d_j}(\mathcal{T})}{d_j}$$

with

$$Tr_d(\mathcal{T})$$

$$
\begin{aligned}
&= (m-1)\Bigg\{(t^d_{11\ldots1}+t^d_{22\ldots2}) + \sum_{h+k=d,h,k\geq1}\sum_{s=1}^{w}\frac{s(h+k)}{hk(m-1)} \\
&\cdot\Bigg[\sum_{(a_1,\ldots,a_h)\in\mathbb{D}^s,\,(b_1,\ldots,b_k)\in\mathbb{E}^s}\prod_{p=1}^{h}\prod_{q=1}^{k}\Bigg(\sum_{|\mathbb{H}_1(\{i_2,\ldots,i_m\})|=m-1-a_p,\,|\mathbb{H}_2(\{i_2,\ldots,i_m\})|=a_p} t_{1i_2\ldots i_m}\Bigg) \\
&\cdot\Bigg(\sum_{|\mathbb{H}_1(\{j_2,\ldots,j_m\})|=b_p,\,|\mathbb{H}_2(\{j_2,\ldots,j_m\})|=m-1-b_p} t_{2j_2\ldots j_m}\Bigg)\Bigg]\Bigg\}
\end{aligned}
$$

for $d\in\{1,\ldots,2(m-1)\}$.

Note that as $\mathrm{Edet}(\mathcal{T})=(-1)^{n(m-1)^{n-1}}\psi(0)$ by Theorem 8.1 and Corollary 2.2, when $n=2$, we get an explicit formula for $\mathrm{Edet}(\mathcal{T})$ as

$$
\mathrm{Edet}(\mathcal{T})=\sum_{i=1}^{2(m-1)}\frac{1}{i!}\sum_{d_j>0,\sum_{j=1}^{i}d_j=2(m-1)}\prod_{j=1}^{i}\frac{-\mathrm{Tr}_{d_j}(\mathcal{T})}{d_j}.
$$

When $m=3$, we have the following corollary.

Corollary 9.2 *Let* $\mathcal{T}\in\mathbb{T}(\mathbb{C}^2,3)$. *We have*

$$
\begin{aligned}
\psi(\lambda) &= \lambda^4-\lambda^3 Tr_1(\mathcal{T})+\frac{1}{2}\lambda^2\left([Tr_1(\mathcal{T})]^2-Tr_2(\mathcal{T})\right) \\
&+ \frac{1}{12}\lambda\left(-2\,[Tr_1(\mathcal{T})]^3+6\,Tr_1(\mathcal{T})\,Tr_2(\mathcal{T})-4\,Tr_3(\mathcal{T})\right) \\
&+ \frac{1}{24}\left([Tr_1(\mathcal{T})]^4-6\,[Tr_1(\mathcal{T})]^2\,Tr_2(\mathcal{T})+8\,Tr_1(\mathcal{T})\,Tr_3(\mathcal{T})+3\,[Tr_2(\mathcal{T})]^2-6\,Tr_4(\mathcal{T})\right)
\end{aligned}
$$

and

$$
Edet(\mathcal{T})=\frac{1}{24}\left([Tr_1(\mathcal{T})]^4-6\,[Tr_1(\mathcal{T})]^2\,Tr_2(\mathcal{T})+8\,Tr_1(\mathcal{T})\,Tr_3(\mathcal{T})+3\,[Tr_2(\mathcal{T})]^2-6\,Tr_4(\mathcal{T})\right)
$$

with

$$
\begin{aligned}
Tr_1(\mathcal{T}) &= 2(t_{111}+t_{222}), \\
Tr_2(\mathcal{T}) &= 2(t_{111}^2+t_{222}^2)+2(t_{112}+t_{121})(t_{212}+t_{221})+4(t_{122}t_{211}), \\
Tr_3(\mathcal{T}) &= 2(t_{111}^3+t_{222}^3) \\
&\quad +\frac{3}{2}(t_{112}+t_{121})\left[(t_{212}+t_{221})t_{222}\right]+3t_{122}\left[(t_{212}+t_{221})^2+t_{211}t_{222}\right] \\
&\quad +\frac{3}{2}(t_{212}+t_{221})\left[(t_{112}+t_{121})t_{111}\right]+3t_{211}\left[(t_{112}+t_{121})^2+t_{122}t_{111}\right], \\
Tr_4(\mathcal{T}) &= 2(t_{111}^4+t_{222}^4)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4}{3}(t_{112}+t_{121})\left[t_{222}^2(t_{212}+t_{221})\right]+\frac{8}{3}t_{122}\left[t_{222}(t_{212}+t_{221})^2+t_{222}^2t_{211}\right]\\
&+\frac{4}{3}(t_{212}+t_{221})\left[t_{122}^2(t_{112}+t_{121})\right]+\frac{8}{3}t_{211}\left[t_{111}(t_{112}+t_{121})^2+t_{111}^2t_{122}\right]\\
&+t_{111}(t_{121}+t_{121})t_{222}(t_{212}+t_{221})+3\left[t_{111}(t_{121}+t_{112})\right]\left[t_{211}(t_{221}+t_{212})\right]\\
&\quad+2\left[(t_{121}+t_{112})^2+t_{122}t_{111}\right]\left[t_{211}t_{222}+(t_{212}+t_{221})^2\right]+4t_{122}^2t_{211}^2.
\end{aligned}
$$

10 Inequalities of the E-Determinant

In this section, we generalize several inequalities of the determinants for matrices to the E-determinants for tensors.

10.1 Hadamard's Inequality

We generalize Hadamrd's inequality for matrices to tensors in this subsection. We assume that m is an even integer in this subsection.

Lemma 10.1 *Let $\mathcal{T} \in \mathbb{S}(\mathbb{R}^n, m)$ and $\mathcal{U}$ be a sub-tensor of $\mathcal{T}$ associated to any nonempty subset of $\{1,\dots,n\}$. If $\mathcal{T}$ is positive semidefinite, then Edet$(\mathcal{U}) \geq 0$.*

Proof. It follows from the definitions of positive semidefiniteness and sub-tensors (Definition 5.1), and [26, Proposition 7]. □

The following assumption is involved.

Assumption 10.1 *Let $\mathcal{T} \in \mathbb{S}(\mathbb{R}^n, m)$. Suppose that $\sigma(\mathcal{T})$ consists of only real numbers.*

Lemma 10.2 *Let $\mathcal{T} \in \mathbb{S}(\mathbb{R}^n, m)$, and suppose that Assumptions 8.1 and 10.1 hold. If $\mathcal{T}$ is positive semidefinite and $t_{i\dots i} \leq 1$ for all $i \in \{1,\dots,n\}$, then $1 \geq$ Edet$(\mathcal{T}) \geq 0$.*

Proof. By Lemma 10.1, $\mathrm{Edet}(\mathcal{T}) \geq 0$. Furthermore,

$$
\begin{aligned}
n(m-1)^{n-1} &\geq (m-1)^{n-1}\sum_{i=1}^n t_{ii\dots i}\\
&= \sum_{\lambda_i\in\sigma(\mathcal{T})} m_i\lambda_i\\
&\geq n(m-1)^{n-1}\left(\Pi_{\lambda_i\in\sigma(\mathcal{T})}\lambda_i^{m_i}\right)^{\frac{1}{n(m-1)^{n-1}}}\\
&= n(m-1)^{n-1}\left(\mathrm{Edet}(\mathcal{T})\right)^{\frac{1}{n(m-1)^{n-1}}},
\end{aligned}
$$

where m_i is the algebraic multiplicity of eigenvalue λ_i. Here the first inequality follows from the assumption that $t_{i\ldots i} \leq 1$ for all $i \in \{1, \ldots, n\}$; the first equality from Corollary 8.2; and, the last inequality from Assumptions 8.1 and 10.1 (which, together with the positive semidefiniteness of $\mathcal{T}$, imply that $\mathcal{T}$ has only nonnegative eigenvalues) and the arithmetic geometry mean inequality. This, together with $\mathrm{Edet}(\mathcal{T}) \geq 0$, implies $\mathrm{Edet}(\mathcal{T}) \leq 1$. The proof is complete. $\square$

Note again that the tensor in [26, Example 2] satisfies all the hypotheses in Lemma 10.2.

Theorem 10.1 *Let $\mathcal{T} \in \mathbb{S}(\mathbb{R}^n, m)$, $D := Diag\{t_{1\ldots1}, \ldots, t_{n\ldots n}\}$ be the diagonal matrix consisting of diagonal elements $t_{1\ldots1}, \ldots, t_{n\ldots n}$, and $\mathcal{K} := D^{-\frac{1}{m}} \circ \mathcal{T} \circ D^{-\frac{1}{m}}$, where D is invertible. Suppose that Assumptions 8.1 and 10.1 hold for both $\mathcal{T}$ and $\mathcal{K}$. If $\mathcal{T}$ is positive semidefinite, then $0 \leq Edet(\mathcal{T}) \leq (\Pi_{i=1}^n t_{i\ldots i})^{(m-1)^{n-1}}$.*

Proof. If $t_{i\ldots i} = 0$ for some i, then $\mathcal{T}\mathbf{e}_i^m := \mathbf{e}_i^T(\mathcal{T}\mathbf{e}_i^{m-1}) = 0$. Consequently, $\mathcal{T}$ has a zero eigenvalue, since it is positive semidefinite and symmetric (see proof of [26, Theorem 5]). This, together with Corollary 8.2, further implies $\mathrm{Edet}(\mathcal{T}) = 0$. Then, the results follow trivially.

Now, suppose that matrix D is invertible. By the definition of positive semidefiniteness, it is clear that $\mathcal{K}$ is positive semidefinite as well. Since it also satisfies Assumptions 8.1 and 10.1, consequently, $0 \leq \mathrm{Edet}(\mathcal{K}) \leq 1$ by Lemma 10.2. Furthermore, by Propositions 4.3 and 4.4, it is easy to see that

$$\mathrm{Edet}(\mathcal{K}) = \left(\mathrm{Det}(D^{-\frac{1}{m}})\right)^{m(m-1)^{n-1}} \mathrm{Edet}(\mathcal{T}).$$

Hence,

$$\mathrm{Edet}(\mathcal{T}) = (\Pi_{i=1}^n t_{i\ldots i})^{(m-1)^{n-1}} \mathrm{Edet}(\mathcal{K}) \leq (\Pi_{i=1}^n t_{i\ldots i})^{(m-1)^{n-1}}.$$

The proof is complete. $\square$

Remark 10.1 When $m = 2$, Assumptions 8.1 and 10.1 are satisfied by matrices, so Theorem 10.1 reduces to the well-known Hadamard's inequality [16, Theorem 2.5.4].

10.2 Geršgorin's Inequality

We generalize Geršgorin's inequality for matrices [15, Problem 6.1.3] to tensors in this subsection.

Lemma 10.3 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ and $\rho(\mathcal{T}) := \max_{\lambda \in \sigma(\mathcal{T})} |\lambda|$ be its spectral radius. Then,*

$$\rho(\mathcal{T}) \leq \max_{1 \leq i \leq n} \left(\sum_{i_2, \ldots, i_m = 1}^{n} |t_{ii_2 \ldots i_m}| \right).$$

Proof. The result follows from [26, Theorem 6] immediately. □

Proposition 10.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. Then,*

$$|Edet(\mathcal{T})| \leq \prod_{1 \leq i \leq n} \left(\sum_{i_2, \ldots, i_m = 1}^{n} |t_{ii_2 \ldots i_m}| \right)^{(m-1)^{n-1}}. \tag{32}$$

Proof. If $\sum_{i_2, \ldots, i_m = 1}^{n} |t_{ii_2 \ldots i_m}| = 0$ for some $i \in \{1, \ldots, n\}$, then $\mathrm{Edet}(\mathcal{T}) = 0$ by Proposition 2.1 (i). Consequently, (32) follows trivially.

Now, suppose that $\sum_{i_2, \ldots, i_m = 1}^{n} |t_{ii_2 \ldots i_m}| \neq 0$ for all $i \in \{1, \ldots, n\}$. Let tensor $\mathcal{U} \in \mathbb{T}(\mathbb{C}^n, m)$ be defined as

$$u_{ii_2 \ldots i_m} := \frac{t_{ii_2 \ldots i_m}}{\sum_{i_2, \ldots, i_m = 1}^{n} |t_{ii_2 \ldots i_m}|}, \ \forall i, i_2, \ldots, i_m = 1, \ldots, n. \tag{33}$$

Then, by Lemma 10.3, we have that $\rho(\mathcal{U}) \leq 1$. This, together with Corollary 8.2, further implies that

$$|\mathrm{Edet}(\mathcal{U})| \leq 1.$$

Moreover, by Proposition 2.1(ii) and (33), it is clear that

$$|\mathrm{Edet}(\mathcal{U})| = \frac{|\mathrm{Edet}(\mathcal{T})|}{\prod_{1 \leq i \leq n} \left(\sum_{i_2, \ldots, i_m = 1}^{n} |t_{ii_2 \ldots i_m}| \right)^{(m-1)^{n-1}}}.$$

Consequently, (32) follows. The proof is complete. □

10.3 Minikowski's Inequality

We give a partial generalization of Minikowski's inequality for matrices [15, Theorem 7.8.8] to tensors in this subsection. We present the following lemma first.

Lemma 10.4 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. Then,*

$$\lambda \in \sigma(\mathcal{T}) \quad \Longleftrightarrow \quad 1 + \lambda \in \sigma(\mathcal{E} + \mathcal{T}).$$

Proof. It follows from Definition 2.1 directly. $\square$

Proposition 10.2 *Let $\mathcal{T} \in \mathbb{S}(\mathbb{R}^n, m)$ and m be even, and suppose that Assumptions 8.1 and 10.1 hold. If $\mathcal{T}$ is positive semidefinite, then*

$$[Edet(\mathcal{E} + \mathcal{T})]^{\frac{1}{n(m-1)^{n-1}}} \geq 1 + [Edet(\mathcal{T})]^{\frac{1}{n(m-1)^{n-1}}}.$$

Proof. Suppose that $0 \leq \lambda_1 \leq \ldots \leq \lambda_{n(m-1)^{n-1}}$ are the eigenvalues of tensor $\mathcal{T}$. Then, the arithmetic geometric mean inequality implies that

$$\Pi_{i=1}^{n(m-1)^{n-1}}(1+\lambda_i) \geq \left(1 + \sqrt[n(m-1)^{n-1}]{\Pi_{i=1}^{n(m-1)^{n-1}} \lambda_i}\right)^{n(m-1)^{n-1}}.$$

This, together with Corollary 8.2 and Lemma 10.4, implies the result. $\square$

11 Final Remarks

In this paper, we introduced the E-determinant of a tensor and investigated its various properties. The simple application in Section 6 demonstrates that the E-determinant theory is applicable and worth further exploring.

Certainly, it is worth exploring if Conjecture 4.1 is true or not and what is the right expression of the E-determinant of the composition if the conjecture is not true. There are also many other issues for further research in the theory of the E-determinant, which is surely one of the foundations of the eigenvalue theory of tensors. For example,

- How to remove Assumption 8.1 in Proposition 8.1, Assumptions 8.1 and 10.1 in Proposition 10.2, Lemma 10.2 and Theorem 10.1, or to derive which class of tensors satisfies these assumptions.
- More explicit formulae for the characteristic polynomials of general tensors, like those in Theorem 9.1 and Corollary 9.2.
- More properties about the E-determinants. For example, given a matrix A, Laplace's formula [15, Page 7] reads:

  $$\mathrm{Det}(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \tag{34}$$

  with minor M_{ij} being defined to be the determinant of the $(n-1)\times(n-1)$ matrix that results from A by removing the i-th row and the j-th column. If a generalization of

(34) can be derived for the E-determinant of a tensor, many other useful inequalities, for example, Oppenheim's inequality, can be proved for the E-determinants.

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