Classical solutions and steady states of an attraction-repulsion chemotaxis in one dimension

Jia Liu
Department of Mathematics and Statistics
University of West Florida
Pensacola, FL 32514
jliu@uwf.edu

Zhi-An Wang*
Department of Applied Mathematics
Hong Kong Polytechnic University
Hung Hom, Kowloon, Hong Kong
mawza@polyu.edu.hk

Abstract: We establish the existence of global classical solutions and non-trivial steady states of an one-dimensional attraction-repulsion chemotaxis model subject to Neumann boundary conditions. The results are derived based on the method of energy estimates and the phase plane analysis.

Key words: Chemotaxis, attraction-repulsion, classical solutions, steady states

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1 Introduction

Chemotaxis describes the directed migration of cells along the concentration gradient of the chemical which is produced by cells. It is a leading mechanism to account for the morphogenesis and self-organization of many biological system. The prototype of the population-based chemotaxis model, known as the Keller-Segel model, was first proposed by Keller and Segel in the 1970s [5] to describe the aggregation of cellular slime molds Dictyostelium discoideum. The rudimental structure of the Keller-Segel model is a system of parabolic partial differential equations as follows

$$\begin{aligned} u_t &= D_u \Delta u - \nabla(\chi u \nabla v), \\ v_t &= D_v \Delta v + f(u, v), \end{aligned} \tag{1.1}$$

where $u(x,t)$ denotes the cell density and $v(x,t)$ is the chemical concentration, D_u and D_v are positive diffusion coefficients and $\chi > 0$ is called the chemotactic coefficient measuring the strength of influence of the chemical on cells. The Keller-Segel model (1.1) describes the cell chemotactic movement toward a single chemical (i.e. chemoattractant) and has

*Corresponding author. Supported by the start-up funding from the Hong Kong Polytechnic University.

been extensively studied in the past four decades from various perspectives ([8, 14, 15]). However in many biological processes, the cells may interact with a combination of repulsive and attractive signalling chemicals to produce various interesting biological patterns, such as the formation of nigrostriatal circuits during development [7], the chick primitive streak formation [3], and many others (e.g., see [4]). In this paper, we shall consider the following attraction-repulsion chemotaxis model

$$\begin{aligned} u_t &= D_u \Delta u - \nabla(\chi_v u \nabla v) + \nabla(\chi_w u \nabla w), \\ v_t &= D_v \Delta v + \alpha u - \beta v, \\ w_t &= D_w \Delta w + \gamma u - \delta w, \end{aligned} \tag{1.2}$$

where $D_u, D_v, D_w > 0$ are diffusion coefficients, $\chi_v > 0, \chi_w > 0$ are chemotactic coefficients, $\alpha, \gamma > 0$ and $\beta, \delta \geq 0$. The model (1.2) was proposed in [10] to describe the aggregation of microglia observed in Alzheimer's disease and in [12] to describe the quorum effect in the chemotactic process. In their approaches, it is assumed that there exists a secondary chemical, denoted by w, which behaves as a chemo-repellent to mediate the chemotactic response to the chemoattractant v accordingly. To the best of our knowledge, there is rigorous result by now on the chemotaxis model with two opposite chemicals (i.e. chemo-attractant and chemo-repellent). The purpose of this paper is to establish the global existence of classical solutions and steady states of (1.2) in one dimension with Neumann boundary conditions. The results for higher dimensions still remains open.

In one dimension, the system (1.2) reads

$$\begin{aligned} u_t &= D_u u_{xx} - (\chi_v u v_x)_x + (\chi_w u w_x)_x, \\ v_t &= D_v v_{xx} + \alpha u - \beta v, \\ w_t &= D_w w_{xx} + \gamma u - \delta w. \end{aligned} \tag{1.3}$$

With the following scalings

$$\tilde{t} = D_u t, \tilde{v} = \frac{\chi_v}{D_u} v, \tilde{w} = \frac{\chi_w}{D_u} w, \tilde{u} = \frac{\chi_v}{D_u^2} u, \tilde{\gamma} = \frac{\gamma \chi_w}{\alpha \chi_v}, \ (\tilde{D}_v, \tilde{D}_w, \tilde{\beta}, \tilde{\delta}) = \frac{1}{D_u}(D_v, D_w, \beta, \delta),$$

system (1.3) can be reduced to the following system

$$\begin{aligned} u_t &= u_{xx} - (u v_x)_x + (u w_x)_x, \\ v_t &= D_v v_{xx} + u - \beta v, \\ w_t &= D_w w_{xx} + \gamma u - \delta w, \end{aligned} \tag{1.4}$$

where the tilde superscripts have been suppressed for readability.

Letting Ω be a bounded open interval in $\mathbb{R} = (-\infty, \infty)$, we prescribe the initial conditions

$$u(x,0) = u_0(x), \ v(x,0) = v_0(x), \ w(x,0) = w_0(x), \tag{1.5}$$

and Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \ x \in \partial\Omega, \tag{1.6}$$

where ν denotes the unit outward normal vector to the boundary $\partial\Omega$.

In the present paper, we shall prove the existence of global classical solutions to the model (1.4), (1.5) and (1.6) based on Amann's theory and the method of energy estimates. We also show the existence of non-trivial steady states of (1.4) subject to the Neumann boundary conditions (1.6) for the case $\beta D_w = \delta D_v$ by the phase plane analysis.

Notations. Throughout the paper, Ω denotes a bounded open interval in $\mathbb{R}$ unless otherwise specified and C denotes a generic constant which can change from one line to another. $L^p = L^p(\Omega)(1 \le p \le \infty)$ denotes the usual Lebesgue space in a bounded open interval $\Omega \subset \mathbb{R} = (-\infty, \infty)$ with norm $\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p}$ for $1 \le p < \infty$ and $\|f\|_{L^\infty} = \operatorname{ess\,sup}_{x \in \Omega} |f(x)|$. When $p = 2$, we write $\|f\|_{L^2} = \|f\|$ for notational convenience. H^l denotes the l-th order Sobolev space $W^{l,2}$ with norm $\|f\|_{H^l} = \|f\|_l = \left(\sum_{i=0}^l \|\partial_x^i f\|^2 \right)^{1/2}$. For simplicity, $\|f(\cdot,t)\|_{L^p}$ and $\|f(\cdot,t)\|_l$ will be denoted by $\|f(t)\|_{L^p}$ and $\|f(t)\|_l$, respectively. Moreover, we denote $\|(f,g)\|_{L^p} = \|f\|_{L^p} + \|g\|_{L^p}$ for $1 \le p \le \infty$ and $\|(f,g)\|_{H^l} = \|f\|_{H^l} + \|g\|_{H^l}$ for $l = 1, 2, 3, \cdots$.

2 Preliminaries

In this section, we present some inequalities which will be used to derive the required estimates. First we recall the Gagliardo-Nirenberg inequality for functions that do not vanish at the boundary of Ω (see Theorem 1 in [11]).

Lemma 2.1. *Let Ω be a open bounded domain in $\mathbb{R}^n$ with smooth boundary. Then for any $q \ge 1$, there exists a positive constant C_q, which depends on n, q, Ω, such that for all $f \in W^{1,2}(\Omega)$,*

$$\|f\|_{L^q} \le C_q \left(\|\nabla f\|_2^a \, \|f\|_{L^1}^{1-a} + \|f\|_{L^1} \right) \tag{2.1}$$

where $a = (1 - \frac{1}{q})/(\frac{1}{n} + \frac{1}{2})$ and $0 \le a < 1$.

Letting $n = 1, q = 4, \alpha = 1/2$ in (2.1) and using the inequality $(a+b)^2 \le 2(a^2 + b^2)$ for any $a, b \in \mathbb{R}$, we obtain the following inequality

$$\|f\|_{L^4}^2 \le C(\|f_x\| \|f\|_{L^1} + \|f\|_{L^1}^2). \tag{2.2}$$

The following Gronwall's type inequality [16] will be used later.

Proposition 2.2. *Let $\eta(\cdot)$ be a nonnegative differentiable function on $[0, \infty)$ satisfying the differential inequality $\eta'(t) + l\eta(t) \le \omega(t)$, where l is a constant and $\omega(t)$ is a nonnegative continuous functions on $[0, \infty)$. Then*

$$\eta(t) \le \left(\eta(0) + \int_0^t e^{l\tau} \omega(\tau) d\tau \right) e^{-lt}. \tag{2.3}$$

Alternatively, if for $t \ge 0$, $\phi(t) \ge 0$ and $\psi(t) \ge 0$ are continuous function such that the inequality $\phi(t) \le C \exp(rt) + L \int_0^t \psi(s)\phi(s) ds$ holds on $t \ge 0$ with C and L positive constants, then

$$\phi(t) \le C \exp(rt) \exp\left(L \int_0^t \psi(s) ds \right). \tag{2.4}$$

3 Global existence of classical solutions

In this section, we shall establish the global existence of classical solutions of the system (1.4), (1.5) and (1.6). The main result is the following:

Theorem 3.1. *Let* $(u_0, v_0, w_0) \in H^2(\Omega)$*. Then there exists a unique global solution* (u, v, w) *to the system (1.4), (1.5) and (1.6) such that* $(u, v, w) \in [C^0(\overline{\Omega} \times [0, \infty); \mathbb{R}^3)]^3 \cap [C^{2,1}(\overline{\Omega} \times (0, \infty); \mathbb{R}^3)]^3$*. Moreover* $u, v, w \geq 0$ *if* $u_0, v_0, w_0 \geq 0$*.*

Remark 3.1. *Theorem 3.1 does not exclude the possibility that the solution may blow up at infinity time.*

Theorem 3.1 will be proved by the local existence and the *a priori* estimates as given below.

3.1 Local existence

In this section, we shall apply Amann's theory [1] to establish the local existence of solutions.

Theorem 3.2. (local existence). *Let* Ω *be a bounded open interval in* $\mathbb{R}$*. Then*

(i) For any initial data $(u_0, v_0, w_0) \in [H^1(\Omega)]^3$*, there exists a maximal existence time constant* $T_0 \in (0, \infty]$ *depending on the initial data* (u_0, v_0, w_0)*, such that the problem (1.4), (1.5) and (1.6) has a unique maximal solution* (u, v, w) *defined on* $\Omega \times [0, T_0)$ *satisfying*

$$(u, v, w) \in [C^0(\overline{\Omega} \times [0, T_0); \mathbb{R}^3)]^3 \cap [C^{2,1}(\overline{\Omega} \times (0, T_0); \mathbb{R}^3)]^3.$$

(ii) If $\sup\limits_{0<t<T_0 \cap T} \|(u, v, w)(\cdot, t)\|_{L^\infty} < \infty$ *for each* $T > 0$*, then* $T_0 = \infty$*, namely,* (u, v, w) *is a global classical solution of the system (1.4), (1.5) and (1.6). Moreover* $u \geq 0, v \geq 0, w \geq 0$ *if* $u_0 \geq 0, v_0 \geq 0, w_0 \geq 0$*.*

Proof. Define $\eta = (u, v, w) \in \mathbb{R}^3$. Then the system (1.4) with (1.5) and (1.6) can be rewritten as

$$\begin{cases} \eta_t - \nabla \cdot (a(\eta)\nabla\eta) = \mathcal{F}(\eta), & \text{in } \Omega \times [0, +\infty), \\ \dfrac{\partial \eta}{\partial \nu} = 0, & \text{on } \partial\Omega \times [0, +\infty), \\ \eta(\cdot, 0) = (u_0, v_0, w_0), & \text{in } \Omega, \end{cases} \tag{3.1}$$

where

$$a(\eta) = \begin{pmatrix} 1 & -u & u \\ 0 & D_v & 0 \\ 0 & 0 & D_w \end{pmatrix}, \quad \mathcal{F}(\eta) = \begin{pmatrix} 0 \\ u - \beta v \\ \gamma u - \delta w \end{pmatrix}.$$

It is clear that the eigenvalues of $a(\eta)$ are all positive and hence system (1.4) is normally elliptic. Then the local existence result of assertion (i) follows from [1, Theorem 14.6], and (ii) is a consequence of [1, Theorem 15.3]. Finally the positivity of solutions follows from [1, Theorem 15.1].

□

3.2 A *priori* estimates

In this section, we are devoted to deriving the *a priori* estimates of solutions obtained in Theorem 3.1 to establish the global existence of solutions. First of all, we notice that the first equation of (1.4) is a conservation equation. If we denote

$$\int_\Omega u_0(x)dx =: m \tag{3.1}$$

then by integrating the first equation of (1.4) and using the Neumann boundary conditions (1.6), we have

$$\|u(t)\|_{L^1} = \int_\Omega u(x,t)dx = m. \tag{3.2}$$

Lemma 3.3. *Let* $(v_0, w_0) \in [L^2(\Omega)]^2$ *and (1.6) hold. Let* (u, v, w) *be a solution of the problem (1.4)-(1.6). Then for any* $T > 0$*, there is a constant* C *such that the following inequality holds for any* $0 < t < T$

$$\|(v,w)(t)\|^2 \le C, \ \int_0^t \|(v,w)(\tau)\|^2 d\tau + \int_0^t \|(v_x, w_x)(\tau)\|^2 d\tau \le C(1+t). \tag{3.3}$$

Proof. Multiplying the second equation of (1.4) by v and integrating the resulting equation with respect to x over Ω gives rise to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 dx + \beta\int_\Omega v^2 dx + D_v\int_\Omega v_x^2 dx = \int_\Omega uvdx \le m\|v\|_{L^\infty}.$$

Applying the Sobolev embedding $H^1 \hookrightarrow L^\infty$, one has that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 dx + \beta\int_\Omega v^2 dx + D_v\int_\Omega v_x^2 dx \le Cm(\|v\| + \|v_x\|) \le \frac{D_v}{2}\|v_x\|^2 + \frac{\beta}{2}\|v\|^2 + C,$$

where the Young inequality has been used. Then it follows that

$$\frac{d}{dt}\|v\|^2 + \beta\|v\|^2 + D_v\|v_x\|^2 \le C. \tag{3.4}$$

Applying Gronwall's inequality (2.3) to (3.4) yields that

$$\begin{aligned}\|v\|^2 &\le \left(\|v_0\|^2 + C\int_0^t e^{\beta\tau}d\tau\right)e^{-\beta t}\\ &\le (\|v_0\|^2 - C/\beta)e^{-\beta t} + C/\beta \le C.\end{aligned}$$

Furthermore the integration of (3.4) with respect to t over $[0,t]$ gives

$$\int_0^t \|v(\tau)\|^2 d\tau + \int_0^t \|v_x(\tau)\|^2 d\tau \le C(1+t). \tag{3.5}$$

Applying the same procedure to w, we finish the proof.

□

Lemma 3.4. *Let* $u_0 \in L^2(\Omega), (v_0, w_0) \in [H^1(\Omega)]^2$ *and* (u, v, w) *be a solution of the problem (1.4)-(1.6). Then for any* $T > 0$*, there is a positive constant* C *such that for any* $0 < t < T$ *it follows that*

$$\|u(t)\|^2 + \int_0^t \|u_x(\tau)\|^2 d\tau + \|(v,w)(t)\|_1^2 + \int_0^t \|(v,w)(\tau)\|_2^2 d\tau \le C\big(1 + e^{Ct}\big). \tag{3.6}$$

Proof. We multiply the first equation of (1.4) by u and integrate the resulting equation by parts. Then by (2.2), the Hölder inequality and the Young inequality, we derive that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx+\int_\Omega u_x^2dx&=-\frac{1}{2}\int_\Omega u^2v_{xx}dx+\frac{1}{2}\int_\Omega u^2w_{xx}dx\\&\le\frac{1}{2}\|u\|_{L^4}^2(\|v_{xx}\|+\|w_{xx}\|)\\&\le C(m\|u_x\|+m^2)(\|v_{xx}\|+\|w_{xx}\|)\\&\le\frac{1}{2}\|u_x\|^2+C(\|v_{xx}\|+\|w_{xx}\|+\|v_{xx}\|^2+\|w_{xx}\|^2),\end{aligned}$$

where the Young inequality has been used. Using the Cauchy-Schwarz inequality to derive $C(\|v_{xx}\|+\|w_{xx}\|)\le 2C^2+\|v_{xx}\|^2+\|w_{xx}\|^2$, we have

$$\frac{d}{dt}\int_\Omega u^2dx+\int_\Omega u_x^2dx\le C(1+\|v_{xx}\|^2+\|w_{xx}\|^2).\tag{3.7}$$

Next we estimate the right hand side of (3.7). To this end, we multiply the second equation of (1.4) by $-v_{xx}$ and integrate the resulting equation to obtain that

$$\frac{d}{dt}\int_\Omega v_x^2dx+\int_\Omega v_x^2dx+\int_\Omega v_{xx}^2dx\le C\int_\Omega u^2dx.\tag{3.8}$$

Similar procedure applied to the third equation of (1.4) leads to

$$\frac{d}{dt}\int_\Omega w_x^2dx+\int_\Omega w_x^2dx+\int_\Omega w_{xx}^2dx\le C\int_\Omega u^2dx.\tag{3.9}$$

Combining (3.8) and (3.9) we have

$$\frac{d}{dt}\int_\Omega(v_x^2+w_x^2)dx+\int_\Omega(v_x^2+w_x^2)dx+\int_\Omega(v_{xx}^2+w_{xx}^2)dx\le C\int_\Omega u^2dx.\tag{3.10}$$

Then integrating (3.10) with respect to t yields that

$$\begin{aligned}\|(v_x,w_x)\|^2+\int_0^t\|(v_x,w_x)(\tau)\|^2d\tau+\int_0^t\|(v_{xx},w_{xx})(\tau)\|^2d\tau\\\le C\Big(1+\int_0^t\|u(\tau)\|^2d\tau\Big).\end{aligned}\tag{3.11}$$

Now we integrate (3.7) with respect to t and obtain that

$$\|u(t)\|^2+\int_0^t\|u_x(\tau)\|^2d\tau\le C(1+t)+C\int_0^t\|(v_{xx},w_{xx}(\tau)\|^2d\tau.\tag{3.12}$$

By using the Gronwall's inequality (2.4) to (3.12), one has that

$$\|u(t)\|^2\le C(1+t)e^{Ct}.\tag{3.13}$$

Therefore substituting (3.13) back to (3.11) gives

$$\begin{aligned}&\|(v_x,w_x)\|^2+\int_0^t\|(v_x,w_x)(\tau)\|^2d\tau+\int_0^t\|(v_{xx},w_{xx})(\tau)\|^2d\tau\\&\qquad\le C(1+t)e^{Ct}\le Ce^{Ct}\end{aligned}\tag{3.14}$$

where we have used the fact that $0<t\le e^{Ct}$ for $C\ge 1$.

Then the combination of (3.13) and (3.14) with (3.5) gives (3.6).

□

With Lemma 3.4, we can derive the following estimates.

Lemma 3.5. *If $(v_0,w_0)\in[H^2(\Omega)]^2$. Let (u,v,w) be a solution of the problem (1.4)-(1.6). Then for any $T>0$, there is a positive constant C such that for any $0<t<T$ it holds that*

$$\|(v_{xx},w_{xx})(t)\|^2+\int_0^t\|(v_{xx},w_{xx})(\tau)\|^2d\tau+\int_0^t\|(v_{xxx},w_{xxx})(\tau)\|^2d\tau\le C(1+e^{Ct}).$$

Proof. Differentiating the second question of (1.4) with respect to x twice and then multiplying the result by v_{xx}, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_{xx}^2dx+\beta\int_\Omega v_{xx}^2dx+D_v\int_\Omega v_{xxx}^2dx=\int_\Omega u_{xx}v_{xx}dx=-\int_\Omega u_xv_{xxx}dx.$$

By the Cauchy-Schwarz inequality, we have $-u_xv_{xxx}\le\frac{D_v}{2}v_{xxx}^2+\frac{2}{D_v}u_x^2$, which is applied to the above identity yields

$$\|v_{xx}\|^2+\int_0^t\|v_{xx}(\tau)\|^2d\tau+\int_0^t\|v_{xxx}(\tau)\|^2d\tau\le C\int_0^t\|u_x(\tau)\|^2d\tau.$$

Then the application of Lemma 3.4 to the above inequality gives the estimate for v. By the same procedure, we can derive the similar estimates for w and complete the proof. □

Combining Lemma 3.5 and Lemma 3.4 and using the Sobolev embedding $H^1\hookrightarrow L^\infty$, we derive that

$$\|v_x\|_{L^\infty}+\|w_x\|_{L^\infty}\le C(1+e^{Ct}).\tag{3.15}$$

Then we can derive the H^1-estimates for u.

Lemma 3.6. *Let $u_0\in H^1(\Omega)$. Assume that (u,v,w) is a solution of the problem (1.4)-(1.6). Then for any $T>0$, there is a positive constant C such that for any $0<t<T$ it has*

$$\|u_x\|^2+\int_0^t\|u_{xx}(\tau)\|^2d\tau\le C(1+e^{Ct}).$$

Proof. Multiplying the first equation of (1.4) by $(-u_{xx})$ and integrating the result yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_x^2dx+\int_\Omega u_{xx}^2dx=\int_\Omega u_{xx}(uv_x)_xdx-\int_\Omega u_{xx}(uw_x)_xdx\tag{3.16}$$

Next we estimates the terms on the right hand side terms of (3.16). To this end, we first differentiate the second equation of (1.4) thrice and then multiply the resulting equation by v_{xxx}. After integrating the result with respect to x and t, we have

$$\|v_{xxx}\|^2+\int_0^t\|v_{xxx}(\tau)\|^2d\tau+\int_0^t\|v_{xxxx}(\tau)\|^2d\tau\le C_0\int_0^t\|u_{xx}(\tau)\|^2d\tau\tag{3.17}$$

where we have used the Cauchy-Schwarz inequality and C_0 is a constant. In virtue of the integration by parts, we deduce that

$$\begin{aligned}\int_\Omega u_{xx}(uv_x)_x dx &= \frac{1}{2}\int_\Omega u^2 v_{xxxx} dx + 3\int_\Omega v_x u_x u_{xx} dx\\ &\le \|u\|_{L^4}^2\|v_{xxxx}\| + 3\|v_x\|_{L^\infty}\int_\Omega |u_x u_{xx}| dx\end{aligned}$$

where the Hölder inequality has been used. Then by the Cauchy-Schwarz inequality and using (3.17), (3.6), (2.2) and (3.15), one has

$$\begin{aligned}\int_0^t\int_\Omega u_{xx}(uv_x)_x dx &\le C\int_0^t (\|u(\tau)\|_{L^4}^2)^2 d\tau + \frac{1}{8C_0}\int_0^t \|v_{xxxx}(\tau)\|^2 d\tau\\ &\quad + C\int_0^t (1+e^{C\tau})\int_\Omega u_x^2 dx d\tau + \frac{1}{8}\int_0^t\int_\Omega u_{xx}^2 dx d\tau\\ &\le C\int_0^t (1+\|u_x(\tau)\|)^2 d\tau + C(1+e^{Ct})\int_0^t \|u_x(\tau)\|^2 d\tau\\ &\quad + \frac{1}{4}\int_0^t \|u_{xx}(\tau)\|^2 d\tau\\ &\le C(1+e^{Ct}) + \frac{1}{4}\int_0^t \|u_{xx}\|^2 d\tau.\end{aligned} \tag{3.18}$$

The same argument as above applied to w also gives rise to

$$\int_0^t\int_\Omega u_{xx}(uw_x)_x dx \le C(1+e^{Ct}) + \frac{1}{4}\int_0^t \|u_{xx}\|^2 d\tau. \tag{3.19}$$

Then integrating (3.16) with respect to t and applying the inequalities (3.18) and (3.19), we complete the proof. □

3.3 Proof of Theorem 3.1

By Lemma 3.4 and Lemma 3.6 as well as Sobolev embedding $H^1 \hookrightarrow L^\infty$, we have

$$\sup_{0<t<T_0\cap T} \|(u,v,w)(t)\|_{L^\infty} \le C(1+e^{Ct})$$

for any $T>0$. That is for any finite time t with $0<t<T_0\cap T$, $\|(u,v,w)(t)\|_{L^\infty}$ is bounded. By the statement (ii) of Theorem 3.2, the maximal existence time constant T_0 of the classical solution obtained in Theorem 3.2 must be infinite. The non-negativity of the solution follows from (ii) of Theorem 3.2 directly. Then the proof of theorem 3.1 is finished.

4 Steady States

In this section, we study the non-trivial steady states of (1.4) with homogeneous boundary conditions (1.6). Steady states of (1.4) satisfies the system

$$\begin{aligned}&u_{xx} - (uv_x)_x + (uw_x)_x = 0,\\ &D_v v_{xx} + u - \beta v = 0,\\ &D_w w_{xx} + \gamma u - \delta w = 0.\end{aligned} \tag{4.1}$$

The non-trivial steady state of (1.4) is defined as the solution of (4.1) where none of u, v and w is a constant. In this paper, we only consider the simple case $\frac{\beta}{D_v} = \frac{\delta}{D_w} = \mu$ which indicates that both the chemoattractant and chemo-repellent have the same death rate relative to their diffusions, respectively. The result of the non-trivial steady state for the general system (4.1) still remains open. By defining $\phi = v - w$, the system (4.1) can be transformed as

$$\begin{aligned} &u_{xx} - (u\phi_x)_x = 0 \\ &\phi_{xx} + \lambda u - \mu\phi = 0 \end{aligned} \tag{4.2}$$

where $\lambda = \frac{1}{D_v} - \frac{\gamma}{D_w}$. Then integrating the first equation of (4.2) and using the homogeneous boundary conditions (1.6), we have

$$u = \eta e^{\phi}$$

where η is a positive constant.

We substitute the expression for u into the second equation of (4.2) and obtain an elliptic equation for the steady states:

$$\phi_{xx} = \mu\phi - \lambda\eta e^{\phi}. \tag{4.3}$$

This equation of steady states has been extensively investigated when $\phi \geq 0$ and $\lambda > 0$. e.g., see [13, 8]. However in our model both the variable ϕ and the constant λ can be non-positive. We write (4.3) as a first order Hamiltonian system

$$\begin{aligned} &\phi_x = y, \\ &y_x = \mu\phi - \lambda\eta e^{\phi}. \end{aligned} \tag{4.4}$$

Without loss of generality we assume $\Omega = (0, L)$ with $L > 0$. Then the Neumann boundary conditions (1.6) becomes

$$y(0) = y(L) = 0. \tag{4.5}$$

Let (ϕ^*, y^*) be an equilibrium point of (4.4). Then the coefficient matrix of the linearized system of (4.4) about (ϕ^*, y^*) is

$$M = \begin{bmatrix} 0 & 1 \\ \mu - \lambda\eta e^{\phi^*} & 0 \end{bmatrix}.$$

It is straightforward to see that the equilibria of (4.4) satisfies $y = 0$ and

$$\mu\phi = \lambda\eta e^{\phi}. \tag{4.6}$$

Then there are two cases to consider:

(1) When $\lambda \leq 0$, namely $D_w \leq \gamma D_v$, the equation (4.6) always has a unique solution $\phi^* < 0$. The equilibrium $(\phi^*, 0)$ is a saddle point for the linearized system due to $\det M = -\mu + \lambda\eta e^{\phi^*} < 0$. It is also a saddle for the full nonlinear system (4.4) by Hartman-Grobman theorem. Since the nonlinear system has the Hamiltonian functional $H(\phi, y) = \frac{y^2}{2} - \frac{\mu}{2}\phi^2 + \lambda\eta e^{\phi}$, there are no non-trivial steady state solutions satisfying the boundary condition (4.5) by simple phase plane analysis.

(2) When $\lambda > 0$, namely $D_w > \gamma D_v$, the equation (4.6) can have zero, one or two solutions depending on the parameters. It is straightforward to check that only the case of two solutions yields the non-trivial steady states. The equation (4.6) has two solutions $0 < \phi_1^* < \phi_2^*$ if and only if $\mu > \lambda\eta e$, where ϕ_1^* satisfies $\mu > \lambda\eta e^{\phi_1^*}$ and ϕ_2^* satisfies $\mu < \lambda\eta e^{\phi_2^*}$. It is trivial to check

that the equilibrium $(\phi_1^*, 0)$ is a saddle and the equilibrium $(\phi_2^*, 0)$ is a center. Since the interval $[0, L]$ is bounded and the system (4.4) is Hamiltonian, by the standard phase plane analysis, we can readily show that for each L, there is a non-trivial solution of (4.4)-(4.5) which is a closed orbit. The non-trivial steady states are nested around the center $(\phi_2^*, 0)$.

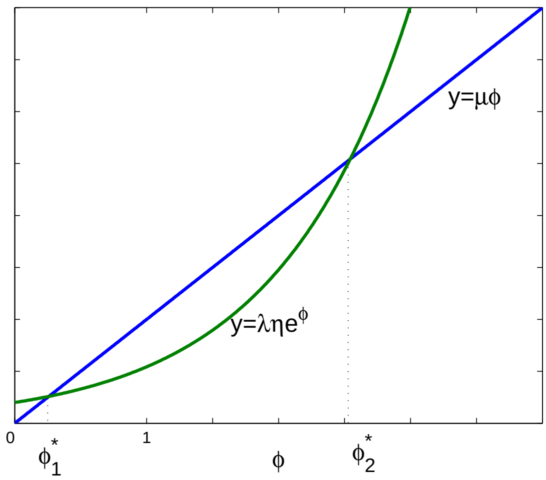

Figure 1: An illustration of the two solutions of the equation (4.6).

Therefore when $D_w > \gamma D_v$, there is a non-trivial smooth solution to the equation (4.3) which satisfies the Neumann boundary condition $\frac{\partial \phi}{\partial \nu} = 0$ at $x = 0, L$. Hence the steady state solution $u = \eta e^{\phi}$ exists. Substituting it into the second equation of (4.1) yields with (1.6)

$$\begin{aligned} -v_{xx} + \mu v &= \frac{\eta}{D_v} e^{\phi(x)}, \ 0 < x < L \\ \frac{\partial v}{\partial \nu} &= 0, \ x = 0 \text{ or } L \end{aligned} \tag{4.7}$$

which is a linear elliptic equation with Neumann boundary condition. Since the non-homogeneous term e^{ϕ} is smooth for $x \in (0, L)$, the smooth solution of (4.7) exists (e.g. see [2]). By the same argument, the solution w of the third equation of (4.1) with Neumann boundary condition can be obtained. In summary, we have the following theorem about the steady states of system (1.4) subject to the boundary condition (1.6).

Theorem 4.1. *Let $\Omega = (0, L)$. Assume $\frac{\beta}{D_v} = \frac{\delta}{D_w}$. If $D_w \le \gamma D_v$, the system (1.4) with Neumann boundary conditions (1.6) has no non-trivial steady states. If $D_w > \gamma D_v$, then a non-trivial steady state of the system (1.4) subject to (1.6) exists for each $L > 0$.*

From the above analysis and results, we see that the existence of non-trivial steady states of (1.4) depends on the sign of parameter λ which relates the diffusion coefficients D_v and D_w. Hence if other parameters are fixed, the relative diffusivity of the chemo-attractant to chemo-repellent play a prominent role in determining the nature of the steady states.

5 Summary

In this paper, we establish the existence of global classical solutions and steady states to an attraction-repulsion chemotaxis model in one dimension. Our result does not exclude the possibility that the solution may blow up at infinity time. From the analysis of steady state, we find that the existence of non-trivial steady states depends on the ratio of the chemo-attractant diffusion to the chemo-repellent diffusion. The existence of global solutions of the attraction-repulsion chemotaxis model in multi-dimensional spaces still remains open although it is more interesting to investigate. The pattern formation of the attraction-repulsion chemotaxis mode is also an interesting issue worthwhile to be studied in the future. In particular, the difference of the solution behavior between the classical Keller-Segel model (i.e. attraction chemotaxis model) and attraction-repulsion chemotaxis model needs to investigated both analytically and numerically.

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