LARGE-TIME BEHAVIOR OF A PARABOLIC-PARABOLIC CHEMOTAXIS MODEL WITH LOGARITHMIC SENSITIVITY IN ONE DIMENSION

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ABSTRACT. This paper deals with the chemotaxis system

$$\begin{cases} u_t = Du_{xx} - \chi[u(\ln v)_x]_x, & x \in (0,1),\ t > 0, \\ v_t = \varepsilon v_{xx} + uv - \mu v, & x \in (0,1),\ t > 0, \end{cases}$$

under Neumann boundary condition, where $\chi < 0$, $D > 0$, $\varepsilon > 0$ and $\mu > 0$ are constants.

It is shown that for any sufficiently smooth initial data (u_0, v_0) fulfilling $u_0 \geq 0$, $u_0 \not\equiv 0$ and $v_0 > 0$, the system possesses a unique global smooth solution that enjoys exponential convergence properties in $L^\infty(\Omega)$ as time goes to infinity, which depend on the sign of $\mu - \bar{u}_0$, where $\bar{u}_0 := \int_0^1 u_0 dx$. Moreover, we prove that the constant pair $(\mu, (\frac{\mu}{\lambda})^{\frac{D}{\chi}})$ (where $\lambda > 0$ is an arbitrary constant) is the only positive stationary solution. The biological implications of our results will be given in the paper.

1. **Introduction.** Directional cell migration, namely chemotaxis, plays a central role in a wide spectrum of physiological and pathological processes, including embryo development, wounding healing, immunity, and cancer metastasis. The process of chemotaxis is characterized by the sustained migration of cells in the direction of an increasing concentration of chemoattractant or decreasing concentration of chemorepellent, where the former is referred to as attractive chemotaxis and the latter to repulsive chemotaxis. The prototype of the population-based chemotaxis model was proposed by Keller and Segel in the 1970s [15] to describe the aggregation

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of cellular slime molds *Dictyostelium discoideum* in response to the chemical cyclic adenosine monophosphate (cAMP). A general form of Keller-Segel model reads

$$
\begin{cases}
u_t = D\Delta u - \nabla \cdot (\chi u \nabla \phi(v)), \\
v_t = \varepsilon\Delta v + g(u,v),
\end{cases}
\tag{1.1}
$$

where u and v denote the cell density and chemical concentration, respectively. $D > 0$ and $\varepsilon \geq 0$ are cell and chemical diffusion coefficients, respectively. The chemotaxis is called to be attractive if $\chi > 0$ and repulsive if $\chi < 0$ with $|\chi|$ measuring the strength of the chemical signal. The potential function $\phi(v)$, also called chemotactic sensitivity function, describes the signal detection mechanism and $g(u,v)$ characterizes the chemical growth and degradation. Most of studies on chemotaxis deal with the classical attractive chemotaxis model where $\chi > 0, \phi(v) = v, g(u,v) = u - v$, see [14]; we also note that when $\varepsilon > 0, \chi > 0, \phi(v) = \ln v$ and $g(u,v) = u - v$, (1.1) with Neumann boundary conditions possesses the spike-layer steady states, see a review paper [27]. In contrast, the studies of repulsive chemotaxis were much less. A few results on repulsive chemotaxis have been developed recently, see [6, 26, 35] and references therein. In this paper, we consider a chemotaxis model with logarithmic sensitivity

$$
\begin{cases}
u_t = D\Delta u - \nabla \cdot (\chi u \nabla \ln v), \\
v_t = \varepsilon\Delta v + uv - \mu v,
\end{cases}
\tag{1.2}
$$

which was proposed in [18, 32] to model the reinforced random walk. The logarithmic sensitivity $\phi(v) = \ln v$ indicates that cell chemotactic response to the chemical signal follows the Weber-Fechner law which had prominent specific applications in biological modelings, cf. [16], [33] and [13]. Since $\nabla \ln v = \frac{\nabla v}{v}$, the logarithmic sensitivity means that cell chemotactic movement is inhibited by the high chemical concentration. The term uv entails that the chemical grows exponentially [32] where the rate depends on cell density u, which is much faster than the linear growth in the classical chemotaxis model. Here we further note that the migration of cells is a fundamental process in health and disease. Migratory cells *in vivo* adhere to surrounding extracellular matrix (ECM) molecules via specific receptors such as integrins, together with cytokine and growth-factor signals, to produce and secrete proteases ([30], [31]). The nonlinear signal production term, such as uv in (1.2), reflects the fact that protease production *in vivo* is tightly confined to the immediate pericellular environment through signals transduced by the interaction of ECM with specific cellular receptors (cf. [30], [31] and the references therein).

When $\varepsilon = 0$ and $\chi > 0$, the dynamical behaviors of model (1.2) including the aggregation, blow up and collapse was extensively discussed in [18] and the solvability was subsequently followed in [43, 44]. When $\varepsilon = 0$ and $\chi < 0$, the global existence of classical solution to (1.2) and convergence to constant states for small perturbations were established in [8]. When $\chi < 0$, the existence and nonlinear stability of traveling wave solutions with small perturbations of model (1.2) were recently studied in [37, 21, 22] for $\varepsilon = 0$ and in [23] for $\varepsilon > 0$, based on a Hopf-Cole type transformation

$$
\mathbf{w} = \frac{\nabla v}{v} = \nabla \ln v
$$

and scalings $\tilde{t} = -\chi t/D, \tilde{x} = \sqrt{-\chi}x, \tilde{\mathbf{w}} = \mathbf{w}/\sqrt{-\chi}$, which transform (1.2) into a system of conservation laws

$$\begin{cases} u_t - \nabla \cdot (u\mathbf{w}) = \Delta u, \\ \mathbf{w}_t - \nabla(\varepsilon\mathbf{w}^2 + u) = \varepsilon\Delta\mathbf{w}. \end{cases} \tag{1.3}$$

When $\varepsilon = 0$, the initial-boundary value problem and Cauchy problem of (1.3) in one dimension was studied in [45] and in [12], respectively. Furthermore the Cauchy problem of (1.3) in multi-dimensional spaces for initial data being sufficiently close to some constant ground states was investigated in [19], and the large-time behavior of classical solutions for the initial-boundary value problem of (1.3) in one space dimension with large initial data and in multi-dimensional spaces for small initial data were established in [20]. The results of model (1.3) for $\varepsilon > 0$ largely remain open. Recently the authors of [38] consider the initial-boundary value problem of one-dimensional model (1.3), as follows

$$\begin{cases} u_t - (uw)_x = u_{xx}, & x \in (0,1), \ t > 0, \\ w_t - (\varepsilon w^2 + u)_x = \varepsilon w_{xx}, & x \in (0,1), \ t > 0, \\ (u,w)(x,0) = (u_0, w_0)(x), & x \in [0,1], \\ u_x|_{x=0,1} = w|_{x=0,1} = 0, & t > 0. \end{cases} \tag{1.4}$$

The global well-posedness and large time behavior of model (1.4) were established in [38] for small $\varepsilon > 0$ based on a series of L^2-energy estimates. In the present paper, we shall: (1) by employing a Lyapunov functional approach inspired by [6], remove the smallness assumption of ε in [38] and simplify the proof of the theorem on the asymptotic behavior of solutions; (2) show that the repulsive chemotaxis model (1.2) ($\chi < 0$) has only constant positive steady states. Our first main result is the following:

Theorem 1.1. *Assume that $u_0 \geq 0, u_0 \not\equiv 0$ and w_0 are two functions in $W^{2,p_0}((0, 1))$ for some $p_0 > 3$. Then, for any $\varepsilon > 0$, there exists a unique pair (u,w) of bounded functions from $C^0([0,1] \times [0,\infty)) \cap C^{2,1}([0,1] \times (0,\infty))$ solving (1.4) classically in $(0,1) \times (0,\infty)$. Moreover, $u > 0$ in $(0,1) \times (0,\infty)$ and*

$$\lim_{t \to +\infty} (u,w)(\cdot, t) = (\bar{u}_0, 0) \quad \text{exponentially in } L^\infty((0,1)) \times L^\infty((0,1))$$

with $\bar{u}_0 = \int_0^1 u_0 dx$.

Transferring the above result back to the original chemotaxis model (1.2), we have the following result.

Theorem 1.2 (Long-time dynamics)**.** *Consider the following initial-boundary value problem for the one-dimensional chemotaxis model (1.2)*

$$\begin{cases} u_t = Du_{xx} - \chi[u(\ln v)_x]_x, & x \in (0,1), t > 0, \\ v_t = \varepsilon v_{xx} + uv - \mu v, & x \in (0,1), t > 0, \\ (u,v)(x,0) = (u_0, v_0)(x), & x \in [0,1], \\ u_x|_{x=0,x=1} = v_x|_{x=0,x=1} = 0, & t > 0, \end{cases} \tag{1.5}$$

where $\chi < 0$, $\mu > 0$. Suppose that the initial data satisfy $u_0 \geq 0, u_0 \not\equiv 0$, $v_0 > 0$. Then for any $\varepsilon > 0$, there exists a unique global-in-time classical solution (u,v) to (1.5) such that as $t \to \infty$:

$$\|u(\cdot, t) - \bar{u}_0\|_{L^\infty((0,1))} \to 0,$$

and

$$\|v(\cdot,t)\|_{L^\infty((0,1))} \to 0, \qquad\qquad \text{if } \bar{u}_0 < \mu,$$
$$\inf_{x\in(0,1)} v(x,t) \to +\infty, \qquad\quad \text{if } \bar{u}_0 > \mu,$$
$$\|v(\cdot,t) - v^*\|_{L^\infty((0,1))} \to 0, \qquad \text{if } \bar{u}_0 = \mu,$$

where the convergence rates are exponential in time and $v^* \geq \exp(\int_0^1 \ln v_0)$ is some constant.

Theorem 1.2 provides precise conclusions on the final distributions of cells and chemicals, which essentially depends on the strength of the chemical decay rate μ and the cell mass $\bar{u}_0$. The following statement addresses the biological implications of Theorem 1.2.

Corollary 1. *(i) if the chemical decay rate is the same as the cell mass (i.e. $\bar{u}_0 = \mu$), then both the chemical and the cells will eventually evolve to a constant steady state;*
(ii) if the decay rate of the chemical is strong compared with $\bar{u}_0$ (i.e. $\bar{u}_0 < \mu$), then the chemical will eventually vanish and the cells distribute uniformly in the domain;
(iii) if the chemical growth rate is stronger than the decay rate (i.e. $\bar{u}_0 > \mu$), then the chemical will be produced exponentially by cells, which eventually leads to the blow up of the chemical concentration. However the cells are still eventually uniformly distributed in the domain due to the repulsion.

Since the above theorem asserts the asymptotic behavior of solutions, it is very relevant to study the stationary solutions of (1.5) for $\chi < 0$ and explore the relationship between the large-time behavior of time-dependent solution and the stationary solutions. Observe that in (1.5), the total cell mass is conserved (cf. Lemma 2.2). To consider the steady sates of (1.5), we consider the following elliptic system:

$$\begin{cases} Du_{xx} - \chi[u(\ln v)_x]_x = 0, & 0 < x < 1, \\ \varepsilon v_{xx} + uv - \mu v = 0, & 0 < x < 1, \\ u_x = 0 = v_x, & x = 0,1 \\ \int_0^1 u(x)dx = \bar{u}_0, & 0 < x < 1 \end{cases} \qquad (1.6)$$

for which we have the following result.

Theorem 1.3. *Suppose that $\chi < 0$. If $\bar{u}_0 = \mu$, then $(\mu, (\frac{\mu}{\lambda})^{\frac{D}{\chi}})$ is the only positive solution to (1.6), where $\lambda > 0$ is an arbitrary constant. If $\bar{u}_0 \neq \mu$, then (1.6) has neither constant solution nor non-constant solution.*

Theorem 1.3 implies that if $\chi < 0$, then (1.6) does not possess any non-constant solution, and therefore Theorem 1.3 indicates that the stationary distributions of cells and the chemical are uniform within the spatial domain for the repulsion case. On the other hand, if $u = u_c = $ constant, then the condition $\int_0^1 udx = \bar{u}_0$ yields that $u_c = \bar{u}_0$. To better understand the above last case, we further underline that, as afore-mentioned, the logarithmic sensitivity means that cell movement towards higher chemical concentration is intrinsically inhibited by the high chemical concentration due to the fact that $(\ln v)_x = \frac{v_x}{v}$. Mathematically, if v_x is bounded and $v \to \infty$, then $\frac{v_x}{v} \to 0$ which indicates that cell movement is eventually governed by the diffusion process only and therefore cells tend to distribute uniformly (i.e. $u \to \bar{u}_0$). However in the case of $\bar{u}_0 \neq \mu$, the asymptotic states of both cells and the chemical are not the steady states, which implies the eventual distribution of cells and the chemical critically depends on the initial cell mass.

Before concluding this section, we should mention some works on attraction chemotaxis models with logarithmic sensitivity but with linear chemical production and degradation $g(u,v) := u - v$: all classical solutions are global in time when $n = 1$ [29], or when $n = 2$ and $\chi > 0$ is small [3], or when $n \geq 3$ and $\chi > 0$ is small [40]; moreover, when $n \geq 2$, global-in-time weak solutions were recently shown to exist regardless of the size of $\chi > 0$ in [34].

2. **Local existence, an extensibility criterion and preliminaries.** To deal with the nonlinear term $(w^2)_x$ and the term u_x in the second equation in (1.4) and prove the local well-posedness of (1.4), we shall need a regularity assumption on initial data which is stronger than that for the local solvability of classical chemotaxis models (cf. [5], [41] and [42], for instance). A proof of the local existence of (1.4) based on a straightforward fixed point argument can be found in the appendix.

Lemma 2.1. *Assume that $u_0 \geq 0, u_0 \not\equiv 0$ and w_0 are two functions in $W^{2,p_0}((0,1))$ for some $p_0 > 3$. Then there exist $T_{max} \in (0,\infty]$ and a unique pair (u,w) of bounded functions from $C^0([0,1] \times [0,T_{max})) \cap C^{2,1}([0,1] \times (0,T_{max}))$ solving (1.4) classically in $(0,1) \times (0,T_{max})$. Moreover, $u > 0$ in $(0,1) \times (0,T_{max})$ and*

$$\text{if } T_{max} < \infty, \text{ then } \|u(\cdot,t)\|_{W^{2,p_0}((0,1))} + \|w(\cdot,t)\|_{W^{2,p_0}((0,1))} \to \infty \qquad \text{as } t \nearrow T_{max}. \tag{2.1}$$

The following important property on mass can be easily derived.

Lemma 2.2. *The solution (u,w) of (1.4) satisfies the following property*

$$\|u(\cdot,t)\|_{L^1((0,1))} = \|u_0\|_{L^1((0,1))} \qquad \text{for all } t \in (0,T_{max}). \tag{2.2}$$

Proof. Integrating the first equation of (1.4) with respect to $x \in [0,1]$, we get that $\frac{d}{dt}\int_0^1 u \equiv 0$ for $t \in (0,T_{max})$, which yields (2.2). $\square$

The proof of our main result (Theorems 1.1) will be based on some *a priori* estimates. To derive these estimates, we shall need to use the following Gagliardo-Nirenberg interpolation inequality: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, let $p, q \geq 1$ satisfying $(n-q)p \leq nq$, and let $r \in (0,p)$. Then, for any $u(x) \in W^{1,q}(\Omega) \cap L^r(\Omega)$,

$$\|u\|_{L^p(\Omega)} \leq c_1 \|\nabla u\|_{L^q(\Omega)}^a \|u\|_{L^r(\Omega)}^{1-a} + c_2 \|u\|_{L^r(\Omega)} \tag{2.3}$$

with $a \in (0,1)$ satisfying

$$\frac{n}{p} = a\left(\frac{n}{q} - 1\right) + \frac{n}{r}(1-a). \tag{2.4}$$

(In fact, the classical version in Theorem I.10.1 in [9] is stated only for $r \geq 1$, but this restriction can easily be removed upon an application of Hölder's inequality; cf. [39, Lemma 3.2] or [28], for instance).

To derive our desired *a priori* estimates, we shall also need the following Gronwall's lemma (cf. [7, p. 624]).

Lemma 2.3. *Let $f(t)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies for a.e. t the differential inequality*

$$f'(t) \leq h(t)f(t) + g(t),$$

where $h(t)$ and $g(t)$ are nonnegative, summable functions on $[0,T]$. Then

$$f(t) \leq e^{\int_0^t h(s)ds}\left(f(0) + \int_0^t g(s)ds\right).$$

To derive the L^∞ estimate on u, we shall employ the following simplified version of the Moser-Alikakos iteration technique (cf. [1] or [36, Lemma 4.1]).

Lemma 2.4. *Suppose that* $T \in (0, \infty]$, *that* $b(x, t) \in L^\infty((0, T); L^q((0, 1)))$ *for some* $q > 3$, *and that* $u \in L^\infty((0, T); L^1((0, 1)))$. *Then if* $u \in C^0([0, 1] \times [0, T)) \cap C^{2,1}([0, 1] \times (0, T))$ *is a nonnegative function satisfying*

$$\begin{cases} u_t = u_{xx} + (b(x, t))_x, & x \in (0, 1), \ t > 0, \\ u_x|_{x=0,1} = 0, & t > 0, \end{cases}$$

then there exists $c > 0$, *only depending on* $\|b\|_{L^\infty((0,T);L^q((0,1)))}$, $\|u\|_{L^\infty((0,T);L^1((0,1)))}$ *and* $\|u_0\|_{L^\infty([0,1])}$, *such that*

$$\|u(t)\|_{L^\infty([0,1])} \le c \qquad \text{for all } t \in (0, T).$$

3. Global dynamics.

3.1. Boundedness. Notation.
Throughout the remainder of this paper, the norm in the space $L^p([0, 1])$, $0 \le p \le \infty$, is simply denoted by $\| \cdot \|_{L^p}$.

We observe that (1.4) possesses a Lyapunov functional

$$F(u, w) := \int_0^1 \left(u \ln u + \frac{w^2}{2} \right) dx,$$

which is the cornerstone of our analysis.

Lemma 3.1. *The classical solution* (u, w) *to* (1.4) *satisfies the equality*

$$\frac{d}{dt} F(u(t), w(t)) = -E(u(t), w(t)) \qquad \text{for all } t \in (0, T_{max}), \tag{3.1}$$

where

$$E(u, w) := \int_0^1 \left(\frac{u_x^2}{u} + \varepsilon w_x^2 \right) dx.$$

Proof. By the first two equations in (1.4), straightforward computation yields

$$\begin{aligned}
\frac{d}{dt} F(u(t), w(t)) &= \int_0^1 \left((\ln u + 1) u_t + w w_t \right) dx \\
&= -\int_0^1 \frac{u_x^2}{u} dx - \int_0^1 u_x w dx - \varepsilon \int_0^1 w_x^2 dx + \frac{2}{3} \varepsilon \int_0^1 (w^3)_x dx \\
&\quad + \int_0^1 u_x w dx \\
&= -\int_0^1 \left(\frac{u_x^2}{u} + \varepsilon w_x^2 \right) dx + \frac{2}{3} \varepsilon w^3 |_{x=0}^1 \\
&= -\int_0^1 \left(\frac{u_x^2}{u} + \varepsilon w_x^2 \right) dx,
\end{aligned}$$

where we have used the boundary conditions $u_x|_{x=0,1} = w|_{x=0,1} = 0$. This completes the proof of (3.1). $\qquad\square$

The following statement is an immediate consequence of (3.1).

Corollary 2. *The classical solution (u, w) to (1.4) has the property*

$$\int_0^t \int_0^1 \frac{u_x^2}{u} dx ds \leq F(u_0, w_0) + \frac{1}{e} \qquad \text{for all } t \in (0, T_{max}), \tag{3.2}$$

$$\int_0^t \int_0^1 w_x^2 dx ds \leq c_0(\varepsilon) := \frac{1}{\varepsilon}\left(F(u_0, w_0) + \frac{1}{e}\right) \quad \text{for all } t \in (0, T_{max}), \tag{3.3}$$

$$\int_0^1 w^2 dx \leq C_0 := 2F(u_0, w_0) + \frac{2}{e} \qquad \text{for all } t \in (0, T_{max}). \tag{3.4}$$

Proof. Integrating (3.1) over $t \in (0, T_{max})$ we obtain

$$\int_0^1 \frac{w^2}{2} dx + \int_0^t \int_0^1 \left(\frac{u_x^2}{u} + \varepsilon w_x^2\right) dx ds \leq F(u_0, w_0) - \int_0^1 u(\cdot, t) \ln u(\cdot, t) dx$$

for all $t \in (0, T_{max})$. Since $-\xi \ln \xi \leq \frac{1}{e}$ for all $\xi > 0$, this proves (3.2), (3.3) and (3.4). $\square$

With the estimate (3.3) at hand, we now turn to establish L^p-estimate on u for any given $p \geq 2$.

Lemma 3.2. *For any given $p \geq 2$, there exists some $c(\varepsilon, p) > 0$ such that the classical solution (u, w) to (1.4) satisfies*

$$\int_0^1 u^p(x, t) dx \leq c(\varepsilon, p) \qquad \text{for all } t \in (0, T_{max}). \tag{3.5}$$

Proof. Multiplying the first equation in (1.4) by pu^{p-1}, integrating over $[0, 1]$ and using the Hölder inequality, we obtain

$$\begin{aligned}
\frac{d}{dt} \int_0^1 u^p dx &= -\frac{4(p-1)}{p} \int_0^1 |(u^{\frac{p}{2}})_x|^2 dx - p(p-1) \int_0^1 u^{p-1} u_x w dx \\
&\leq -2 \int_0^1 |(u^{\frac{p}{2}})_x|^2 dx - (p-1) \int_0^1 (u^p)_x w dx \\
&= -2 \int_0^1 |(u^{\frac{p}{2}})_x|^2 dx + (p-1) \int_0^1 u^p w_x dx \\
&\leq -2 \int_0^1 |(u^{\frac{p}{2}})_x|^2 dx + (p-1)\left(\int_0^1 u^{2p} dx\right)^{\frac{1}{2}} \cdot \left(\int_0^1 w_x^2 dx\right)^{\frac{1}{2}} \tag{3.6}
\end{aligned}$$

for all $t \in (0, T_{max})$, where we have used the fact that $w|_{x=0,1} = 0$. The Gagliardo-Nirenberg inequality provides $c_1 > 0$ such that

$$\left(\int_0^1 u^{2p} dx\right)^{\frac{1}{2}} = \|u^{\frac{p}{2}}\|_{L^4}^2 \leq c_1\left(\|(u^{\frac{p}{2}})_x\|_{L^2}^{\frac{1}{2}} \cdot \|u^{\frac{p}{2}}\|_{L^2}^{\frac{3}{2}} + \|u^{\frac{p}{2}}\|_{L^2}^2\right)$$

where we have used the fact that the spatial dimension $n = 1$. Then employing the Cauchy inequality, we find some $c_2 > 0$ such that

$$
\begin{aligned}
& (p-1)\left(\int_0^1 u^{2p}dx\right)^{\frac{1}{2}} \cdot \left(\int_0^1 w_x^2 dx\right)^{\frac{1}{2}} \\
= \ & (p-1)\|u^{\frac{p}{2}}\|_{L^4}^2 \cdot \|w_x\|_{L^2} \\
\leq \ & (p-1)c_1\left(\|(u^{\frac{p}{2}})_x\|_{L^2}^{\frac{1}{2}}\|u^{\frac{p}{2}}\|_{L^2}^{\frac{1}{2}} \cdot \|u^{\frac{p}{2}}\|_{L^2}\|w_x\|_{L^2} + \|u^{\frac{p}{2}}\|_{L^2} \cdot \|u^{\frac{p}{2}}\|_{L^2}\|w_x\|_{L^2}\right) \\
\leq \ & \|(u^{\frac{p}{2}})_x\|_{L^2}^2 + \|u^{\frac{p}{2}}\|_{L^2}^2 + c_2\|u^{\frac{p}{2}}\|_{L^2}^2\|w_x\|_{L^2}^2 \qquad (3.7)
\end{aligned}
$$

for all $t \in (0, T_{max})$. Adding $\int_0^1 u^p dx$ in both sides of (3.6) and using (3.7), we obtain

$$
\frac{d}{dt}\|u^{\frac{p}{2}}\|_{L^2}^2 + \|u^{\frac{p}{2}}\|_{L^2}^2 \leq -\|(u^{\frac{p}{2}})_x\|_{L^2}^2 + 2\|u^{\frac{p}{2}}\|_{L^2}^2 + c_2\|u^{\frac{p}{2}}\|_{L^2}^2\|w_x\|_{L^2}^2 \qquad (3.8)
$$

for all $t \in (0, T_{max})$. Again, the Gagliardo-Nirenberg inequality provides $c_3 > 0$ such that

$$
2\|u^{\frac{p}{2}}\|_{L^2}^2 \leq c_3\left(\|(u^{\frac{p}{2}})_x\|_{L^2}^{\frac{2(p-1)}{p+1}} \cdot \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}}^{\frac{4}{p+1}} + \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}}^2\right).
$$

Noting $\|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}} = \left(\int_0^1 u dx\right)^{\frac{p}{2}} = \left(\int_0^1 u_0 dx\right)^{\frac{p}{2}}$ by (2.2) and using the Young inequality, we can pick some $c_4(p) > 0$ and $c_5(p) > 0$ such that

$$
\begin{aligned}
2\|u^{\frac{p}{2}}\|_{L^2}^2 &\leq c_4(p)\|(u^{\frac{p}{2}})_x\|_{L^2}^{2\cdot\frac{(p-1)}{p+1}} + c_4(p) \\
&\leq \|(u^{\frac{p}{2}})_x\|_{L^2}^2 + c_5(p).
\end{aligned}
$$

This in conjunction with (3.8) entails that

$$
\frac{d}{dt}\|u^{\frac{p}{2}}\|_{L^2}^2 + \|u^{\frac{p}{2}}\|_{L^2}^2 \leq c_2\|w_x\|_{L^2}^2 \cdot \|u^{\frac{p}{2}}\|_{L^2}^2 + c_5(p) \qquad (3.9)
$$

for all $t \in (0, T_{max})$. Thus, $y(t) := e^t\|u^{\frac{p}{2}}\|_{L^2}^2$ satisfies the differential inequality

$$
y'(t) \leq c_2\|w_x\|_{L^2}^2 \cdot y(t) + c_5 e^t \qquad \text{for all } t \in (0, T_{max}).
$$

This, along with the Gronwall Lemma 2.3 and (3.3), yields

$$
\begin{aligned}
y(t) &\leq e^{c_2 \int_0^t \|w_x(\cdot,s)\|_{L^2}^2 ds}\left(y(0) + \int_0^t c_5 e^s ds\right) \\
&\leq e^{c_2 \cdot c_0(\varepsilon)}\left(\|u_0^{\frac{p}{2}}\|_{L^2}^2 + c_5 e^t\right) \\
&:= c_6 + c_7 e^t \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}
$$

Thus,

$$
\int_0^1 u^p dx = \|u^{\frac{p}{2}}\|_{L^2}^2 = e^{-t}y(t) \leq c_6 e^{-t} + c_7 \leq c_6 + c_7 \qquad \text{for all } t \in (0, T_{max}).
$$

This proves (3.5). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

To derive the L^∞ estimate on u, we need to establish some L^q estimate on w with $q > n + 2 = 3$ because $n = 1$ for our present setting.

Lemma 3.3. *There exists some $c(\varepsilon) > 0$ such that the classical solution (u, w) to (1.4) satisfies*

$$\int_0^1 w^4(x, t) dx \leq c(\varepsilon) \qquad \text{for all } t \in (0, T_{max}). \tag{3.10}$$

Proof. Multiplying the second equation in (1.4) by w^3, integrating over $[0, 1]$ and using the Cauchy inequality, we can pick some $c_1 > 0$ and $c_2(\varepsilon) > 0$ such that

$$
\begin{aligned}
\frac{1}{4} \frac{d}{dt} \int_0^1 w^4 dx &= \varepsilon \int_0^1 w^3 w_{xx} dx + \int_0^1 w^3 (\varepsilon w^2 + u)_x dx \\
&= -3\varepsilon \int_0^1 w^2 w_x^2 dx - 3\varepsilon \int_0^1 w^4 w_x dx - 3 \int_0^1 w^2 w_x u dx \\
&= -\frac{3\varepsilon}{4} \int_0^1 |(w^2)_x|^2 dx - 3\varepsilon \int_0^1 w w_x \cdot w^3 dx - 3 \int_0^1 w w_x \cdot w u dx \\
&\leq -\frac{\varepsilon}{4} \int_0^1 |(w^2)_x|^2 dx + c_1 \int_0^1 w^6 dx + c_2 \int_0^1 w^2 u^2 dx \\
&\leq -\frac{\varepsilon}{4} \int_0^1 |(w^2)_x|^2 dx + c_1 \int_0^1 w^6 dx \\
&\quad + c_2 \int_0^1 w^4 dx + c_2 \int_0^1 u^4 dx
\end{aligned}
\tag{3.11}
$$

for all $t \in (0, T_{max})$. Adding $\frac{1}{4} \int_0^1 w^4 dx$ in both sides of (3.11) and using (3.5) and the Young inequality we can find some $c_3(\varepsilon) > 0$ such that

$$\frac{d}{dt} \int_0^1 w^4 dx + \int_0^1 w^4 dx \leq -\varepsilon \int_0^1 |(w^2)_x|^2 dx + c_3 \int_0^1 w^6 dx + c_3 \tag{3.12}$$

for all $t \in (0, T_{max})$. The Gagliardo-Nirenberg inequality, the estimate (3.4) and the Young inequality yield some $c_4 > 0, c_5 > 0$ and $c_6(\varepsilon) > 0$ such that

$$
\begin{aligned}
c_3 \int_0^1 w^6 dx = c_3 \|w^2\|_{L^3}^3 &\leq c_4 \left(\|(w^2)_x\|_{L^2}^{\frac{4}{3}} \cdot \|w^2\|_{L^1}^{\frac{5}{3}} + \|w^2\|_{L^1}^3 \right) \\
&\leq c_5 \|(w^2)_x\|_{L^2}^{\frac{4}{3}} + c_5 \\
&\leq \varepsilon \|(w^2)_x\|_{L^2}^2 + c_6(\varepsilon)
\end{aligned}
\tag{3.13}
$$

for all $t \in (0, T_{max})$. Combining (3.12) and (3.13) yields some $c_7 > 0$ such that

$$\frac{d}{dt} \int_0^1 w^4 dx + \int_0^1 w^4 dx \leq c_7 \qquad \text{for all } t \in (0, T_{max}). \tag{3.14}$$

Hence, (3.10) holds. $\qquad \square$

We are now in the position to derive the uniform-in-time boundedness of u.

Lemma 3.4. *There exists some $c(\varepsilon) > 0$ such that the classical solution (u, w) to (1.4) has the property*

$$\|u(\cdot, t)\|_{L^\infty} \leq c(\varepsilon) \qquad \text{for all } t \in (0, T_{max}). \tag{3.15}$$

Proof. For each $q \in (3, 4)$, it follows from Lemma 3.2, Lemma 3.3 and the Hölder inequality that there exists some $c_1(q, \varepsilon) > 0$ such that

$$\|uw\|_{L^q} \leq \|u\|_{L^{\frac{4q}{4-q}}} \|w\|_{L^4} \leq c_1(q, \varepsilon) \qquad \text{for all } t \in (0, T_{max}). \tag{3.16}$$

This in conjunction with Moser-Alikakos iteration technique (Lemma 2.4) proves
(3.15).   □

With (3.10) and (3.15) at hand, we now can further improve the estimate on w.

Lemma 3.5. *For any $p > 6$, there exists some $c(\varepsilon, p) > 0$ such that the classical
solution (u, w) to (1.4) satisfies*

$$\int_0^1 w^p(x, t)dx \leq c(\varepsilon, p) \qquad \text{for all } t \in (0, T_{max}). \tag{3.17}$$

Proof. Multiplying the second equation in (1.4) by pw^{p-1}, integrating over $[0, 1]$,
using the Young inequality and employing the estimate (3.15), we can pick some
$c_1(p) > 0$, $c_2(\varepsilon, p) > 0$ and $c_3(\varepsilon, p) > 0$ such that

$$
\begin{aligned}
\frac{d}{dt}\int_0^1 w^p dx &= \varepsilon p \int_0^1 w^{p-1}w_{xx}dx + p\int_0^1 w^{p-1}(\varepsilon w^2 + u)_x dx \\
&= -\varepsilon p(p-1)\int_0^1 w^{p-2}w_x^2 dx - \varepsilon p(p-1)\int_0^1 w^p w_x dx \\
&\quad -p(p-1)\int_0^1 w^{p-2}w_x u dx \\
&\leq -\frac{4\varepsilon(p-1)}{p}\int_0^1 |(w^{\frac{p}{2}})_x|^2 dx - \varepsilon p(p-1)\int_0^1 w^{\frac{p}{2}+1}\cdot w^{\frac{p}{2}-1}w_x dx \\
&\qquad\qquad\qquad\qquad\qquad +c_1\int_0^1 |w|^{\frac{p}{2}-1}\cdot|w|^{\frac{p}{2}-1}|w_x|dx \\
&\leq -\frac{2\varepsilon(p-1)}{p}\int_0^1 |(w^{\frac{p}{2}})_x|^2 dx + c_2\int_0^1 w^{p+2}dx + c_2\int_0^1 w^{p-2}dx + c_2 \\
&\leq -\varepsilon\int_0^1 |(w^{\frac{p}{2}})_x|^2 dx + c_3\int_0^1 w^{p+2}dx + c_3 \tag{3.18}
\end{aligned}
$$

for all $t \in (0, T_{max})$. Adding $\int_0^1 w^p dx$ in both sides of (3.18) and using the Young
inequality we can find some $c_4(\varepsilon, p) > 0$ such that

$$\frac{d}{dt}\int_0^1 w^p dx + \int_0^1 w^p dx \leq -\varepsilon\int_0^1 |(w^{\frac{p}{2}})_x|^2 dx + c_4\int_0^1 w^{p+2}dx + c_4 \tag{3.19}$$

for all $t \in (0, T_{max})$. The Gagliardo-Nirenberg inequality, the estimate (3.10) and
the Young inequality yield some $c_5(p) > 0, c_6(\varepsilon, p) > 0$ and $c_7(\varepsilon, p) > 0$ such that

$$
\begin{aligned}
c_4\int_0^1 w^{p+2}dx &= c_4\|w^{\frac{p}{2}}\|_{L^{\frac{2(p+2)}{p}}}^{\frac{2(p+2)}{p}} \\
&\leq c_5\left(\|(w^{\frac{p}{2}})_x\|_{L^2}^{\frac{2(p+2)}{p}\cdot\frac{p(p-2)}{(p+2)(p+4)}}\cdot\|w^{\frac{p}{2}}\|_{L^{\frac{8}{p}}}^{\frac{2(p+2)}{p}\cdot\frac{8(p+1)}{(p+2)(p+4)}} + \|w^{\frac{p}{2}}\|_{L^{\frac{8}{p}}}^{\frac{2(p+2)}{p}}\right) \\
&\leq c_6\|(w^{\frac{p}{2}})_x\|_{L^2}^{2\cdot\frac{p-2}{p+4}} + c_6 \\
&\leq \varepsilon\|(w^{\frac{p}{2}})_x\|_{L^2}^2 + c_7 \qquad \text{for all } t \in (0, T_{max}),
\end{aligned}
$$

where we have used the fact that $\|w^{\frac{p}{2}}\|_{L^{\frac{8}{p}}} = \left(\int_0^1 w^4 dx \right)^{\frac{p}{8}} \leq (c(\varepsilon))^{\frac{p}{8}}$. This in conjunction with (3.19) yields some $c_8(\varepsilon, p) > 0$ such that

$$\frac{d}{dt} \int_0^1 w^p dx + \int_0^1 w^p dx \leq c_8 \qquad \text{for all } t \in (0, T_{max}). \tag{3.20}$$

Thus, (3.17) holds. $\qquad \square$

Lemma 3.6. *There exists some $c(\varepsilon) > 0$ such that the classical solution (u, w) to (1.4) has the property*

$$\|w(\cdot, t)\|_{L^\infty} \leq c(\varepsilon) \qquad \text{for all } t \in (0, T_{max}). \tag{3.21}$$

Proof. For any $q > 3$, it follows from Lemma 3.4 and Lemma 3.5 that there exists some $c_1(q, \varepsilon) > 0$ such that

$$\|\varepsilon w^2 + u\|_{L^q} \leq c_1(q, \varepsilon) \qquad \text{for all } t \in (0, T_{max}). \tag{3.22}$$

This in conjunction with Moser-Alikakos iteration technique (Lemma 2.4) proves (3.21). $\qquad \square$

3.2. **Decay estimate.** Our proof of the convergence result is inspired by an argument developed in [6]. We modify the Lyapunov functional $F(u, w)$ a little bit and define

$$G(u, w) := \int_0^1 \left(u \ln \frac{u}{\bar{u}} + \frac{1}{2} w^2 \right) dx,$$

where, thanks to (2.2), $\bar{u} := \int_0^1 u dx = \int_0^1 u_0 dx = \bar{u}_0$.

To prove the decay estimate of u, we first establish the decay estimate of the functional G.

Lemma 3.7. *Suppose that (u, w) is the classical solution to (1.4). Then the functional G satisfies the following decay property*

$$0 \leq G(u(t), w(t)) \leq G(u_0, w_0) e^{-\alpha t} \qquad \text{for all } t \in (0, T_{max}), \tag{3.23}$$

where the positive constant α depends only on u_0 and ε.

Proof. For clarity, we divide the proof into four steps.
Step 1.   We prove the non-negativity of G.
Since the function $s \ln s$ is convex for $s > 0$ and $\bar{u} = \bar{u}_0$, it follows from Jensen's inequality ([7, p. 621]) that

$$\begin{aligned}
\int_0^1 u \ln \frac{u}{\bar{u}} dx &= \bar{u}_0 \cdot \int_0^1 \frac{u}{\bar{u}} \ln \frac{u}{\bar{u}} dx \\
&\geq \bar{u}_0 \cdot \left(\int_0^1 \frac{u}{\bar{u}} dx \right) \ln \left(\int_0^1 \frac{u}{\bar{u}} dx \right) \\
&= 0.
\end{aligned}$$

Thus, $G(u, w) \geq 0$.
Step 2.   We derive a functional identity.
Proceeding the computations as in the proof of Lemma 3.1, we obtain

$$\frac{d}{dt} G(u(t), w(t)) = -E(u(t), w(t)). \tag{3.24}$$

Step 3.   We chain $E(u, w)$ to $G(u, w)$.

Applying the preliminary inequality

$$r \ln r \leq \begin{cases} 0, & \text{for } 0 \leq r < 1 \\ r - 1 + \frac{1}{2}(r-1)^2 & \text{for } r \geq 1 \end{cases}$$

with $r = u/\bar{u}$, noting $\int_0^1 (u/\bar{u} - 1)dx \equiv 0$ and using the Poincaré inequality ([7, p. 275]) we find some $c_1 > 0$ such that

$$\begin{aligned} \int_0^1 u(t) \ln \frac{u(t)}{\bar{u}} dx = \bar{u} \int_0^1 \frac{u(t)}{\bar{u}} \ln \frac{u(t)}{\bar{u}} dx & = \bar{u} \int_0^1 \left[\frac{u(t)}{\bar{u}} \ln \frac{u(t)}{\bar{u}} - \left(\frac{u(t)}{\bar{u}} - 1 \right) \right] dx \\ & \leq \bar{u} \int_0^1 \frac{1}{2} \left(\frac{u(t)}{\bar{u}} - 1 \right)^2 dx \\ & = \frac{1}{2\bar{u}} \int_0^1 (u - \bar{u})^2 dx \\ & \leq c_1 \int_0^1 [(u - \bar{u})_x]^2 dx \\ & = c_1 \int_0^1 u_x^2 dx \end{aligned}$$

for all $t \in (0, T_{max})$. This in conjunction with (3.15) yields some $c_2(\varepsilon) > 0$ such that

$$\begin{aligned} \int_0^1 u(t) \ln \frac{u(t)}{\bar{u}} dx & \leq c_1 \int_0^1 u_x^2 dx \leq c_1 \|u\|_{L^\infty} \int_0^1 \frac{u_x^2}{u} dx \\ & \leq c_2(\varepsilon) \int_0^1 \frac{u_x^2}{u} dx \qquad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{3.25}$$

On the other hand, since $w|_{x=0,1} = 0$, Poincaré's inequality provides some $c_3 > 0$ such that

$$\int_0^1 w^2 dx \leq c_3 \int_0^1 w_x^2 dx \qquad \text{for all } t \in (0, T_{max}). \tag{3.26}$$

Collecting (3.25)-(3.26) and noting the definitions of G and E, we find

$$G(u(t), w(t)) \leq \max \left(c_2(\varepsilon), \frac{c_3}{2\varepsilon} \right) \cdot E(u(t), w(t)). \tag{3.27}$$

Step 4.   We prove the decay estimate (3.23).

Denote

$$\alpha := \frac{1}{\max \left(c_2(\varepsilon), \frac{c_3}{2\varepsilon} \right)} > 0.$$

Then, combining (3.24) and (3.27) entails that

$$\frac{d}{dt} G(u(t), w(t)) \leq -\alpha G(u(t), w(t)) \qquad \text{for all } t \in (0, T_{max}).$$

This yields (3.23).   □

Lemma 3.8. *The only stationary solution (u_s, w_s) with $u_s > 0$ to (1.4) in $W^{2,p_0}((0, 1))$ for $p_0 > 3$ are the constant pairs $(\bar{u}, 0)$ for $\bar{u} \in (0, \infty)$, where $\bar{u}$ denotes the cell mass.*

Proof. Assume that $(u_s, w_s) \in \left(W^{2,p_0}((0,1))\right)^2$ for $p_0 > 3$ with $u_s > 0$ is a stationary solution to (1.4). Noting that (u_s, w_s) is also a solution to the time-dependent problem (1.4), we have

$$0 = \frac{d}{dt} G(u_s, w_s) = -\int_0^1 \left(\frac{[(u_s)_x]^2}{u_s} + \varepsilon[(w_s)_x]^2 \right) dx$$

which indicates that

$$u_s = C_1, \quad \text{and} \quad w_s = C_2$$

since $u_s > 0$, where C_1 and C_2 are both constant. The boundary condition of w immediately implies that $C_2 = 0$ and the cell mass $\bar{u} = \int_0^1 u_s dx$ entails that $C_1 = \bar{u}$. This completes the proof. □

3.3. **Proof of Theorem 1.1.** We are now in the position to prove Theorem 1.1.

Proof (of Theorem 1.1). <u>Global existence</u>. For any given $T_1 > 0$, let $T := \min\{T_1, T_{max}\}$. From (1.4) we find that u solves

$$\begin{cases} u_t - u_{xx} - wu_x = uw_x, & x \in (0,1), \ t \in (0,T), \\ u_x|_{x=0,1} = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,1]. \end{cases} \tag{3.28}$$

From (3.10), (3.3) and (3.15) we readily find some $c_1 > 0$ such that

$$\|w(\cdot,t)\|_{L^4((0,1))} \leq c_1 \quad \text{for all } t \in (0,T), \tag{3.29}$$

$$\|uw_x\|_{L^2((0,1)\times(0,T))} \leq c_1, \tag{3.30}$$

where $4 > n + 2 = 3$ due to the fact that $n = 1$. Then we can apply parabolic L^p-theory (cf. [17, Theorem IV.9.1 and Section V.7]) to obtain some $c_2(T) > 0$ such that

$$\|u\|_{W^{2,1,2}(Q_T)} \leq c_2(T). \tag{3.31}$$

This in conjunction with the fact that $n = 1$ and the Sobolev imbedding Theorem ([17, Lemma II.3.3] yields some $c_3 > 0$ such that

$$\|u_x\|_{L^6(Q_T)} \leq c_3\|u\|_{W^{2,1,2}(Q_T)} \leq c_3 \cdot c_2(T). \tag{3.32}$$

From (1.4) we find that w solves

$$\begin{cases} w_t - \varepsilon w_{xx} - 2\varepsilon w \cdot w_x = u_x, & x \in (0,1), \ t \in (0,T), \\ w|_{x=0,1} = 0, & t \in (0,T), \\ w(x,0) = w_0(x), & x \in [0,1]. \end{cases} \tag{3.33}$$

Noting (3.29), (3.32) and $w_0 \in W^{2,p_0}$ for some $p_0 > 3$ and applying parabolic L^p-theory as above, we obtain some $c_4(T) > 0$ such that

$$\|w\|_{W^{2,1,p_0}(Q_T)} \leq c_4(T). \tag{3.34}$$

This, together with the Sobolev imbedding Theorem ([17, Lemma II.3.3]), yields some $c_5(T) > 0$ such that

$$\|w\|_{C^{1+\theta,(1+\theta)/2}(\bar{Q}_T)} \leq c_5(T) \qquad \text{where } \theta := 1 - \frac{3}{p_0}. \tag{3.35}$$

Using this and the classical regularity of parabolic equations ([17, Theorem V.6.1]) we can obtain some $c_6(T) > 0$ such that

$$\|u(x,t)\|_{C^{2+\theta,(1+\theta)/2}([0,1]\times[\eta,T])} + \|w(x,t)\|_{C^{2+\theta,(1+\theta)/2}([0,1]\times[\eta,T])} \leq c_6(T)$$

for all $\eta \in (0, T)$. Particularly,

$$\|u(\cdot, t)\|_{W^{2,p_0}((0,1))} + \|w(\cdot, t)\|_{W^{2,p_0}((0,1))} \le c_7(T) \qquad \text{for all } t \in [\eta, T] \qquad (3.36)$$

for some $c_7(T) > 0$. This in conjunction with the extensibility criterion (2.1) yields $T_{max} = +\infty$.

Convergence.   For clarity, we divide the proof into three steps.

Step 1.   We prove the exponential convergence of u in $L^1((0,1))$. From (3.23) and the definition of G we infer that

$$\int_0^1 u \ln \frac{u}{\bar{u}} dx \quad \le \quad G(u_0, w_0) e^{-\alpha t}, \qquad (3.37)$$

$$\int_0^1 w^2 dx \quad \le \quad G(u_0, w_0) e^{-\alpha t}. \qquad (3.38)$$

Then inequality (3.37), along with the Csiszár-Kullback-Pinsker inequality (cf. [4])

$$\frac{1}{2\bar{u}} \|u - \bar{u}\|_{L^1((0,1))}^2 \le \int_0^1 u \ln \frac{u}{\bar{u}} dx,$$

yields

$$\|u - \bar{u}\|_{L^1((0,1))}^2 \le 2\bar{u}_0 G(u_0, w_0) e^{-\alpha t}. \qquad (3.39)$$

Step 2.   We prove the boundedness of $\|(u - \bar{u})_x\|_{L^2((0,1))}$. First, (3.2) in conjunction with (3.15) yields $c_8 > 0$ such that

$$\int_0^t \int_0^1 u_x^2 dx ds \le \|u\|_{L^\infty} \int_0^t \int_0^1 \frac{u_x^2}{u} dx ds \le c_8 \qquad \text{for all } t > 0. \qquad (3.40)$$

Then, since $\bar{u}$ is a constant, $(u - \bar{u})$ satisfies

$$\begin{cases} (u - \bar{u})_t = (u - \bar{u})_{xx} + (uw)_x, & x \in (0, 1), \ t > 0, \\ (u - \bar{u})(x, 0) = u_0(x) - \bar{u}, & x \in [0, 1], \\ (u - \bar{u})_x|_{x=0,1} = 0, & t > 0. \end{cases} \qquad (3.41)$$

Testing the first equation in (3.41) against $-(u-\bar{u})_{xx}$ and using Cauchy's inequality, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^1 (u - \bar{u})_x^2 dx + \int_0^1 (u - \bar{u})_{xx}^2 dx \quad = \quad -\int_0^1 (u - \bar{u})_{xx}(uw)_x dx$$

$$\le \quad \frac{1}{2} \int_0^1 (u - \bar{u})_{xx}^2 dx + \frac{1}{2} \int_0^1 (uw)_x^2 dx$$

for all $t > 0$. That is

$$\frac{d}{dt} \int_0^1 (u - \bar{u})_x^2 dx + \int_0^1 (u - \bar{u})_{xx}^2 dx \le \int_0^1 (uw)_x^2 dx \qquad \text{for all } t > 0.$$

This, along with the basic fact that $(uw)_x^2 \le 2(w^2 u_x^2 + u^2 w_x^2)$ and the boundedness of u and w in (3.15) and (3.21), yields $c_9 > 0$ such that

$$\frac{d}{dt} \int_0^1 (u - \bar{u})_x^2 dx + \int_0^1 (u - \bar{u})_{xx}^2 dx \le c_9 \int_0^1 (u_x^2 + w_x^2) dx \qquad \text{for all } t > 0.$$

Upon integration over the time t, we find

$$\int_0^1 (u - \bar{u})_x^2 dx \le \int_0^1 (u_0 - \bar{u})_x^2 dx + c_9 \int_0^t \int_0^1 (u_x^2 + w_x^2) dx ds \qquad \text{for all } t > 0. \quad (3.42)$$

From this, (3.40) and (3.3), we obtain $c_{10} > 0$ such that

$$\int_0^1 (u - \bar{u})_x^2 dx \leq c_{10} \qquad \text{for all } t > 0. \tag{3.43}$$

Step 3. We prove the exponential convergence of u in $L^\infty((0,1))$. The Gagliardo-Nirenberg inequality yields $c_{11} > 0$ such that

$$\|u - \bar{u}\|_{L^\infty((0,1))} \leq c_{11} \|(u - \bar{u})_x\|_{L^2((0,1))}^{\frac{2}{3}} \cdot \|u - \bar{u}\|_{L^1((0,1))}^{\frac{1}{3}} + c_{11} \|u - \bar{u}\|_{L^1((0,1))}.$$

This, together with (3.39) and (3.43), yields some $c_{12} > 0$ such that

$$\|u - \bar{u}\|_{L^\infty((0,1))} \leq c_{12} e^{-\frac{\alpha}{3} t} \qquad \text{for all } t > 0. \tag{3.44}$$

This completes the proof of u in $L^\infty((0,1))$.

Finally, the exponential convergence of w in $L^\infty((0,1))$ can be similarly proved as above. $\qquad \square$

3.4. Results for original model. Proof of Theorem 1.2. From Theorem 1.1 we obtain some constants α_1 and $c > 0$ such that

$$\|u - \bar{u}\|_{L^\infty((0,1))} \leq c e^{-\alpha_1 t} \qquad \text{for all } t > 0. \tag{3.45}$$

This will be the starting point towards the proof of the convergence for v.

We begin with proving the convergence of v when $\mu \neq \bar{u}_0$.

Lemma 3.9. *The solution component v of (1.5) has the property*

$$\begin{aligned} \|v(\cdot, t)\|_{L^\infty((0,1))} \to \quad & 0, \qquad \text{if } \bar{u}_0 < \mu, \\ \inf_{x \in (0,1)} v(x, t) \to \quad & +\infty, \qquad \text{if } \bar{u}_0 > \mu. \end{aligned}$$

Proof. Noticing that $\frac{v_{xx}}{v} = w_x + w^2$, we can rewrite the second equation of (1.5) as

$$(\ln v)_t = u - \mu + \varepsilon w_x + \varepsilon w^2$$

where $w = (\ln v)_x$. Integrating the above equation over $[0,1] \times [0, t]$ we get

$$\int_0^1 \ln v \, dx = \int_0^1 \ln v_0 \, dx + (\bar{u}_0 - \mu)t + \varepsilon \int_0^t \|w(\cdot, \tau)\|_{L^2}^2 d\tau,$$

where we have used (2.2) and the boundary condition $w|_{x=0,1} = 0$. Define

$$\xi(x, t) = \ln v - \int_0^1 \ln v_0 \, dx - (\bar{u}_0 - \mu)t - \varepsilon \int_0^t \|w(\cdot, \tau)\|_{L^2}^2 d\tau. \tag{3.46}$$

It is straightforward to check that

$$\xi_x = w, \quad \text{and} \quad \int_0^1 \xi(x, t) dx = 0.$$

Thus, by the Poincaré inequality we have $\|\xi\|_{L^2}^2 \leq \|w\|_{L^2}^2$. This, along with $\|\xi_x\|_{L^2}^2 = \|w\|_{L^2}^2$, $n = 1$, Theorem 1.1 and the Sobolev inequality, yields

$$\|\xi(t)\|_{L^\infty} \leq c_1 e^{-\beta t} \tag{3.47}$$

for some positive constants c_1 and β which are independent of t.

Now from (3.46) we see that

$$v(x, t) = \exp\left\{\xi(x, t) + \int_0^1 \ln v_0 \, dx + \varepsilon \int_0^t \|w(\cdot, \tau)\|_{L^2}^2 d\tau\right\} \cdot \exp\left\{(\bar{u}_0 - \mu)t\right\}. \tag{3.48}$$

If $\bar{u}_0 \neq \mu$, it then follow from Lemma 3.7, the non-negativity of $\int_0^1 u \ln \frac{u}{\bar{u}} dx$ and (3.47) that

$$c_2 \exp\{(\bar{u}_0 - \mu)t\} \leq v(x,t) \leq c_3 \exp\{(\bar{u}_0 - \mu)t\}$$

for some positive constants c_2 and c_3 which are independent of t. Thus

$$\|v(\cdot,t)\|_{L^\infty((0,1))} \to 0 \qquad \text{as} \qquad t \to \infty, \quad \text{when} \quad \bar{u}_0 < \mu,$$
$$\inf_{x \in (0,1)} v(x,t) \to +\infty \qquad \text{as} \qquad t \to \infty, \quad \text{when} \quad \bar{u}_0 > \mu.$$

This completes the proof of Lemma 3.9. □

We next turn to prove the convergence of v when $\mu = \bar{u}_0$. The proof in this case appears to be very interesting and involves serval technical steps. We first establish positive lower and upper bounds of v for sufficiently large t, which will be used twice in Lemmas 3.12 and 3.13 below.

Lemma 3.10. *If $\bar{u}_0 = \mu$, then there exists a sufficiently large $T > 0$ such that the solution component v of (1.5) has positive lower and supper bounds*

$$\frac{1}{2} \exp\left\{\int_0^1 \ln v_0 dx\right\} \leq v(x,t) \leq \exp\left\{c + \int_0^1 \ln v_0 dx + \frac{\varepsilon}{\alpha} G(u_0, v_0)\right\}, \qquad (3.49)$$

for all $x \in (0,1)$ and $t > T$, with some constant $c > 0$.

Proof. If $\bar{u}_0 = \mu$, then again from Lemma 3.7, the non-negativity of $\int_0^1 u \ln \frac{u}{\bar{u}} dx$, (3.46) and (3.47) we have

$$\exp\left\{-c_1 e^{-\beta t} + \int_0^1 \ln v_0 dx\right\} \leq v(x,t) \leq$$

$$\exp\left\{c_1 e^{-\beta t} + \int_0^1 \ln v_0 dx + \varepsilon G(u_0, v_0) \int_0^t e^{-\alpha s} ds\right\}. \quad (3.50)$$

This, along with the fact that $\exp\{-c_1 e^{-\beta t}\} \to 1$ as $t \to \infty$, yields (3.49). □

For $t > T + 2$, we set $Q_1 := [0,1] \times [t-1, t+1]$ and $Q_2 := [0,1] \times [t-2, t+2]$. Suppose $(x,s), (y,s)$ and $(y,\tau) \in Q_1$, we denote

$$C^{\gamma, \gamma/2}(Q_1) := \left\{h(x,t) \in C(Q_1) \quad \middle| \quad |h_x(x,s) - h_x(y,s)| \leq c|x-y|^\gamma\right.$$

$$\left. \text{and } |h(y,s) - h(y,\tau)| \leq c|s-\tau|^{\gamma/2}\right\}$$

where $\gamma > 0, c > 0$ are constants. To proceed our proof, we need the following technical lemma concerning local Hölder estimates for linear parabolic equations.

Lemma 3.11. *Suppose that $h(x,t)$ solves the equation*

$$h_t = \varepsilon h_{xx} + f_x, \qquad x \in (0,1), \ t > 0, \qquad (3.51)$$

with zero Dirichlet boundary condition or homogeneous Neumann boundary condition, where $f(x,t)$ is a given function.
(i) If $f \in L^\infty(Q_2)$, then there exist some $0 < \gamma < 1$ and $c_1 > 0$ depending only on ε such that

$$[h]_{C^{\gamma, \gamma/2}(Q_1)} \leq c_1 \left(\|h\|_{L^2(Q_2)} + \|f\|_{L^\infty(Q_2)}\right); \qquad (3.52)$$

(ii) If $f \in C^{\gamma, \gamma/2}(Q_2)$, then there exists some $c_2 > 0$ depending only on ε such that

$$[h_x]_{C^{\gamma, \gamma/2}(Q_1)} \leq c_2 \left(\|h\|_{L^\infty(Q_2)} + \|f\|_{C^{\gamma, \gamma/2}(Q_2)}\right). \qquad (3.53)$$

Proof. (3.52) follows from [24, Theorem 6.44] and it is also implicitly proved in [17, Theorem V.1.1], whereas (3.53) is implicitly proved in [2, Theorem 5.4].   □

With the estimate (3.45) at hand, we now can establish a decay estimate on v_x in $C^{\gamma,\gamma/2}(Q_1)$.

Lemma 3.12. *If $\bar{u}_0 = \mu$, then the solution component v of (1.5) has the property*

$$[v_x]_{C^{\gamma,\gamma/2}(Q_1)} \leq ce^{-\alpha_2 t} \qquad \text{for all } t > T, \tag{3.54}$$

with some constants $0 < \gamma < 1$, $c > 0$ and $\alpha_2 > 0$. In particular, we have

$$\|v_x\|_{L^\infty((0,1))} \leq ce^{-\alpha_2 t} \qquad \text{for all } t > T. \tag{3.55}$$

Proof. Taking derivative with respect to x in both sides of the second equation in (1.5) and using $\mu = \bar{u}_0 = \bar{u}$ we obtain

$$\begin{cases} (v_x)_t = \varepsilon(v_x)_{xx} + \big((u - \bar{u})v\big)_x, & x \in (0,1), \ t > 0, \\ v_x = 0, & x = 0, 1. \end{cases} \tag{3.56}$$

Testing the first equation in (3.56) by v_x, using Cauchy's inequality and Poincaré inequality: $\int_0^1 v_x^2 \leq c_1 \int_0^1 v_{xx}^2$ for some $c_1 > 0$ thanks to $v_x|_{x=0,1} = 0$, and employing (3.45) and (3.49), we obtain $c_2 > 0$ and $c_3 > 0$ such that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 v_x^2 dx &= -\varepsilon\int_0^1 v_{xx}^2 dx - \int_0^1 (u-\bar{u})v v_{xx} dx \\ &\leq -\frac{\varepsilon}{2}\int_0^1 v_{xx}^2 dx + c_2\int_0^1 (u-\bar{u})^2 v^2 dx \\ &\leq -\frac{\varepsilon}{2c_1}\int_0^1 v_x^2 dx + c_3 e^{-2\alpha_1 t} \qquad \text{for all } t > 0. \end{aligned}$$

From this we can obtain some $c_4 > 0$ such that

$$\int_0^1 v_x^2 dx \leq c_4 e^{-c_5 t} \qquad \text{for all } t > 0, \tag{3.57}$$

where $c_5 := \min\{\frac{\varepsilon}{c_1}, 2\alpha_1\}$. We now apply Lemma 3.11 (see (3.52)) to (3.56) to obtain $c_6 > 0$ such that

$$[v_x]_{C^{\gamma,\gamma/2}(Q_1)} \leq c_6\Big(\|v_x\|_{L^2(Q_2)} + \|(u - \bar{u})v\|_{L^\infty(Q_2)}\Big) \qquad \text{for all } t > T.$$

This, along with (3.45), (3.49) and (3.57), yields $c_7 > 0$ such that

$$\begin{aligned} [v_x]_{C^{\gamma,\gamma/2}(Q_1)} &\leq c_6\left\{\left(\int_{t-2}^{t+2}\int_0^1 v_x^2 dx ds\right)^{\frac{1}{2}} + c_7 e^{-\alpha_1 t}\right\} \\ &\leq c_6\left(2c_4^{\frac{1}{2}} e^{-\frac{c_5}{2}t} + c_7 e^{-\alpha_1 t}\right) \\ &\leq c_6\big(2c_4^{\frac{1}{2}} + c_7\big)e^{-\min(\frac{c_5}{2},\alpha_1)t} \qquad \text{for all } t > T. \end{aligned}$$

Hence, (3.54) holds with $\alpha_2 := \min(\frac{c_5}{2}, \alpha_1)$.   □

With the estimate (3.54) at hand, we can improve the estimate (3.45).

Lemma 3.13. *If $\bar{u}_0 = \mu$, then the solution component u of (1.5) has the property*

$$[u - \bar{u}]_{C^{\gamma,\gamma/2}(Q_1)} \leq ce^{-\alpha_3 t} \qquad \text{for all } t > T, \tag{3.58}$$

with some constants $c > 0$ and $\alpha_3 > 0$.

Proof. Since $\bar{u}_t = \bar{u}_{xx} = 0$, we can rewrite the first equation in (1.5) as

$$(u - \bar{u})_t = D(u - \bar{u})_{xx} - \chi\left(\frac{u}{v}v_x\right)_x, \qquad x \in (0,1),\ t > 0. \qquad (3.59)$$

We apply Lemma 3.11 (see (3.52)) to (3.59) to find some $c_1 > 0$ such that

$$[u - \bar{u}]_{C^{\gamma,\gamma/2}(Q_1)} \le c_1\left(\|u - \bar{u}\|_{L^2(Q_2)} + \|\frac{u}{v}v_x\|_{L^\infty(Q_2)}\right).$$

Using this in conjunction with (3.45) and (3.55) and noting v has a positive lower bound for all $t > T$ (see (3.49)), we obtain $c_2 > 0$ and $c_3 > 0$ such that

$$
\begin{aligned}
[u - \bar{u}]_{C^{\gamma,\gamma/2}(Q_1)} &\le c_2 e^{-\alpha_1 t} + c_2 e^{-\int_0^1 \ln v_0 dx} \cdot \|u\|_{L^\infty((0,1))} \cdot \|v_x\|_{L^\infty(Q_2)} \\
&\le c_2 e^{-\alpha_1 t} + c_3 e^{-\alpha_2 t} \\
&\le (c_2 + c_3)e^{-\min(\alpha_1,\alpha_2)t} \qquad \text{for all } t > T.
\end{aligned}
$$

Thus, (3.58) holds with $\alpha_3 := \min(\alpha_1, \alpha_2)$. $\qquad\square$

With the estimates (3.54) and (3.58), we can further improve the estimate on v_x.

Lemma 3.14. *If $\bar{u}_0 = \mu$, then the solution component v of (1.5) has the property*

$$[v_{xx}]_{C^{\gamma,\gamma/2}(Q_1)} \le ce^{-\alpha_4 t} \qquad \text{for all } t > T, \qquad (3.60)$$

with some constants $c > 0$ and $\alpha_4 > 0$.

Proof. We go back to the problem (3.56) and apply Lemma 3.11 (see (3.53)) to obtain $c_1 > 0$ such that

$$[(v_x)_x]_{C^{\gamma,\gamma/2}(Q_1)} \le c_1\left(\|v_x\|_{L^\infty(Q_2)} + [(u - \bar{u})v_x]_{C^{\gamma,\gamma/2}(Q_2)}\right).$$

This in conjunction with (3.55), (3.54) and (3.58) yields some $c_2 > 0$ such that

$$
\begin{aligned}
[(v_x)_x]_{C^{\gamma,\gamma/2}(Q_1)} &\le c_2 e^{-\alpha_2 t} + c_2 e^{-\min(\alpha_2,\alpha_3)t} \\
&\le 2c_2 e^{-\min(\alpha_2,\alpha_3)t} \qquad \text{for all } t > T.
\end{aligned}
$$

This proves (3.60) with $\alpha_4 := \min(\alpha_2, \alpha_3)$. $\qquad\square$

Next, we can establish the exponential convergence of v when $\bar{u}_0 = \mu$.

Lemma 3.15. *If $\bar{u}_0 = \mu$, then there exists constants $c > 0$ and $v^* \ge \exp\{\int_0^1 \ln v_0\} > 0$ such that the solution component v of (1.5) has the property*

$$\|v - v^*\|_{L^\infty((0,1))} \le ce^{-\alpha_4 t} \qquad \text{for all } t > T. \qquad (3.61)$$

Proof. From the second equation in (1.5) and $\mu = \bar{u}_0 = \bar{u}$ we infer that

$$v_t - (u - \bar{u})v = \varepsilon v_{xx}, \qquad x \in (0,1),\ t > 0. \qquad (3.62)$$

By this and the fact that $u(x,t) - \bar{u} \to 0$ uniformly for $x \in (0,1)$ as $t \to \infty$ thanks to (3.45), we find that

$$\left(e^{\int_t^\infty (u(\cdot,s)-\bar{u})ds}v\right)_t = \varepsilon e^{\int_t^\infty (u(\cdot,s)-\bar{u})ds}v_{xx}. \qquad (3.63)$$

This in conjunction with (3.45) and (3.60) yields $c_1 > 0$ and $c_2 > 0$ such that

$$
\begin{aligned}
\left\|\left(e^{\int_t^\infty (u(\cdot,s)-\bar{u})ds}v\right)_t\right\|_{L^\infty((0,1))} &\le \varepsilon e^{\int_t^\infty c_1 e^{-\alpha_1 s}ds} \cdot c_2 e^{-\alpha_4 t} \\
&\le c_2 \varepsilon e^{\frac{c_1}{\alpha_1}} \cdot e^{-\alpha_4 t} \\
&:= c_4 e^{-\alpha_4 t} \qquad \text{for all } t > T. \qquad (3.64)
\end{aligned}
$$

For notational simplicity, we set $g(x,t) := e^{\int_t^\infty (u(\cdot,s)-\bar{u})ds} v$ for each fixed $x \in (0,1)$ and all $t > T$. By (3.64) we find that for any $t+1 > s > t > T$ there holds

$$
\begin{aligned}
\|g(\cdot,s) - g(\cdot,t)\|_{L^\infty((0,1))} &= \|g_t(\cdot,\tilde{t}) \cdot (s-t)\|_{L^\infty((0,1))} \\
&\leq \|g_t(\cdot,\tilde{t})\|_{L^\infty((0,1))} \\
&\leq c_4 e^{-\alpha_4 \tilde{t}} \\
&\leq c_4 e^{-\alpha_4 t} \to 0 \qquad \text{as } t \to \infty, \qquad (3.65)
\end{aligned}
$$

thanks to $0 < s - t < 1$, where $\tilde{t} \in (t,s) \subset (t,t+1)$. From (3.65) we assert that for any time sequence $\{t_n\}$: $T < t_1 < t_2 < \cdots < t_n \to +\infty$ satisfying $t_n - t_{n-1} < 1$, $g(x,t_n)$ is a Cauchy's sequence for each fixed $x \in (0,1)$. Therefore, there exists a function $\bar{v}(x)$ such that

$$
\|g(x,t) - \bar{v}(x)\|_{L^\infty((0,1))} \to 0 \qquad \text{as } t \to +\infty,
$$

that is,

$$
\|e^{\int_t^\infty (u(x,s)-\bar{u})ds} v(x,t) - \bar{v}(x)\|_{L^\infty((0,1))} \to 0 \qquad \text{as } t \to +\infty. \qquad (3.66)
$$

On the other hand, from (3.45) we obtain $c_5 > 0$ such that

$$
1 \leftarrow e^{-\frac{c_5}{\alpha_1} e^{-\alpha_1 t}} \leq \|e^{\int_t^\infty (u(x,s)-\bar{u})ds}\|_{L^\infty((0,1))} \leq e^{\frac{c_5}{\alpha_1} e^{-\alpha_1 t}} \to 1 \qquad \text{as } t \to \infty,
$$

which implies

$$
e^{\int_t^\infty (u(x,s)-\bar{u})ds} \to 1 \quad \text{uniformly for } x \in (0,1), \qquad \text{as } t \to \infty. \qquad (3.67)
$$

Combining (3.66) and (3.67) entails that

$$
\|v(x,t) - \bar{v}(x)\|_{L^\infty((0,1))} \to 0 \qquad \text{as } t \to +\infty. \qquad (3.68)
$$

Now it remains to prove that $\bar{v}(x) = const.$. To this end, we infer from (3.55) that for any $x_1, x_2 \in (0,1)$, there holds

$$
\begin{aligned}
|v(x_1,t) - v(x_2,t)| &= |v_x(\tilde{x},t)(x_1 - x_2)| \\
&\leq \|v_x(\cdot,t)\|_{L^\infty((0,1))} \\
&\leq c_5 e^{-\alpha_2 t} \to 0 \qquad \text{as } t \to \infty, \qquad (3.69)
\end{aligned}
$$

where $c_5 > 0$ is a constant and $\tilde{x}$ is between x_1 and x_2. This in conjunction with (3.68) yields

$$
\begin{aligned}
|\bar{v}(x_1) - \bar{v}(x_2)| &\leq |v(x_1,t) - \bar{v}(x_1)| + |v(x_2,t) - v(x_1,t)| + |\bar{v}(x_2) - v(x_2,t)| \\
&\to 0 \qquad \text{as } t \to +\infty.
\end{aligned}
$$

Hence,

$$
\bar{v}(x_1) = \bar{v}(x_2) \qquad \text{for any } x_1, x_2 \in (0,1)
$$

and thus

$$
\bar{v}(x) = \text{const.} := v^*. \qquad (3.70)
$$

Finally, (3.61) is an immediate consequence of (3.68), (3.70), (3.65) and (3.50). Moreover, from (3.46), (3.48), (3.68) and (3.70) we infer that

$$
v^* = e^{\int_0^1 \ln v_0 dx + \varepsilon \int_0^\infty \|w(\cdot,\tau)\|_{L^2((0,1))}^2 d\tau}.
$$

$\square$

We are now in the position to prove Theorem 1.2.

Proof (of Theorem 1.2). Theorem 1.2 is an immediate consequence of (3.45), Lemmas 3.9 and 3.15. □

3.5. **Stationary solutions.** *Proof (of Theorem 1.3).* Using the boundary conditions in (1.6), we can solve the first equation in (1.6) to obtain

$$u(x) = \lambda v^{\frac{\chi}{D}}(x) \tag{3.71}$$

where $\lambda > 0$ is arbitrary.

Inserting (3.71) into the second equation in (1.6) gives

$$\begin{cases} \varepsilon v_{xx} - \mu v + \lambda v^{1+\frac{\chi}{D}} = 0, & 0 < x < 1, \\ v_x = 0, & x = 0, 1, \end{cases} \tag{3.72}$$

where $\chi < 0$. Setting $\tilde{\varepsilon} = \frac{\varepsilon}{\mu}, \eta = (\frac{\lambda}{\mu})^{\frac{D}{\chi}}$ and $\tilde{v} = \eta v$, we see that

$$\begin{cases} \tilde{v}_{xx} - \tilde{v} + \tilde{v}^p = 0, & 0 < x < 1, \\ \tilde{v}_x = 0, & x = 0, 1, \end{cases} \tag{3.73}$$

where the tilde over ε has been dropped without confusion, and thanks to $\chi < 0$,

$$p := 1 + \frac{\chi}{D} < 1. \tag{3.74}$$

It is easy to show that $\tilde{v}_c = 1$ is the unique positive constant solution to (3.73). Hence,

$$v_c = \frac{1}{\eta}\tilde{v}_c = \frac{1}{\eta} = \left(\frac{\mu}{\lambda}\right)^{\frac{D}{\chi}}$$

and thus by (3.71),

$$u_c = \lambda v_c^{\frac{\chi}{D}} = \lambda \cdot \left[\left(\frac{\mu}{\lambda}\right)^{\frac{D}{\chi}}\right]^{\frac{\chi}{D}} = \lambda \cdot \frac{\mu}{\lambda} = \mu.$$

Actually, $\tilde{v}_c = 1$ is the only positive solution to (3.73). To prove this assertion, we distinguish the following two cases.

<u>Case 1:</u> $0 < p < 1$.   In this case, the assertion was already given in [25]. However, for completeness, we here present a direct proof.

Suppose, on the contrary, that there exists a positive non-constant classical solution $\tilde{v}(x)$ to (3.73). Denote $\tilde{v}(x_0) := \min_{x \in [0,1]} \tilde{v}(x) > 0$. Since

$$\begin{cases} -\varepsilon \tilde{v}_{xx} + \tilde{v} = \tilde{v}^p > 0, & 0 < x < 1, \\ \tilde{v}_x = 0, & x = 0, 1, \end{cases} \tag{3.75}$$

from the strong maximum principle we infer that $x_0 \neq 0, 1$. Thus $x_0 \in (0, 1)$ and therefore

$$\tilde{v}_{xx}(x_0) \geq 0. \tag{3.76}$$

We first assert that

$$\tilde{v}(x_0) = \min_{x \in [0,1]} \tilde{v}(x) \geq 1. \tag{3.77}$$

In fact, if $0 < \tilde{v}(x_0) < 1$, then this in conjunction with (3.76) and the fact that $0 < p < 1$ yields

$$\varepsilon \tilde{v}_{xx}(x_0) - \tilde{v}(x_0) + \tilde{v}^p(x_0) \geq \tilde{v}^p(x_0) - \tilde{v}(x_0) = \tilde{v}^p(x_0)(1 - \tilde{v}^{1-p}(x_0)) > 0,$$

which contradicts the first equation in (3.73). So, (3.77) holds.

Next, we define $\tilde{v}(x^*) := \max_{x \in [0,1]} \tilde{v}(x) > 0$. When $x^* \in (0, 1)$, we claim that

$$\tilde{v}(x^*) = \max_{x \in [0,1]} \tilde{v}(x) \leq 1. \tag{3.78}$$

In fact, if $\tilde{v}(x^*) > 1$, then using this and noting $\tilde{v}_{xx}(x^*) \leq 0$ and the fact that $0 < p < 1$ we obtain

$$\varepsilon\tilde{v}_{xx}(x^*) - \tilde{v}(x^*) + \tilde{v}^p(x^*) \leq -\tilde{v}(x^*) + \tilde{v}^p(x^*) = -\tilde{v}^p(x^*)(\tilde{v}^{1-p}(x^*) - 1) < 0,$$

which contradicts the first equation in (3.73) again. Hence,

$$\tilde{v}(x) \leq 1 \qquad \text{for all } x \in (0, 1). \tag{3.79}$$

This, along the continuity of $\tilde{v}(x)$ on $[0, 1]$, proves (3.78). Combining (3.77) and (3.78), we have

$$\tilde{v}(x) \equiv 1,$$

which contradicts the assumption that $\tilde{v}(x)$ is non-constant.

<u>Case 2</u>: $p \leq 0$. Suppose that $\tilde{v}_1$ and $\tilde{v}_2$ are two positive solutions to (1.6). The mean value theorem yields some $\xi(x)$ satisfying $0 < \min\{\tilde{v}_1(x), \tilde{v}_2(x)\} \leq \xi(x) \leq \max\{\tilde{v}_1(x), \tilde{v}_2(x)\}$ such that

$$\tilde{v}_1^p(x) - \tilde{v}_2^p(x) = p\xi^{\frac{\chi}{D}}(x)(\tilde{v}_1(x) - \tilde{v}_2(x)).$$

This, along with a straightforward computation, yields that $\bar{v} := \tilde{v}_1 - \tilde{v}_2$ solves

$$\begin{cases} \varepsilon\bar{v}_{xx} - \left[1 - p\xi^{\frac{\chi}{D}}(x)\right]\bar{v} = 0, & 0 < x < 1, \\ \bar{v}_x = 0, & x = 0, 1. \end{cases} \tag{3.80}$$

Thanks to $p \leq 0$ and $\xi(x) > 0$, we see that

$$1 - p\xi^{\frac{\chi}{D}}(x) \geq 1 > 0. \tag{3.81}$$

Hence, we can apply the elliptic maximum principle to (3.80) to conclude that $\bar{v} \equiv 0$ which indicates that $\tilde{v}_1 \equiv \tilde{v}_2$. This proves that $\tilde{v}_c \equiv 1$ is the only positive solution to (3.73). Considering the constant positive solution $(\mu, (\frac{\mu}{\lambda})^{\frac{D}{\chi}})$ of (1.6) must satisfy the restriction $\int_0^1 u(x)dx = \bar{u}_0$, the proof of Theorem 1.3 is complete. $\qquad\square$

3.6. **Numerical simulations.** The chemotaxis model (1.5) is very difficult to solve using the routine numerical scheme due to the singularity occurring in the flux term $(\ln v)_x = v_x/v$. Noticing that the u in (1.5) is the same as one in the transformed system (1.4), which no longer has logarithm's singularity. Hence we can solve system (1.4) to obtain the solution u of the original chemotaxis model (1.5) for $\chi < 0$.

We shall implement the finite-difference based Matlab PDE solver to solve systems (1.4) in $(0, 1)$, where the time step size $\Delta t = 0.01$ and spatial step size $\Delta x = 0.01$. In Fig. 1, we show the large time behavior of the solution u to system (1.4), where Fig. 1(a) plots the initial distribution $u_0(x) = 1 + \cos(4\pi x)$ which is a perturbation of cell mass $\bar{u}_0 = 1$. Fig. 1(b) shows the time evolution of $u(x, t)$ at $x = 0.4$ where we see that the solution u quickly converges to the cell mass $\bar{u}_0 = 1$ in time $t < 1$. This indicates that the convergence rate may be exponential and illustrate our results in Theorem 1.1.

4. **Appendix.** *Proof of Lemma 2.1.* Denote

$$R := \|u_0\|_{W^{2,p_0}((0,1))} + \|w_0\|_{W^{2,p_0}((0,1))} + \|u_0\|_{C^1([0,1])} + \|w_0\|_{C^1([0,1])} + 1.$$

With this R and $T \in (0, 1)$ to be specified below, in the Banach space

$$X := C^{1,0}(\bar{Q}_T) \times C^{1,0}(\bar{Q}_T),$$

we consider the closed convex set

$$S_T := \left\{(u, w) \in X \,\middle|\, \|u(\cdot, t)\|_{C^{1,0}(\bar{Q}_T)} \leq R \quad \text{and} \quad \|w(\cdot, t)\|_{C^{1,0}(\bar{Q}_T)} \leq R\right\}$$
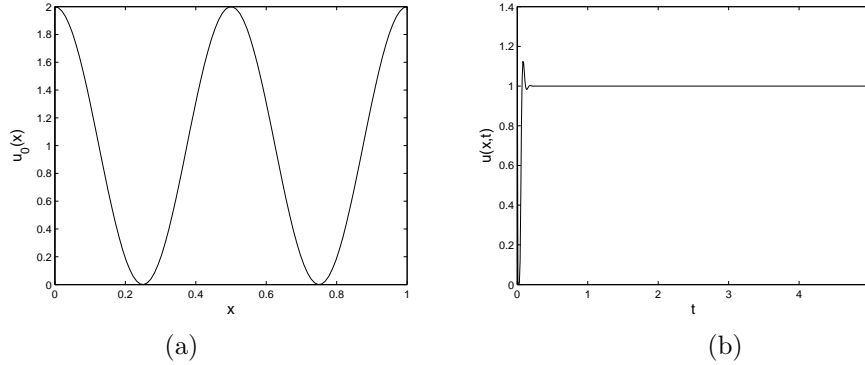
FIGURE 1.   Numerical solutions to the transformed repulsive chemotaxis model (1.4) with initial data $u_0(x) = 1 + \cos(4\pi x), w_0(x) = \sin(2\pi x)$, where $\varepsilon = 1$. (a) plots the initial distribution $u_0(x)$ and (b) plots the time evolution of solution $u(x,t)$ at spatial position $x = 0.4$.

and introduce a mapping $\Phi : S_T \longmapsto S_T$ such that given $(\tilde{u}, \tilde{w}) \in S_T, \Phi((\tilde{u}, \tilde{w})) = (u, w)$ where u is the solution to

$$\begin{cases} u_t - u_{xx} - wu_x - w_x u = 0, & x \in (0,1),\ t \in (0,T), \\ u_x|_{x=0,1} = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,1], \end{cases} \quad (4.1)$$

and w defined the solution of

$$\begin{cases} w_t - \varepsilon w_{xx} = (\varepsilon \tilde{w}^2 + \tilde{u})_x, & x \in (0,1),\ t \in (0,T), \\ w|_{x=0,1} = 0, & t \in (0,T), \\ w(x,0) = w_0(x), & x \in [0,1]. \end{cases} \quad (4.2)$$

We shall show that for T small enough Φ has a unique fixed point.

Let $Q_T := (0,1) \times (0,T)$. For consistency, throughout the remainder of this paper we denote

$$W^{2,1,p}(Q_T) \quad := \quad \{u \mid u, u_x, u_{xx}, u_t \in L^p(Q_T)\}$$

for $p \geq 1$, equipped with the norm

$$\|u\|_{W^{2,1,p}(Q_T)} \quad = \quad \|u\|_{L^p(Q_T)} + \|u_x\|_{L^p(Q_T)} + \|u_{xx}\|_{L^p(Q_T)} + \|u_t\|_{L^p(Q_T)}.$$

Since $w_0(x) \in W^{2,p_0}((0,1))$ and $\|(\varepsilon\tilde{w}^2 + \tilde{u})_x\|_{L^\infty(Q_T)} \leq R(1 + 2\varepsilon R)$ thanks to $(\tilde{u}, \tilde{w}) \in S_T$, from parabolic L^p-theory (cf. [10, Theorem 2.3] and [17, Theorem IV.9.1]) we infer that there exists a unique solution $w(x,t) \in W^{2,1,p_0}(Q_T)$ to (4.2) and that there exists some constant $c_1 > 0$ such that

$$\begin{aligned} \|w\|_{W^{2,1,p_0}(Q_T)} \quad &\leq \quad c_1(\|(\varepsilon\tilde{w}^2 + \tilde{u})_x\|_{L^{p_0}(Q_T)} + \|w_0\|_{L^{p_0}(Q_T)} \\ &\qquad + \|(w_0)_x\|_{L^{p_0}(Q_T)} + \|(w_0)_{xx}\|_{L^{p_0}(Q_T)}) \\ &\leq \quad c_1 T^{\frac{1}{p_0}} \cdot (\|(\varepsilon\tilde{w}^2 + \tilde{u})_x\|_{L^\infty(Q_T)} + \|w_0\|_{W^{2,p_0}((0,1))}) \\ &\leq \quad c_1 T^{\frac{1}{p_0}} \cdot [R(1 + 2\varepsilon R) + R] \\ &\leq \quad c_2(R) := c_1 \cdot [R(1 + 2\varepsilon R) + R] \qquad (4.3) \end{aligned}$$

thanks to the facts that $T \in (0,1)$ and $\|w_0\|_{W^{2,p_0}((0,1))} < R$. This in conjunction with the Sobolev imbedding Theorem ([17, Lemma II. 3.3]) yields some $c_3 > 0$ such that

$$\|w\|_{C^{1+\theta,(1+\theta)/2}(\bar{Q}_T)} \leq c_3\|w\|_{W^{2,1,p_0}(Q_T)} \leq c_4(R) := c_3 \cdot c_2(R) \qquad (4.4)$$

where $\theta := 1 - \frac{3}{p_0}$ and particularly

$$\|w\|_{C^{1,0}(\bar{Q}_T)} \leq c_4(R). \qquad (4.5)$$

We next turn to consider (4.1). Since $\|u_0\|_{W^{2,p_0}((0,1))} < R$ and $\|w\|_{L^\infty(Q_T)} \leq c_4(R), \|w_x\|_{L^\infty(Q_T)} \leq c_4(R)$ by (4.5), from parabolic L^p-theory (cf. [10, Theorem 2.3] and [17, Theorem IV.9.1 and Section V.7]) we infer that there exists a unique solution $u(x,t) \in W^{2,1,p_0}(Q_T)$ to (4.1). Moreover, proceeding as in the derivation of (4.4), we can obtain some $c_5(R) > 0$ such that

$$\|u\|_{C^{1+\theta,(1+\theta)/2}(\bar{Q}_T)} \leq c_5(R). \qquad (4.6)$$

Thus,

$$\begin{aligned}
\|u\|_{C^{1,0}(\bar{Q}_T)} &\leq \|u(x,t) - u(x,0)\|_{C^{1,0}(\bar{Q}_T)} + \|u(x,0)\|_{C^{1,0}(\bar{Q}_T)} \\
&\leq T^{\frac{1+\theta}{2}} \cdot \|u\|_{C^{0,(1+\theta)/2}(\bar{Q}_T)} + \|u_0\|_{C^1([0,1])} \\
&\leq T^{\frac{1+\theta}{2}} \cdot c_5(R) + \|u_0\|_{C^1([0,1])}.
\end{aligned}$$

From this we deduce that if we take $T > 0$ sufficiently small that $T \leq \left(\frac{1}{c_5(R)}\right)^{\frac{2}{1+\theta}}$, then we have

$$\|u\|_{C^{1,0}(\bar{Q}_T)} \leq 1 + \|u_0\|_{C^1([0,1])} \leq R. \qquad (4.7)$$

Similarly, if T is further diminished, say

$$T \leq T_0 = T_0(R) \leq \left(\frac{1}{c_5(R)}\right)^{\frac{2}{1+\theta}}, \qquad (4.8)$$

then we also have

$$\|w\|_{C^{1,0}(\bar{Q}_T)} \leq R. \qquad (4.9)$$

This, along with (4.7), proves $(u,w) \in S_T$. Thus, Φ maps S_T into itself. By a straightforward adaptation of the above reasoning, one can easily deduce that if T is further diminished then Φ in fact becomes a contraction on S_T. For such T we therefore conclude from the contraction mapping principle ([11, Theorem 5.1]) that there exists a unique $(u,w) \in S_T$ such that $\Phi((u,w)) = (u,w)$.

By (4.1), $w, w_x \in C^{\theta,\theta/2}(\bar{Q}_T)$ and the classical regularity of parabolic equations ([17, Theorem V. 6.1]) we obtain

$$u(x,t) \in C^{2+\theta,(1+\theta)/2}([0,1] \times [\eta,T]) \qquad \text{for all } \eta \in (0,T_0].$$

Similarly,

$$w(x,t) \in C^{2+\theta,(1+\theta)/2}([0,1] \times [\eta,T]) \qquad \text{for all } \eta \in (0,T_0].$$

This proves the regularity of the solution (u,w) to (1.4). The solution may be prolonged in the interval $[0,T_{max})$ with either $T_{max} = \infty$ or $T_{max} < \infty$, where in the latter case

$$\|u(\cdot,t)\|_{W^{2,p_0}((0,1))} + \|w(\cdot,t)\|_{W^{2,p_0}((0,1))} \to \infty \qquad \text{as } t \nearrow T_{max},$$

because T_0 depends only on R by (4.8), and $\|u_0\|_{C^1([0,1])} \leq c_6\|u_0\|_{W^{2,p_0}((0,1))}$ and $\|w_0\|_{C^1([0,1])} \leq c_6\|w_0\|_{W^{2,p_0}((0,1))}$ for some $c_6 > 0$ by the Sobolev embedding:

$W^{2,p_0}((0,1)) \hookrightarrow C^1([0,1])$ for $p_0 > 3$. Finally, the positivity of u results from the strong parabolic maximum principle, because $u_0 \not\equiv 0$ ensures that $u \not\equiv 0$. □

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