Linear Algebra and its Applications 451 (2014) 1–14

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Infinite and finite dimensional Hilbert tensors ☆

Yisheng Song [a], Liqun Qi [b,*]

[a] *School of Mathematics and Information Science, Henan Normal University, Xinxiang, Henan 453007, PR China*
[b] *Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong*

A R T I C L E I N F O

Article history:
Received 21 January 2014
Accepted 10 March 2014
Available online xxxx
Submitted by R. Brualdi

MSC:
47H15
47H12
34B10
47A52
47J10
47H09
15A48
47H07

Keywords:
Hilbert tensor
Positively homogeneous
Eigenvalue
Spectral radius

A B S T R A C T

For an m-order n-dimensional Hilbert tensor (hypermatrix) $\mathcal{H}_n = (\mathcal{H}_{i_1 i_2 \cdots i_m})$,

$$\mathcal{H}_{i_1 i_2 \cdots i_m} = \frac{1}{i_1 + i_2 + \cdots + i_m - m + 1},$$
$$i_1, \ldots, i_m = 1, 2, \ldots, n$$

its spectral radius is not larger than $n^{m-1} \sin \frac{\pi}{n}$, and an upper bound of its E-spectral radius is $n^{\frac{m}{2}} \sin \frac{\pi}{n}$. Moreover, its spectral radius is strictly increasing and its E-spectral radius is nondecreasing with respect to the dimension n. When the order is even, both infinite and finite dimensional Hilbert tensors are positive definite. We also show that the m-order infinite dimensional Hilbert tensor (hypermatrix) $\mathcal{H}_\infty = (\mathcal{H}_{i_1 i_2 \cdots i_m})$ defines a bounded and positively $(m-1)$-homogeneous operator from l^1 into l^p $(1 < p < \infty)$, and the norm of corresponding positively homogeneous operator is smaller than or equal to $\frac{\pi}{\sqrt{6}}$.

☆ The work was supported by the Hong Kong Research Grant Council (Grant Nos. PolyU 502510, 502111, 501212, 501913), the first author was supported partly by the National Natural Science Foundation of China (Grant Nos. 11171094, 11271112) and by the Research Projects of Department of Science and Technology of Henan Province (Grant Nos. 122300410414, 132300410432).

* Corresponding author.
E-mail addresses: songyisheng1@gmail.com (Y. Song), maqilq@polyu.edu.hk (L. Qi).

http://dx.doi.org/10.1016/j.laa.2014.03.023
0024-3795/© 2014 Published by Elsevier Inc.

1. Introduction

In linear algebra, an n-dimensional Hilbert matrix $H_n = (H_{ij})$ is a square matrix with entries being the unit fractions, i.e.,

$$H_{ij} = \frac{1}{i+j-1}, \quad i,j = 1,2,\dots,n,$$

which was introduced by Hilbert [5]. Clearly, an n-dimensional Hilbert matrix is symmetric and positive definite, and is a compact linear operator on finite dimensional space. Many nice properties of n-dimensional Hilbert matrix have been investigated by Frazer [4] and Taussky [15]. An infinite dimensional Hilbert matrix

$$H_\infty = \left(\frac{1}{i+j-1}\right), \quad i,j = 1,2,\dots,n,\dots$$

can be regarded as a bounded linear operator from Hilbert space l^2 into itself (here, l^p $(0 < p < \infty)$ is a space consisting of all sequences $x = (x_i)_{i=1}^\infty$ satisfying $\sum_{i=1}^\infty |x_i|^p < +\infty$), but not compact operator (Choi [3] and Ingham [6]). The spectral properties of infinite dimensional Hilbert matrix have been studied by Magnus [9] and Kato [7].

As a natural extension of a Hilbert matrix, the entries of an m-order n-dimensional Hilbert tensor (hypermatrix) $\mathcal{H}_n = (\mathcal{H}_{i_1 i_2 \cdots i_m})$ are defined by

$$\mathcal{H}_{i_1 i_2 \cdots i_m} = \frac{1}{i_1 + i_2 + \cdots + i_m - m + 1}, \quad i_1, i_2, \dots, i_m = 1,2,\dots,n.$$

The entries of an m-order infinite dimensional Hilbert tensor (hypermatrix) $\mathcal{H}_\infty = (\mathcal{H}_{i_1 i_2 \cdots i_m})$ are defined by

$$\mathcal{H}_{i_1 i_2 \cdots i_m} = \frac{1}{i_1 + i_2 + \cdots + i_m - m + 1}, \quad i_1, i_2, \dots, i_m = 1,2,\dots,n,\dots.$$

The Hilbert tensor may be regarded as derived from the integral

$$\mathcal{H}_{i_1 i_2 \cdots i_m} = \int_0^1 t^{i_1 + i_2 + \cdots + i_m - m} \, dt. \tag{1.1}$$

Clearly, both $\mathcal{H}_n$ and $\mathcal{H}_\infty$ are positive $(\mathcal{H}_{i_1 i_2 \cdots i_m} > 0)$ and symmetric ($\mathcal{H}_{i_1 i_2 \cdots i_m}$ are invariant for any permutation of the indices), and an m-order n-dimensional Hilbert tensor $\mathcal{H}_n$ is a Hankel tensor with $v = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{nm})$ (Qi [12]), and an m-order infinite dimensional Hilbert tensor $\mathcal{H}_\infty$ is a Hankel tensor with $v = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots)$.

For a vector $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, $\mathcal{H}_n x^{m-1}$ is a vector defined by

$$\left(\mathcal{H}_n x^{m-1}\right)_i = \sum_{i_2,\dots,i_m=1}^n \frac{x_{i_2} \cdots x_{i_m}}{i + i_2 + \cdots + i_m - m + 1}, \quad i = 1,2,\dots,n. \tag{1.2}$$

Then $x^T(\mathcal{H}_n x^{m-1})$ is a homogeneous polynomial, denoted by $\mathcal{H}_n x^m$, i.e.,

$$\mathcal{H}_n x^m = x^T\left(\mathcal{H}_n x^{m-1}\right) = \sum_{i_1,i_2,\ldots,i_m=1}^{n} \frac{x_{i_1}x_{i_2}\cdots x_{i_m}}{i_1+i_2+\cdots+i_m-m+1}, \tag{1.3}$$

where x^T is the transposition of x.

For a real vector $x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) \in l^1$ (here l^1 is a space of sequences whose series is absolutely convergent), $\mathcal{H}_\infty x^{m-1}$ is an infinite dimensional vector defined by

$$\left(\mathcal{H}_\infty x^{m-1}\right)_i = \sum_{i_2,\ldots,i_m=1}^{\infty} \frac{x_{i_2}\cdots x_{i_m}}{i+i_2+\cdots+i_m-m+1}, \quad i = 1, 2, \ldots. \tag{1.4}$$

Accordingly, $\mathcal{H}_\infty x^m$ is given by

$$\mathcal{H}_\infty x^m = \lim_{n\to\infty} \mathcal{H}_n x^m = \sum_{i_1,i_2,\ldots,i_m=1}^{\infty} \frac{x_{i_1}x_{i_2}\cdots x_{i_m}}{i_1+i_2+\cdots+i_m-m+1}. \tag{1.5}$$

Then $\mathcal{H}_\infty x^m$ is exactly a real number for each real vector $x \in l^1$, i.e., $\mathcal{H}_\infty x^m < \infty$. In fact, since $\sum_{i=1}^{\infty} |x_i| < \infty$ for $x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) \in l^1$, we have

$$\begin{aligned}
\mathcal{H}_\infty x^m &= \lim_{n\to\infty} \mathcal{H}_n x^m = \lim_{n\to\infty} \sum_{i_1,i_2,\ldots,i_m=1}^{n} \frac{x_{i_1}x_{i_2}\cdots x_{i_m}}{i_1+i_2+\cdots+i_m-m+1} \\
&\leqslant \lim_{n\to\infty} \sum_{i_1,i_2,\ldots,i_m=1}^{n} \frac{|x_{i_1}x_{i_2}\cdots x_{i_m}|}{\underbrace{1+1+\cdots+1}_{m}-m+1} \\
&= \lim_{n\to\infty} \sum_{i_1,i_2,\ldots,i_m=1}^{n} |x_{i_1}||x_{i_2}|\cdots|x_{i_m}| \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} |x_i|\right)^m = \left(\sum_{i=1}^{\infty} |x_i|\right)^m < \infty.
\end{aligned}$$

In Section 2 we will prove that $\mathcal{H}_\infty x^{m-1}$ is well defined, i.e., $\mathcal{H}_\infty x^{m-1} \in l^p$ $(1 < p < \infty)$ for all real vector $x \in l^1$.

Both infinite and finite dimensional Hilbert tensors $\mathcal{H}_n$ and $\mathcal{H}_\infty$ are positive tensors. Thus, they are strictly copositive, i.e.,

$$\mathcal{H}_n x^m > 0 \quad \text{for all } x \in \mathbb{R}^n_+ \setminus \{\theta\}$$

and

$$\mathcal{H}_\infty x^m > 0 \quad \text{for all real nonnegative vector } x \in l^1 \setminus \{\theta\},$$

where θ is zero vector with all entries being 0 and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n;\ x_i \geqslant 0,\ i = 1, 2, \dots, n\}$. The concept of (strictly) copositive tensors was introduced and used by Qi [11].

When the order m is even, both infinite and finite dimensional Hilbert tensors are positive definite. The concept of positive (semi-)definite tensors was introduced by Qi [10].

Theorem 1.1. *Let m, n be two positive integers and m be even. Then both m-order Hilbert tensors $\mathcal{H}_n$ and $\mathcal{H}_\infty$ are positive definite, i.e.,*

$$\mathcal{H}_n x^m > 0 \quad \textit{for all } x \in \mathbb{R}^n \setminus \{\theta\}$$

and

$$\mathcal{H}_\infty x^m > 0 \quad \textit{for all real vector } x \in l^1 \setminus \{\theta\}.$$

Proof. By (1.1), for each positive integer n and $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
\mathcal{H}_n x^m &= \sum_{i_1, i_2, \dots, i_m = 1}^{n} \int_0^1 t^{i_1 + i_2 + \cdots + i_m - m} x_{i_1} x_{i_2} \cdots x_{i_m} \, dt \\
&= \int_0^1 \sum_{i_1, i_2, \dots, i_m = 1}^{n} \left(\prod_{j=1}^{m} t^{i_j - 1} x_{i_j} \right) dt \\
&= \int_0^1 \left(\sum_{i=1}^{n} t^{i-1} x_i \right)^m dt \\
&\geqslant 0.
\end{aligned}$$

This shows that $\mathcal{H}_n$ is positive semi-definite.

Now we assume that $\mathcal{H}_n$ is not positive definite. Then there exists $\bar{x} \in \mathbb{R}^n \setminus \{\theta\}$ such that $\mathcal{H}_n x^m = 0$. Then from the derivation in the last paragraph, we see that

$$\int_0^1 \left(\sum_{i=1}^{n} t^{i-1} \bar{x}_i \right)^m dt = 0.$$

By the continuity, we have

$$\sum_{i=1}^{n} t^{i-1} \bar{x}_i \equiv 0 \quad \text{for all } t \in [0, 1].$$

Letting $t = 0$, we have $\bar{x}_1 = 0$ and so,

$$t(\bar{x}_2 + t\bar{x}_1 + \cdots + t^{n-2}\bar{x}_n) = 0 \quad \text{for all } t \in [0, 1].$$

So, for all $t \in (0, 1]$, we have

$$\bar{x}_2 + t\bar{x}_1 + \cdots + t^{n-2}\bar{x}_n = 0.$$

Again by continuity, we see that

$$\bar{x}_2 + t\bar{x}_1 + \cdots + t^{n-2}\bar{x}_n = 0 \quad \text{for all } t \in [0, 1].$$

Letting $t = 0$, we see that $\bar{x}_2 = 0$. Repeating this process, we see that

$$\bar{x}_i = 0 \quad \text{for all } i = 1, \ldots, n.$$

Therefore, $\bar{x} = \theta$, which forms a contradiction. Hence $\mathcal{H}_n$ is positive definite.

Similarly, we can show that $\mathcal{H}_\infty$ is positive definite. □

In the remainder of this paper, we will investigate some other nice properties of infinite and finite dimensional Hilbert tensors such as spectral radius and operator norm and so on.

In Section 2, we will prove that the m-order infinite dimensional Hilbert tensor (hypermatrix) $\mathcal{H}_\infty = (\mathcal{H}_{i_1 i_2 \cdots i_m})$ defines a bounded and positively $(m-1)$-homogeneous operator from l^1 into l^p $(1 < p < \infty)$. When $(\mathcal{H}_\infty x^{m-1})^{[\frac{1}{m-1}]}$ is well defined for all real vector $x \in l^1$, let

$$F_\infty x = \left(\mathcal{H}_\infty x^{m-1}\right)^{[\frac{1}{m-1}]} \quad \text{and} \quad T_\infty x = \begin{cases} \|x\|_1^{2-m} \mathcal{H}_\infty x^{m-1}, & x \neq \theta \\ \theta, & x = \theta, \end{cases} \tag{1.6}$$

where $x^{[\frac{1}{m-1}]} = (x_1^{\frac{1}{m-1}}, x_2^{\frac{1}{m-1}}, \ldots, x_n^{\frac{1}{m-1}}, \ldots)$ and θ is zero vector with entries being all 0. We will show that T_∞ is a bounded, continuous and positively homogeneous operator from l^1 into l^p $(1 < p < \infty)$ and F_∞ is a bounded, continuous and positively homogeneous operator from l^1 into l^p $(m - 1 < p < \infty)$. Furthermore, their norms are at most $\frac{\pi}{\sqrt{6}}$.

In Section 3, we will study the spectral properties of an m-order n-dimensional Hilbert tensor $\mathcal{H}_n$. With the help of the finite dimensional Hilbert inequality, the largest H-eigenvalue (spectral radius) of Hilbert tensor $\mathcal{H}_n$ is at most $n^{m-1} \sin \frac{\pi}{n}$, and the largest Z-eigenvalue (E-spectral radius) of $\mathcal{H}_n$ is at most $n^{\frac{m}{2}} \sin \frac{\pi}{n}$. Furthermore, the spectral radius of Hilbert tensor $\mathcal{H}_n$ is strictly increasing with respect to the dimensionality n and its E-spectral radius is nondecreasing with respect to the dimensionality n.

2. Infinite dimensional Hilbert tensors

For $0 < p < \infty$, l^p is the space consisting of all sequences $x = (x_i)$ satisfying

$$\sum_{i=1}^{\infty} |x_i|^p < \infty.$$

If $p \geqslant 1$, then a norm on l^p is defined by

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}.$$

In fact, the space $(l^p, \|\cdot\|_p)$ is a Banach space for $p \geqslant 1$.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces, and $T : X \to Y$ be an operator and r is a real number. T is called

- *r-homogeneous* if $T(tx) = t^r Tx$ for each $t \in \mathbb{K}$ and all $x \in X$;
- *positively homogeneous* if $T(tx) = tTx$ for each $t > 0$ and all $x \in X$;
- *bounded* if there is a real number $M > 0$ such that

$$\|Tx\|_Y \leqslant M\|x\|_X, \quad \text{for all } x \in X.$$

Let T be a bounded, continuous and positively homogeneous operator from X into Y. Then the norm of T can be defined by

$$\|T\| = \sup\bigl\{\|Tx\|_Y\colon \|x\|_X = 1\bigr\}. \tag{2.1}$$

When $(\mathcal{H}_\infty x^{m-1})^{[\frac{1}{m-1}]}$ is well defined for all real vector $x \in l^1$, let

$$F_\infty x = \bigl(\mathcal{H}_\infty x^{m-1}\bigr)^{[\frac{1}{m-1}]} \tag{2.2}$$

and

$$T_\infty x = \begin{cases} \|x\|_1^{2-m}\mathcal{H}_\infty x^{m-1}, & x \neq \theta \\ \theta, & x = \theta, \end{cases} \tag{2.3}$$

where $x^{[\frac{1}{m-1}]} = (x_1^{\frac{1}{m-1}}, x_2^{\frac{1}{m-1}}, \ldots, x_n^{\frac{1}{m-1}}, \ldots)$ and θ is zero vector with entries being all 0. Clearly, both F_∞ and T_∞ are continuous and positively homogeneous. With the help of the well known series

$$\sum_{i=1}^{\infty} \frac{1}{i^p} < \infty \quad \text{for } \infty > p > 1 \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6},$$

now we discuss properties of the infinite dimensional Hilbert tensor.

Theorem 2.1. *Let F_∞ and T_∞ be defined by Eqs. (2.2) and (2.3), respectively. Then*

(i) *if $x \in l^1$, then $T_\infty x \in l^p$ for $1 < p < \infty$;*
(ii) *if $x \in l^1$, then $F_\infty x \in l^p$ for $m - 1 < p < \infty$.*

Furthermore, T_∞ is a bounded, continuous and positively homogeneous operator from l^1 into l^p $(1 < p < \infty)$ and F_∞ is a bounded, continuous and positively homogeneous operator from l^1 into l^p $(m-1 < p < \infty)$. In particular,

$$\|T_\infty\| = \sup_{\|x\|_1=1} \|T_\infty x\|_2 \leqslant \frac{\pi}{\sqrt{6}}$$

and

$$\|F_\infty\| = \sup_{\|x\|_1=1} \|F_\infty x\|_{2(m-1)} \leqslant \frac{\pi}{\sqrt{6}}.$$

Proof. For $x \in l^1$,

$$\begin{aligned}
\left|\left(\mathcal{H}_\infty x^{m-1}\right)_i\right| &= \lim_{n\to\infty}\left|\sum_{i_2,\dots,i_m=1}^{n} \frac{x_{i_2}\cdots x_{i_m}}{i+i_2+\cdots+i_m-m+1}\right| \\
&\leqslant \lim_{n\to\infty}\sum_{i_2,\dots,i_m=1}^{n} \frac{|x_{i_2}\cdots x_{i_m}|}{i+\underbrace{1+\cdots+1}_{m-1}-m+1} \\
&= \frac{1}{i}\lim_{n\to\infty}\sum_{i_2,\dots,i_m=1}^{n} |x_{i_2}||x_{i_3}|\cdots|x_{i_m}| \\
&= \frac{1}{i}\lim_{n\to\infty}\left(\sum_{k=1}^{n}|x_k|\right)^{m-1} \\
&= \frac{1}{i}\left(\sum_{k=1}^{\infty}|x_k|\right)^{m-1} \\
&= \frac{1}{i}\|x\|_1^{m-1}.
\end{aligned}$$

Then (i) for $p > 1$, it follows from the definition of T_∞ that

$$\begin{aligned}
\sum_{i=1}^{\infty}\left|(T_\infty x)_i\right|^p &= \sum_{i=1}^{\infty}\left|\left(\|x\|_1^{2-m}\mathcal{H}_\infty x^{m-1}\right)_i\right|^p \\
&= \|x\|_1^{(2-m)p}\sum_{i=1}^{\infty}\left|\left(\mathcal{H}_\infty x^{m-1}\right)_i\right|^p \\
&\leqslant \|x\|_1^{(2-m)p}\sum_{i=1}^{\infty}\left(\frac{1}{i}\|x\|_1^{m-1}\right)^p \\
&= \|x\|_1^p\sum_{i=1}^{\infty}\frac{1}{i^p} < \infty
\end{aligned}$$

since $\sum_{i=1}^{\infty}\frac{1}{i^p}<\infty$ for $p>1$, and thus $T_\infty x\in l^p$ for all $x\in l^1$. Moreover, we also have

$$\|T_\infty x\|_p=\left(\sum_{i=1}^{\infty}\left|(T_\infty x)_i\right|^p\right)^{\frac{1}{p}}\leqslant M\|x\|_1, \tag{2.4}$$

where $M=(\sum_{i=1}^{\infty}\frac{1}{i^p})^{\frac{1}{p}}>0$. So, T_∞ is a bounded operator from l^1 into l^p $(1<p<\infty)$. In particular, take $p=2$, $M=(\sum_{i=1}^{\infty}\frac{1}{i^2})^{\frac{1}{2}}=\frac{\pi}{\sqrt{6}}$. It follows from (2.1) and (2.4) that

$$\|T_\infty\|=\sup_{\|x\|_1=1}\|T_\infty x\|_2\leqslant\frac{\pi}{\sqrt{6}}.$$

(ii) for $p>m-1$, it follows from the definition of F_∞ that

$$\begin{aligned}\sum_{i=1}^{\infty}\left|(F_\infty x)_i\right|^p&=\sum_{i=1}^{\infty}\left|\left(\mathcal{H}_\infty x^{m-1}\right)_i\right|^{\frac{p}{m-1}}\\&\leqslant\sum_{i=1}^{\infty}\left(\frac{1}{i}\|x\|_1^{m-1}\right)^{\frac{p}{m-1}}\\&=\|x\|_1^p\sum_{i=1}^{\infty}\frac{1}{i^{\frac{p}{m-1}}}<\infty\end{aligned}$$

since $\sum_{i=1}^{\infty}\frac{1}{i^{\frac{p}{m-1}}}<\infty$ for $p>m-1$, and hence $F_\infty x\in l^p$ for all $x\in l^1$. Moreover, we also have

$$\|T_\infty x\|_p=\left(\sum_{i=1}^{\infty}\left|(F_\infty x)_i\right|^p\right)^{\frac{1}{p}}\leqslant C\|x\|_1, \tag{2.5}$$

where $C=(\sum_{i=1}^{\infty}\frac{1}{i^{\frac{p}{m-1}}})^{\frac{m-1}{p}}>0$. So, F_∞ is a bounded operator from l^1 into l^p $(m-1<p<\infty)$. Similarly, take $p=2(m-1)$, $C=\frac{\pi}{\sqrt{6}}$. It follows from (2.1) and (2.5) that

$$\|F_\infty\|=\sup_{\|x\|_1=1}\|T_\infty x\|_{2(m-1)}\leqslant\frac{\pi}{\sqrt{6}}.$$

This completes the proof. □

It follows from the definition (1.4) that $\mathcal{H}_\infty x^{m-1}$ is continuous, positively $(m-1)$-homogeneous, and so from the proof of Theorem 2.1, it also is bounded.

Theorem 2.2. *Let $\mathcal{H}_\infty$ be an m-order infinite dimensional Hilbert tensor and $f(x)=\mathcal{H}_\infty x^{m-1}$. Then f is a bounded, continuous and positively $(m-1)$-homogeneous operator from l^1 into l^p $(1<p<\infty)$.*

Remark 1. It is well known that Hilbert matrix H_∞ is a bounded linear operator from l^2 into l^2 and

$$\|H_\infty\|_2 = \sup_{\|x\|_2=1} \|H_\infty x\|_2 = \pi.$$

For more details, see [3]. Then if the Hilbert matrix H_∞ is regarded as a bounded linear operator from l^1 into l^2, whether $\|H_\infty\| = \sup_{\|x\|_1=1} \|H_\infty x\|_2$ is exactly equal to $\frac{\pi}{\sqrt{6}}$ or another number? Furthermore, may the values of $\|T_\infty\| = \sup_{\|x\|_1=1} \|T_\infty x\|_2$ and $\|F_\infty\| = \sup_{\|x\|_1=1} \|F_\infty x\|_{2(m-1)}$ be worked out exactly?

3. Finite dimensional Hilbert tensors

For $x \in \mathbb{R}^n$ and $\infty > p \geqslant 1$, it is known well that

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$$

is the norm defined on $\mathbb{R}^n$ for each $p \geqslant 1$ and

$$\|x\|_p \leqslant \|x\|_r \leqslant n^{\frac{1}{r}-\frac{1}{p}} \|x\|_p \quad \text{for } p > r. \tag{3.1}$$

Then for a continuous, positively homogeneous $T : \mathbb{R}^n \to \mathbb{R}^n$, it is obvious that

$$\|T\|_p = \max_{\|x\|_p=1} \|Tx\|_p$$

is the operator norm of T for each $p \geqslant 1$ (Song and Qi [13]). When $(\mathcal{H}_n x^{m-1})^{[\frac{1}{m-1}]}$ is well defined for all $x \in \mathbb{R}^n$, let

$$F_n x = \left(\mathcal{H}_n x^{m-1}\right)^{[\frac{1}{m-1}]}$$

and

$$T_n x = \begin{cases} \|x\|_2^{2-m} \mathcal{H}_n x^{m-1}, & x \neq \theta \\ \theta, & x = \theta, \end{cases} \tag{3.2}$$

where $x^{[\frac{1}{m-1}]} = (x_1^{\frac{1}{m-1}}, x_2^{\frac{1}{m-1}}, \ldots, x_n^{\frac{1}{m-1}})^T$ and $\theta = (0, 0, \ldots, 0)^T$. Clearly, both F_n and T_n are continuous and positively homogeneous. The following Hilbert inequality is well known (Frazer [4]).

Lemma 3.1. *Let $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. Then*

$$\sum_{i=1}^n \sum_{j=1}^n \frac{|x_i||x_j|}{i+j-1} \leqslant \left(n \sin\frac{\pi}{n}\right) \sum_{k=1}^n x_k^2 = \|x\|_2^2 n \sin\frac{\pi}{n}. \tag{3.3}$$

Recall that $\lambda \in \mathbb{C}$ is called an *eigenvalue of* $\mathcal{H}_n$, if there is a vector $x \in \mathbb{R}^n \setminus \{\theta\}$ such that

$$\mathcal{H}_n x^{m-1} = \lambda x^{[m-1]}, \tag{3.4}$$

where $x^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})^T$, and call x an *eigenvector* associated with λ. We call such an eigenvalue *H-eigenvalue* if it is real and has a real eigenvector x, and call such a real eigenvector x an *H-eigenvector*. A number $\mu \in \mathbb{C}$ is called *E-eigenvalue of* $\mathcal{H}_n$, if there is a vector $x \in \mathbb{R}^n \setminus \{\theta\}$ such that

$$\begin{cases} \mathcal{H}_n x^{m-1} = \mu x \\ x^T x = 1, \end{cases} \tag{3.5}$$

and call a vector x an *E-eigenvector* associated with μ. If x is real, then μ is also real. In this case, μ and x are called *Z-eigenvalue* of $\mathcal{H}_n$ and *Z-eigenvector* associated with μ, respectively. These concepts were first introduced by Qi [10] for the higher order symmetric tensors. Lim [8] independently introduced the notion of eigenvalue for higher order tensors but restricted x to be a real vector and λ to be a real number.

Since Hilbert tensor is positive (all entries are positive) and symmetric, then the following conclusions (i) were easily obtained from Chang, Pearson and Zhang [2], Qi [11], Song and Qi [14] and Yang and Yang [17,16], and the conclusions (ii) can be obtained from Chang, Pearson and Zhang [1] and Song and Qi [13].

Lemma 3.2. *Let $\rho(F_n)$ and $\rho(T_n)$ respectively denote the largest modulus of the eigenvalues of operators F_n and T_n. Then*

(i) *$(\rho(F_n))^{m-1}$ is a positive H-eigenvalue of $\mathcal{H}_n$ with a positive H-eigenvector, i.e. all components are positive and*

$$\rho(F_n)^{m-1} = \max\{\mathcal{H}_n x^m;\ x \in \mathbb{R}^n_+,\ \|x\|_m = 1\}; \tag{3.6}$$

(ii) *$\rho(T_n)$ is a positive Z-eigenvalue of $\mathcal{H}_n$ with a nonnegative Z-eigenvector and*

$$\rho(T_n) = \max\{\mathcal{H}_n x^m;\ x \in \mathbb{R}^n,\ \|x\|_2 = 1\}. \tag{3.7}$$

Now we give the upper bounded of the eigenvalues of operators F_n and T_n.

Theorem 3.3. *Let $\mathcal{H}_n$ be an m-order n-dimensional Hilbert tensor. Then*

(i) *for all E-eigenvalues (Z-eigenvalues) μ of Hilbert tensor $\mathcal{H}_n$,*

$$|\mu| \leqslant \rho(T_n) \leqslant n^{\frac{m}{2}} \sin\frac{\pi}{n};$$

(ii) *for all eigenvalues (H-eigenvalues) λ of Hilbert tensor $\mathcal{H}_n$,*

$$|\lambda| \leqslant \rho(F_n)^{m-1} \leqslant n^{m-1}\sin\frac{\pi}{n}.$$

Proof. For $x \in \mathbb{R}^n \setminus \{\theta\}$, it follows from Lemma 3.1 that

$$\begin{aligned}
\left|\mathcal{H}_n x^m\right| &= \left|\sum_{i_1,i_2,\ldots,i_m=1}^{n} \frac{x_{i_1}x_{i_2}\cdots x_{i_m}}{i_1+i_2+\cdots+i_m-m+1}\right| \\
&\leqslant \sum_{i_1,i_2,\ldots,i_m=1}^{n} \frac{|x_{i_1}x_{i_2}\cdots x_{i_m}|}{i_1+i_2+\underbrace{1+\cdots+1}_{m-2}-m+1} \\
&= \sum_{i_1,i_2,\ldots,i_m=1}^{n} \frac{|x_{i_1}||x_{i_2}|\cdots|x_{i_m}|}{i_1+i_2-1} \\
&= \left(\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\frac{|x_{i_1}||x_{i_2}|}{i_1+i_2-1}\right)\sum_{i_3,i_4,\ldots,i_m=1}^{n}|x_{i_3}||x_{i_4}|\cdots|x_{i_m}| \\
&\leqslant \left(\|x\|_2^2 n\sin\frac{\pi}{n}\right)\left(\sum_{i=1}^{n}|x_i|\right)^{m-2} \\
&= \left(n\sin\frac{\pi}{n}\right)\|x\|_2^2\|x\|_1^{m-2}.
\end{aligned} \tag{3.8}$$

(i) From (3.1), it follows that $\|x\|_1 \leqslant \sqrt{n}\|x\|_2$ for $x \in \mathbb{R}^n$. Then we have

$$\begin{aligned}
\mathcal{H}_n x^m &\leqslant \left(n\sin\frac{\pi}{n}\right)\|x\|_2^2\|x\|_1^{m-2} \\
&\leqslant \left(n\sin\frac{\pi}{n}\right)n^{\frac{m-2}{2}}\|x\|_2^m \\
&= \left(n^{\frac{m}{2}}\sin\frac{\pi}{n}\right)\|x\|_2^m,
\end{aligned}$$

and hence, for $x \in \mathbb{R}^n \setminus \{\theta\}$,

$$\mathcal{H}_n\left(\frac{x}{\|x\|_2}\right)^m = \frac{1}{\|x\|_2^m}\mathcal{H}_n x^m \leqslant n^{\frac{m}{2}}\sin\frac{\pi}{n}.$$

It follows from (3.7) of Lemma 3.2(ii) that

$$\rho(T_n) \leqslant n^{\frac{m}{2}}\sin\frac{\pi}{n}.$$

(ii) From (3.1), it follows that

$$\|x\|_2 \leqslant n^{\frac{1}{2}-\frac{1}{m}}\|x\|_m \quad \text{and} \quad \|x\|_1 \leqslant n^{1-\frac{1}{m}}\|x\|_m$$

for $m \geqslant 2$ and $x \in \mathbb{R}^n$. Then by (3.8), we have

$$\begin{aligned}
\mathcal{H}_n x^m &\leqslant \left(n \sin\frac{\pi}{n}\right)\|x\|_2^2\|x\|_1^{m-2} \\
&\leqslant \left(n \sin\frac{\pi}{n}\right)\left(n^{1-\frac{2}{m}}\|x\|_m^2\right)\left(n^{(1-\frac{1}{m})(m-2)}\|x\|_2^{m-2}\right) \\
&= \left(n \sin\frac{\pi}{n}\right)\left(n^{m-2}\|x\|_m^m\right) \\
&= \left(n^{m-1} \sin\frac{\pi}{n}\right)\|x\|_m^m,
\end{aligned}$$

and hence, for $x \in \mathbb{R}^n \setminus \{\theta\}$,

$$\mathcal{H}_n\left(\frac{x}{\|x\|_m}\right)^m = \frac{1}{\|x\|_m^m}\mathcal{H}_n x^m \leqslant n^{m-1}\sin\frac{\pi}{n}.$$

It follows from (3.6) of Lemma 3.2(i) that

$$\rho(F_n)^{m-1} \leqslant n^{m-1}\sin\frac{\pi}{n}.$$

This completes the proof. □

Recall that an m-order r-dimensional $\mathcal{B}$ is called *a principal sub-tensor* of an m-order n-dimensional tensor $\mathcal{A} = (\mathcal{A}_{i_1\cdots i_m})$ $(r \leqslant n)$, if $\mathcal{B}$ consists of r^m elements in $\mathcal{A} = (a_{i_1\cdots i_m})$: for a set $\mathcal{N}$ that composed of r elements in $\{1, 2, \ldots, n\}$,

$$\mathcal{B} = (\mathcal{A}_{i_1\cdots i_m}), \quad \text{for all } i_1, i_2, \ldots, i_m \in \mathcal{N}.$$

The concept was first introduced and used by Qi [10] for the higher order symmetric tensor. Clearly, an m-order n_1-dimensional Hilbert tensor $\mathcal{H}_{n_1}$ is a principal sub-tensor of m-order n_2-dimensional Hilbert tensor $\mathcal{H}_{n_2}$ if $n_1 \leqslant n_2$.

Theorem 3.4. *If $n < k$, then*

$$\rho(F_n) < \rho(F_k) \quad \textit{and} \quad \rho(T_n) \leqslant \rho(T_k).$$

Proof. Since $n < k$, then $\mathcal{H}_n$ is a principal sub-tensor of $\mathcal{H}_k$. It follows from Lemma 3.2(i) that $\rho(F_n)^{m-1}$ is a positive eigenvalue of $\mathcal{H}_n$ with positive eigenvector $x_{(n)} = (x_1^{(n)}, \ldots, x_n^{(n)})$, and hence $\rho(F_n)^{m-1}$ is an eigenvalue of $\mathcal{H}_k$ with corresponding eigenvector $x' = (x_1^{(n)}, \dots, x_n^{(n)}, \underbrace{0, \dots, 0}_{k-n})$. Since $\rho(F_k)^{m-1}$ is positive eigenvalue of $\mathcal{H}_k$ with positive eigenvector $x_{(k)} = (x_1^{(k)}, \dots, x_k^{(k)})$ by Lemma 3.2(i), then

$$\rho(F_n)^{m-1} < \rho(F_k)^{m-1},$$

and hence, $\rho(F_n) < \rho(F_k)$.

Similarly, applying Lemma 3.2(ii), we also have

$$\rho(T_n) \leqslant \rho(T_k).$$

The desired conclusion follows. □

Remark 2.

(i) In Theorem 3.4, the monotonicity of the spectral radius with respect to the dimensionality n is proved. Then whether or not the eigenvector $x_{(n)}$ associated with the spectral radius is the same monotonicity.
(ii) In Theorem 3.3, the upper bounds of two classes of spectral radii are established. It is not clear whether these two upper bounds may be attained or only one of these two upper bounds may be attained or both cannot be attained.

Acknowledgements

The authors would like to thank Dr. Guoyin Li, Prof. Yimin Wei and Mr. Weiyang Ding for their valuable suggestions. In particular, in the original draft, we only proved that $\mathcal{H}_n$ and $\mathcal{H}_\infty$ are positive semi-definite. Dr. Guoyin Li suggested the current proof for showing that $\mathcal{H}_n$ and $\mathcal{H}_\infty$ are positive definite.

References

[1] K.C. Chang, K. Pearson, T. Zhang, Some variational principles for Z-eigenvalues of nonnegative tensors, Linear Algebra Appl. 438 (2013) 4166–4182.
[2] K.C. Chang, K. Pearson, T. Zhang, Perron–Frobenius theorem for nonnegative tensors, Commun. Math. Sci. 6 (2008) 507–520.
[3] Man-Duen Choi, Tricks for treats with the Hilbert matrix, Amer. Math. Monthly 90 (1983) 301–312.
[4] H. Frazer, Note on Hilbert's inequality, J. Lond. Math. Soc. 21 (1) (1946) 7–9.
[5] D. Hilbert, Ein Beitrag zur Theorie des Legendre'schen Polynoms, Acta Math. 18 (1894) 155–159.
[6] A.E. Ingham, A note on Hilbert's inequality, J. Lond. Math. Soc. 11 (3) (1936) 237–240.
[7] T. Kato, On the Hilbert matrix, Proc. Amer. Math. Soc. 8 (1) (1957) 73–81.
[8] L.H. Lim, Singular values and eigenvalues of tensors: a variational approach, in: Proc. 1st IEEE International Workshop on Computational Advances of Multi-tensor Adaptive Processing, December 13–15, 2005, pp. 129–132.
[9] W. Magnus, On the spectrum of Hilbert's matrix, Amer. J. Math. 72 (4) (1950) 699–704.
[10] L. Qi, Eigenvalues of a real supersymmetric tensor, J. Symbolic Comput. 40 (2005) 1302–1324.
[11] L. Qi, Symmetric nonnegative tensors and copositive tensors, Linear Algebra Appl. 439 (2013) 228–238.

[12] L. Qi, Hankel tensors: associated Hankel matrices and Vandermonde decomposition, Commun. Math. Sci. (2014), in press, arXiv:1310.5470.
[13] Y. Song, L. Qi, Positive eigenvalue-eigenvector of nonlinear positive mappings, Front. Math. China 9 (1) (2014) 181–199.
[14] Y. Song, L. Qi, Spectral properties of positively homogeneous operators induced by higher order tensors, SIAM J. Matrix Anal. Appl. 34 (4) (2013) 1581–1595.
[15] O. Taussky, A remark concerning the characteristic roots of the finite segments of the Hilbert matrix, Q. J. Math., Oxford Ser. 20 (1949) 80–83.
[16] Q. Yang, Y. Yang, Further results for Perron–Frobenius theorem for nonnegative tensors II, SIAM J. Matrix Anal. Appl. 32 (4) (2011) 1236–1250.
[17] Y. Yang, Q. Yang, Further results for Perron–Frobenius theorem for nonnegative tensors, SIAM J. Matrix Anal. Appl. 31 (5) (2010) 2517–2530.