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Cored hypergraphs, power hypergraphs and their Laplacian H-eigenvalues ☆

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ABSTRACT

In this paper, we introduce the class of cored hypergraphs and power hypergraphs, and investigate the properties of their Laplacian H-eigenvalues. From an ordinary graph, one may generate a k-uniform hypergraph, called the kth power hypergraph of that graph. Power hypergraphs are cored hypergraphs, but not vice versa. Sunflowers, loose paths and loose cycles are power hypergraphs, while squids, generalized loose s-paths and loose s-cycles for $2 \leqslant s < \frac{k}{2}$ are cored hypergraphs, but not power graphs in general. We show that the largest Laplacian H-eigenvalue of an even-uniform cored hypergraph is equal to its largest signless Laplacian H-eigenvalue. Especially, we find out these largest H-eigenvalues for even-uniform squids. Moreover, we show that the largest Laplacian H-eigenvalue of an odd-uniform squid, loose path and loose cycle is equal to the maximum degree, i.e., 2. We also compute the Laplacian H-spectra of the class of sunflowers. When k is odd, the Laplacian H-spectra of the loose cycle of size 3 and the loose path of length 3 are characterized as well.

1. Introduction

A natural definition for the Laplacian tensor and the signless Laplacian tensor of a k-uniform hypergraph for $k \geqslant 3$ was introduced in [20]. See Definition 2.2 of this paper. Recently, Hu, Qi and Xie [11] studied the largest Laplacian and signless Laplacian eigenvalues of a k-uniform hypergraph, and

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generalized some classical results of spectral graph theory to spectral hypergraph theory, in particular, when k is even.

One classical result in spectral graph theory [2,29] is that the largest Laplacian eigenvalue of a graph is always less than or equal to the largest signless Laplacian eigenvalue, and when the graph is connected, equality holds if and only if the graph is bipartite.

In [11], it was shown that the largest Laplacian H-eigenvalue of a k-uniform hypergraph is always less than or equal to the largest signless Laplacian H-eigenvalue, and when the hypergraph is connected, equality holds if and only if the hypergraph is odd-bipartite. A k-uniform hypergraph is called odd-bipartite if k is even, and the vertex set of the hypergraph can be divided to two parts, such that each edge has odd number of vertices in each of these two parts.

Hu, Qi and Xie [11] showed that the largest signless Laplacian H-eigenvalue of a loose cycle [12,13,15] of a k-uniform hypergraph is computable and an even-order loose cycle is odd-bipartite, they showed that the largest Laplacian H-eigenvalue of a loose cycle is computable when k is even.

Another classical result in spectral graph theory [2,6,29] is that the largest Laplacian eigenvalue of a graph is always greater than or equal to the maximum degree of that graph, plus one. The lower bound is attained if there exists a vertex adjacent to all the other vertices of that graph.

Hu, Qi and Xie [11] showed that when k is even the largest Laplacian H-eigenvalue of a k-uniform hypergraph is always greater than or equal to the maximum degree of that hypergraph, plus $\alpha(k)$, where $\alpha(k) > 0$ and $\alpha(2) = 1$. They showed that the lower bound is attained if the hypergraph is an $S(n, 1, k)$ sunflower [5] (later we simply refer an $S(n, 1, k)$ sunflower as a sunflower).

When k is odd, the situation is very different. It was shown in [11] that in this case the largest Laplacian H-eigenvalue is always strictly less than the largest signless Laplacian H-eigenvalue, and the lower bound of the largest Laplacian H-eigenvalue is the maximum degree itself, attained by any sunflower.

The results of [11] raised interests to study the Laplacian H-eigenvalues of a k-uniform hypergraph. Actually, they posed several questions for further research. First, when k is odd, is a sunflower the only example of a k-uniform hypergraph whose largest Laplacian H-eigenvalue is equal to the maximum degree? Second, can we calculate all Laplacian H-eigenvalues for some special k-uniform hypergraphs, such as sunflowers and loose cycles? This is useful if one wishes to study the second smallest Laplacian H-eigenvalue (with multiplicity) of a k-uniform hypergraph, as the second smallest Laplacian eigenvalue of a graph plays a key role in spectral graph theory [2,3,29].

Motivated by these questions, we study Laplacian H-eigenvalues of some special k-uniform hypergraphs in this paper.

Using the same method as [11], we may generalize an arbitrary graph G to a k-uniform hypergraph G^k, called the kth power of G. See Definition 2.4 of this paper. We refer to any hypergraph obtained in this way as a *power* hypergraph. In particular, paths are generalized to loose paths [15]. We will see that when k is even, a power hypergraph is odd-bipartite. We further conclude this for a broader class of k-uniform hypergraphs. We call such hypergraphs cored hypergraphs. See Definition 2.3 of this paper. A power hypergraph is a cored hypergraph but not vice versa. In particular, we introduce a special subclass of cored hypergraphs, called squids. See Definition 3.1 of this paper. A squid is not a power hypergraph in general. We also identify that generalized loose s-paths and loose s-cycles [18] for $2 \leqslant s < \frac{k}{2}$ are cored hypergraphs but not power hypergraphs in general. We show that when k is even, a cored hypergraph is odd-bipartite. Thus, when k is even, the largest Laplacian H-eigenvalue and the largest signless Laplacian H-eigenvalue of a cored hypergraph is the same. This enhances our understanding on odd-bipartite hypergraphs and their largest Laplacian eigenvalues. We will show that the largest Laplacian H-eigenvalue of an even-order squid is computable.

Then, when k is odd, we will show that the largest Laplacian H-eigenvalue of an odd-uniform squid, loose path and loose cycle is equal to the maximum degree, i.e., 2. This shows that for a very broad class of hypergraphs, when k is odd, the largest Laplacian H-eigenvalue is equal to the maximum degree of the hypergraph.

Finally, we compute the Laplacian H-spectra of the class of sunflowers, the loose cycle of size 3 and the loose path of length 3. This will be useful for research on the second smallest Laplacian H-eigenvalue of k-uniform hypergraphs.

For discussion on the eigenvectors of the zero Laplacian and signless Laplacian eigenvalues of a k-uniform hypergraph, see [9]. For discussion on eigenvalues of adjacency tensors and the other types of Laplacian tensors of k-uniform hypergraphs, see [4,8,14,16,17,22–24,26,27] and references therein.

The rest of this paper is organized as follows. Definitions on eigenvalues of tensors and uniform hypergraphs are presented in the next section. Cored hypergraphs and power hypergraphs are introduced there. We discuss in Section 3 some properties of cored hypergraphs. An even-uniform cored hypergraph has equality for the largest Laplacian and the signless Laplacian H-eigenvalues. Squids are introduced and investigated in Subsection 3.2. We compute the largest Laplacian H-eigenvalues of even-uniform squids and prove that they are equal to the maximum degrees, i.e., 2, for odd-uniform squids. Generalized loose s-paths and loose s-cycles [18] for $2 \leqslant s < \frac{k}{2}$ are discussed in Subsection 3.3. We show in Subsection 4.1 that the largest Laplacian H-eigenvalues of odd-uniform loose paths and loose cycles are equal to the maximum degrees, i.e., 2. We make a conjecture in Subsection 4.2 that the largest H-eigenvalues of even-uniform power hypergraphs with respect to the same underlying usual graph are strictly decreasing as k increasing. This conjecture is proved to be true for sunflowers and loose cycles. In Section 5, we compute all the Laplacian H-eigenvalues of sunflowers, the loose path of length 3 and the loose cycle of size 3. Some final remarks are made in the last section.

2. Preliminaries

2.1. H-eigenvalues of tensors

In this subsection, some definitions of H-eigenvalues of tensors are presented. For comprehensive references, see [19,7] and references therein. Especially, for spectral hypergraph theory oriented facts on eigenvalues of tensors, please see [20,9].

Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}^n$ the n-dimensional real space. $\mathbb{R}^n_+$ denotes the nonnegative orthant of $\mathbb{R}^n$. For integers $k \geqslant 3$ and $n \geqslant 2$, a real tensor $\mathcal{T} = (t_{i_1 \ldots i_k})$ of order k and dimension n refers to a multidimensional array (also called hypermatrix) with entries $t_{i_1 \ldots i_k}$ such that $t_{i_1 \ldots i_k} \in \mathbb{R}$ for all $i_j \in [n] := \{1, \ldots, n\}$ and $j \in [k]$. Tensors are always referred to kth order real tensors in this paper, and the dimensions will be clear from the content. Given a vector $\mathbf{x} \in \mathbb{R}^n$, $\mathcal{T}\mathbf{x}^{k-1}$ is defined as an n-dimensional vector such that its ith element is $\sum_{i_2,\ldots,i_k \in [n]} t_{ii_2\ldots i_k} x_{i_2} \cdots x_{i_k}$ for all $i \in [n]$. Let $\mathcal{I}$ be the identity tensor of appropriate dimension, e.g., $i_{i_1 \ldots i_k} = 1$ if and only if $i_1 = \cdots = i_k \in [n]$, and zero otherwise when the dimension is n. The following definition was introduced by Qi [19].

Definition 2.1. Let $\mathcal{T}$ be a kth order n-dimensional real tensor. For some $\lambda \in \mathbb{R}$, if polynomial system $(\lambda\mathcal{I} - \mathcal{T})\mathbf{x}^{k-1} = 0$ has a solution $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$, then λ is called an H-eigenvalue and $\mathbf{x}$ an H-eigenvector.

H-eigenvalues are real numbers, by Definition 2.1. By [7,19], we have that the number of H-eigenvalues of a real tensor is finite. By [20], we have that all the tensors considered in this paper have at least one H-eigenvalue. Hence, we can denote by $\lambda(\mathcal{T})$ as the largest H-eigenvalue of a real tensor $\mathcal{T}$.

For a subset $S \subseteq [n]$, we denoted by $|S|$ its cardinality.

2.2. Uniform hypergraphs

In this subsection, we present some essential notions of uniform hypergraphs which will be used in the sequel. Please refer to [1,3,2,9,20] for comprehensive references.

In this paper, unless stated otherwise, a hypergraph means an undirected simple k-uniform hypergraph G with vertex set V, which is labeled as $[n] = \{1, \ldots, n\}$, and edge set E. By k-uniformity, we mean that for every edge $e \in E$, the cardinality $|e|$ of e is equal to k. Throughout this paper, $k \geqslant 3$ and $n \geqslant k$. Moreover, since the trivial hypergraph (i.e., $E = \emptyset$) is of less interest, we consider only hypergraphs having at least one edge (i.e., nontrivial) in this paper.

For a subset $S \subset [n]$, we denote by E_S the set of edges $\{e \in E \mid S \cap e \neq \emptyset\}$. For a vertex $i \in V$, we simplify $E_{\{i\}}$ as E_i. It is the set of edges containing the vertex i, i.e., $E_i := \{e \in E \mid i \in e\}$. The cardinality

$|E_i|$ of the set E_i is defined as the *degree* of the vertex i, which is denoted by d_i. Two different vertices i and j are *connected* to each other (or the pair i and j is connected), if there is a sequence of edges $(e_1, \ldots, e_m)$ such that $i \in e_1$, $j \in e_m$ and $e_r \cap e_{r+1} \neq \emptyset$ for all $r \in [m-1]$. A hypergraph is called *connected*, if every pair of vertices of G is connected. Let $S \subseteq V$, the hypergraph with vertex set S and edge set $\{e \in E \mid e \subseteq S\}$ is called the *sub-hypergraph* of G induced by S. We will denote it by G_S. A hypergraph is *regular* if $d_1 = \cdots = d_n = d$. A hypergraph $G = (V, E)$ is *complete* if E consists of all the possible edges. In this case, G is regular of degree $d = \binom{n-1}{k-1}$.

For the sake of simplicity, we mainly consider connected hypergraphs in the subsequent analysis. By the techniques in [20,9], the conclusions on connected hypergraphs can be easily generalized to general hypergraphs.

The following definition for the Laplacian tensor and signless Laplacian tensor was proposed by Qi [20].

Definition 2.2. Let $G = (V, E)$ be a k-uniform hypergraph. The *adjacency tensor* of G is defined as the kth order n-dimensional tensor $\mathcal{A}$ whose $(i_1 \ldots i_k)$-entry is:

$$a_{i_1 \ldots i_k} := \begin{cases} \frac{1}{(k-1)!} & \text{if } \{i_1, \ldots, i_k\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}$ be a kth order n-dimensional diagonal tensor with its diagonal element $d_{i \ldots i}$ being d_i, the degree of vertex i, for all $i \in [n]$. Then $\mathcal{L} := \mathcal{D} - \mathcal{A}$ is the *Laplacian tensor* of the hypergraph G, and $\mathcal{Q} := \mathcal{D} + \mathcal{A}$ is the *signless Laplacian tensor* of the hypergraph G.

By [20], zero is always the smallest H-eigenvalue of $\mathcal{L}$, and we have $\lambda(\mathcal{L}) \leqslant \lambda(\mathcal{Q}) \leqslant 2d$, where d is the maximum degree of G.

In the following, we introduce the class of cored hypergraphs.

Definition 2.3. Let $G = (V, E)$ be a k-uniform hypergraph. If for every edge $e \in E$, there is a vertex $i_e \in e$ such that the degree of the vertex i_e is one, then G is a *cored hypergraph*. A vertex with degree one is a *core vertex*, and a vertex with degree larger than one is an *intersection vertex*.

Let $G = (V, E)$ be an ordinary graph. For every $k \geqslant 3$, we can introduce a hypergraph by blowing up the edges of G.

Definition 2.4. Let $G = (V, E)$ be a 2-uniform graph. For any $k \geqslant 3$, the kth power of G, $G^k := (V^k, E^k)$ is defined as the k-uniform hypergraph with the set of edges $E^k := \{e \cup \{i_{e,1}, \ldots, i_{e,k-2}\} \mid e \in E\}$, and the set of vertices $V^k := V \cup \{i_{e,1}, \ldots, i_{e,k-2},\ e \in E\}$.

It is easy to see that the class of power hypergraphs is a subclass of cored hypergraphs. The classes of sunflowers and loose cycles are studied in [11]. It can be seen that the classes of sunflowers and loose cycles are subclasses of power hypergraphs. Actually, a k-uniform sunflower (respectively loose cycle) is the kth power of a star (respectively cycle) graph.

We present in Fig. 1 an example of an ordinary graph and its 3rd and 4th power hypergraphs.

For completeness, we include the definitions for sunflowers and loose cycles in Definitions 2.5 and 2.6 respectively.

Definition 2.5. Let $G = (V, E)$ be a k-uniform hypergraph. If there is a disjoint partition of the vertex set V as $V = V_0 \cup V_1 \cup \cdots \cup V_d$ such that $|V_0| = 1$ and $|V_1| = \cdots = |V_d| = k - 1$, and $E = \{V_0 \cup V_i \mid i \in [d]\}$, then G is called a *sunflower*. The degree d of the vertex in V_0, which is called the *heart*, is the *size* of the sunflower. The edges of G are *leaves*, and the vertices other than the heart are vertices of leaves.

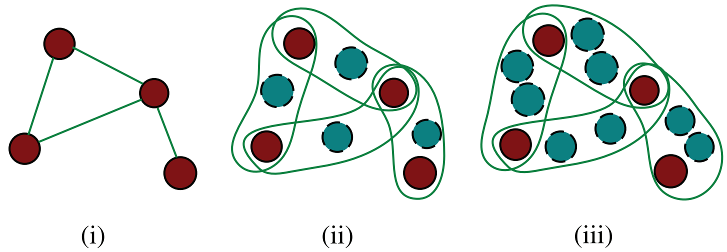

Fig. 1. (i) is an example of a usual graph. (ii) and (iii) are respectively the 3-uniform and 4-uniform power hypergraphs of the graph in (i). A solid disk represents a vertex. For the usual graph, a line connecting two points is an edge. For hypergraphs, an edge is pictured as a closed curve with the containing solid disks the vertices in that edge. The newly added vertices are in different color (also in dashed margins).

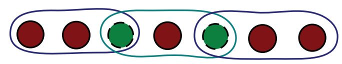

Fig. 2. An example of a 3-uniform loose path of length 3. The intersection vertices are in red (also in dashed margins). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Definition 2.6. Let $G=(V,E)$ be a k-uniform nontrivial hypergraph. If we can number the vertex set V as $V:=\{i_{1,1},\dots,i_{1,k-1},\dots,i_{d,1},\dots,i_{d,k-1}\}$ for some positive integer d such that $E=\{\{i_{1,1},\dots,i_{1,k-1},i_{2,1}\},\{i_{2,1},\dots,i_{2,k-1},i_{3,1}\},\dots,\{i_{d-1,1},\dots,i_{d-1,k-1},i_{d,1}\},\{i_{d,1},\dots,i_{d,k-1},i_{1,1}\}\}$, then G is called a *loose cycle*. d is the *size* of the loose cycle.

It is easy to see that a k-uniform sunflower of size $s>0$ has $n=s(k-1)+1$ vertices, a k-uniform loose cycle of size $s>0$ has $n=s(k-1)$ vertices, and they are both connected.

Besides sunflowers and loose cycles, power hypergraphs contain loose paths [12,13,15]. Loose paths are power hypergraphs of usual paths. We state it in the next definition.

Definition 2.7. Let $G=(V,E)$ be a k-uniform hypergraph. If we can number the vertex set V as $V:=\{i_{1,1},\dots,i_{1,k},i_{2,2},\dots,i_{2,k},\dots,i_{d-1,2},\dots,i_{d-1,k},i_{d,2},\dots,i_{d,k}\}$ for some positive integer d such that $E=\{\{i_{1,1},\dots,i_{1,k}\},\{i_{1,k},i_{2,2},\dots,i_{2,k}\},\dots,\{i_{d-1,k},i_{d,2},\dots,i_{d,k}\}\}$, then G is a *loose path*. d is the *length* of the loose path.

Fig. 2 is an example of a 3-uniform loose.

The notions of odd-bipartite and even-bipartite even-uniform hypergraphs are introduced in [10].

Definition 2.8. Let k be even and $G=(V,E)$ be a k-uniform hypergraph. It is called *odd-bipartite* if either it is trivial (i.e., $E=\emptyset$) or there is a partition of the vertex set V as $V=V_1\cup V_2$ such that $V_1,V_2\neq\emptyset$ and every edge in E intersects V_1 with exactly an odd number of vertices.

3. Cored hypergraphs

Some facts about the H-eigenvalues and H-eigenvectors of the Laplacian tensors of cored hypergraphs are established in this section.

3.1. General cored hypergraphs

In this subsection, we establish some facts that all cored hypergraphs share.

Our first lemma shows some structure exhibited by the H-eigenvectors of cored hypergraphs.

Lemma 3.1. *Let $G=(V,E)$ be a k-uniform cored hypergraph and $\mathbf{x}\in\mathbb{R}^n$ be an H-eigenvector of its Laplacian tensor $\mathcal{L}$ corresponding to an H-eigenvalue $\lambda\neq 1$. If there are two core vertices i,j in an edge $e\in E$, then $|x_i|=|x_j|$. Moreover, $x_i=x_j$ when k is an odd number.*

Proof. By the definition of H-eigenvalues and the fact that i and j are core vertices, we have

$$\lambda x_i^{k-1} = \left(\mathcal{L}\mathbf{x}^{k-1}\right)_i = x_i^{k-1} - x_j \prod_{s \in e \setminus \{i,j\}} x_s, \qquad \lambda x_j^{k-1} = \left(\mathcal{L}\mathbf{x}^{k-1}\right)_j = x_j^{k-1} - x_i \prod_{s \in e \setminus \{i,j\}} x_s.$$

Hence,

$$(\lambda - 1)x_i^k = (\lambda - 1)x_j^k.$$

Since $\lambda \neq 1$, we have that $|x_i| = |x_j|$. Moreover, when k is odd, we see that $x_i = x_j$. $\square$

By [20, Theorem 3.2], we have the following lemma.

Lemma 3.2. *Let $G = (V, E)$ be a k-uniform hypergraph with its maximum degree $d > 0$ and $\mathcal{L}$ be its Laplacian tensor. Then $\lambda(\mathcal{L}) \geqslant d$.*

Lemma 3.3. *Let $G = (V, E)$ be a k-uniform cored hypergraph and $\mathbf{x} \in \mathbb{R}^n$ be an H-eigenvector of its Laplacian tensor $\mathcal{L}$ corresponding to $\lambda \geqslant 1$. Then, $\prod_{s \in e} x_s \leqslant 0$ for all $e \in E$ when k is even; and $\prod_{s \in e \setminus \{i_e\}} x_s \leqslant 0$ for all $e \in E$ when k is odd. Here $i_e \in e$ is a core vertex.*

Proof. Suppose that i is a core vertex of an arbitrary but fixed edge $e \in E$. If $\lambda = 1$, then

$$x_i^{k-1} = \lambda x_i^{k-1} = x_i^{k-1} - \prod_{s \in e \setminus \{i\}} x_s$$

implies that $\prod_{s \in e \setminus \{i\}} x_s = 0$. We are done.

In the following, suppose that $\lambda > 1$. Then,

$$(\lambda - 1)x_i^{k-1} = -\prod_{s \in e \setminus \{i\}} x_s.$$

When k is odd, $x_i^{k-1} \geqslant 0$. Then the result follows. When k is even, we have $\prod_{s \in e} x_s \leqslant 0$ since $(\lambda - 1)x_i^k = -\prod_{s \in e} x_s$. $\square$

By Lemmas 3.2 and 3.3, we get the next proposition.

Proposition 3.1. *Let $G = (V, E)$ be a k-uniform cored hypergraph and $\mathbf{x} \in \mathbb{R}^n$ be an H-eigenvector of its Laplacian tensor $\mathcal{L}$ corresponding to $\lambda(\mathcal{L})$. Then, $\prod_{s \in e} x_s \leqslant 0$ for all $e \in E$ when k is even; and $\prod_{s \in e \setminus \{i_e\}} x_s \leqslant 0$ for all $e \in E$ when k is odd. Here $i_e \in e$ is a core vertex.*

By [11, Theorem 5.1], we can get the next proposition.

Proposition 3.2. *Let k be even and $G = (V, E)$ be a k-uniform cored hypergraph. Let $\mathcal{L}$ and $\mathcal{Q}$ be the Laplacian tensor and signless Laplacian tensor of G respectively. Then G is odd-bipartite, and hence $\lambda(\mathcal{L}) = \lambda(\mathcal{Q})$.*

Proof. For all $e \in E$, let $i_e \in e$ be a core vertex. Set $V_1 := \{i_e \mid e \in E\}$ and $V_2 := V \setminus V_1$. Then it is easy to see that $V = V_1 \cup V_2$ is an odd-bipartition (Definition 2.8). Thus, the result follows from [11, Theorem 5.1]. $\square$

Actually, we can get the next proposition.

Proposition 3.3. *Let k be even and $G = (V, E)$ be a k-uniform cored hypergraph. Let $\mathcal{L}$ and $\mathcal{Q}$ be the Laplacian tensor and signless Laplacian tensor of G respectively. For every $e \in E$, let $i_e \in e$ be a core vertex.*

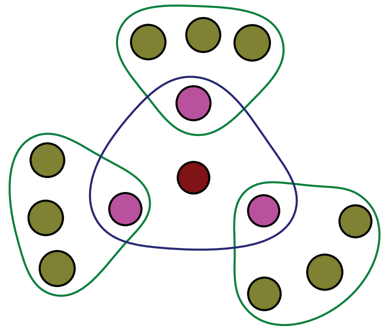

Fig. 3. An example of the 4-uniform squid.

(i) *If* $\mathbf{x} \in \mathbb{R}^n_+$ *is an H-eigenvector of* $\mathcal{Q}$ *corresponding to* $\lambda(\mathcal{Q})$*, then* $\mathbf{y} \in \mathbb{R}^n$ *is an H-eigenvector of* $\mathcal{L}$ *corresponding to* $\lambda(\mathcal{L})$ *with* $y_{i_e} = -x_{i_e}$ *for all* $e \in E$ *and* $y_j = x_j$ *for the others.*
(ii) *If* $\mathbf{x} \in \mathbb{R}^n$ *is an H-eigenvector of* $\mathcal{L}$ *corresponding to* $\lambda(\mathcal{L})$*, then* $\mathbf{y} \in \mathbb{R}^n_+$ *is an H-eigenvector of* $\mathcal{Q}$ *corresponding to* $\lambda(\mathcal{Q})$ *with* $y_i = |x_i|$ *for all* $j \in [n]$*.*

Proof. The results follow from Definition 2.1 and Proposition 3.1. □

3.2. Squids

It is easy to see that not all cored hypergraphs are power hypergraphs. We investigate one such class, the squids.

Definition 3.1. Let $G = (V, E)$ be a k-uniform hypergraph. If we can number the vertex set V as $V := \{i_{1,1}, \dots, i_{1,k}, \dots, i_{k-1,1}, \dots, i_{k-1,k}, i_k\}$ such that the set of edges $E = \{\{i_{1,1}, \dots, i_{1,k}\}, \dots, \{i_{k-1,1}, \dots, i_{k-1,k}\}, \{i_{1,1}, \dots, i_{k-1,1}, i_k\}\}$, then G is a *squid.*

Note that the squid for every positive integer k is unique, in the sense that by a possible renumbering the vertices two k-uniform squids are the same. Fig. 3 is an example of the 4-uniform squid.

The next proposition calculates the largest H-eigenvalue of the Laplacian tensor of an even-uniform squid.

Proposition 3.4. *Let* k *be even and* $G = (V, E)$ *be the* k*-uniform squid. Let* $\mathcal{L}$ *be the Laplacian tensor of* G*. Then* G *is odd-bipartite and* $\lambda(\mathcal{L})$ *is the unique root of* $(\mu - 2) - (\frac{1}{\mu-1})^{\frac{1}{k-1}} - (\frac{1}{\mu-1})^{k-1} = 0$ *in the interval* $(2, 4)$*.*

Proof. Suppose that $V = \{i_{1,1}, \dots, i_{1,k}, \dots, i_{k-1,1}, \dots, i_{k-1,k}, i_k\}$, and the set of edges is $E = \{\{i_{1,1}, \dots, i_{1,k}\}, \dots, \{i_{k-1,1}, \dots, i_{k-1,k}\}, \{i_{1,1}, \dots, i_{k-1,1}, i_k\}\}$. Let $\mathcal{Q}$ be the signless Laplacian tensor of G. By Proposition 3.2, G is odd-bipartite and $\lambda(\mathcal{L}) = \lambda(\mathcal{Q})$.

By [28, Theorem 3.20], [21, Theorem 4] and [17, Lemma 3.1] (see also [9, Lemmas 2.2 and 2.3]), if we can find a positive H-eigenvector $\mathbf{x} \in \mathbb{R}^n$ of $\mathcal{Q}$ corresponding to an H-eigenvalue μ, then $\mu = \lambda(\mathcal{Q})$.

Let $x_{i_k} = \alpha > 0$, $x_{i_{j,1}} = 1$, and $x_{i_{j,2}} = \cdots = x_{i_{j,k}} = \gamma > 0$ for all $j \in [k-1]$. Suppose that $\mathbf{x}$ is an H-eigenvector of $\mathcal{Q}$ corresponding to the H-eigenvalue $\mu = \lambda(\mathcal{Q})$. By Definition 2.1, we have

$$(\mu - 1)\alpha^{k-1} = 1, \qquad (\mu - 2) = \alpha + \gamma^{k-1}, \quad \text{and} \quad (\mu - 1)\gamma^{k-1} = \gamma^{k-2}.$$

By Lemma 3.2, we have $\mu \geqslant 2$. Thus, the first and the third equalities imply that $\alpha^{k-1} = \gamma$. Hence,

$$(\mu - 2) = \left(\frac{1}{\mu - 1}\right)^{\frac{1}{k-1}} + \left(\frac{1}{\mu - 1}\right)^{k-1}.$$

Let $f(\mu) := (\mu-2) - (\frac{1}{\mu-1})^{\frac{1}{k-1}} - (\frac{1}{\mu-1})^{k-1}$. We have that $f(2) = -2 < 0$ and

$$f(4) = 2 - \left(\frac{1}{3}\right)^{\frac{1}{k-1}} - \frac{1}{3^{k-1}} > 0.$$

Thus, $f(\mu) = 0$ does have a root in the interval $(2, 4)$. Since $\mathcal{Q}$ has a unique positive H-eigenvector (see [9, Lemmas 2.2 and 2.3]), the equation $(\mu-2) - (\frac{1}{\mu-1})^{\frac{1}{k-1}} - (\frac{1}{\mu-1})^{k-1} = 0$ has a unique positive solution which is in the interval $(2, 4)$. Hence, the result follows. $\square$

The next proposition says that the largest H-eigenvalue of the Laplacian tensor of an odd-uniform squid is equal to the maximum degree, i.e., 2.

Proposition 3.5. *Let k be odd and $G = (V, E)$ be the k-uniform squid. Let $\mathcal{L}$ be the Laplacian tensor of G. Then $\lambda(\mathcal{L}) = 2$.*

Proof. Suppose that $V = \{i_{1,1}, \dots, i_{1,k}, \dots, i_{k-1,1}, \dots, i_{k-1,k}, i_k\}$, and the set of edges is $E = \{\{i_{1,1}, \dots, i_{1,k}\}, \dots, \{i_{k-1,1}, \dots, i_{k-1,k}\}, \{i_{1,1}, \dots, i_{k-1,1}, i_k\}\}$. Let $\mathbf{w} \in \mathbb{R}^n$ be an H-eigenvector of $\mathcal{L}$ corresponding to $\lambda(\mathcal{L})$. Then, we have

$$\big(\lambda(\mathcal{L}) - 1\big) w_{i_{j,s}}^{k-1} = -w_{i_{j,1}} \prod_{t \in \{2,\dots,k\} \setminus \{s\}} w_{i_{j,t}}, \quad \forall j \in [k-1],\ s \in \{2, \dots, k\}.$$

Thus, $(\lambda(\mathcal{L}) - 1) w_{i_{j,s}}^k = -w_{i_{j,1}} \prod_{t \in \{2,\dots,k\}} w_{i_{j,t}}$. Hence, $w_{i_{j,2}} = \dots = w_{i_{j,k}} =: z_j$ for all $j \in [k-1]$, since $\lambda(\mathcal{L}) \geqslant 2$ by Lemma 3.2. Let $x := w_{i_k}$, and $y_j := w_{i_{j,1}}$ for all $j \in [k-1]$. Then, we have $y_j = (1 - \lambda(\mathcal{L})) z_j$ if $z_j \neq 0$, for each $j \in [k-1]$. Moreover, by Definition 2.1, we have

$$\big(\lambda(\mathcal{L}) - 2\big) y_j^{k-1} = -x \prod_{s \in [k-1] \setminus \{j\}} y_s - z_j^{k-1}, \quad \forall j \in [k-1].$$

Consequently, we have

$$\Big[\big(\lambda(\mathcal{L}) - 2\big)\big(1 - \lambda(\mathcal{L})\big)^{k-1} + \big(1 - \lambda(\mathcal{L})\big)\Big] z_j^k = -x \prod_{s \in [k-1]} y_s, \quad \forall j \in [k-1].$$

Thus, either $z_1 = \dots = z_{k-1} =: z$ since k is odd, or $(\lambda(\mathcal{L}) - 2)(1 - \lambda(\mathcal{L}))^{k-1} + (1 - \lambda(\mathcal{L})) = 0$ and $x \prod_{s \in [k-1]} y_s = 0$.

If $(\lambda(\mathcal{L}) - 2)(1 - \lambda(\mathcal{L}))^{k-1} + (1 - \lambda(\mathcal{L})) = 0$, then $(\lambda(\mathcal{L}) - 2)(1 - \lambda(\mathcal{L}))^{k-2} + 1 = 0$ since $\lambda(\mathcal{L}) \geqslant 2$. We can assume that $\lambda(\mathcal{L}) > 2$, otherwise we are done. On the other hand, if all y_t with $t \in [k-1]$ are zero, we get that $\mathbf{w} = 0$ which is a contradiction. Hence, we must have that $x \prod_{s \in [k-1] \setminus \{j\}} y_s = 0$ and $y_j \neq 0$ for some $j \in [k-1]$, since $x \prod_{s \in [k-1]} y_s = 0$. These two facts will contradict the fact that

$$\big(\lambda(\mathcal{L}) - 2\big) y_j^{k-1} = -x \prod_{s \in [k-1] \setminus \{j\}} y_s - z_j^{k-1} = -z_j^{k-1},$$

since $k-1$ is even. Thus, this situation can never happen.

If $z_1 = \dots = z_{k-1} = z \neq 0$, then $y_1 = \dots = y_{k-1} =: y \neq 0$ since $y_j = (1 - \lambda(\mathcal{L})) z_j$. By Definition 2.1, we have $0 \leqslant (\lambda(\mathcal{L}) - 1) x^{k-1} = -y^{k-1} \leqslant 0$. Hence $x = y = 0$, which is a contradiction. Then, we must have that $z = 0$. The equations of the H-eigenvalue $\lambda(\mathcal{L})$ become

$$\big(\lambda(\mathcal{L}) - 1\big) x^{k-1} = -\prod_{s \in [k-1]} y_s, \qquad \big(\lambda(\mathcal{L}) - 2\big) y_j^{k-1} = -x \prod_{s \in [k-1] \setminus \{j\}} y_s, \quad \forall j \in [k-1].$$

If $\lambda(\mathcal{L}) > 2$, we must have $y_1 = \dots = y_{k-1} =: y$, since $(\lambda(\mathcal{L}) - 2) y_j^k = -x \prod_{s \in [k-1]} y_s$ for all $j \in [k-1]$. Similarly, we must have $x = y = 0$ which is a contradiction. Hence, $\lambda(\mathcal{L}) = 2$. An H-eigenvector would be $y_1 = 1$ and the rest are zero. $\square$

3.3. Generalized loose paths and cycles

In [18], generalized loose paths and cycles were discussed. Let $G=(V,E)$ be a k-uniform hypergraph. Suppose $1\leqslant s\leqslant k-1$. If $V=\{i_1,i_2,\ldots,i_{s+d(k-s)}\}$ such that $\{i_{1+j(k-s)},\ldots,i_{s+(j+1)(k-s)}\}$ is an edge of G for $j=0,\ldots,d-1$, then G is called an s-path. In [15], G is called a *loose path* if $s=1$, and a *tight path* if $s=k-1$. The definition of the loose path here is the same as we defined. In [18], G is also called a loose path for $2\leqslant s\leqslant\frac{k}{2}$ and a tight path for $\frac{k}{2}<s\leqslant k-2$. To avoid confusion, we call G a *generalized loose path* and a *generalized tight path* respectively. We see that if $2\leqslant s<\frac{k}{2}$, then G is a cored hypergraph, but not a power hypergraph in general. Note that this excludes the case that $s=\frac{k}{2}$, though that case was divided to the "loose" category in [18]. Similarly, if in the above, $i_1,\ldots,i_{d(k-s)}$ are distinct but $i_{d(k-s)+j}=i_j$ for $j=1,\ldots,s$, then G is called an s-cycle. According to [12,13,15], if $s=1$, G is called a *loose cycle*, and if $s=k-1$, G is called a *tight cycle*. The definition of the loose cycle here is the same as we defined. We may call G a *generalized loose cycle* for $2\leqslant s\leqslant\frac{k}{2}$, and a *generalized tight cycle* for $\frac{k}{2}<s\leqslant k-2$, by [18]. Again, we see that if $2\leqslant s<\frac{k}{2}$, then G is a cored hypergraph, but not a power hypergraph in general.

4. Power hypergraphs

Some facts about the H-eigenvalues and H-eigenvectors of the Laplacian tensor of a power hypergraph are investigated in this section.

4.1. Odd-uniform power hypergraphs

In this subsection, we show that the largest H-eigenvalue of the Laplacian tensor of an odd-uniform loose cycle (loose path) is equal to the maximum degree, i.e., 2.

The next lemma is useful.

Lemma 4.1. *Let k be odd, $G=(V,E)$ be a k-uniform power hypergraph and $\mathbf{x}\in\mathbb{R}^n$ be an H-eigenvalue of its Laplacian tensor $\mathcal{L}$ corresponding to $\lambda\neq 1$. Let $e\in E$ be an arbitrary but fixed edge.*

(i) *If e has only one intersection vertex i, and $x_s\neq 0$ for some core vertex $s\in e$, then $(1-\lambda)x_s=x_i$.*
(ii) *If e has two intersection vertices i and j, and $x_s\neq 0$ for some core vertex $s\in e$, then $x_ix_j=(1-\lambda)x_s^2$.*

Proof. For (i), by Definition 2.1 and Lemma 3.1, we have

$$\lambda x_s^{k-1}=x_s^{k-1}-x_s^{k-2}x_i.$$

Thus, $x_i=(1-\lambda)x_s$.

For (ii), by Definition 2.1 and Lemma 3.1, we have

$$\lambda x_s^{k-1}=x_s^{k-1}-x_s^{k-3}x_ix_j.$$

Thus, $x_ix_j=(1-\lambda)x_s^2$. $\square$

The next corollary is a direct consequence of Lemma 4.1.

Corollary 4.1. *Let k be odd and $G=(V,E)$ be a k-uniform power hypergraph and $\mathbf{x}\in\mathbb{R}^n$ be an H-eigenvalue of its Laplacian tensor $\mathcal{L}$ corresponding to $\lambda>1$. Let $e\in E$ be an arbitrary but fixed edge.*

(i) *If e has only one intersection vertex i, and $x_s\neq 0$ for some core vertex $s\in e$, then $x_ix_s<0$.*
(ii) *If e has two intersection vertices i and j, and $x_s\neq 0$ for some core vertex $s\in e$, then $x_ix_j<0$.*

The next proposition is for loose cycles.

Proposition 4.1. *Let k be odd and $G = (V, E)$ be a k-uniform loose cycle with size $r \geqslant 2$. Let $\mathcal{L}$ be its Laplacian tensor. Then $\lambda(\mathcal{L}) = 2$.*

Proof. Suppose that $\lambda(\mathcal{L}) \neq 2$. Then $\lambda(\mathcal{L}) > 2$ by Lemma 3.2. Let

$$E = \big\{\{j_1, i_{1,1} \dots, i_{1,k-2}, j_2\}, \{j_2, i_{2,1} \dots, i_{2,k-2}, j_3\}, \dots,$$
$$\{j_{r-1}, i_{r-1,1} \dots, i_{r-1,k-2}, j_r\}, \{j_r, i_{r,1} \dots, i_{r,k-2}, j_1\}\big\}.$$

Let $\mathbf{z} \in \mathbb{R}^{r(k-1)}$ be an H-eigenvector of $\mathcal{L}$ corresponding to $\lambda(\mathcal{L})$. Let $y_s := z_{j_s}$ for $s \in [r]$. By Lemma 3.1, we have that $z_{i_{s,1}} = \dots = z_{i_{s,k-2}}$. Let $x_s := z_{i_{s,1}}$ for $s \in [r]$.

(I). If $x_s \neq 0$ for all $s \in [r]$, by Lemma 4.1, we have that $y_s y_{s+1} < 0$ with $y_{r+1} := y_1$ for all $s \in [r]$. Thus, if r is odd, we get a contradiction by the rule of signs. In the following, we assume that r is even. Since $\lambda(\mathcal{L}) > 2$, by Definition 2.1, we have

$$\big(\lambda(\mathcal{L}) - 2\big) y_s^k = -x_{s-1}^{k-2} y_{s-1} y_s - x_s^{k-2} y_s y_{s+1}, \quad \forall s \in [r] \tag{4.1}$$

with the convention that $x_0 = x_r$, $x_{r+1} = x_1$, and $y_0 = y_r$, $y_{r+1} = y_1$. Thus, we have

$$y_1^k - y_2^k + y_3^k - \dots + y_{r-1}^k - y_r^k = 0.$$

Since $y_s y_{s+1} < 0$ and k is odd, we get a contradiction, since $y_1^k - y_2^k + y_3^k - \dots + y_{r-1}^k - y_r^k$ should be positive (respectively negative) if we let y_1 be positive (respectively negative). Consequently, $\lambda(\mathcal{L}) = 2$.

(II). Suppose that $x_s = 0$ for some $s \in [r]$. If $x_s = 0$ for all $s \in [r]$. Then by $\lambda(\mathcal{L}) > 2$ and Eq. (4.1), we would have $y_s = 0$ for all $s \in [r]$, which means that z is a zero vector, a contradiction. Also, by Lemma 4.1, we have that $y_s y_{s+1} < 0$ whenever $x_s \neq 0$.

Without loss of generality, we assume that $x_r = 0$ and $x_1 > 0$. By taking $s = 1$ in (4.1), we have

$$\big(\lambda(\mathcal{L}) - 2\big) y_1^{k-1} = -x_1^{k-2} y_2.$$

Since $x_1 > 0$, we have that $y_2 < 0$. Since $y_1 y_2 < 0$, we have $y_1 > 0$. Also, by taking $s = 2$ in (4.1), we have

$$\big(\lambda(\mathcal{L}) - 2\big) y_2^{k-1} = -x_1^{k-2} y_1 - x_2^{k-2} y_3.$$

Then, we must have $x_2 < 0$. By using induction, we will finally get $x_r \neq 0$, a contradiction. Consequently, we obtain $\lambda(\mathcal{L}) = 2$. □

The next proposition is on loose paths.

Proposition 4.2. *Let k be odd and $G = (V, E)$ be a k-uniform loose path with length $r \geqslant 3$. Let $\mathcal{L}$ be its Laplacian tensor. Then $\lambda(\mathcal{L}) = 2$.*

Proof. Suppose that $\lambda(\mathcal{L}) \neq 2$. Then $\lambda(\mathcal{L}) > 2$ by Lemma 3.2. Let

$$E = \big\{\{i_{1,1}, \dots, i_{1,k-1}, j_1\}, \{j_1, i_{2,1}, \dots, i_{2,k-2}, j_2\}, \dots,$$
$$\{j_{r-2}, i_{r-1,1}, \dots, i_{r-1,k-2}, j_{r-1}\}, \{j_{r-1}, i_{r,1}, \dots, i_{r,k-1}\}\big\}.$$

Let $\mathbf{z} \in \mathbb{R}^{r(k-1)+1}$ be an H-eigenvector of $\mathcal{L}$ corresponding to $\lambda(\mathcal{L})$. Let $y_s := z_{j_s}$ for $s \in [r-1]$. By Lemma 3.1, we have that $z_{i_{s,1}} = \dots = z_{i_{s,k-2}}$ for $s \in \{2, \dots, r-1\}$, and $z_{i_{i,1}} = \dots = z_{i_{1,k-1}}$, $z_{i_{r,1}} = \dots = z_{i_{r,k-1}}$. Let $x_s := z_{is,1}$ for $s \in [r]$. We assume that $\lambda(\mathcal{L}) > 2$ and derive a contradiction case by case. The proof is in the same spirit of and similar to that for Proposition 4.1, we include it here for completeness.

(I). If both x_1 and x_r are zero, then the proof is same as that for the proof (II) in Proposition 4.1, since we always have a piece of the loose with $y_t, x_{t+1}, \dots, y_m \neq 0$ for some $m \geqslant t \geqslant 1$.

(II). If $x_1 \neq 0$ and $x_m = 0$ for some $m \leqslant r$, then $y_1 \neq 0$. We can assume that $y_1 > 0$. By Lemma 4.1, we have $x_1 < 0$. By Definition 2.1, we have

$$\big(\lambda(\mathcal{L}) - 2\big) y_1^{k-1} = -x_1^{k-1} - x_2^{k-2} y_2.$$

Since $y_2 < 0$ by Lemma 4.1, we have that $x_2 > 0$. Also,

$$\big(\lambda(\mathcal{L}) - 2\big) y_2^{k-1} = -x_2^{k-2} y_1 - x_3^{k-2} y_3.$$

Then, we must have $x_3 > 0$. Inductively, we have $x_s y_s < 0$. Hence, $x_m y_m < 0$, contradicting the hypothesis that $x_m = 0$. Consequently, $\lambda(\mathcal{L}) = 2$.

(III). The proof for the case $x_r \neq 0$ and $x_m = 0$ for some $m < r$ is similar. Actually, it follows from (II) immediately by renumbering the indices.

(IV). If $x_s \neq 0$ for all $s \in [r]$. Similar to (II), we have that $x_s y_s < 0$. Particularly, we have $x_{r-1} y_{r-1} < 0$ which implies that $x_{r-1} y_{r-2} > 0$. By Definition 2.1, we have

$$\big(\lambda(\mathcal{L}) - 2\big) y_{r-1}^{k-1} = -x_{r-1}^{k-2} y_{r-2} - x_r^{k-1} < 0.$$

Thus, a contradiction is derived since k is odd. Consequently, $\lambda(\mathcal{L}) = 2$. □

By Propositions 5.6, 4.1 and 4.2, we get that [11, Conjecture 3.1] has a negative answer.

4.2. Even-uniform power hypergraphs

We have a conjecture for even-uniform hypergraphs.

Conjecture 4.1. *Let $G = (V, E)$ be a usual graph, $k = 2r$ be even and $G^k = (V^k, E^k)$ be the k-power hypergraph of G. Let $\mathcal{L}^k$ and $\mathcal{Q}^k$ be the Laplacian and signless Laplacian tensors of G^k respectively. Then $\{\lambda(\mathcal{L}^k) = \lambda(\mathcal{Q}^k)\}$ is a strictly decreasing sequence.*

By [11, Theorem 3.1 and Corollary 5.1], we have the next proposition.

Proposition 4.3. *Conjecture 4.1 is true for sunflowers and loose cycles.*

Proof. For the case of sunflowers, by [11, Theorem 3.1], we have that $\lambda(\mathcal{L}^k)$ is the unique root of

$$(1 - \lambda)^{k-1}(\lambda - d) + d = 0,$$

where $k = 2r$ is even, and d is the size of the sunflower. Let $f_k(\lambda) := (1 - \lambda)^{k-1}(\lambda - d) + d$. We have $f_{k+2}(d) = d > 0$ and

$$f_{k+2}\big(\lambda(\mathcal{L}^k)\big) = f_{k+2}\big(\lambda(\mathcal{L}^k)\big) - f_k\big(\lambda(\mathcal{L}^k)\big) = \big(1 - \lambda(\mathcal{L}^k)\big)^{k-1}\big(\lambda(\mathcal{L}^k) - d\big)\big(\big(1 - \lambda(\mathcal{L}^k)\big)^2 - 1\big).$$

Since $\lambda(\mathcal{L}^k) > d \geqslant 2$ and k is even, we have $f_{k+2}(\lambda(\mathcal{L}^k)) < 0$. Thus we have $\lambda(\mathcal{L}^{k+2}) \in (d, \lambda(\mathcal{L}^k))$.

The proof for the loose cycle is similar. □

5. Laplacian H-spectra of special power hypergraphs

We compute all the Laplacian H-eigenvalues of some special power hypergraphs in this section.

5.1. Sunflowers

Let $G = (V, E)$ be a k-uniform sunflower with $k \geqslant 3$ and the size $d \geqslant 2$, where $V = [n]$, $E = \{e_1, \ldots, e_d\}$, and $d_1 = d$ (i.e., the vertex 1 is the heart). Let $\mathcal{L} = \mathcal{D} - \mathcal{A}$ be the Laplacian tensor of G. Then it is easy to see that the eigenvalue equations $(\lambda\mathcal{I} - \mathcal{L})\mathbf{x}^{k-1} = 0$ are equivalent to the following set of relations:

$$(\lambda - d)x_1^{k-1} = -\sum_{i=1}^{d} \prod_{s \in e_i \setminus \{1\}} x_s, \tag{1}$$

and

$$(\lambda - 1)x_j^{k-1} = -\prod_{s \in e(j)\setminus\{j\}} x_s, \quad \forall j \in \{2, \dots, n\}, \tag{2}$$

where $e(j)$ denotes the unique edge containing the vertex j for $j \geqslant 2$.

The next lemma strengthens Lemma 4.1 for the case of sunflowers.

Lemma 5.1. *Let G, k, d and $\mathcal{L}$ be as above. Suppose that (λ, x) is an H-eigenpair of $\mathcal{L}$ with $\lambda \neq 1$. Then we have*:

(i) *If $i, j \geqslant 2$ and i, j are adjacent (there is an edge containing both i and j), then $x_i = x_j$ when k is odd, and $|x_i| = |x_j|$ when k is even.*
(ii) *If $i, j \geqslant 2$ and x_i, x_j are both nonzero, then $x_i = x_j$ when k is odd, and $|x_i| = |x_j|$ when k is even.*

Proof. (i) follows from Lemma 4.1.

(ii) **Case 1: k is odd**. If $j \geqslant 2$ and $x_j \neq 0$, by (2) and the result (i) of this lemma we also have $x_1 = (1-\lambda)x_j$.

Similarly for $i \geqslant 2$ and $x_i \neq 0$, we also have $x_1 = (1-\lambda)x_i$. From this we obtain that $x_i = x_j$.

Case 2: k is even. If $j \geqslant 2$ and $x_j \neq 0$, by taking the absolute values of the both sides of Eq. (2) and using the result (1) of this lemma we also have $|x_1| = |(1-\lambda)x_j|$.

Similarly for $i \geqslant 2$ and $x_i \neq 0$, we also have $|x_1| = |(1-\lambda)x_i|$. From this we obtain that $|x_i| = |x_j|$. □

From Lemma 5.1 we can obtain the set of all distinct H-eigenvalues and all corresponding H-eigenvectors of the Laplacian tensor $\mathcal{L}$ of the sunflower G (except for the eigenvalue 1) in the following Proposition 5.1 (for the case when k is odd) and Proposition 5.2 (for the case when k is even). The set of all eigenvectors corresponding to the eigenvalue 1 will be given in Proposition 5.3.

Proposition 5.1. *Let $G = (V, E)$ be a k-uniform sunflower with odd $k \geqslant 3$ and the size $d \geqslant 2$, where $V = [n]$, $E = \{e_1, \dots, e_d\}$, and $d_1 = d$ (i.e., the vertex 1 is the heart). Let $\mathcal{L} = \mathcal{D} - \mathcal{A}$ be the Laplacian tensor of G. Let*

$$f_r(\lambda) = (\lambda - d)(1-\lambda)^{k-1} + r \quad \text{for } r = 0, 1, \dots, d.$$

Then we have:

(1) *$\lambda \neq 1$ is an H-eigenvalue of $\mathcal{L}$ if and only if it is a real root of the polynomial $f_r(\lambda)$ for some $r \in \{0, 1, \dots, d\}$.*
(2) *If $\lambda \neq 1$ is a real root of the polynomial $f_r(\lambda)$, then we can construct all the H-eigenvectors of $\mathcal{L}$ corresponding to λ (up to a constant multiple) by going through the following procedure*:
Step 1: *Take $x_1 = 1 - \lambda$.*
Step 2: *Choose any r edges of G, take the x-values of all the pendant vertices of these r edges to be 1.*
Step 3: *Take the x-values of all the other vertices of G to be zero.*

Proof. (i) Necessity. Let $(\lambda, \mathbf{x})$ is an H-eigenpair of $\mathcal{L}$ with $\lambda \neq 1$.

According to the results of Lemma 5.1, we call an edge e as x-nonzero, if the common x-value of all the pendant vertices of e is nonzero. Otherwise this edge is called x-zero.

Let r be the number of x-nonzero edges of G. Then we have $0 \leqslant r \leqslant d$. If $r = 0$, then $\mathbf{x} = (1, 0, \dots, 0)^T$ is an H-eigenvector corresponding to the eigenvalue d, which is the unique root of $f_0(\lambda)$ other than 1. So in the following, we may assume that $1 \leqslant r \leqslant d$.

By result (ii) of Lemma 5.1, we may assume that $x_i = 1$ for all $i \geqslant 2$ with $x_i \neq 0$ (up to a constant multiple). In this case, we also have $x_1 = 1 - \lambda$.

Now from (1) we further have $(\lambda - d)x_1^{k-1} = -r$. Combining this with $x_1 = 1 - \lambda$ we obtain that $(\lambda - d)(1-\lambda)^{k-1} + r = 0$, which means that λ is a real root of the polynomial $f_r(\lambda)$.

Sufficiency part of (i) follows directly from the constructive procedure of result (ii).

(ii) It is not difficult to verify that any vector x obtained after going through the steps 1–3 will satisfy the (1) and (2), so it is an H-eigenvector corresponding to the H-eigenvalue λ. $\square$

Now we consider the case when k is even.

Proposition 5.2. *Let $G = (V, E)$ be a k-uniform sunflower with even $k \geqslant 4$ and the size $d \geqslant 2$, where $V = [n]$, $E = \{e_1, \ldots, e_d\}$, and $d_1 = d$. Let $\mathcal{L} = \mathcal{D} - \mathcal{A}$ be the Laplacian tensor of G. Then we have*:

(i) *$\lambda \neq 1$ is an H-eigenvalue of $\mathcal{L}$ if and only if it is a real root of the polynomial $f_r(\lambda)$ for some $r \in \{0, 1, \ldots, d\}$.*

(ii) *If $\lambda \neq 1$ is a real root of the polynomial $f_r(\lambda)$, then we can construct all the H-eigenvectors of $\mathcal{L}$ corresponding to λ (up to a constant multiple) by going through the following procedure*:

Step 1: *Take $x_1 = 1 - \lambda$.*

Step 2: *Choose any r edges of G, take the x-values of all the pendant vertices of these r edges to be ± 1, where the number of -1 value in each edge is even.*

Step 3: *Take the x-values of all the other vertices of G to be zero.*

Proof. (i) Necessity. Let (λ, x) be an H-eigenpair of $\mathcal{L}$ with $\lambda \neq 1$.

Let r be the number of x-nonzero edges of G. Then we have $0 \leqslant r \leqslant d$.

By result (ii) of Lemma 5.1, we may assume that $x_i = \pm 1$ for all $i \geqslant 2$ with $x_i \neq 0$ (up to a constant multiple). In this case, we also have $x_1 = \pm(1 - \lambda)$ by (2). We now consider the following two cases:

Case 1: $x_1 = (1 - \lambda)$. Then from (2) we have $\prod_{s \in e(j) \setminus \{1\}} x_s = 1$ for $j \geqslant 2$ and $x_j \neq 0$. Thus from (1) we further have $(\lambda - d)x_1^{k-1} = -r$. Combining this with $x_1 = 1 - \lambda$ we obtain that $(\lambda - d)(1 - \lambda)^{k-1} + r = 0$, which means that λ is a real root of the polynomial $f_r(\lambda)$.

Case 2: $x_1 = -(1 - \lambda)$. Then from (2) we have $\prod_{s \in e(j) \setminus \{1\}} x_s = -1$ for $j \geqslant 2$ and $x_j \neq 0$. Thus from (1) we further have $(\lambda - d)x_1^{k-1} = r$. Combining this with $x_1 = -(1 - \lambda)$ and the hypothesis that k is even, we also obtain that $(\lambda - d)(1 - \lambda)^{k-1} + r = 0$, which means that λ is a real root of the polynomial $f_r(\lambda)$.

Notice that the eigenvectors x constructed in Case 1 and Case 2 only differ by a multiple -1, so we only need to consider Case 1.

Sufficiency part of (i) follows directly from the constructive procedure of result (ii).

(ii) It is not difficult to verify that any vector x obtained after going through the steps 1–3 will satisfy the (1) and (2), so it is an H-eigenvector corresponding to the H-eigenvalue λ. $\square$

Now we construct all the eigenvectors of $\mathcal{L}$ corresponding to the eigenvalue 1.

Proposition 5.3. *Let $G = (V, E)$ be a k-uniform sunflower with $k \geqslant 3$ and the size $d \geqslant 2$, where $V = [n]$, $E = \{e_1, \ldots, e_d\}$, and $d_1 = d$. Let $\mathcal{L} = \mathcal{D} - \mathcal{A}$ be the Laplacian tensor of G. Then a nonzero vector x is an eigenvector corresponding to the eigenvalue 1 if and only if $x_1 = 0$ and the x-values of all the pendant vertices of G satisfy the following relation*:

$$\sum_{i=1}^{d} \left(\prod_{s \in e_i \setminus \{1\}} x_s \right) = 0. \tag{3}$$

Proof. When $\lambda = 1$, (2) becomes

$$\prod_{s \in e(j) \setminus \{j\}} x_s = 0 \quad (j \geqslant 2). \tag{4}$$

Necessity. Suppose that $\mathbf{x}$ is an eigenvector corresponding to the eigenvalue 1. If $x_1 \neq 0$, then from (4) we see that each edge of G contains at least two pendant vertices whose x-values are zero. From this and the (1), we would have $d = 1$, a contradiction. So we have that $x_1 = 0$.

Now $x_1 = 0$ means that (1) becomes (3). This proves the necessity part.

Sufficiency. It is easy to verify that if $x_1 = 0$ and the x-values of all the pendant vertices of G satisfy the relation (3), then x satisfies (1) and (2) for $\lambda = 1$. Thus x is an eigenvector corresponding to the eigenvalue 1. □

5.2. Loose paths

In this subsection, we consider a loose path of length 3 when k is odd.

The next corollary, which is a direct consequence of Proposition 5.5, says that $\lambda = 1$ is also an H-eigenvalue of the Laplacian tensor of a loose path.

Corollary 5.1. *Let $G = (V, E)$ be a k-uniform loose path with length $s \geqslant 3$ and $\mathcal{L}$ be its Laplacian tensor. Then $\lambda = 1$ is an H-eigenvalue of $\mathcal{L}$.*

The next lemma follows from [20, Theorem 3.1(d)].

Lemma 5.2. *Let $G = (V, E)$ be a k-uniform hypergraph and $\mathcal{L}$ be its Laplacian tensor. If $\lambda \in \mathbb{R}$ is an H-eigenvalue of $\mathcal{L}$, then $\lambda \geqslant 0$.*

Proposition 5.4. *Let k be odd and $G = (V, E)$ be a k-uniform loose path with length 3. Let $\mathcal{L}$ be its Laplacian tensor. Then $\lambda \neq 1$ is an H-eigenvalue of $\mathcal{L}$ if and only if one of the following four cases happens:*

(i) $\lambda = 2$ *or* $\lambda = 0$,
(ii) λ *is the unique root of the equation* $(\lambda - 2)(1 - \lambda)^{k-1} + 1 = 0$, *which is in* $(0, 1)$,
(iii) λ *is the unique root of the equation* $(\lambda - 2)^2(1 - \lambda)^{k-2} - 1 = 0$, *which is in* $(0, 1)$, *and*
(iv) λ *is a real root of the equation* $(\lambda - 2)^2(1 - \lambda)^{k-1} + 2\lambda - 3 = 0$ *in* $(0, 2)$.

Proof. Suppose that $E = \{\{1, \ldots, k\}, \{k, \ldots, 2k-1\}, \{2k-1, \ldots, 3k-2\}\}$, and $\mathbf{x} \in \mathbb{R}^n$ be an H-eigenvector of $\mathcal{L}$ corresponding to $\lambda \neq 1$.

Let $x_k = \alpha$ and $x_{2k-1} = \beta$. By Lemmas 3.1 and 4.1, we have $x_1 = \cdots = x_{k-1} = \frac{1}{1-\lambda}\alpha$ if there are nonzero; $x_{k+1} = \cdots = x_{2k-2} = \pm\sqrt{\frac{\alpha\beta}{1-\lambda}}$ if there are nonzero; and $x_{2k} = \cdots = x_{3k-2} = \frac{1}{1-\lambda}\beta$ if there are nonzero.

The proof is divided into two cases, which contain several sub-cases respectively.

Case 1. We assume that $x_{k+1} = 0$.

(I). If $x_1 = 0$, then we must have that either $\lambda = 2$ or $\alpha = 0$, since $(\lambda - 2)\alpha^{k-1} = 0$. If $\alpha = 0$, then we can assume that $\beta = 1$. Thus, either $(\lambda - 2) = -(\frac{1}{1-\lambda})^{k-1}$ whenever $x_{2k} \neq 0$ or $\lambda = 2$. Hence, we have that either $\lambda = 2$ or it is a root of the equation $(\lambda - 2)(1 - \lambda)^{k-1} = -1$. Let $f(\lambda) = (\lambda - 2)(1 - \lambda)^{k-1} + 1$. We see that $f(0) = -1 < 0$ and $f(1) = 1 > 0$. Moreover, f is a strictly increasing function in $(-\infty, 1)$ and $(2, \infty)$. We have that $f(\lambda) > 0$ in $(2, \infty)$, since $f(2) = 1 > 0$. Obviously, $f = 0$ does not have a root in $[1, 2]$. Thus, it has a unique root, which is in the interval $(0, 1)$.

(II). If $x_1 \neq 0$, then we have

$$(\lambda - 2)\alpha^{k-1} = -\left(\frac{1}{1-\lambda}\alpha\right)^{k-1}.$$

Since $\alpha \neq 0$ in this case by Lemma 4.1, λ should be the unique root of the equation $(\lambda - 2)(1 - \lambda)^{k-1} + 1 = 0$.

The discussion for the cases (i) $x_{2k} = 0$, and (ii) $x_{2k} \neq 0$ are similar, and either $\lambda = 2$ or it is the unique root of the equation $(\lambda - 2)(1 - \lambda)^{k-1} + 1 = 0$.

Case 2. We assume that $x_{k+1} \neq 0$.

(I). If $x_1 = 0$ and $x_{2k} = 0$, then we have

$$(\lambda - 2)\alpha^{k-1} = -\left(\pm\sqrt{\frac{\alpha\beta}{1-\lambda}}\right)^{k-2}\beta \quad \text{and} \quad (\lambda - 2)\beta^{k-1} = -\left(\pm\sqrt{\frac{\alpha\beta}{1-\lambda}}\right)^{k-2}\alpha. \tag{5}$$

Multiplying the first equality by α and the second by β, we get that

$$(\lambda - 2)\left(\alpha^k - \beta^k\right) = 0. \tag{6}$$

If $\lambda > 1$, then we have that $\alpha\beta < 0$ by Corollary 4.1. Thus, the only possibility would be $\lambda = 2$ in this case. But $\lambda = 2$ contradicts (5). Hence, in this case we should have that $\lambda < 1$. Then, by (6), we must have $\alpha = \beta \neq 0$ since k is odd. By (5), we get that x_{k+1} should be $\sqrt{\frac{\alpha\beta}{1-\lambda}}$ since k is odd and $\lambda < 1$. Thus, λ should be a root of the equation $(\lambda-2)^2(1-\lambda)^{k-2} - 1 = 0$ in $(0, 1)$. With a similar discussion as that in (I) of Case 1, we have that $(\lambda-2)^2(1-\lambda)^{k-2} - 1 = 0$ has a unique root, which is in $(0, 1)$.

(II). If $x_1 = 0$ and $x_{2k} \neq 0$, then we have

$$\begin{aligned}
(\lambda - 2)\alpha^{k-1} &= -\left(\pm\sqrt{\frac{\alpha\beta}{1-\lambda}}\right)^{k-2}\beta, \\
(\lambda - 2)\beta^{k-1} &= -\left(\pm\sqrt{\frac{\alpha\beta}{1-\lambda}}\right)^{k-2}\alpha - \left(\frac{1}{1-\lambda}\beta\right)^{k-1}.
\end{aligned} \tag{7}$$

Thus, we have

$$(\lambda - 2)\alpha^k = \left[(\lambda - 2) + \left(\frac{1}{1-\lambda}\right)^{k-1}\right]\beta^k.$$

Let $t := \frac{\alpha}{\beta}$. Then

$$(\lambda - 2)(1-\lambda)^{k-1}t^k = (\lambda - 2)(1-\lambda)^{k-1} + 1. \tag{8}$$

We must have that $\lambda < 2$, since $\lambda \leqslant 2$ by Proposition 4.2 and $\lambda = 2$ cannot be a solution of (8) for any $t \in \mathbb{R}$.

By squaring the both sides of (7), we get that

$$(\lambda - 2)^2(1-\lambda)^{k-2}t^{2k-2} = t^{k-2}. \tag{9}$$

By (8) and (9), (λ, t) should be a common solution pair of the polynomial equations

$$\begin{aligned}
&(\lambda - 2)(1-\lambda)^{k-1}t^k - (\lambda - 2)(1-\lambda)^{k-1} - 1 = 0, \\
&(\lambda - 2)^2(1-\lambda)^{k-2}t^k - 1 = 0.
\end{aligned}$$

Since k is odd, solve t from the second equation, we get that $(\lambda-2)^2(1-\lambda)^{k-1} + 2\lambda - 3 = 0$. This, together with Lemma 5.2, implies the result (iv).

The discussion for the case $x_{k+1} \neq 0$, $x_1 \neq 0$ and $x_{2k} = 0$ is similar, and the result is the same as the above case.

(III). If $x_1 \neq 0$ and $x_{2k} \neq 0$, then we have

$$\begin{aligned}
(\lambda - 2)\alpha^{k-1} &= -\left(\pm\sqrt{\frac{\alpha\beta}{1-\lambda}}\right)^{k-2}\beta - \left(\frac{1}{1-\lambda}\alpha\right)^{k-1}, \\
(\lambda - 2)\beta^{k-1} &= -\left(\pm\sqrt{\frac{\alpha\beta}{1-\lambda}}\right)^{k-2}\alpha - \left(\frac{1}{1-\lambda}\beta\right)^{k-1}.
\end{aligned} \tag{10}$$

Thus, we have

$$\left[(\lambda-2)+\left(\frac{1}{1-\lambda}\right)^{k-1}\right](\alpha^k-\beta^k)=0.$$

If $\lambda>1$, then we have that $\alpha\beta<0$ by Corollary 4.1. Hence, we must have $(\lambda-2)(1-\lambda)^{k-1}+1=0$. But $(\lambda-2)(1-\lambda)^{k-1}+1=0$ has a unique solution in $(0,1)$. Consequently, we must have $\lambda<1$ in this case.

If $\alpha\neq\beta$, then $(\lambda-2)(1-\lambda)^{k-1}+1=0$. From (10), we have

$$(\lambda-2)(1-\lambda)^{k-1}\alpha^{k-1}+\alpha^{k-1}=-(1-\lambda)^{k-1}\left(\pm\sqrt{\frac{\alpha\beta}{1-\lambda}}\right)^{k-2}\beta.$$

Consequently, $(1-\lambda)^{k-1}(\pm\sqrt{\frac{\alpha\beta}{1-\lambda}})^{k-2}\beta=0$. Hence, $1-\lambda=0$ which is a contradiction to $\lambda<1$. Thus, this case does not happen.

If $\alpha=\beta$, then by (10) we have that

$$\left[(\lambda-2)(1-\lambda)^{k-1}+1\right]^2(1-\lambda)^k=1. \tag{11}$$

Note that if $\lambda<0$, then $1-\lambda>1$ and $(\lambda-2)(1-\lambda)^{k-1}+1<-1$, then it cannot be a root of the equation in (11). If $\lambda\in(0,1)$, then $1-\lambda\in(0,1)$ and $(\lambda-2)(1-\lambda)^{k-1}+1\in(-1,1)$, then the equation in (11) does not have a root in $(0,1)$. Thus, the unique solution should be $\lambda=0$. □

The next lemma says that the degree d_i of the vertex i is a Laplacian H-eigenvalue for all $i\in[n]$.

Lemma 5.3. *Let $G=(V,E)$ be a k-uniform hypergraph and $\mathcal{L}$ be its Laplacian tensor. Then $\lambda=d_i$ is an H-eigenvalue of $\mathcal{L}$ for all $i\in[n]$.*

Proof. For any $i\in[n]$, let $\mathbf{x}\in\mathbb{R}^n$ such that $x_i=1$ and $x_j=0$ for the others. By Definition 2.1, we have the result. □

The next proposition, which follows from Lemma 5.3, says that $\lambda=1$ is an H-eigenvalue of the Laplacian tensor of a cored hypergraph.

Proposition 5.5. *Let $G=(V,E)$ be a k-uniform cored hypergraph and $\mathcal{L}$ be its Laplacian tensor. Then $\lambda=1$ is an H-eigenvalue of $\mathcal{L}$.*

5.3. Loose cycles

In this subsection, we consider a loose cycle of length 3 when k is odd.

The next corollary, which is a direct consequence of Proposition 5.5, says that $\lambda=1$ is also an H-eigenvalue of the Laplacian tensor of a loose cycle.

Corollary 5.2. *Let $G=(V,E)$ be a k-uniform loose cycle with size $s\geqslant 2$ and $\mathcal{L}$ be its Laplacian tensor. Then $\lambda=1$ is an H-eigenvalue of $\mathcal{L}$.*

Proposition 5.6. *Let k be odd and $G=(V,E)$ be a k-uniform loose cycle with size 3. Let $\mathcal{L}$ be its Laplacian tensor. Then $\lambda\neq 1$ is an H-eigenvalue of $\mathcal{L}$ if and only if one of the following four cases happens*:

(i) $\lambda=2$,
(ii) *λ is the unique root of the equation $(\lambda-2)^2(1-\lambda)^{k-2}-1=0$, which is in $(0,1)$,*
(iii) *λ is the unique root of the equation $(\lambda-2)^2(1-\lambda)^{k-2}-2\sqrt[k]{4}=0$, which is in $(0,1)$, and*
(iv) *λ is a real root of the equation $[(\lambda-2)+(\pm\sqrt{1-\lambda})^{k-2}](2-\lambda)+2=0$ in $[0,1)$.*

Proof. Suppose that $E = \{\{1, \ldots, k\}, \{k, \ldots, 2k-1\}, \{2k-1, \ldots, 3k-3, 1\}\}$, and $\mathbf{x} \in \mathbb{R}^n$ be an H-eigenvector of $\mathcal{L}$ corresponding to $\lambda \neq 1$.

Let $x_1 = \alpha$, $x_k = \beta$ and $x_{2k-1} = \gamma$. By Lemmas 3.1 and 4.1, we have $x_2 = \cdots = x_{k-1} = \pm\sqrt{\frac{\alpha\beta}{1-\lambda}}$ if there are nonzero; $x_{k+1} = \cdots = x_{2k-2} = \pm\sqrt{\frac{\beta\gamma}{1-\lambda}}$ if there are nonzero; and $x_{2k} = \cdots = x_{3k-3} = \pm\sqrt{\frac{\alpha\gamma}{1-\lambda}}$ if there are nonzero.

(I). If $x_2 \neq 0$, $x_{k+1} = 0$ and $x_{2k} = 0$, then we have

$$(\lambda-2)\alpha^{k-1} = -\left(\pm\sqrt{\frac{\alpha\beta}{1-\lambda}}\right)^{k-2}\beta \quad \text{and} \quad (\lambda-2)\beta^{k-1} = -\left(\pm\sqrt{\frac{\alpha\beta}{1-\lambda}}\right)^{k-2}\alpha. \tag{12}$$

Multiplying the first equality by α and the second by β, we get that

$$(\lambda-2)\left(\alpha^k - \beta^k\right) = 0. \tag{13}$$

If $\lambda > 1$, then we have that $\alpha\beta < 0$ by Corollary 4.1. Thus, the only possibility would be $\lambda = 2$ in this case, since $\lambda \leqslant 2$. But $\lambda = 2$ contradicts (12). Hence, in this case we should have that $\lambda < 1$. Then, by (13), we must have $\alpha = \beta$ since k is odd. Without loss of generality, we assume that $\alpha = \beta > 0$. By (12), we get that x_{k+1} should be $\sqrt{\frac{\alpha\beta}{1-\lambda}}$ since k is odd and $\lambda < 1$. Thus, λ should be a root of the equation $(\lambda-2)^2(1-\lambda)^{k-2} - 1 = 0$. With a similar discussion as that in (I) of Case 1 in Proposition 5.4, we have that $(\lambda-2)^2(1-\lambda)^{k-2} - 1 = 0$ has a unique root, which is in $(0, 1)$.

(II). If $x_2 \neq 0$, $x_{k+1} \neq 0$ and $x_{2k} = 0$, then we have

$$\begin{aligned}
(\lambda-2)\alpha^{k-1} &= -\left(\pm\sqrt{\frac{\alpha\beta}{1-\lambda}}\right)^{k-2}\beta, \\
(\lambda-2)\beta^{k-1} &= -\left(\pm\sqrt{\frac{\alpha\beta}{1-\lambda}}\right)^{k-2}\alpha - \left(\pm\sqrt{\frac{\beta\gamma}{1-\lambda}}\right)^{k-2}\gamma, \\
(\lambda-2)\gamma^{k-1} &= -\left(\pm\sqrt{\frac{\beta\gamma}{1-\lambda}}\right)^{k-2}\beta.
\end{aligned} \tag{14}$$

Let $\beta = 1$, and $s := \frac{\alpha}{\beta}$ and $t := \frac{\gamma}{\beta}$. We have

$$(\lambda-2)s^{k-1} = -\left(\sqrt{\frac{s}{1-\lambda}}\right)^{k-2}, \tag{15}$$

$$(\lambda-2) = -\left(\sqrt{\frac{s}{1-\lambda}}\right)^{k-2}s - \left(\sqrt{\frac{t}{1-\lambda}}\right)^{k-2}t,$$

$$(\lambda-2)t^{k-1} = -\left(\sqrt{\frac{t}{1-\lambda}}\right)^{k-2}. \tag{16}$$

Multiplying the first by s and the last by t, we have either $\lambda = 2$ or $s^k + t^k = 1$. $\lambda = 2$ contradicts (14). If $\lambda > 1$, then $s^k + t^k = 1$ contradicts to the fact that $s < 0$ and $t < 0$ by Corollary 4.1. Thus, $\lambda < 1$, $s = t > 0$ by (15) and (16). Since $s^k + t^k = 1$, we have $s = t = \sqrt[k]{\frac{1}{2}}$. By (15), we have that λ should be a root of $(\lambda-2)^2(1-\lambda)^{k-2} - 2\sqrt[k]{4} = 0$. It can be seen that $(\lambda-2)^2(1-\lambda)^{k-2} - 2\sqrt[k]{4} = 0$ has a unique root, which is in $(0, 1)$.

(III). If $x_2 \neq 0$, $x_{k+1} \neq 0$ and $x_{2k} \neq 0$, then we have

$$\begin{aligned}
(\lambda-2)\alpha^{k-1} &= -\left(\pm\sqrt{\frac{\alpha\beta}{1-\lambda}}\right)^{k-2}\beta - \left(\pm\sqrt{\frac{\alpha\gamma}{1-\lambda}}\right)^{k-2}\gamma, \\
(\lambda-2)\beta^{k-1} &= -\left(\pm\sqrt{\frac{\alpha\beta}{1-\lambda}}\right)^{k-2}\alpha - \left(\pm\sqrt{\frac{\beta\gamma}{1-\lambda}}\right)^{k-2}\gamma,
\end{aligned}$$

$$(\lambda-2)\gamma^{k-1}=-\left(\pm\sqrt{\frac{\beta\gamma}{1-\lambda}}\right)^{k-2}\beta-\left(\pm\sqrt{\frac{\alpha\gamma}{1-\lambda}}\right)^{k-2}\alpha.$$

If $\lambda>1$, then $\alpha\beta<0$, $\beta\gamma<0$ and $\gamma\alpha<0$ by Corollary 4.1. This is a contradiction. Hence, $\lambda<1$.

Let $\beta=1$, and $s:=\frac{\alpha}{\beta}$ and $t:=\frac{\gamma}{\beta}$. By Lemma 4.1, we have $s>0$ and $t>0$. Without loss of generality, we assume that $s\leqslant 1$ and $t\leqslant 1$. We have

$$(\lambda-2)s^{k-1}=-\left(\pm\sqrt{\frac{s}{1-\lambda}}\right)^{k-2}-\left(\pm\sqrt{\frac{st}{1-\lambda}}\right)^{k-2}t, \tag{17}$$

$$(\lambda-2)=-\left(\pm\sqrt{\frac{s}{1-\lambda}}\right)^{k-2}s-\left(\pm\sqrt{\frac{t}{1-\lambda}}\right)^{k-2}t, \tag{18}$$

$$(\lambda-2)t^{k-1}=-\left(\pm\sqrt{\frac{t}{1-\lambda}}\right)^{k-2}-\left(\pm\sqrt{\frac{st}{1-\lambda}}\right)^{k-2}s. \tag{19}$$

Multiplying the first by s and the last by t, we have that

$$(\lambda-2)s^k+\left(\pm\sqrt{\frac{s}{1-\lambda}}\right)^{k-2}s=(\lambda-2)t^k+\left(\pm\sqrt{\frac{t}{1-\lambda}}\right)^{k-2}t. \tag{20}$$

This, together with (18), implies that

$$(\lambda-2)\left(s^k-1\right)=(\lambda-2)t^k+2\left(\pm\sqrt{\frac{t}{1-\lambda}}\right)^{k-2}t.$$

Since $s\leqslant 1$, $\lambda<1$ and k is odd, we have $x_{k+1}=\sqrt{\frac{t}{1-\lambda}}$. Similarly, we have $x_2=\sqrt{\frac{s}{1-\lambda}}$. Thus, (20) becomes

$$(\lambda-2)s^k+\left(\sqrt{\frac{s}{1-\lambda}}\right)^{k-2}s=(\lambda-2)t^k+\left(\sqrt{\frac{t}{1-\lambda}}\right)^{k-2}t.$$

It is equivalent to

$$\left(s^{\frac{k}{2}}-t^{\frac{k}{2}}\right)\left[(\lambda-2)\left(s^{\frac{k}{2}}+t^{\frac{k}{2}}\right)+\left(\sqrt{\frac{1}{1-\lambda}}\right)^{k-2}\right]=0.$$

On the other hand, (18) implies that

$$2-\lambda=\left(\sqrt{\frac{1}{1-\lambda}}\right)^{k-2}\left(s^{\frac{k}{2}}+t^{\frac{k}{2}}\right). \tag{21}$$

Hence,

$$\left(s^{\frac{k}{2}}-t^{\frac{k}{2}}\right)\left[1-\left(s^{\frac{k}{2}}+t^{\frac{k}{2}}\right)\right]\left(\sqrt{\frac{1}{1-\lambda}}\right)^{k-2}=0.$$

Thus, either $s=t$ or $s^{\frac{k}{2}}+t^{\frac{k}{2}}=1$.

If $s=t$, then (17) and (21) imply that λ and s should be a solution pair of

$$(2-\lambda)^2(1-\lambda)^{k-2}-4s^k=0,$$
$$\left[(\lambda-2)+\left(\pm\sqrt{\frac{1}{1-\lambda}}\right)^{k-2}\right]s^k+\left(\sqrt{\frac{1}{1-\lambda}}\right)^{k-2}s^{\frac{k}{2}}=0.$$

Since $s>0$, we can solve s from the first equation. Then λ should be a real root of the equation $[(\lambda-2)+(\pm\sqrt{1-\lambda})^{k-2}](2-\lambda)2+2=0$ in $[0,1)$, since $\lambda<1$ and $\lambda\geqslant 0$ by Lemma 5.2.

If $s^{\frac{k}{2}}+t^{\frac{k}{2}}=1$, then λ should be a root of $(2-\lambda)^2(1-\lambda)^{k-2}-1=0$, which is (ii). $\square$

6. Final remarks

In this paper, we studied Laplacian H-eigenvalues of cored hypergraphs, power hypergraphs, and some of their subclasses, such as sunflowers, loose paths, loose cycles and squids. As the kth power of a tree graph, we have a k-uniform hypertree. In 2003, Stevanović [25] presented an upper bound for the largest Laplacian eigenvalue of a tree in terms of the maximum degree. We wonder if this result can be generalized to hypertrees or not.

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