H\(^+\)-Eigenvalues of Laplacian and Signless Laplacian Tensors

Liqun Qi *

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Abstract

We propose a simple and natural definition for the Laplacian and the signless Laplacian tensors of a uniform hypergraph. We study their H\(^+\)-eigenvalues, i.e., H-eigenvalues with nonnegative H-eigenvectors, and H\(^++\)-eigenvalues, i.e., H-eigenvalues with positive H-eigenvectors. We show that each of the Laplacian tensor, the signless Laplacian tensor and the adjacency tensor has at most one H\(^++\)-eigenvalue, but has several other H\(^+\)-eigenvalues. We identify their largest and smallest H\(^+\)-eigenvalues, and establish some maximum and minimum properties of these H\(^+\)-eigenvalues. We then define algebraic connectivity of a uniform hypergraph and discuss its application in edge connectivity.

Key words: Laplacian tensor, signless Laplacian tensor, uniform hypergraph, H\(^+\)-eigenvalue

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1 Introduction

Recently, several papers appeared on spectral hypergraph theory via tensors \[3, 6, 10, 11, 12, 15, 17, 18, 19\]. These works are all on uniform hypergraphs \[1\]. In 2008, Lim \[12\] proposed to study spectral hypergraph theory via eigenvalues of tensors. In 2009, Bulò and Pelillo \[3\] gave new bounds on the clique number of a graph based on analysis of the largest eigenvalue of the adjacency tensor of a uniform hypergraph. In 2012, Hu and Qi \[10\] proposed a definition for the Laplacian tensor of an even uniform hypergraph, and

*Email: maqilq@polyu.edu.hk. Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong. This author’s work was supported by the Hong Kong Research Grant Council (Grant No. PolyU 501909, 502510, 502111 and 501212).
analyzed its connection with edge and vertex connectivity. In the same year, Cooper and Dutle [6] analyzed the eigenvalues of the adjacency tensor of a uniform hypergraph, and proved a number of natural analogs of basic results in spectral graph theory. Li, Qi and Yu [11] proposed another definition for the Laplacian tensor of an even uniform hypergraph, established a variational formula for its second smallest Z-eigenvalue, and used it to provide lower bounds for the bipartition width of the hypergraph. Recently, in [17, 19], Xie and Chang proposed a definition for the signless Laplacian tensor of an even uniform hypergraph, studied its largest and smallest H-eigenvalues and Z-eigenvalues, and its applications in the edge cut and the edge connectivity of the hypergraph. They also studied the largest and the smallest Z-eigenvalues of the adjacency tensor of a uniform hypergraph in [18]. Very recently, Pearson and Zhang [15] studied the H-eigenvalues and the Z-eigenvalues of the adjacency tensor of a uniform hypergraph.

A uniform hypergraph is also called a $k$-graph [1, 2]. Let $G = (V, E)$ be a $k$-graph, where $V = \{1, 2, \ldots, n\}$ is the vertex set, $E = \{e_1, e_2, \ldots, e_m\}$ is the edge set, $e_p \subset V$ and $|e_p| = k$ for $p = 1, \ldots, m$, and $k \geq 2$. If $k = 2$, then $G$ is an ordinary graph. We assume that $e_p \neq e_q$ if $p \neq q$. The adjacency tensor $A = A(G)$ of $G$, is a $k$th order $n$-dimensional symmetric tensor, with $A = (a_{i_1i_2\ldots i_k})$, where $a_{i_1i_2\ldots i_k} = \frac{1}{(k-1)!}$ if $(i_1, i_2, \ldots, i_k) \in E$, and 0 otherwise. Thus, $a_{i_1i_2\ldots i_k} = 0$ if two of its indices are the same. For $i \in V$, its degree $d(i)$ is defined as $d(i) = |\{e_p : i \in e_p \in E\}|$. We assume that every vertex has at least one edge. Thus, $d(i) > 0$ for all $i$. The degree tensor $D = D(G)$ of $G$, is a $k$th order $n$-dimensional diagonal tensor, with its $i$th diagonal element as $d(i)$. We denote the maximum degree, the minimum degree and the average degree of $G$ by $\Delta$, $d_{\min}$ and $\bar{d}$ respectively. If $\bar{d} = \Delta = d$, then $G$ is a regular graph, called a $d$-regular $k$-graph.

The definition of the adjacency tensor is natural. It was studied in [3, 6, 18]. On the other hand, the definitions of Laplacian and signless Laplacian tensors in [10, 11, 17, 19] are based upon some forms of sums of $k$-th powers. They are not simple and natural, and only work when $k$ is even.

In this paper, we propose a simple and natural definition for the Laplacian and the signless Laplacian tensors of a $k$-graph $G$. Recall that when $k = 2$, the Laplacian matrix and the signless Laplacian matrix of $G$ are defined as $L = D - A$ and $Q = D + A$ [2]. Many results of spectral graph theory are based upon this definition. Thus, for $k \geq 3$, we propose to define the Laplacian tensor and the signless Laplacian tensor of $G$ simply by $L = D - A$ and $Q = D + A$. This definition is simple and natural, and is closely related with the adjacency tensor $A$. Furthermore, the signless Laplacian tensor $Q$ is a symmetric nonnegative tensor, while the Laplacian tensor $L$ is the limit of symmetric $M$-tensors in the sense of [22]. $M$-tensors are closely related with nonnegative tensors [22]. Thus, we may use the recently developed theory and algorithms on eigenvalues of nonnegative tensors [4, 5, 8, 9, 13, 14, 20, 21] to study $L$ and $Q$. 

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We discover that $L$ and $Q$ have very nice spectral properties. They are not irreducible in the sense of [4]. But they are weakly irreducible in the sense of [8] if $G$ is connected. Each of them has at least $n + 1$ H-eigenvalues with nonnegative H-eigenvectors. We call such H-eigenvalues $H^+$-eigenvalues. Furthermore, each of them has at most one $H^+$-eigenvalue with a positive eigenvector. We call such an $H^+$-eigenvalue an $H^{++}$-eigenvalue.

The remainder of this paper is distributed as follows. In the next section, we review the definition and properties of eigenvalues and H-eigenvalues of tensors, and introduce $H^+$-eigenvalues and $H^{++}$-eigenvalues. We study $H^+$-eigenvalues of $A$, $L$ and $Q$ in Section 3. We show that each of $A$, $L$ and $Q$ has at most one $H^{++}$-eigenvalue, but has several other $H^+$-eigenvalues. In Sections 4, we study the smallest H-eigenvalue of $L$, and its link with connectedness of $G$. We identify the largest $H^+$-eigenvalue of $L$, and establish a maximum property of this $H^+$-eigenvalue in Section 5. We establish some maximum properties of the largest H-eigenvalues of $Q$ and $A$, and discuss their computational methods in Section 6. In Section 7, we identify the smallest $H^+$-eigenvalue of $Q$, establish a minimum property of this $H^+$-eigenvalue, and discuss its applications in edge connectivity and max imum cut. In Section 8, we define algebraic connectivity of $G$ as a minimum quantity related with $L$, and discuss its application in edge connectivity. Some final remarks are made in Section 9.

Denote by $l$ the all 1 $n$-dimensional vector, $l_j = 1$ for $j = 1, \cdots, n$. Denote by $l^{(i)}$ the $i$th unit vector in $\mathbb{R}^n$, i.e., $l^{(i)}_j = 1$ if $i = j$ and $l^{(i)}_j = 0$ if $i \neq j$, for $i, j = 1, \cdots, n$. For a vector $x$ in $\mathbb{R}^n$, we define its support as supp$(x) = \{i \in V : x_i \neq 0\}$. Denote the set of all nonnegative vectors in $\mathbb{R}^n$ by $\mathbb{R}^n_+$ and the set of all positive vectors in $\mathbb{R}^n$ by $\mathbb{R}^n_+$. For a $k$th order $n$-dimensional tensor with all of its elements being 1.

## 2 $H^+$-Eigenvalues and $H^{++}$-Eigenvalues

In this section, we will review the definition and properties of eigenvalues and H-eigenvalues of tensors in [16], introduce $H^+$-eigenvalues and $H^{++}$-eigenvalues, and review the Perron-Frobenius Theorem for nonnegative tensors in [4, 8, 20]. We also discuss the reducibility and weak irreducibility of $L$ and $Q$ in this section.

Consider a real $k$th order $n$-dimensional tensor $T = (t_{i_1 \cdots i_k})$. Let $x \in \mathbb{R}^n$. Then

$$T x^k = \sum_{i_1, \cdots, i_k = 1}^n t_{i_1 \cdots i_k} x_{i_1} \cdots x_{i_k},$$
and $\mathcal{T}x^{k-1}$ is a vector in $C^n$, with its $i$th component defined by

$$(\mathcal{T}x^{k-1})_i = \sum_{i_2,\ldots,i_k = 1}^{n} t_{i_1\ldots i_k}x_{i_2}\cdots x_{i_k}.$$ 

Let $r$ be a positive integer. Then $x^{[r]}$ is a vector in $C^n$, with its $i$th component defined by $x_i^r$. We say that $\mathcal{T}$ is symmetric if its entries $t_{i_1,\ldots,i_k}$ are invariant for any permutation of its indices.

Suppose that $x \in C^n$, $x \neq 0$, $\lambda \in C$, $x$ and $\lambda$ satisfy

$$\mathcal{T}x^{k-1} = \lambda x^{[k-1]},$$ 

(1)

Then we call $\lambda$ an eigenvalue of $\mathcal{T}$, and $x$ its corresponding eigenvector. From (1), we may see that if $\lambda$ is an eigenvalue of $\mathcal{T}$ and $x$ is its corresponding eigenvector, then

$$\lambda = \frac{(\mathcal{T}x^{k-1})_j}{x^{k-1}_j},$$ 

(2)

for some $j$ with $x_j \neq 0$. In particular, if $x$ is real, then $\lambda$ is also real. In this case, we say that $\lambda$ is an H-eigenvalue of $\mathcal{T}$ and $x$ is its corresponding H-eigenvector. If $x \in \mathbb{R}_+^n$, then we say that $\lambda$ is an $\mathbb{H}^+$-eigenvalue of $\mathcal{T}$. If $x \in \mathbb{R}_{++}^n$, then we say that $\lambda$ is an $\mathbb{H}^{++}$-eigenvalue of $\mathcal{T}$. If $\lambda$ is an $\mathbb{H}^+$-eigenvalue but not an $\mathbb{H}^{++}$-eigenvalue of $\mathcal{T}$, then we say that $\lambda$ is a genuine $\mathbb{H}^+$-eigenvalue of $\mathcal{T}$.

We say that $\mathcal{T}$ is positive definite (semi-definite) if $\mathcal{T}x^k > 0$ ($\mathcal{T}x^k \geq 0$) for all $x \in \mathbb{R}^n, x \neq 0$. Clearly, this is meaningful only if $k$ is even.

Note that (1) is a homogeneous system of $x$, with $n$ variables and $n$ equations. According to algebraic geometry [7], the resultant of (1) is a polynomial of coefficients of (1), hence a polynomial of $\lambda$, which vanishes if and only if (1) has a nonzero solution $x$. Denote this polynomial by $\phi_{\mathcal{T}}(\lambda)$, and call it the characteristic polynomial of $\mathcal{T}$.

The main properties of eigenvalues and H-eigenvalues of a real $k$th order $n$-dimensional symmetric tensor in [16] are summarized in the following theorem.

**Theorem 1 (Eigenvalues of Real Symmetric Tensors) (Qi 2005)**

We have the following conclusions on eigenvalues of a real $k$th order $n$-dimensional symmetric tensor $\mathcal{T}$:

(a). A number $\lambda \in C$ is an eigenvalue of $\mathcal{T}$ if and only if it is a root of the characteristic polynomial $\phi_{\mathcal{T}}$.

(b). The number of eigenvalues of $\mathcal{T}$ is $n(k - 1)^{n-1}$. Their product is equal to $\det(\mathcal{T})$, the resultant of $\mathcal{T}x^{k-1} = 0$.

(c). The sum of all the eigenvalues of $\mathcal{T}$ is

$$(k - 1)^{n-1}\text{tr}(\mathcal{T}),$$
where \( \text{tr}(\mathcal{T}) \) denotes the sum of the diagonal elements of \( \mathcal{T} \).

\[ \text{(d).} \] If \( k \) is even, then \( \mathcal{T} \) always has \( H \)-eigenvalues. \( \mathcal{T} \) is positive definite (positive semidefinite) if and only if all of its \( H \)-eigenvalues are positive (nonnegative).

\[ \text{(e).} \] The eigenvalues of \( \mathcal{T} \) lie in the following \( n \) disks:

\[
|\lambda - t_{ii\cdots i_k}| \leq \sum \{|t_{i_2\cdots i_k}| : i_2, \cdots, i_k = 1, \cdots, n, \{i_2, \cdots, i_k\} \neq \{i, \cdots, i\}\},
\]

for \( i = 1, \cdots, n \).

A major part of this theorem is still true when \( \mathcal{T} \) is not symmetric. As we only concern with real symmetric tensors, we do not go to this in detail.

We call \( \sum \{|t_{i_2\cdots i_k}| : i_2, \cdots, i_k = 1, \cdots, n, \{i_2, \cdots, i_k\} \neq \{i, \cdots, i\}\} \) the \( i \)th off-diagonal sum of \( \mathcal{T} \).

The set of eigenvalues of \( \mathcal{T} \) are called the spectrum of \( \mathcal{T} \). The largest modulus of the eigenvalues of \( \mathcal{T} \) is called the spectral radius of \( \mathcal{T} \), denoted by \( \rho(\mathcal{T}) \).

By \[4\], \( \mathcal{T} \) is called reducible if there exists a proper nonempty subset \( I \) of \( \{1, \cdots, n\} \) such that

\[ t_{i_1\cdots i_k} = 0, \forall i_1 \in I, \forall i_2, \cdots, i_k \not\in I. \]

If \( \mathcal{T} \) is not reducible, then we say that \( \mathcal{T} \) is irreducible. Let \( I = \{1, \cdots, n-1\} \). We may see that both \( \mathcal{L} \) and \( \mathcal{Q} \) are reducible.

Suppose that \( \mathcal{T} = (t_{i_1\cdots i_k}) \) is a \( k \)th order \( n \)-dimensional tensor. Construct a graph \( \hat{\mathcal{G}}(\mathcal{T}) = (\hat{\mathcal{V}}, \hat{\mathcal{E}}) \), where \( \hat{\mathcal{V}} = \bigcup_{j=1}^{n} V_j \), \( V_j \) is a copy of \( \{1, \cdots, n\} \), for \( j = 1, \cdots, n \). Assume that \( i_j \in V_j, i_l \in V_l, j \neq l \). The edge \( (i_j, i_l) \in \hat{\mathcal{E}} \) if and only if \( t_{i_1\cdots i_k} \neq 0 \) for some \( k-2 \) indices \( \{i_1, \cdots, i_k\} \setminus \{i_j, i_l\} \). The tensor \( \mathcal{T} \) is called weakly irreducible if \( \hat{\mathcal{G}}(\mathcal{T}) \) is connected. The original definition in [8] for weakly irreducible tensors are only for nonnegative tensors. Here we remove the nonnegativity restriction. As observed in [8], an irreducible tensor is always weakly irreducible. Very recently, Pearson and Zhang [15] proved that the adjacency tensor \( \mathcal{A} \) is weakly irreducible if and only if the \( k \)-graph \( \mathcal{G} \) is connected. Clearly, if the adjacency tensor \( \mathcal{A} \) is weakly irreducible, then \( \mathcal{L} \) and \( \mathcal{Q} \) are weakly irreducible. This shows that if \( \mathcal{G} \) is connected, then \( \mathcal{A}, \mathcal{L} \) and \( \mathcal{Q} \) are weakly irreducible.

If the entries \( t_{i_1\cdots i_k} \) are nonnegative, \( \mathcal{T} \) is called a nonnegative tensor. There is a rich theory on eigenvalues of a nonnegative tensor [4, 5, 8, 13, 14, 20, 21]. We now summarize the Perron-Frobenius theorem for nonnegative tensors, established in [4, 8, 20]. With the new definitions of \( H^+ \)-eigenvalues and \( H^{++} \)-eigenvalues, this theorem can be stated concisely.

**Theorem 2 (The Perron-Frobenius Theorem for Nonnegative Tensors)**

1. \( \text{(Yang and Yang 2010)} \) If \( \mathcal{T} \) is a nonnegative tensor of order \( k \) and dimension \( n \), then \( \rho(\mathcal{T}) \) is an \( H^+ \)-eigenvalue of \( \mathcal{T} \).
2. (Friedland, Gaubert and Han 2011) If furthermore $\mathcal{T}$ is weakly irreducible, then $\rho(\mathcal{T})$ is the unique $H^+-$eigenvalue of $\mathcal{T}$, with the unique eigenvector $x \in \mathbb{R}^n_+$, up to a positive scaling coefficient.

3. (Chang, Pearson and Zhang 2008) If moreover $\mathcal{T}$ is irreducible, then $\rho(\mathcal{T})$ is the unique $H^+-$eigenvalue of $\mathcal{T}$.

The tensors $\mathcal{L}$ and $\mathcal{Q}$ are reducible. This makes room that they have some genuine $H^+$ eigenvalues. In the next five sections, we will study their $H^+$ eigenvalues.

3 $H^+$-Eigenvalues of $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{Q}$

By Theorems 1, we have some basic properties of eigenvalues of the adjacency tensor $\mathcal{A}$, the Laplacian tensor $\mathcal{L}$ and the signless Laplacian tensors $\mathcal{Q}$. Note that they are all real $k$th order $n$-dimensional symmetric tensors. Both $\mathcal{A}$ and $\mathcal{Q}$ are nonnegative tensors. The diagonal elements of $\mathcal{A}$ are zero. The $i$th diagonal element of $\mathcal{L}$ and $\mathcal{Q}$ is $d_i > 0$. All the off-diagonal elements of $\mathcal{A}$ and $\mathcal{Q}$ are nonnegative. All the off-diagonal elements of $\mathcal{L}$ are non-positive. The $i$th off-diagonal sum of $\mathcal{A}$ and $\mathcal{Q}$ is $d_i$. The $i$th off-diagonal sum of $\mathcal{L}$ is $-d_i$.

**Theorem 3 (Basic Properties of Eigenvalues of $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{Q}$)**

Assume that $k \geq 3$. We have the following conclusions on eigenvalues of $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{Q}$.

(a). A number $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}/\mathcal{L}/\mathcal{Q}$ if and only if it is a root of the characteristic polynomial $\phi_{\mathcal{A}}/\phi_{\mathcal{L}}/\phi_{\mathcal{Q}}$.

(b). The number of eigenvalues of $\mathcal{A}/\mathcal{L}/\mathcal{Q}$ is $n(k-1)^{n-1}$. Their product is equal to $\det(\mathcal{A})/\det(\mathcal{L})/\det(\mathcal{Q})$.

(c). The sum of all the eigenvalues of $\mathcal{A}$ is zero. The sum of all the eigenvalues of $\mathcal{L}/\mathcal{Q}$ is $(k-1)^{n-1} \sum_{i=1}^{n} d_i = k(k-1)^{n-1}m$.

(d). The eigenvalues of $\mathcal{A}$ lie in the disk $|\lambda| \leq \Delta$. The eigenvalues of $\mathcal{L}$ and $\mathcal{Q}$ lie in the disk $|\lambda - \Delta| \leq \Delta$.

(e). $\mathcal{L}$ and $\mathcal{Q}$ are positive semidefinite when $k$ is even.

**Proof.** The conclusions (a), (b), (c) and (d) follow directly from Theorem 1 (a), (b), (c) and (e), and the basic structure of $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{Q}$. By (d), the real parts of all the eigenvalues of $\mathcal{L}$ and $\mathcal{Q}$ are nonnegative. Then (e) follows from Theorem 1 (d).

We now discuss $H^+$-eigenvalues of $\mathcal{L}$.

**Theorem 4 (H$^+$-Eigenvalues of $\mathcal{L}$)** Assume that $k \geq 3$. For $j = 1, \cdots, n$, $d_j$ is a genuine $H^+$-eigenvalue of $\mathcal{L}$ with an $H$-eigenvector $l^{(j)}$. Zero is the unique $H^{++}$-eigenvalue of $\mathcal{L}$ with an $H$-eigenvector $l$, and is the smallest $H$-eigenvalue of $\mathcal{L}$.
Proof. A real number $\mu$ is an H-eigenvalue of $\mathcal{L}$, with an H-eigenvector $x$, if and only if $x \in \mathbb{R}^n$, $x \neq 0$, and $\mathcal{L}x^{k-1} = \mu x^{[k-1]}$, i.e.,

$$d_i x_i^{k-1} - \sum \left\{ \frac{1}{(k-1)!} x_{i_2} \cdots x_{i_k} : (i, i_2, \ldots, i_k) \in E \right\} = \mu x_i^{k-1}, \quad (3)$$

for $i = 1, \ldots, n$. We now may easily verify that for $j = 1, \ldots, n$, $d_j$ is an $H^+$-eigenvalue of $\mathcal{L}$ with an H-eigenvector $l^{(j)}$, and zero is an $H^{++}$-eigenvalue of $\mathcal{L}$ with an H-eigenvector $l$. By Theorem 3(d), the real parts of all the eigenvalues of $\mathcal{L}$ are nonnegative. Thus, zero is the smallest H-eigenvalue of $\mathcal{L}$. Assume that $x$ is a positive H-eigenvector of $\mathcal{L}$, associated with an H-eigenvalue $\mu$. By Theorem 3(d), $\mu \geq 0$. Let $x_j = \min_i \{x_i\}$. By (3), we have

$$\mu = d_j - \sum \left\{ \frac{1}{(k-1)!} x_{i_2} \cdots x_{i_k} : (j, i_2, \ldots, i_k) \in E \right\} \leq d_j - d_j = 0.$$ 

This shows that $\mu = 0$. Thus, zero is the unique $H^{++}$ eigenvalue of $\mathcal{L}$, and $d_j$ is a genuine $H^+$ eigenvalue of $\mathcal{L}$, for $j = 1, \ldots, n$. \[\square\]

As in spectral graph theory [2], we may call eigenvalues / H-eigenvalue / $H^+$-eigenvalue / $H^{++}$-eigenvalue / spectrum / spectral radius of $A$ as eigenvalues / H-eigenvalue / $H^+$-eigenvalue / $H^{++}$-eigenvalue / spectrum / spectral radius of the $k$-graph $G$, or simply eigenvalues / H-eigenvalue / $H^+$-eigenvalue / $H^{++}$-eigenvalue / spectrum / spectral radius if the context is clear. Similarly, we may call eigenvalues / H-eigenvalues / $H^+$-eigenvalue / $H^{++}$-eigenvalue / spectrum / spectral radius of $\mathcal{L}$ and $Q$ as Laplacian and signless Laplacian eigenvalues / H-eigenvalues / $H^+$-eigenvalue / $H^{++}$-eigenvalue / spectrum / spectral radius of $G$, or simply Laplacian and signless Laplacian eigenvalues / H-eigenvalues / $H^+$-eigenvalue / $H^{++}$-eigenvalue / spectrum / spectral radius if the context is clear.

Theorem 3.1 of [6] is on the spectrum of the union of two disjoint hypergraphs. Checking its proof, it also holds for Laplacian and signless Laplacian spectra. This will be useful for our further discussion. We state it here but omit its proof as the proof is the same as the proof of Theorem 3.1 of [6].

Theorem 5 (The Union of Two Disjoint Hypergraphs) Suppose $G = (V, E)$ is the union of two disjoint hypergraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where $|V_1| = n_1, |V_2| = n_2, n_1 + n_2 = n = |V|$. Then the spectrum / the Laplacian spectrum / the signless Laplacian spectrum of $G$ is the union of the spectra / the Laplacian spectra / the signless Laplacian spectra of $G_1$ and $G_2$, where, as multisets, an eigenvalue with multiplicity $r$ in the spectrum / the Laplacian spectrum / the signless Laplacian spectrum of $G_1$ contributes it to the spectrum / the Laplacian spectrum / the signless Laplacian spectrum of $G$ with multiplicity $r(k-1)^{n_2}$.

In general, $G$ may be decomposed to some components $G_r = (V_r, E_r)$ for $r = 1, \ldots, s$. If $s = 1$, then $G$ is connected. Denote the adjacency tensor and the signless Laplacian tensor
of $G_r$ by $\mathcal{A}(G_r)$ and $\mathcal{Q}(G_r)$ respectively, for $r = 1, \cdots, s$. Then by Theorem 5,

$$\rho(\mathcal{A}) = \max_{r = 1, \cdots, s} \{ \rho(\mathcal{A}(G_r)) \}, \quad \rho(\mathcal{Q}) = \max_{r = 1, \cdots, s} \{ \rho(\mathcal{Q}(G_r)) \}.$$ 

With the above discussion, we are now ready to study $H^+$-eigenvalues of $\mathcal{Q}$ and $\mathcal{A}$.

**Theorem 6 (H$^+$-Eigenvalues of $\mathcal{Q}$) Assume that $k \geq 3$. Suppose that $G$ has $s$ components $G_r = (V_r, E_r)$ for $r = 1, \cdots, s$. For $j = 1, \cdots, n$, $d_j$ is a genuine $H^+$-eigenvalue of $\mathcal{Q}$ with an $H$-eigenvector $l^{(j)}$. Let $\nu_1 = \rho(\mathcal{Q})$. If $\nu_1 \equiv \rho(\mathcal{Q}(G_r))$ for $r = 1, \cdots, s$, then $\nu_1$ is the unique $H^{++}$-eigenvalue of $\mathcal{Q}$. Otherwise, $\mathcal{Q}$ has no $H^{++}$-eigenvalue, and for $r = 1, \cdots, s$, $\rho(\mathcal{Q}(G_r))$ is a genuine $H^+$-eigenvalue of $\mathcal{Q}$.

**Proof.** A real number $\nu$ is an $H$-eigenvalue of $\mathcal{Q}$, with an $H$-eigenvector $x$, if and only if $x \in \mathbb{R}^n$, $x \neq 0$, and $\mathcal{Q}x^{-k} = \nu x^{[k-1]}$, i.e.,

$$d_i x_i^{-k+1} + \sum \left\{ \frac{1}{(k-1)!} x_{i_2} \cdots x_{i_k} : (i, i_2, \cdots, i_k) \in E \right\} = \nu x_i^{k-1}, \quad (4)$$

for $i = 1 \cdots, n$. Then, we may easily verify that for $j = 1, \cdots, n$, $d_j$ is an $H^+$-eigenvalue of $\mathcal{Q}$ with an $H$-eigenvector $l^{(j)}$.

For $r = 1, \cdots, s$, as $G_r$ is connected, $\mathcal{Q}(G_r)$ is weakly irreducible by [15]. By Theorem 2, $\rho(\mathcal{Q}(G_r))$ is the unique $H^{++}$-eigenvalue of $\mathcal{Q}(G_r)$, with a positive $H$-eigenvector $x^{(r)} \in \mathbb{R}^{|V_r|}$. In (4), let $\nu = \rho(\mathcal{Q}(G_r))$, $x_i = x_i^{(r)}$ if $i \in V_r$ and $x_i = 0$ if $i \notin V_r$. Then we see that (4) is satisfied for $i = 1, \cdots, n$. This shows that for $r = 1, \cdots, s$, $\rho(\mathcal{Q}(G_r))$ is an $H^+$-eigenvalue of $\mathcal{Q}$.

Assume that $\nu$ is an $H^{++}$-eigenvalue of $\mathcal{Q}$ with a positive $H$-eigenvector $x$. For $r = 1, \cdots, s$, define $x^{(r)} \in \mathbb{R}^{|V_r|}$ by $x_i^{(r)} = x_i$ for $i \in V_r$. Then $x^{(r)}$ is a positive $H$-eigenvector in $\mathbb{R}^{|V_r|}$. By (4), $\nu$ is an $H^+$-eigenvalue of $\mathcal{Q}(G_r)$. Since $\mathcal{Q}(G_r)$ is weakly irreducible, by Theorem 2, $\nu = \rho(\mathcal{Q}(G_r))$. Thus, if $\mathcal{Q}$ has an $H^{++}$-eigenvalue, then it must be $\nu_1 = \rho(\mathcal{Q}) \equiv \rho(\mathcal{Q}(G_r))$ for $r = 1, \cdots, s$. This completes our proof. \qed

**Theorem 7 (H$^+$-Eigenvalues of $\mathcal{A}$) Assume that $k \geq 3$. Then zero is a genuine $H^+$-eigenvalue of $\mathcal{A}$. Suppose that $G$ has $s$ components $G_r = (V_r, E_r)$ for $r = 1, \cdots, s$. Let $\lambda_1 = \rho(\mathcal{A})$. If $\lambda_1 \equiv \rho(\mathcal{A}(G_r))$ for $r = 1, \cdots, s$, then $\lambda_1$ is the unique $H^{++}$-eigenvalue of $\mathcal{A}$. Otherwise, $\mathcal{A}$ has no $H^{++}$-eigenvalue, and for $r = 1, \cdots, s$, $\rho(\mathcal{A}(G_r))$ is a genuine $H^+$-eigenvalue of $\mathcal{A}$.

**Proof.** Zero is an $H$-eigenvalue of $\mathcal{A}$, with an $H$-eigenvector $x$, if and only if $x \in \mathbb{R}^n$, $x \neq 0$, and $\mathcal{A}x^{-k} = 0$, i.e.,

$$\sum \left\{ \frac{1}{(k-1)!} x_{i_2} \cdots x_{i_k} : (i, i_2, \cdots, i_k) \in E \right\} = 0,$$
for $i = 1 \cdots, n$. Let $x$ be a vector in $\mathbb{R}_+^n$ with $1 \leq \text{supp}(x) \leq k - 2$. Then we see that $x$ is a nonnegative $H$-eigenvector of $A$, corresponding to the zero $H$-eigenvalue. Thus, zero is an $H^+$-eigenvalue of $A$. The proof of the remaining conclusions of this theorem is similar to the last part of the proof of the last theorem. We omit it.

For some $k$-graph $G$, $L$, $Q$ and $A$ may have more genuine $H^+$-eigenvalues. For example, let $k = 3$, $n = 8$, $m = 8$, and $E = \{(1, 2, 3), (1, 4, 5), (2, 4, 5), (3, 4, 5), (4, 5, 6), (4, 5, 7), (4, 5, 8), (6, 7, 8)\}$. Then $d_1 = d_2 = d_3 = d_6 = d_7 = d_8 = 2$ and $d_4 = d_5 = 6$ are genuine $H^+$-eigenvalues of $L$ and $Q$, 0 is a genuine $H^+$-eigenvalue of $A$. It is easy to verify that $\mu = 1$, $\nu = 3$ and $\lambda = 1$ are also genuine $H^+$-eigenvalues of $L$, $Q$ and $A$, with an $H$-eigenvector $(1, 1, 1, 0, 0, 0, 0, 0)$.

We will not identify all genuine $H^+$-eigenvalues of $L$, $Q$ and $A$, but we will identify the largest and the smallest $H^+$-eigenvalues of $L$ and $Q$, and establish their maximum or minimum properties in the next few sections. They are the most important $H^+$-eigenvalues of $L$ and $Q$.

There are also $H$-eigenvalues of $L$ and $Q$, which are not $H^+$-eigenvalues. We will give such an example in Sections 5 and 7.

Theorems 4, 6 and 7 say that each of $L$, $Q$ and $A$ has at most one $H^{++}$-eigenvalue. Actually, a real symmetric matrix has at most one $H^{++}$-eigenvalue. By Theorem 2, a weakly irreducible nonnegative tensor has at most one $H^{++}$-eigenvalue. By extending the proof of Theorem 6, probably this is also true for a general nonnegative tensor. We may also show that this is true for a real diagonal tensor. However, by numerical experiments, we found that this is not true for some real symmetric tensors. Thus, we ask the following question.

**Question 1.** Is there a reasonable class of real symmetric tensors, which includes the above cases, such that any tensor in this class has at most one $H^{++}$-eigenvalue?

### 4 The Smallest Laplacian H-Eigenvalue

The smallest Laplacian $H$-eigenvalue of $G$ is $\mu_1 = 0$. By Theorem 4, $l$ is an $H$-eigenvector of $L$, associated with the $H^{++}$-eigenvalue $\mu_1 = 0$. We say that $x \in \mathbb{R}^n$ is a **binary vector** if $x_i$ is either 0 or 1 for $i = 1, \cdots, n$. Thus, $l$ is a binary $H$-eigenvector of $L$, associated with the $H$-eigenvalue $\mu_1 = 0$. We say that a binary $H$-eigenvector $x$ of $L$, associated with an $H$-eigenvalue $\mu$, is a **minimal binary $H$-eigenvector** of $L$, associated with $\mu$, if there does not exist another binary $H$-eigenvector $y$ of $L$, associated with $\mu$, such that $\text{supp}(y)$ is a proper subset of $\text{supp}(x)$.

For $e_p = (i_1, \cdots, i_k) \in E$, define a $k$th order $n$-dimensional symmetric tensor $L(e_p)$ by

$$L(e_p)x^k = \sum_{j=1}^{k} x_{i_j}^k - k x_{i_1} \cdots x_{i_k}$$
for any $x \in C^n$. Then, for any $x \in C^n$, we have

$$\mathcal{L}x^k = \sum_{e_p \in E} \mathcal{L}(e_p)x^k.$$  

**Theorem 8 (The Smallest Laplacian H-Eigenvalue)** For a $k$-graph $G$, we have the following conclusions.

(a) For any $x \in \mathbb{R}^+$, $\mathcal{L}x^k \geq 0$. We have

$$0 = \min \{\mathcal{L}x^k : x \in \mathbb{R}^+_n, \sum_{i=1}^n x_i^k = 1\}.$$  

(b) A binary vector $x \in \mathbb{R}^n$ is a minimal binary H-eigenvector of $\mathcal{L}$, associated with the H-eigenvalue $\mu_1 = 0$, if and only if $\text{supp}(x)$ is the vertex set of a component of $G$.

(c) A vector $x \in \mathbb{R}^n$ is an H-eigenvector of $\mathcal{L}$, associated with the H-eigenvalue $\mu_1 = 0$, if it is a nonzero linear combination of such minimal binary H-eigenvectors of $\mathcal{L}$, associated with the H-eigenvalue $\mu_1 = 0$.

**Proof.** (a). For any $e_p = (i_1, \cdots, i_k) \in E$ and $x \in \mathbb{R}^+_n$, we know that the arithmetic mean of $x_{i_1}^k, \cdots, x_{i_k}^k$ is greater than or equal to their geometric mean, i.e.,

$$\frac{1}{k} \sum_{j=1}^k x_{i_j}^k \geq x_{i_1} \cdots x_{i_k}.$$  

This implies that $\mathcal{L}(e_p)x^k \geq 0$. Thus, $\mathcal{L}x^k \geq 0$ for any $x \in \mathbb{R}^+_n$. As $\mathcal{L}y^k = 0$, where $y = \frac{1}{n^k}$, we have

$$0 = \min \{\mathcal{L}x^k : x \in \mathbb{R}^+_n, \sum_{i=1}^n x_i^k = 1\}.$$  

(b). A nonzero vector $x \in \mathbb{R}^n$ is an H-eigenvector of $\mathcal{L}$, associated with the H-eigenvalue $\mu_1 = 0$, if and only $\mathcal{L}x^{k-1} = 0$, i.e.,

$$d_ix_i^{k-1} = \sum \left\{ \frac{1}{(k-1)!} x_{i_2} \cdots x_{i_k} : (i, i_2, \cdots, i_k) \in E \right\},$$  

for $i = 1, \cdots, n$.

Suppose that $x$ is a binary vector and $\text{supp}(x)$ is the vertex set of a component of $G$. Then the equation (5) is $d_i = d_i$ if $i \in \text{supp}(x)$, and $0 = 0$ if $i \not\in \text{supp}(x)$. Thus, $x$ is a binary H-eigenvector of $\mathcal{L}$, associated with the H-eigenvalue $\mu_1 = 0$. Suppose that $y$ is a binary vector and $\text{supp}(y)$ is a proper subset of $\text{supp}(x)$. Then there are $i \in \text{supp}(y)$ and an edge $(i, i_2, \cdots, i_k) \in E$ such that one of the indices $\{i_2, \cdots, i_k\}$ not in $\text{supp}(y)$. Then, for this $i$, the left hand side of (5) is $d_i$, while the right hand side of (5) is strictly less than $d_i$, i.e., (5) does not hold. This shows that $y$ cannot be a binary H-eigenvector $x$ of $\mathcal{L}$, associated with
the H-eigenvalue \( \mu_1 = 0 \), i.e., \( x \) is a minimal binary H-eigenvector of \( \mathcal{L} \), associated with the H-eigenvalue \( \mu_1 = 0 \).

On the other hand, suppose that \( x \) is a binary H-eigenvector of \( \mathcal{L} \), associated with the H-eigenvalue \( \mu_1 = 0 \). Let \( i \in \text{supp}(x) \). Then, in order that the equation (5) holds for \( i \), for any \((i, i_2, \ldots, i_k) \in E\), we must have \( i_2, \ldots, i_k \in \text{supp}(x) \). This shows that \( \text{supp}(x) \) is either the vertex set of a component of \( G \), or the union of the vertex sets of several components of \( G \). This proves (b).

(c). Let \( \{y^{(1)}, \ldots, y^{(s)}\} \) be the set of such binary H-eigenvectors of \( \mathcal{L} \), associated with the H-eigenvalue \( \mu_1 = 0 \).

Suppose that \( x \) is a nonzero linear combination of \( y^{(1)}, \ldots, y^{(s)} \), \( x = \sum_{r=1}^{s} \alpha_r y^{(r)} \), where \( \alpha_r \) are real numbers. If \( i \in \text{supp}(y^{(r)}) \) for some \( r \), then the equation (5) is \( \alpha_r^{k-1}d_i = \alpha_r^{k-1}d_i \). Otherwise, the equation (5) is \( 0 = 0 \). Thus, \( x \) is an H-eigenvector of \( \mathcal{L} \), associated with the H-eigenvalue \( \mu_1 = 0 \). This proves (c).

\[ \square \]

**Corollary 9** The following two statements are equivalent.

(a). The \( k \)-graph \( G \) is connected.

(b). The vector \( l \) is the unique minimal binary H-eigenvector of \( \mathcal{L} \), associated with the H-eigenvalue \( \mu_1 = 0 \).

5 The Largest Laplacian \( H^+ \)-Eigenvalue

In Section 3, we show that zero is the unique Laplacian \( H^+ \)-eigenvalue of \( G \), and \( d_j \) is a genuine \( H^+ \)-eigenvalue of \( G \), for \( j = 1, \ldots, n \). We now identify the largest Laplacian \( H^+ \)-eigenvalue of \( G \), and establish a maximum property of this Laplacian \( H^+ \)-eigenvalue.

**Theorem 10 (The Largest Laplacian \( H^+ \)-Eigenvalue)** Assume that \( k \geq 3 \). The largest Laplacian \( H^+ \)-eigenvalue of \( G \) is \( \Delta = \max \{d_i\} \). We have

\[ \Delta = \max \{\mathcal{L}x^k : x \in \mathbb{R}_+^n, \sum_{i=1}^{n} x_i^k = 1\}. \]  

(6)

**Proof.** Suppose that \( \mu \) is a Laplacian \( H^+ \)-eigenvalue of \( G \), associated with a nonnegative H-eigenvector \( x \). Assume that \( x_j > 0 \). By (3), we have

\[ \mu x_j^{k-1} = d_j x_j^{k-1} - \sum \left\{ \frac{1}{(k-1)!} x_{i_2} \cdots x_{i_k} : (i, i_2, \ldots, i_k) \in E \right\} \leq d_j x_j^{k-1}. \]

This implies that

\[ \mu \leq d_j \leq \Delta. \]

By Theorem 4, \( \Delta \) is an \( H^+ \)-eigenvalue of \( \mathcal{L} \). Thus, \( \Delta \) is the largest \( H^+ \)-eigenvalue of \( \mathcal{L} \).
Suppose that $\Delta = d_j$. Let $x = l^{(j)}$. Then $x$ is a feasible point of the maximization problem in (6). We have

$$\mathcal{L} x^k = \sum_{i=1}^{n} \left[ d_i x_i^k - \sum \left\{ \frac{1}{(k-1)!} x_i x_{i_2} \cdots x_{i_k} : (i, i_2, \ldots, i_k) \in E \right\} \right] = \Delta.$$ 

This shows that

$$\Delta \leq \max \{ \mathcal{L} x^k : x \in \mathbb{R}_+^n, \sum_{i=1}^{n} x_i^k = 1 \}.$$ 

On the other hand, suppose $x^*$ is a maximizer of the maximization problem in (6). As the feasible set is compact, and the objective function is continuous, such a maximizer exists. By optimization theory, for $i = 1, \ldots, n$, either $x_i^* = 0$ and

$$d_i (x_i^*)^{k-1} - \sum \left\{ \frac{1}{(k-1)!} x_{i_2}^* \cdots x_{i_k}^* : (i, i_2, \ldots, i_k) \in E \right\} \geq \mu(x_i^*)^{k-1},$$ 

or $x_i^* > 0$ and

$$d_i (x_i^*)^{k-1} - \sum \left\{ \frac{1}{(k-1)!} x_{i_2}^* \cdots x_{i_k}^* : (i, i_2, \ldots, i_k) \in E \right\} = \mu(x_i^*)^{k-1},$$ 

where $\mu$ is a Lagrangian multiplier. As $x^*$ is feasible to the maximization problem, (8) holds for at least one $i$. For this $i$, we have

$$d_i (x_i^*)^{k-1} \geq \mu(x_i^*)^{k-1}.$$ 

As $x_i^* > 0$ for this $i$, we have $\mu \leq d_i \leq \Delta$. Multiplying (7) and (8) by $x_i^*$ and summing up them for $i = 1, \ldots, n$, we have

$$\mathcal{L}(x^*)^k = \mu \sum_{i=1}^{n} (x_i^*)^k = \mu.$$ 

Thus,

$$\mu = \max \{ \mathcal{L} x^k : x \in \mathbb{R}_+^n, \sum_{i=1}^{n} x_i^k = 1 \}.$$ 

This shows that

$$\Delta \geq \max \{ \mathcal{L} x^k : x \in \mathbb{R}_+^n, \sum_{i=1}^{n} x_i^k = 1 \}.$$ 

Hence, (6) holds.

In general, $\Delta$ may not be the largest H-eigenvalue of $\mathcal{L}$. For example, let $n = k = 6, m = 1$ and $E = \{(1, 2, 3, 4, 5, 6)\}$. Then $\Delta = 1$, while $\mu = 2$ is an H-eigenvalue of $\mathcal{L}$ with an H-eigenvector $(1, 1, 1, -1, -1, -1)$.
6 The Largest H-Eigenvalue and The Largest Signless Laplacian H-Eigenvalue

The largest H-eigenvalue is \( \lambda_1 = \rho(A) \). The largest signless Laplacian H-eigenvalue is \( \nu_1 = \rho(Q) \). As both \( A \) and \( Q \) are nonnegative tensors, their properties are similar. We thus discuss them together.

When \( k \) is even, by [16], we know that
\[
\lambda_1 = \max \{ Ax^k : x \in \mathbb{R}^n, \sum_{i=1}^{n} x_i^k = 1 \},
\]
and
\[
\nu_1 = \max \{ Qx^k : x \in \mathbb{R}^n, \sum_{i=1}^{n} x_i^k = 1 \}.
\]
The feasible sets of the above two maximization problems are the same. It is a compact set when \( k \) is even. When \( k \) is odd, it is not compact. We intend to establish some maximum properties of \( \lambda_1 \) and \( \nu_1 \), which hold whenever \( k \) is even or odd.

Corollary 3.4 of [6] indicates that when \( G \) is connected,
\[
\lambda_1 = \max \{ Ax^k : x \in \mathbb{R}_{++}^n, \sum_{i=1}^{n} x_i^k = 1 \}. \tag{9}
\]
Using a similar argument, we may show that when \( G \) is connected,
\[
\nu_1 = \max \{ Qx^k : x \in \mathbb{R}_{++}^n, \sum_{i=1}^{n} x_i^k = 1 \}. \tag{10}
\]
We wish to show that (9) and (10) hold even if \( G \) is not connected.

**Theorem 11 (The Largest H-Eigenvalue and the Largest Signless Laplacian H-Eigenvalue)** Assume that \( k \geq 3 \). Then (9) and (10) always hold.

**Proof.** We now prove (9). Suppose that \( G \) is decomposed to some components \( G_r = (V_r, E_r) \) for \( r = 1, \ldots, s \). Then \( \lambda_1 = \max \{ \rho(A(G_r)) : r = 1, \ldots, s \} \), and for \( r = 1, \ldots, s \),
\[
\rho(A(G_r)) = \max \{ A(G_r)(x^{(r)})^k : x^{(r)} \in \mathbb{R}_{+}^{|I_r|}, \sum_{i \in I_r} (x_i^{(r)})^k = 1 \}.
\]
Suppose that \( \lambda_1 = \rho(A(G_j)) \) for some \( j \). Define \( x \in \mathbb{R}_{+}^n \) by \( x_i = x_i^{(j)} \) if \( i \in I_j \) and \( x_i = 0 \) otherwise. Then \( \sum_{i=1}^{n} x_i^k = 1 \), and \( Ax^k = A(G_j)(x^{(j)})^k \). We see that \( \lambda_1 = Ax^k \) and \( x \) is a feasible point of the maximization problem in (9). This shows that
\[
\lambda_1 \leq \max \{ Ax^k : x \in \mathbb{R}_{++}^n, \sum_{i=1}^{n} x_i^k = 1 \}.
\]
On the other hand, suppose that $x_*$ is a maximizer of the maximization problem in (9). Then,
\[
\max\{Ax^k : x \in \mathbb{R}_+^n, \sum_{i=1}^n x_i^k = 1\} = A x_*^k = \sum_{r=1}^s \mathcal{A}(G_r)(\bar{x}^{(r)})^k,
\]
where $\bar{x}^{(r)} \in \mathbb{R}_+^{|I_r|}$ and $\bar{x}^{(r)}_i = (x_*)_i$ for $i \in I_r$, for $r = 1, \ldots, s$. For $r = 1, \ldots, s$, assume that $\alpha_r = \sum_{i \in I_r} (x_*)_i^k$. Then $\alpha_r \geq 0$ for $r = 1, \ldots, s$, and $\sum_{r=1}^s \alpha_r = 1$. If $\alpha_r > 0$, then define $x^{(r)} \in \mathbb{R}_+^{|I_r|}$ by $x^{(r)} = \frac{1}{(\alpha_r)^{1/k}} \bar{x}^{(r)}$. Then $\sum_{i \in I_r} (x^{(r)}_i)^k = 1$. We now have
\[
A x_*^k = \sum \{\mathcal{A}(G_r)(\bar{x}^{(r)})^k : \alpha_r > 0\} = \sum (\alpha_r \mathcal{A}(G_r)(x^{(r)})^k : \alpha_r > 0) \leq \sum (\alpha_r \rho(\mathcal{A}(G_r)) : \alpha_r > 0)
\]
\[
\leq \sum \{\alpha_r \lambda_1 : \alpha_r > 0\} = \lambda_1.
\]
Thus, we have
\[
\lambda_1 \geq \max\{Ax^k : x \in \mathbb{R}_+^n, \sum_{i=1}^n x_i^k = 1\}.
\]
Hence, (9) holds.

Similarly, we may show that (10) holds.

**Corollary 12 (Bounds for $\nu_1$)** We always have
\[
\max\{\Delta, 2\bar{d}\} \leq \nu_1 \leq 2\Delta. \tag{11}
\]

**Proof.** By Theorem 3 (d), we have that
\[
0 \leq \nu_1 \leq 2\Delta.
\]
In (10), letting $x = \frac{l}{n_\mathcal{L}_j}$, we see that $\nu_1 \geq 2\bar{d}$. Assume that $d_j = \Delta$. In (10), letting $x = l^{(j)}$, we see that $\nu_1 \geq \Delta$. Thus, we always have
\[
\nu_1 \geq \max\{\Delta, 2\bar{d}\}.
\]
These prove (11). \qed

It was established in [6] that $\bar{d} \leq \lambda_1 \leq \Delta$.

**Question 2.** Are there some analytical formulas for $\lambda_1$ and $\nu_1$?

We may compare $\nu_1$, $\lambda_1$ and $\rho(\mathcal{L})$. We prove a lemma first.

**Lemma 13** If $\mathcal{C}$ is a nonnegative tensor of order $k$ and dimension $n$, and $\mathcal{B}$ is a tensor of order $k$ and dimension $n$, satisfying $|\mathcal{B}| \leq \mathcal{C}$, then $\rho(\mathcal{B}) \leq \rho(\mathcal{C})$. 

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Proof. Let \( C = C + \epsilon J \), with \( \epsilon > 0 \). Then \( C \) is a positive tensor thus irreducible, \( |B| \leq C_\epsilon \). By Lemma 3.2 of [20], we have \( \rho(B) \leq \rho(C_\epsilon) \). Let \( \epsilon \to 0 \). As the eigenvalues of a tensor are roots of the characteristic polynomial, whose coefficients are elements of that tensor [16], the spectral radius of that tensor should be continuous with its elements. Then we have \( \rho(B) \leq \rho(C) \). 

With this lemma, we immediately have the following proposition.

**Proposition 14** For a k-graph \( G \), we have 

\[
\nu_1 = \rho(Q) \geq \rho(L), \quad \text{and} \quad \nu_1 = \rho(Q) \geq \lambda_1 = \rho(A).
\]

Note that it is possible that \( \nu_1 = \rho(Q) = \rho(L) \). For example, let \( n = k = 6, m = 1 \) and \( E = \{(1, 2, 3, 4, 5, 6)\} \). Then \( G \) is connected. Thus, \( A, L \) and \( Q \) are weakly irreducible. We have \( Lx^6 = \sum_{i=1}^6 x_i^6 - 6x_1 \cdots x_6 \) and \( Qx^6 = \sum_{i=1}^6 x_i^6 - 6x_1 \cdots x_6 \). We see that \( \nu = 2 \) is an \( H_+ \) eigenvalue of \( Q \) with an \( H \)-eigenvector \( l = (1, 1, 1, 1, 1, 1) \). By Theorem 2 (b), we have \( \rho(Q) = 2 \). On the other hand, we see \( \mu = 2 \) is an \( H \)-eigenvalue of \( L \) with an \( H \)-eigenvector \( l = (1, 1, 1, -1, -1, -1) \). By Proposition 14 (d), we have \( \rho(L) = \rho(Q) = 2 \). Thus, it is a research topic to identify the conditions that \( \rho(L) = \rho(Q) \).

We now discuss algorithms for computing \( \nu_1 \). As \( Q \) is a nonnegative tensor, we may use some algorithms for finding the largest eigenvalue of a nonnegative tensor to compute it. However, the convergence of the NQZ algorithm [14] needs the condition that \( Q \) is primitive [5], the convergence of the LZI algorithm needs the condition that \( Q \) is irreducible [13]. These are somewhat strong. The linear convergence of the LZI algorithm needs the condition that \( Q \) is weakly positive [21]. A nonnegative tensor \( T = (t_{i_1, \ldots, i_k}) \) is weakly positive if \( t_{i, j, \ldots, j} > 0 \). We see that \( Q \) cannot be weakly positive. Thus, we do not recommend to use these two algorithms for computing \( \nu_1 \). We suggest to use the HHQ algorithm proposed in [9] to compute \( \mu_1 \). The HHQ algorithm is globally R-linearly convergent if \( Q \) is weakly irreducible in the sense of [8]. As we discussed before, if \( G \) is connected, then \( Q \) is weakly irreducible. Thus, the HHQ algorithm is practicable for computing \( \nu_1 \). This argument is also valid for computing \( \lambda_1 \).

Thus, we may use the HHQ algorithm to compute \( \lambda_1 \) and \( \nu_1 \), and we have globally R-linearly convergence.

### 7 The Smallest Signless Laplacian \( H^+ \)-Eigenvalue

We now identify the smallest signless Laplacian \( H^+ \)-eigenvalue of \( G \), and establish a minimum property of this signless Laplacian \( H^+ \)-eigenvalue.
Theorem 15 (The Smallest Signless Laplacian $H^+$-Eigenvalue) The smallest signless Laplacian $H^+$-eigenvalue of $G$ is $d_{\text{min}}$. We always have

$$d_{\text{min}} = \min\{Qx^k : x \in \mathbb{R}^n_+, \sum_{i=1}^n x_i^k = 1\}. \quad (12)$$

**Proof.** Suppose that $\nu$ is an $H^+$-eigenvalue of $Q$, with a nonnegative $H$-eigenvector $x$. Suppose that $x_j > 0$. By (4), we have

$$d_j x_j^{k-1} + \sum \left\{ \frac{1}{(k-1)!} x_{i_1} \cdots x_{i_k} : (j, i_2, \cdots, i_k) \in E \right\} = \nu x_j^{k-1}.$$ 

This implies that $d_j x_j^{k-1} \leq \nu x_j^{k-1}$, i.e., $\nu \geq d_j \geq d_{\text{min}}$. As $d_{\text{min}}$ is an $H^+$-eigenvalue of $Q$ by Theorem 6, this shows that $d_{\text{min}}$ is the smallest $H^+$-eigenvalue of $Q$.

We now prove (12). Suppose that $d_j = d_{\text{min}}$. Let $x = l^{(j)}$ in (12). Then we have

$$d_{\text{min}} \geq \min\{Qx^k : x \in \mathbb{R}^n_+, \sum_{i=1}^n x_i^k = 1\}. \quad (13)$$

Suppose that $x^*$ is an optimal solution of the minimization problem in (12). By the optimization theory, there are optimal Lagrangian multipliers $u \in \mathbb{R}^n$ and $\nu \in \mathbb{R}$ such that for $i = 1, \cdots, n$,

$$\left( Q(x^*)^{k-1} \right)_i = \nu (x^*_i)^{k-1} + u_i,$$ 

$$x_i^* \geq 0, \; u_i \geq 0, \; x_i^* u_i = 0$$ 

and

$$\sum_{i=1}^n (x_i^*)^k = 1. \quad (15)$$

Let $I = \text{supp}(x^*)$. By (15), $I \neq \emptyset$. Then for $i \in I, u_i = 0$ and for $i \notin I, x_i^* = 0$. Multiplying (14) by $x_i^*$ and summing up it from $i = 1$ to $n$, we have

$$\nu = Q(x^*)^k.$$ 

Now assume that $x_j^* = \max\{x_i^* : i \in I\}$. Then $x_j^* > 0$ and $u_j = 0$. By (14), we have

$$\left( Q(x^*)^{k-1} \right)_j = \nu (x_j^*)^{k-1},$$

which implies that

$$d_j (x_j^*)^{k-1} \leq \nu (x_j^*)^{k-1}.$$

Thus,

$$\nu = Q(x^*)^k \geq d_j \geq d_{\text{min}}.$$
Hence,
\[ d_{\min} \leq \min \{ Qx^k : x \in \mathbb{R}^n_+, \sum_{i=1}^n x_i^k = 1 \}. \]

Combining this with (13), we have (12).

In general, \( d_{\min} \) may not be the smallest H-eigenvalue of \( Q \). For example, let \( n = k = 6, m = 1 \) and \( E = \{(1,2,3,4,5,6)\} \). Then \( d_{\min} = 1 \), while \( \nu = 0 \) is an H-eigenvalue of \( Q \) with an H-eigenvector \((1,1,1,-1,-1,-1)\). We may show that \( Q \) has a zero H-eigenvalue if and only if \( k = 4j + 2 \) for some integer \( j \), and there is a vector \( x \in \mathbb{R}^k \) such that for any edge \( e_p = (i_1, \ldots, i_k) \in E \), a half of \( x_{i_1}, \ldots, x_{i_k} \) are \( j \), and the other half are \(-1\). Hence, if \( k = 4j \) or if \( k = 4j + 2 \) but such an \( x \) does not exist, then \( Q \) is positive definite.

We now give an application of Theorem 15. Suppose that \( S \) is a proper nonempty subset of \( V \). Denote \( \bar{S} = V \setminus S \). Then \( \bar{S} \) is also a proper nonempty subset of \( V \). The edge set \( E \) is now partition into three parts \( E(S), E(\bar{S}) \) and \( E(S, \bar{S}) \). The edge set \( E(S) \) consists of edges whose vertices are all in \( S \). The edge set \( E(\bar{S}) \) consists of edges whose vertices are all in \( \bar{S} \). The edge set \( E(S, \bar{S}) \) consists of edges whose vertices are in both \( S \) and \( \bar{S} \).

We call \( E(S, \bar{S}) \) an **edge cut** of \( G \). If we take out of \( E(S, \bar{S}) \), then \( G \) is separated to two \( k \)-graphs \( G(S) = (S, E(S)) \) and \( G(\bar{S}) = (\bar{S}, E(\bar{S})) \). For a vertex \( i \in S \), we denote its degree at \( G(S) \) by \( d_i(S) \). Similarly, for a vertex \( i \in \bar{S} \), we denote its degree at \( G(\bar{S}) \) by \( d_i(\bar{S}) \). We denote the maximum degrees, the minimum degrees, the average degrees of \( G(S) \) and \( G(\bar{S}) \) by \( \Delta(S), \Delta(\bar{S}), d_{\min}(S), d_{\min}(\bar{S}), \bar{d}(S) \) and \( \bar{d}(\bar{S}) \) respectively. For an edge \( e_p \in E(S, \bar{S}) \), \( t(e_p) \) of its vertices are in \( S \), where \( 1 \leq t(e_p) \leq k - 1 \). For all edges \( e_p \in E(S, \bar{S}) \), the average value of such \( t(e_p) \), is denoted by \( t(S) \). Then \( 1 \leq t(S) \leq k - 1 \). Similarly, we may define \( t(\bar{S}) \). Then \( t(S) + t(\bar{S}) = k \). We call the minimum/maximum cardinality of such an edge cut the **edge connectivity** / maximum **cut** of \( G \), and denote it by \( e(G) / c(G) \).

For \( e_p = (i_1, \ldots, i_k) \in E \), define a \( k \)th order \( n \)-dimensional symmetric tensor \( Q(e_p) \) by
\[ Q(e_p)x^k = \sum_{j=1}^k x_{i_j}^k + kx_{i_1} \cdots x_{i_k} \]
for any \( x \in \mathbb{R}^n \). Then, for any \( x \in \mathbb{R}^n \), we have
\[ Qx^k = \sum_{e_p \in E} Q(e_p)x^k. \]

**Proposition 16** For a \( k \)-graph \( G \), we have the following conclusions.

(a). The edge connectivity \( e(G) \leq d_{\min} \).

(b). If \( n \leq 2k - 1 \), then \( e(G) = d_{\min} \).

(c). We have
\[ c(G) \leq \frac{n}{k}(2 \bar{d} - d_{\min}). \]
**Proof.** (a). Assume that $d_j = d_{\min}$. Let $S = \{j\}$. Then $|E(S, \bar{S})| = d_j = d_{\min}$. This proves (a).

(c). Let $S$ be a nonempty proper subset of $V$. Let $x = \frac{1}{|S|} \sum_{i \in S} l^{(i)}$. For $e_p \in E(S)$, we have

$$Q(e_p)x^k = \frac{2^k}{|S|}.$$ 

For $e_p \in E(\bar{S})$, we have

$$Q(e_p)x^k = 0.$$ 

For $e_p \in E(S, \bar{S})$, we have

$$Q(e_p)x^k = \frac{t(e_p)}{|S|}.$$ 

As

$$Qx^k = \left( \sum_{e_p \in E(S)} + \sum_{e_p \in E(\bar{S})} + \sum_{e_p \in E(S, \bar{S})} \right) Q(e_p)x^k,$$

we have

$$Qx^k = \frac{2^k}{|S|} |E(S)| + \frac{t(S)}{|S|} |E(S, \bar{S})|. \quad (16)$$

Similarly, letting $y = \frac{1}{|\bar{S}|} \sum_{i \in \bar{S}} l^{(i)}$, we have

$$Qy^k = \frac{2^k}{|\bar{S}|} |E(\bar{S})| + \frac{t(\bar{S})}{|\bar{S}|} |E(S, \bar{S})|. \quad (17)$$

By (12) and (16), we have

$$|S|d_{\min} \leq 2^k |E(S)| + t(S) |E(S, \bar{S})|. \quad (18)$$

By (12) and (17), we have

$$|\bar{S}|d_{\min} \leq 2^k |E(\bar{S})| + t(\bar{S}) |E(S, \bar{S})|. \quad (19)$$

Summing up (18) and (19), we have

$$nd_{\min} \leq 2^k (|E(S)| + |E(\bar{S})|) + k|E(S, \bar{S})|,$$

i.e.,

$$nd_{\min} \leq 2^k \left( m - |E(S, \bar{S})| \right) + k|E(S, \bar{S})|,$$

which implies that

$$d_{\min} \leq \frac{2km}{n} - \frac{k}{n} |E(S, \bar{S})|.$$ 

Noticing that $\bar{d} = \frac{km}{n}$, we have

$$|E(S, S)| \leq \frac{n}{k} (2\bar{d} - d_{\min}).$$
This proves (c).

(b). When \(n \leq 2k - 1\), either \(|S| < k\) or \(|\bar{S}| < k\). Without loss of generality, assume that \(|S| < k\). Then \(E(S) = \emptyset\) and \(|E(S)| = 0\). From (18), we have

\[|S|d_{\min} \leq t(S)|E(S, \bar{S})|\]

We always have \(t(S) \leq |S|\). Thus, we have

\[d_{\min} \leq |E(S, \bar{S})|\]

Combining this with Conclusion (a), we have Conclusion (b). \(\square\)

8 Algebraic Connectivity

We define the **algebraic connectivity** \(\alpha(G)\) of the \(k\)-graph \(G\) by

\[\alpha(G) = \min_{j=1, \ldots, n} \min \{Lx^k : x \in \mathbb{R}_+^n, \sum_{i=1}^n x^k_i = 1, x_j = 0\}\]

By Theorem 8, \(Lx^k \geq 0\) for any \(x \in \mathbb{R}_+^n\). Thus, \(\alpha(G) \geq 0\). We first prove the following proposition.

**Proposition 17** The \(k\)-graph \(G\) is connected if and only if the algebraic connectivity \(\alpha(G) > 0\).

**Proof.** Suppose that \(G\) is not connected. Let \(G_1 = (V_1, E_1)\) be a component of \(G\). Then there is a \(j \in V \setminus V_1\). Let \(x = \frac{1}{|V_1|} \sum_{i \in V_1} l^{(i)}\). Then \(x\) is a feasible point of \(\min \{Lx^k : x \in \mathbb{R}_+^n, \sum_{i=1}^n x^k_i = 1, x_j = 0\}\), and we see that \(\min \{Lx^k : x \in \mathbb{R}_+^n, \sum_{i=1}^n x^k_i = 1, x_j = 0\} = 0\). This implies that \(\alpha(G) = 0\).

Suppose that \(\alpha(G) = 0\). There is a \(j\) such that \(\min \{Lx^k : x \in \mathbb{R}_+^n, \sum_{i=1}^n x^k_i = 1, x_j = 0\} = 0\). Suppose that \(x^*\) is a minimizer of this minimization problem. Then \(x^*_j = 0\), \(L(x^*)_k = 0\) and by optimization theory, there are optimal Lagrangian multiplier \(\mu\) such that for \(i = 1, \ldots, n, i \neq j\), either \(x^*_i = 0\) and

\[d_i(x^*_i)^{k-1} - \sum \left\{ \frac{1}{(k-1)!} x^*_{i_2} \cdots x^*_{i_k} : (i, i_2, \ldots, i_k) \in E \right\} \geq \mu(x^*_i)^{k-1}, \quad (20)\]

or \(x^*_i > 0\) and

\[d_i(x^*_i)^{k-1} - \sum \left\{ \frac{1}{(k-1)!} x^*_{i_2} \cdots x^*_{i_k} : (i, i_2, \ldots, i_k) \in E \right\} = \mu(x^*_i)^{k-1}. \quad (21)\]
In (20) and (21), we always have $x^* \in \mathbb{R}^n_+ \sum_{i=1}^n (x^*_i)^k = 1$ and $x^*_j = 0$. Multiplying (20) and (21) with $x^*_i$ and summarizing them together, we have $\mu \sum_{i=1}^n (x^*_i)^k = \mathcal{L}(x^*)^k = 0$, i.e., $\mu = 0$. Then for $i = 1, \ldots, n, i \neq j$, either $x^*_i = 0$ or
diagimage
$d_i (x^*)^{k-1} - \sum \left\{ \frac{1}{(k-1)!} x^*_i \cdots x^*_k : (i, i_2, \ldots, i_k) \in E \right\} = 0. \tag{22}$
Let $x^*_r = \max \{ x^*_i : i = 1, \ldots, n \}$. Then by (22), we have
$0 = d_r - \sum \left\{ \frac{1}{(k-1)!} x^*_i \cdots x^*_k : (r, i_2, \ldots, i_k) \in E \right\}.$
Note that
$d_r = \sum \left\{ \frac{1}{(k-1)!} : (r, i_2, \ldots, i_k) \in E \right\}.$
Thus, we have $x_i = x_r$ as long as $i$ and $r$ are in the same edge. From this, we see that
$x_i = x_r$ as long as $i$ and $r$ are in the same component of $G$. Since $x^*_j = 0$, we see that $j$ and $r$ are in the different
components of $G$, i.e., $G$ is not connected. This proves the proposition.

We now further explore an application of $\alpha(G)$.

**Proposition 18** For a $k$-graph $G$, we have
$e(G) \geq \frac{n}{k} \alpha(G).$

**Proof.** Let $S$ be a nonempty proper subset of $V$. Then there is a $j \not\in S$ such that
$\min \{ \mathcal{L}x^k : x \in \mathbb{R}^n_+, \sum_{i=1}^n x_i^k = 1, x_j = 0 \} \geq \alpha(G). \tag{23}$
Let $x = \frac{1}{|S|} \sum_{i \in S} l^{(i)}$. Then $x$ is a feasible point of the minimization problem in (23). For
$e_p \in E(S)$ and $e_p \in E(\bar{S})$, we have
$\mathcal{L}(e_p)x^k = 0,$
where $\mathcal{L}(e_p)$ is defined in Section 4. For $e_p \in E(S, \bar{S})$, we have
$\mathcal{L}(e_p)x^k = \frac{t(e_p)}{|S|}.$
As
$\mathcal{L}x^k = \left( \sum_{e_p \in E(S)} + \sum_{e_p \in E(\bar{S})} + \sum_{e_p \in E(S, \bar{S})} \right) \mathcal{L}(e_p)x^k,$
we have
$\mathcal{L}x^k = \frac{t(S)}{|S| \cdot |E(S, \bar{S})|}. \tag{24}$
Similarly, letting \( y = \frac{1}{|S|} \sum_{i \in S} l^{(i)} \), we have

\[
\mathcal{L}y^k = \frac{t(\bar{S})}{|S|} |E(S, \bar{S})|.
\]  

By (23) and (24), we have

\[
|S|\alpha(G) \leq t(S)|E(S, \bar{S})|.
\]  

By (23) and (25), we have

\[
|\bar{S}|\alpha(G) \leq t(\bar{S})|E(S, \bar{S})|.
\]  

Summing up (26) and (27), we have

\[
n\alpha(G) \leq k|E(S, \bar{S})|,
\]

i.e.,

\[
\frac{n}{k} \alpha(G) \leq |E(S, \bar{S})|.
\]

This implies that

\[
e(G) \geq \frac{n}{k}\alpha(G).
\]

We now give an upper bound for \( \alpha(G) \).

**Proposition 19** For a k-graph \( G \), we have

\[
0 \leq \alpha(G) \leq d_{\text{min}}.
\]

**Proof.** We know \( \alpha(G) \geq 0 \). It suffices to prove that \( \alpha(G) \leq d_{\text{min}} \). Suppose that \( d_r = d_{\text{min}} \) and \( j \neq r \). Then \( l^{(r)} \) is a feasible point of

\[
\min\{ \mathcal{L}x^k : x \in \mathbb{R}_+^n, \sum_{i=1}^n x_i^k = 1, x_j = 0 \},
\]

and \( \mathcal{L}(l^{(r)})^k = d_{\text{min}} \). This implies that

\[
\alpha(G) \leq \min\{ \mathcal{L}x^k : x \in \mathbb{R}_+^n, \sum_{i=1}^n x_i^k = 1, x_j = 0 \} \leq d_{\text{min}}.
\]  

By Proposition 17, when \( G \) is not connected, \( \alpha(G) = 0 \). Let \( n = k, m = 1, E = \{(1, 2, \ldots, k)\} \). Then \( \mathcal{L}x^k = \sum_{i=1}^k x_i^k - kx_1 \cdots x_k \), and we see that \( \alpha(G) = 1 = d_{\text{min}} \). Thus, both the lower bound 0 and the upper bound \( d_{\text{min}} \) in Proposition 19 are attainable. However, it is possible that \( 0 < \alpha(G) < d_{\text{min}} \). Let \( k = 3, n = 4, m = 2, E = \{(1, 2, 3), (2, 3, 4)\} \). Then
$G$ is connected and $\alpha(G) > 0$. We have $\mathcal{L} x^3 = x_1^3 + 2x_2^3 + 2x_3^3 + x_4^3 - 3x_1 x_2 x_3 - 3x_2 x_3 x_4$.

Consider

$$\min \{ \mathcal{L} x^3 : x \in \mathbb{R}^4_+, \sum_{i=1}^{4} x_i^3 = 1, x_4 = 0 \}$$

$$= \min \{ x_1^3 + 2x_2^3 + 2x_3^3 - 3x_1 x_2 x_3 : x_1^3 + x_2^3 + x_3^3 = 1, x_1, x_2, x_3 \geq 0 \}.$$ 

Let $y = \left( \frac{1}{4} \right)^{\frac{3}{4}} (1, 1, 1)$. Then we see that

$$\alpha(G) \leq \min \{ \mathcal{L} x^3 : x \in \mathbb{R}^4_+, \sum_{i=1}^{4} x_i^3 = 1, x_4 = 0 \} \leq y_1^3 + 2y_2^3 + 2y_3^3 - 3y_1 y_2 y_3 = \frac{2}{3} < 1 = d_{\min}.$$ 

Actually, the exact value of $\alpha(G)$ for this example is $\alpha(G) = 1 - \beta^2$, where $\beta$ satisfies $\beta + \beta^3 = 1$ and $0.5 < \beta < 1$.

**Question 3.** In general, how can we calculate $\alpha(G)$?

### 9 Final Remarks

In this paper, we propose a simple and natural definition for the Laplacian and the signless Laplacian tensors of a uniform hypergraph. We show that they have very nice spectral properties. This sets the base for further exploring their applications in spectral hypergraph theory. Several further questions are raised. We expect that the research on these two Laplacian tensors will also motivate the further development of spectral theory of tensors.

### References


