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GLOBAL DYNAMICS AND DIFFUSION LIMIT OF A ONE-DIMENSIONAL REPULSIVE CHEMOTAXIS MODEL

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Abstract. In the first part of this paper, we investigate the qualitative behavior of classical solutions for a one-dimensional parabolic system derived from a repulsive chemotaxis model on bounded domains. It is shown that classical solutions to the initial-boundary value problem exist globally in time for large data and converge to constant equilibrium states exponentially in time. The results indicate that repulsive chemotaxis exhibits a strong tendency against pattern formation. In the second part, we study diffusion limit and convergence rate of the model toward a non-diffusive problem studied in [11]. It is shown that when the chemical diffusion coefficient ε tends to zero, the solution is convergent in L^∞-norm with respect to ε at order $O(\varepsilon)$.

1. **Introduction.** In contrast to diffusion (random diffusion without orientation), chemotaxis is the biased movement of cells/particles toward the region that contains higher concentration of beneficial or lower concentration of unfavorable chemicals. The former often refers to the attractive chemotaxis and latter to the repulsive chemotaxis. Well known examples of biological species experiencing chemotaxis include the slime mold amoebae *Dictyostelium discoideum*, the flagellated bacteria *Escherichia coli* and *Salmonella typhimurium*, and the human endothelial cells [17].

The prototype of the population-based chemotaxis model was proposed by Keller and Segel in the 1970s [7] to describe the aggregation of cellular slime molds *Dictyostelium discoideum* in response to the chemical cyclic adenosine monophosphate (cAMP). In its general form, Keller-Segel model reads

$$\begin{cases} u_t = \nabla \cdot (D\nabla u - \chi u \nabla \phi(v)), \\ v_t = \varepsilon \Delta v + g(u,v) \end{cases} \tag{1.1}$$

where u and v denote the cell density and chemical concentration, respectively. $D > 0$ and $\varepsilon \geq 0$ are cell and chemical diffusion coefficients, respectively. The chemotaxis is called to be attractive if $\chi > 0$ and repulsive if $\chi < 0$, where $|\chi|$

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measures the strength of the chemical signal. Here $\phi(v)$ is referred to as the chemotactic potential function describing the signal detection mechanism and $g(u,v)$ is a function characterizing the chemical growth and degradation.

The typical examples of chemotactic potential function ϕ includes $\phi(v) = kv$ (linear law), $\phi(v) = k\log v$ (logarithmic law), or $\phi(v) = kv^m/(1+v^m)$ (receptor law), where $k>0$ and $m\in\mathbb{N}$. The system with linear law $\phi(v)=kv$ and $g(u,v) = u-v$ was called the minimum chemotaxis model following the nomenclature of [3], see a review article [6] for the mathematical results of the minimum model. The logarithmic sensitivity $\phi(v) = k\log v$ adapted the Weber-Fechner law to describe cell chemotactic response and had prominent specific applications [8, 2, 1]. The steady states of (1.1) with logarithmic sensitivity and $g(u,v) = u-v$ was studied in [16] and existence of global solutions was recently obtained in [24]. The receptor sensitivity law has been derived and applied in numerous models for chemotaxis, e.g. see [19, 20] and references therein.

In this paper, we shall consider the model (1.1) with logarithmic chemotactic potential function and $g(u,v) = uv-\mu v$. The resulting model reads

$$\begin{cases} u_t = \nabla\cdot(D\nabla u - \chi u\nabla\ln(v)), \\ v_t = \varepsilon\Delta v + uv - \mu v \end{cases} \tag{1.2}$$

where $\mu>0$ is a constant accounting for the degradation rate of the chemical. When the chemical diffusion is neglected (i.e. $\varepsilon=0$), the model (1.2) was the same to the one proposed in [9, 18] as an example of reinforced random walks describing the interaction of particle with a non-diffusive chemical. Based on special choices of initial date and by the asymptotic analysis, a detailed qualitative and numerical analysis was provided in [9] where explicit solutions about aggregation, blow up and collapse were constructed in one-dimensional space. The local and global existence of solutions was subsequently studied in [25].

If $\chi<0$, with a Hopf-Cole type transformation

$$\mathbf{q} = \frac{\nabla v}{v} = \nabla\ln(v) \tag{1.3}$$

and scalings $\tilde{t} = -\chi t/D, \tilde{x} = \sqrt{-\chi}\mathbf{x}, \tilde{q} = \mathbf{q}/\sqrt{-\chi}$, (1.2) can be converted into a system of conservation laws as follows

$$\begin{cases} p_t - \nabla\cdot(p\mathbf{q}) = \Delta p, \\ \mathbf{q}_t - \nabla(\varepsilon\mathbf{q}^2 + p) = \varepsilon\Delta\mathbf{q} \end{cases} \tag{1.4}$$

where $q=u$. The existence and nonlinear stability of traveling wave solutions of (1.4) in one-dimensional domain $\mathbb{R}$ were established in [23, 12, 13] first for $\varepsilon=0$ and then in [14] for $\varepsilon>0$, where the wave strength is allowed to be arbitrary large. When $\varepsilon=0$, global existence of classical solutions for the initial-boundary value problem of (1.4) in one dimension was established in [26] and the Cauchy problem of (1.4) was studied in [5]. Furthermore the Cauchy problem of (1.4) with $\varepsilon=0$ in multi-dimensional spaces was investigated in [10]. Recently the large-time behavior of classical solutions for the initial-boundary value problem of (1.4) with $\varepsilon=0$ in one space dimension with large initial data and in multi-dimensional spaces for small initial data were established in [11].

In the present paper, we shall consider the initial-boundary value problem of (1.4) in one dimension space with $\varepsilon > 0$, namely

$$\begin{cases} p_t - (pq)_x = p_{xx}, \ x \in (0,1), \ t > 0, \\ q_t - (\varepsilon q^2 + p)_x = \varepsilon q_{xx}, \ x \in (0,1), \ t > 0, \\ (p,q)(x,0) = (p_0, q_0)(x), \ x \in [0,1], \\ p_x|_{x=0,x=1} = q|_{x=0,x=1} = 0, \ t > 0. \end{cases} \tag{1.5}$$

We shall develop the estimates in [11] to establish the large-time behavior of classical solutions for $\varepsilon > 0$ in one space dimension with large initial data and the diffusion limits of solutions as $\varepsilon \to 0$. We note that the diffusion limit of traveling wave solutions was previously obtained in [15]. To present our main results, we introduce some notations.

Notation. Throughout this paper, $\|\cdot\|$, $\|\cdot\|_\infty$ and $\|\cdot\|_{H^s}$ denote the norms of the usual Lebesgue measurable function spaces L^2, L^∞ and the usual Hilbert space H^s, respectively. The function spaces under consideration are $C([0,T]; H^s)$ and $L^2([0,T]; H^s)$, equipped with norms $\sup_{0\le t\le T} \|f(\cdot,t)\|_{H^s}$ and $(\int_0^T \|f(\cdot,t)\|_{H^s}^2)^{1/2}$, respectively. Unless specified, C and C_i will denote generic constants which are independent of the unknown functions and time. The values of the constants may vary line by line according to the context.

Our first result is concerned with the asymptotic behavior of classical solutions to (1.5) with *large data*.

Theorem 1.1 (Global dynamics of large solutions)**.** *Consider the initial-boundary value problem (1.5). Suppose that the initial data satisfy $p_0(x) \ge 0$, $\bar{p} = \int_0^1 p_0(x)dx > 0$ and the compatibility conditions: $\partial_x p_0|_{x=0,x=1} = q_0|_{x=0,x=1} = 0$. If $(p_0, q_0) \in H^2([0,1])$ and $\varepsilon > 0$ is small, then there exists a unique global classical solution (p,q) to (1.5) such that*

$$\|(p-\bar{p})(t)\|_{H^2}^2 + \|q(t)\|_{H^2}^2 + \int_0^t \big(\|(p-\bar{p})(\tau)\|_{H^3}^2 + \|q(\tau)\|_{H^2}^2 + \varepsilon\|q(\tau)\|_{H^3}^2\big) d\tau \le C, \quad \forall\, t \ge 0,$$

for some constant $C > 0$ which is independent of t and ε. Moreover, there exist constants $\alpha, \beta > 0$ which are independent of t and ε such that

$$\|(p-\bar{p})(t)\|_{H^2}^2 + \|q(t)\|_{H^2}^2 \le \alpha\big(\|p_0 - \bar{p}\|_{H^2}^2 + \|q_0\|_{H^2}^2\big) e^{-\beta t}, \quad \forall\, t \ge 0.$$

Remark 1. It is generally believed that diffusion has a stabilization/regularization effect. So we expect that the results for $\varepsilon = 0$ can be extended to the case $\varepsilon > 0$. However, the appearance of the convection-like term $(q^2)_x$ brings additional difficulty to the asymptotic analysis. In general, such kind of nonlinearity does not cause any trouble for small solutions, while the scenario is totally different for large amplitude solutions. Indeed, when ε is large, the nonlinear convection can no longer be dominated by the linear diffusion, and the resulting energy estimates are time-dependent which yield no information about the long-time behavior of the solution. This is the main reason that we require ε to be small. Within this regime of the parameter, by adopting the idea in [11] we can establish the uniform-in-time *and* ε-independent energy estimates which lead to the long-time asymptotic behavior of large amplitude solutions, and the uniform convergence rate of the solutions of (1.5) toward those of the non-diffusive problem.

In [11], global well-posedness and long-time behavior of classical solutions to the non-diffusive problem is established. Our next theorem is concerned with the diffusion limit of the solution of (1.5) when $\varepsilon \to 0$, and convergence rate of (1.5) toward the non-diffusive problem.

Theorem 1.2 (Diffusion limit and convergence rate). *Let the conditions of Theorem 1.1 hold. Let $(p^\varepsilon, q^\varepsilon)$ and (p, q) be the unique classical solutions to (1.5) with $\varepsilon > 0$ and $\varepsilon = 0$, respectively, with the same initial data. Then there exists a constant $C > 0$ which is independent of t and ε such that for any $t > 0$*

$$\|(p^\varepsilon - p)(t)\|_{H^1}^2 + \|(q^\varepsilon - q)(t)\|_{H^1}^2 + \int_0^t \left(\|(p^\varepsilon - p)(\tau)\|_{H^2}^2 + \|(q^\varepsilon - q)(\tau)\|_{H^1}^2\right) d\tau \le C\varepsilon,$$

and

$$\|(p^\varepsilon - p)(t)\|_{L^\infty}^2 + \|(q^\varepsilon - q)(t)\|_{L^\infty}^2 \le C\varepsilon.$$

Then applying Theorems 1.1 and 1.2, we have the following results for the original chemotaxis model (1.2).

Corollary 1 (Long-time dynamics). *Consider the following initial-boundary value problem for the one-dimensional chemotaxis model (1.2)*

$$\begin{cases} u_t = (Du_x - \chi u \ln(v)_x)_x, \ x \in (0,1), t > 0, \\ v_t = \varepsilon v_{xx} + uv - \mu v, \ x \in (0,1), \ t > 0, \\ (u, v)(x, 0) = (u_0, v_0)(x), \ x \in [0,1], \\ u_x|_{x=0,x=1} = v_x|_{x=0,x=1} = 0, \ t > 0, \end{cases} \tag{1.6}$$

where $\chi < 0$, $\mu \ge 0$ and $\varepsilon > 0$ are constants. Suppose that the initial data satisfy $u_0(x) \ge 0$, $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx > 0$, $v_0(x) > 0$ and the compatibility conditions: $\partial_x u_0|_{x=0,x=1} = \partial_x v_0|_{x=0,x=1} = 0$. Assume further that $(u_0 - \bar{u}) \in H^2([0,1])$, $\ln(v_0) \in H^3([0,1])$ and ε is small. Then there exists a unique global-in-time classical solution (u, v) to (1.6) such that as $t \to \infty$:

$$\|u(t) - \bar{u}\|_{L^\infty} \to 0,$$

and

$$\|v(t)\|_{L^\infty} \to \begin{cases} 0, & \text{if } \bar{u} < \mu \\ +\infty, & \text{if } \bar{u} > \mu, \end{cases}$$

where the convergence rates are exponential in time.

Remark 2. Corollary 1 indicates that in the process of repulsive chemotaxis, the cell population collapses to its initial average over the domain, while the chemical concentration will vanish if the initial average of the cell population is below the chemical decay rate μ, or it will aggregate if the average exceeds this rate number, as time goes to infinity. The results indicate that repulsive chemotaxis exhibits a strong tendency *against* pattern formation, which is consistent with general results for the classical repulsive chemotaxis models where the chemotactic potential function is linear with respect to chemical concentration, see [4, 22].

Corollary 2 (Diffusion limit). *Let the conditions of Corollary 1 hold. Let $(u^\varepsilon, v^\varepsilon)$ and (u, v) be the unique classical solutions to (1.6) with $\varepsilon > 0$ and $\varepsilon = 0$, respectively, with the same initial data. Then it holds that for any $t > 0$*

$$\|(u^\varepsilon - u)(t)\|_{L^\infty}^2 \le C_1\varepsilon$$

and

$$\|(v^\varepsilon - v)(t)\|_{L^\infty}^2 \le C_2(t)\varepsilon,$$

where the constant $C_1 > 0$ is independent of t and ε and $C_2(t) > 0$ is independent of ε.

The rest of the paper is organized as follows. We give the proof of Theorem 1.1 in Section 2 and study the diffusion limit and convergence rate of (1.5) in Section 3. Corollary 1 and Corollary 2 will be proved in Section 4. Section 5 is devoted to the numerical illustrations of the analytical results obtained in previous sections.

2. **Long-time dynamics of large solutions (proof of Theorem 1.1).** In this section we study the long-time dynamics of classical solutions to (1.5) with large data. We will adopt the energy framework developed in [11] by employing the free energy formulation associated with (1.5) and accurate energy estimates. Let us consider the initial-boundary value problem:

$$\begin{cases} p_t = p_{xx} + (pq)_x, \\ q_t = \varepsilon q_{xx} + \varepsilon (q^2)_x + p_x; \end{cases} \tag{2.1}$$

and

$$\begin{cases} (p,q)(x,0) = (p_0,q_0)(x) \in H^2((0,1)), \\ p_x|_{x=0,1} = q|_{x=0,1} = 0, \\ p_0(x) \geq 0, \quad \bar{p} = \displaystyle\int_0^1 p_0(x)dx > 0, \end{cases} \tag{2.2}$$

where $q = [\ln(e^{\mu t} v)]_x$.

First of all, using standard arguments, such as fixed point argument, one can show that there exists a unique local solution (p,q) to (1.5) such that $(p - \bar{p}, q) \in C([0,t_0); H^2) \cap L^2([0,t_0); H^3)$, and $p(x,t) \geq 0$ for $(x,t) \in (0,1) \times [0,t_0)$ for some finite $t_0 > 0$ under the assumption $(p_0,q_0) \in H^2$. The precise and detailed proof has been given in [21] and we omit the proof here for brevity. Next, we show the *a priori* estimates of the local solution in order to extend it to a global one. We remark that the *a priori* estimates presented below involve a logarithmic function of p. In [11], a regularization procedure (lifting the initial datum p_0 by a small parameter ϵ) was applied to overcome the singularity at $p = 0$ in order to employ the free energy formulation associated with the system. It was further shown in [11] that the *a priori* estimates are completely independent of the regularization parameter ϵ. Then by taking the ϵ limit the solution to the original equations was obtained. We remark that the same procedure applies here. It will become clear that, as the proof proceeds, the *a priori* estimates are independent of the lower bound of the function p. This will allow us to take the limit of the sequence of approximate solutions in order to obtain the one for the original problem (1.5). However, to simplify the presentation, we shall not go through the details of the regularization procedure in this paper.

We now recall an elementary lemma (see e.g. [11]) which will play an important role in getting the lower order estimate of the solution.

Lemma 2.1. *Let $\rho \geq 0$ and $0 < a \leq \rho_* < \infty$ for some constant a. Then there exists a constant $d > 0$ depending on ρ_* such that*

$$0 \leq [\rho \ln(\rho) - \rho] - [\rho_* \ln(\rho_*) - \rho_*] - (\rho - \rho_*) \ln(\rho_*) \leq d(\rho - \rho_*)^2.$$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. The proof is split into several steps of careful energy estimates.

Step 1. *Free energy dissipation.* Due to the conservation of total mass, after taking L^2 inner product of $(2.1)_1$ with $\ln(p)-\ln(\bar{p})$, we have

$$\frac{d}{dt}\left(\int_0^1 \eta(p)-\eta(\bar{p})-\eta'(\bar{p})(p-\bar{p})dx\right)+\int_0^1\left(p_xq+\frac{(p_x)^2}{p}\right)dx=0, \tag{2.3}$$

where $\eta(z)=z\ln(z)-z$ which is a convex function. Taking L^2 inner product of $(2.1)_2$ with q, we have

$$\frac{1}{2}\frac{d}{dt}\|q\|^2-\int_0^1 p_xq dx+\varepsilon\|q_x\|^2=0. \tag{2.4}$$

Adding (2.4) to (2.3), we get

$$\frac{d}{dt}\left(\int_0^1 \eta(p)-\eta(\bar{p})-\eta'(\bar{p})(p-\bar{p})dx+\frac{1}{2}\|q\|^2\right)+\int_0^1\frac{(p_x)^2}{p}dx+\varepsilon\|q_x\|^2=0. \tag{2.5}$$

Integrating (2.5) over $[0,t]$, we have

$$\begin{aligned}&\left(\int_0^1 \eta(p)-\eta(\bar{p})-\eta'(\bar{p})(p-\bar{p})dx+\frac{1}{2}\|q\|^2\right)(t)+\int_0^t\left(\int_0^1\frac{(p_x)^2}{p}dx+\varepsilon\|q_x\|^2\right)d\tau\\&=\left(\int_0^1 \eta(p_0)-\eta(\bar{p})-\eta'(\bar{p})(p_0-\bar{p})dx+\frac{1}{2}\|q_0\|^2\right).\end{aligned} \tag{2.6}$$

Due to the convexity of $\eta(\cdot)$ and the positivity of p, we have

$$\int_0^1 \eta(p)-\eta(\bar{p})-\eta'(\bar{p})(p-\bar{p})dx\geq 0.$$

On the other hand, since $0<\bar{p}<+\infty$ and $p_0\geq 0$, by Lemma 2.1 we have

$$\int_0^1 \eta(p_0)-\eta(\bar{p})-\eta'(\bar{p})(p_0-\bar{p})dx\leq d_1\|p_0-\bar{p}\|^2,$$

where the constant d_1 depends only on $\bar{p}$. Thus, (2.6) implies that

$$\frac{1}{2}\|q(\cdot,t)\|^2+\int_0^t\left(\int_0^1\frac{(p_x)^2}{p}dx+\varepsilon\|q_x\|^2\right)d\tau\leq\frac{1}{2}\|q_0\|^2+d_1\|p_0-\bar{p}\|^2. \tag{2.7}$$

Step 2. L^2 *estimate.* To carry out further energy estimates and the asymptotic analysis, we first reformulate the problem (2.1). Let $\tilde{p}=p-\bar{p}$. Substituting $\tilde{p}$ into (2.1), we have

$$\begin{cases}\tilde{p}_t-(\tilde{p}q)_x-\bar{p}q_x=\tilde{p}_{xx},\\ q_t-\tilde{p}_x=\varepsilon q_{xx}+\varepsilon(q^2)_x.\end{cases} \tag{2.8}$$

Taking L^2 inner products of $(2.8)_1$ with $\tilde{p}$ and $(2.8)_2$ with $\bar{p}q$ and adding the results, we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|\tilde{p}\|^2+\frac{\bar{p}}{2}\|q\|^2\right)+\|\tilde{p}_x\|^2+\varepsilon\bar{p}\|q_x\|^2&=-\int_0^1\tilde{p}q\tilde{p}_x dx\\&\leq\frac{1}{2}\|\tilde{p}_x\|^2+\frac{1}{2}\|\tilde{p}q\|^2\\&\leq\frac{1}{2}\|\tilde{p}_x\|^2+\frac{1}{2}\|\tilde{p}\|_\infty^2\|q\|^2\\&\leq\frac{1}{2}\|\tilde{p}_x\|^2+\left(\frac{1}{2}\|q_0\|^2+d_1\|p_0-\bar{p}\|^2\right)\|\tilde{p}\|_\infty^2,\end{aligned}\tag{2.9}$$

where we have used (2.7) for $\|q(\cdot,t)\|^2$.

For the estimate of $\|\tilde{p}\|_\infty^2$, we observe that

$$|\tilde{p}|=|p-\bar{p}|=\left|\int_{x_t^*}^x\tilde{p}_x dx\right|\leq\left(\int_0^1 p dx\right)^{\frac{1}{2}}\left(\int_0^1\frac{(\tilde{p}_x)^2}{p}dx\right)^{\frac{1}{2}}=\sqrt{\bar{p}}\left(\int_0^1\frac{(\tilde{p}_x)^2}{p}dx\right)^{\frac{1}{2}},$$

where $x_t^*\in[0,1]$ such that $p(x_t^*,t)=\bar{p}$ for any $t\geq 0$. Then we have

$$\|\tilde{p}\|_\infty^2\leq\bar{p}\left(\int_0^1\frac{(\tilde{p}_x)^2}{p}dx\right).\tag{2.10}$$

Substituting (2.10) into (2.9), we have

$$\frac{d}{dt}\left(\frac{1}{2}\|\tilde{p}\|^2+\frac{\bar{p}}{2}\|q\|^2\right)+\frac{1}{2}\|\tilde{p}_x\|^2+\varepsilon\bar{p}\|q_x\|^2\leq C_1\left(\int_0^1\frac{(\tilde{p}_x)^2}{p}dx\right),\tag{2.11}$$

where

$$C_1=\left(\frac{\bar{p}}{2}\|q_0\|^2+d_1\bar{p}\|p_0-\bar{p}\|^2\right).$$

Then the operation $(2.5)\times 2C_1+(2.11)$ gives

$$\frac{d}{dt}G(t)+K(t)\leq 0,\tag{2.12}$$

where

$$\begin{aligned}G(t)&=2C_1\left(\int_0^1\eta(p)-\eta(\bar{p})-\eta'(\bar{p})(p-\bar{p})dx+\frac{1}{2}\|q\|^2\right)+\left(\frac{1}{2}\|\tilde{p}\|^2+\frac{\bar{p}}{2}\|q\|^2\right),\\K(t)&=C_1\left(\int_0^1\frac{(\tilde{p}_x)^2}{p}dx\right)+\frac{1}{2}\|\tilde{p}_x\|^2+\varepsilon(\bar{p}+2C_1)\|q_x\|^2.\end{aligned}$$

Upon integrating (2.12) in time, we have

$$\|\tilde{p}(\cdot,t)\|^2+\int_0^t\|\tilde{p}_x(\cdot,\tau)\|^2+\varepsilon\|q_x(\cdot,\tau)\|^2 d\tau\leq C_2.\tag{2.13}$$

We remark that the constant C_2 depends only on $\bar{p}$ and initial data, but not on ε.

Taking spatial derivative of $(2.8)_2$ and using equation $(2.8)_1$, we have

$$q_{xt}=-(\tilde{p}q)_x-\bar{p}q_x+\tilde{p}_t+\varepsilon q_{xxx}+\varepsilon(q^2)_{xx}.\tag{2.14}$$

Taking L^2 inner product of (2.14) with q_x, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|q_x\|^2\right)+\bar{p}\|q_x\|^2+\varepsilon\|q_{xx}\|^2\\
=&-\int_0^1(\tilde{p}q)_xq_xdx+\int_0^1\tilde{p}_tq_xdx+\varepsilon\int_0^1(q^2)_{xx}q_xdx\\
=&-\int_0^1(\tilde{p}q)_xq_xdx+\frac{d}{dt}\int_0^1\tilde{p}q_xdx-\int_0^1\tilde{p}q_{xt}dx-\varepsilon\int_0^1(q^2)_xq_{xx}dx\\
=&-\int_0^1(\tilde{p}q)_xq_xdx+\frac{d}{dt}\int_0^1\tilde{p}q_xdx+\|\tilde{p}_x\|^2+\varepsilon\int_0^1[q_{xx}+(q^2)_x]\tilde{p}_xdx-2\varepsilon\int_0^1qq_xq_{xx}dx,
\end{aligned}$$

where we have used the relationship $q_{xt}=\tilde{p}_{xx}+\varepsilon q_{xxx}+\varepsilon(q^2)_{xx}$. After rearranging terms, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|q_x\|^2-\int_0^1\tilde{p}q_xdx\right)+\bar{p}\|q_x\|^2+\varepsilon\|q_{xx}\|^2\\
=&-\int_0^1(\tilde{p}q)_xq_xdx+\varepsilon\int_0^1[q_{xx}+2qq_x]\tilde{p}_xdx-2\varepsilon\int_0^1qq_xq_{xx}dx+\|\tilde{p}_x\|^2.
\end{aligned}$$

Next we estimate the first three terms on the right hand side of the above equation as follows

$$-\int_0^1(\tilde{p}q)_xq_xdx\le\frac{\bar{p}}{2}\|q_x\|^2+\frac{1}{\bar{p}}\left(\|\tilde{p}q_x\|^2+\|q\tilde{p}_x\|^2\right),$$

$$\begin{aligned}
\varepsilon\int_0^1[q_{xx}+2qq_x]\tilde{p}_xdx&\le\frac{\varepsilon}{8}\|q_{xx}\|^2+2\varepsilon\|\tilde{p}_x\|^2+\varepsilon\|q\|^2\|q_x\|_\infty^2+\varepsilon\|\tilde{p}_x\|^2\\
&\le\frac{\varepsilon}{8}\|q_{xx}\|^2+3\varepsilon\|\tilde{p}_x\|^2+2\varepsilon(\|q_0\|^2+d_1\|p_0-\bar{p}\|^2)\|q_x\|\|q_{xx}\|\\
&\le\frac{\varepsilon}{4}\|q_{xx}\|^2+3\varepsilon\|\tilde{p}_x\|^2+8\varepsilon(\|q_0\|^2+d_1\|p_0-\bar{p}\|^2)^2\|q_x\|^2,
\end{aligned}$$

and

$$\begin{aligned}
-2\varepsilon\int_0^1qq_xq_{xx}dx&\le\frac{\varepsilon}{8}\|q_{xx}\|^2+8\varepsilon\|q\|^2\|q_x\|_\infty^2\\
&\le\frac{\varepsilon}{4}\|q_{xx}\|^2+512\varepsilon(\|q_0\|^2+d_1\|p_0-\bar{p}\|^2)^2\|q_x\|^2,
\end{aligned}$$

where we have used (2.7) for the estimate of $\|q\|^2$, the fact that $\|q_x\|_\infty^2\le2\|q_x\|\|q_{xx}\|$ due to $\int_0^1q_xdx=0$, and the Cauchy-Schwarz inequality at various places. Therefore we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|q_x\|^2-\int_0^1\tilde{p}q_xdx\right)+\frac{\bar{p}}{2}\|q_x\|^2+\frac{\varepsilon}{2}\|q_{xx}\|^2\\
\le&\frac{1}{\bar{p}}\left(\|\tilde{p}q_x\|^2+\|q\tilde{p}_x\|^2\right)+(3\varepsilon+1)\|\tilde{p}_x\|^2+520\varepsilon(\|q_0\|^2+d_1\|p_0-\bar{p}\|^2)^2\|q_x\|^2.
\end{aligned}$$

When $\varepsilon\le\min\left\{1/12,\bar{p}\left(2080(\|q_0\|^2+d_1\|p_0-\bar{p}\|^2)^2\right)^{-1}\right\}$, it holds that

$$\frac{d}{dt}\left(\frac{1}{2}\|q_x\|^2-\int_0^1\tilde{p}q_xdx\right)+\frac{\bar{p}}{4}\|q_x\|^2+\frac{\varepsilon}{2}\|q_{xx}\|^2\le\frac{1}{\bar{p}}\left(\|\tilde{p}q_x\|^2+\|q\tilde{p}_x\|^2\right)+\frac{5}{4}\|\tilde{p}_x\|^2. \tag{2.15}$$

We notice that by multiplying (2.12) by 4 and adding the result to (2.15), it holds that

$$\frac{d}{dt}L(t) + M(t) \le \frac{1}{\bar{p}}\big(\|\tilde{p}q_x\|^2 + \|q\tilde{p}_x\|^2\big), \tag{2.16}$$

where

$$\begin{aligned} L(t) &= 4G(t) + \left(\frac{1}{2}\|q_x\|^2 - \int_0^1 \tilde{p}q_x dx\right) \\ &= 8C_1\left(\int_0^1 \eta(p) - \eta(\bar{p}) - \eta'(\bar{p})(p-\bar{p})dx + \frac{1}{2}\|q\|^2\right) \\ &\quad + \frac{1}{4}\|q_x\|^2 + \int_0^1 \left(\frac{1}{2}q_x - \tilde{p}\right)^2 dx + \|\tilde{p}\|^2 + 2\bar{p}\|q\|^2, \\ M(t) &= 4K(t) + \frac{\bar{p}}{4}\|q_x\|^2 + \frac{\varepsilon}{2}\|q_{xx}\|^2 - \frac{5}{4}\|\tilde{p}_x\|^2 \\ &= 4C_1\left(\int_0^1 \frac{(\tilde{p}_x)^2}{p}\right) + \frac{3}{4}\|\tilde{p}_x\|^2 + \frac{\bar{p}}{4}\|q_x\|^2 + \frac{\varepsilon}{2}\|q_{xx}\|^2 + 4\varepsilon(\bar{p}+2C_1)\|q_x\|^2. \end{aligned} \tag{2.17}$$

The right hand side of (2.16) can be estimated as

$$\begin{aligned} \|\tilde{p}q_x\|^2 + \|q\tilde{p}_x\|^2 &\le \|\tilde{p}\|_\infty^2\|q_x\|^2 + \|q\|_\infty^2\|\tilde{p}_x\|^2 \\ &\le 2\big(\|\tilde{p}\|\|\tilde{p}_x\|\|q_x\|^2 + \|q\|\|q_x\|\|\tilde{p}_x\|^2\big) \\ &\le C_3\|\tilde{p}_x\|\|q_x\|\big(\|q_x\| + \|\tilde{p}_x\|\big) \\ &\le C_4(\delta)\|\tilde{p}_x\|^2\|q_x\|^2 + \delta\big(\|\tilde{p}_x\|^2 + \|q_x\|^2\big) \end{aligned} \tag{2.18}$$

for any $\delta > 0$, where we have used the Sobolev inequality $\|f\|_\infty^2 \le 2\|f\|\|f_x\|$ for $f : I = [a,b] \to \mathbb{R}$ satisfying $f|_{\partial I} = 0$ or $\bar{f}_I = 0$, and the uniform estimates of $\|q\|^2$ and $\|\tilde{p}\|^2$ due to (2.7) and (2.13), respectively. By choosing δ small enough, we have from (2.16)

$$\frac{d}{dt}L(t) + N(t) \le C_5\|\tilde{p}_x\|^2\|q_x\|^2 \le 4C_5\|\tilde{p}_x\|^2 L(t), \tag{2.19}$$

where

$$N(t) = 4C_1\int_0^1 \frac{(\tilde{p}_x)^2}{p}dx + \frac{1}{2}\|\tilde{p}_x\|^2 + \frac{\bar{p}}{8}\|q_x\|^2 + \frac{\varepsilon}{2}\|q_{xx}\|^2 + 4\varepsilon(\bar{p}+2C_1)\|q_x\|^2.$$

Applying Gronwall's inequality to (2.19) and using the uniform estimate (2.13), we have

$$\|q_x(\cdot,t)\|^2 + \int_0^t \|q_x(\cdot,\tau)\|^2 + \varepsilon\|q_x(\cdot,\tau)\|_{H^1}^2 d\tau \le C_6. \tag{2.20}$$

Substituting (2.20) back to (2.19), we have

$$\frac{d}{dt}L(t) + N(t) \le C_7\|\tilde{p}_x\|^2. \tag{2.21}$$

We remark that the constants $C_3, ..., C_7$ are independent of t and ε.

Multiplying (2.12) by $4C_7$ and combining the resulting inequality with (2.21) we have

$$\frac{d}{dt}R(t) + S(t) \le 0, \tag{2.22}$$

where

$$\begin{aligned}R(t) &= 4C_7G(t) + L(t)\\&= 8C_1(C_7+1)\bigg(\int_0^1 \eta(p) - \eta(\bar p) - \eta'(\bar p)(p-\bar p)dx + \frac{1}{2}\|q\|^2\bigg)\\&\quad + (2C_7+1)\|\tilde p\|^2 + 2(C_7+1))\bar p\|q\|^2 + \frac{1}{4}\|q_x\|^2 + \int_0^1 \left(\frac{1}{2}q_x - \tilde p\right)^2 dx\\S(t) &= 4C_7K(t) - C_7\|\tilde p_x\|^2 + N(t)\\&= 4C_1(C_7+1)\int_0^1 \frac{(\tilde p_x)^2}{p}dx + \left(C_7 + \frac{1}{2}\right)\|\tilde p_x\|^2 + \frac{\bar p}{8}\|q_x\|^2\\&\quad + 4(C_7+1)\varepsilon(\bar p + 2C_1)\|q_x\|^2 + \frac{\varepsilon}{2}\|q_{xx}\|^2.\end{aligned}$$

Step 3. H^1 *estimate.* Taking spatial derivatives of (2.8), we have

$$\begin{cases}\tilde p_{xt} - (\tilde p q)_{xx} - \bar p q_{xx} = \tilde p_{xxx},\\ q_{xt} - \tilde p_{xx} = \varepsilon q_{xxx} + \varepsilon(q^2)_{xx}.\end{cases} \tag{2.23}$$

Taking L^2 inner products of $(2.23)_1$ with $\tilde p_x$ and $(2.23)_2$ with $\bar p q_x$, adding the results, we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|\tilde p_x\|^2 + \frac{\bar p}{2}\|q_x\|^2\right) + \frac{1}{2}\|\tilde p_{xx}\|^2 + \varepsilon\bar p\|q_{xx}\|^2\\\le&\|\tilde p q_x\|^2 + \|q\tilde p_x\|^2 - 2\varepsilon\bar p\int_0^1 q q_x q_{xx}dx\\\le&C_8\|\tilde p_x\|^2\|q_x\|^2 + \left(\|\tilde p_x\|^2 + \|q_x\|^2\right) + \frac{\varepsilon\bar p}{2}\|q_{xx}\|^2 + 2\varepsilon\bar p\|q\|_\infty^2\|q_x\|^2\\\le&C_8\|\tilde p_x\|^2\|q_x\|^2 + \left(\|\tilde p_x\|^2 + \|q_x\|^2\right) + \frac{\varepsilon\bar p}{2}\|q_{xx}\|^2\\&\quad + 4\varepsilon\bar p[C_6(\|q_0\|^2 + 2d_1\|p_0 - \bar p\|^2)]^{1/2}\|q_x\|^2\\\le&C_9\left(\|\tilde p_x\|^2 + \|q_x\|^2\right) + \frac{\varepsilon\bar p}{2}\|q_{xx}\|^2,\end{aligned} \tag{2.24}$$

where we have used similar argument as in deriving (2.18), the Sobolev inequality: $\|q\|_\infty^2 \le 2\|q\|\|q_x\|$ since $\int_0^1 q dx = 0$, and the uniform estimates of $\|q\|$ and $\|q_x\|^2$ obtained from (2.7) and (2.20), respectively. From above estimate we have

$$\frac{d}{dt}\left(\frac{1}{2}\|\tilde p_x\|^2 + \frac{\bar p}{2}\|q_x\|^2\right) + \frac{1}{2}\|\tilde p_{xx}\|^2 + \frac{\varepsilon\bar p}{2}\|q_{xx}\|^2 \le C_9\left(\|\tilde p_x\|^2 + \|q_x\|^2\right). \tag{2.25}$$

Multiplying (2.22) by $\frac{2C_9}{\min\{(C_7+\frac{1}{2}),\frac{\bar p}{8}\}}$ and combining the resulting inequality with (2.25) we have

$$\frac{d}{dt}V(t) + W(t) \le 0, \tag{2.26}$$

where

$$
\begin{aligned}
V(t) =& \frac{2C_9}{\min\{(C_7+\frac{1}{2}),\frac{\bar{p}}{8}\}}R(t) + \frac{1}{2}\|\tilde{p}_x\|^2 + \frac{\bar{p}}{2}\|q_x\|^2,\\
W(t) =& \frac{2C_9}{\min\{(C_7+\frac{1}{2}),\frac{\bar{p}}{8}\}}S(t) - C_9\big(\|\tilde{p}_x\|^2 + \|q_x\|^2\big) + \frac{1}{2}\|\tilde{p}_{xx}\|^2 + \frac{\varepsilon\bar{p}}{2}\|q_{xx}\|^2\\
\geq& \frac{2C_9(4C_1C_7+4C_1)}{\min\{(C_7+\frac{1}{2}),\frac{\bar{p}}{8}\}}\left(\int_0^1 \frac{(\tilde{p}_x)^2}{p}dx\right) + C_9\big(\|\tilde{p}_x\|^2 + \|q_x\|^2\big) + \frac{1}{2}\|\tilde{p}_{xx}\|^2\\
&+ \frac{\varepsilon\bar{p}}{2}\|q_{xx}\|^2 + \frac{2C_9}{\min\{(C_7+\frac{1}{2}),\frac{\bar{p}}{8}\}}\left(4(C_7+1)\varepsilon(\bar{p}+2C_1)\|q_x\|^2 + \frac{\varepsilon}{2}\|q_{xx}\|^2\right).
\end{aligned}
$$

In particular, integrating (2.26) with respect to t we have

$$
\|\tilde{p}_x(t)\|^2 + \int_0^t \|\tilde{p}_{xx}(\tau)\|^2 d\tau \leq C_{10}. \tag{2.27}
$$

In view of (2.13) and (2.27) we see that $\|\tilde{p}\|_{H^1}^2 \leq C_{11}$, which gives $\|\tilde{p}\|_\infty^2 \leq C_{12}$. Thus, $\|p\|_\infty = \|\tilde{p}+\bar{p}\|_\infty$ is uniformly bounded from above for any $t > 0$. This implies that there exist uniform-in-time constants $d_2, d_3 > 0$ such that

$$
d_2\|\tilde{p}\|^2 \leq \int_0^1 \eta(p) - \eta(\bar{p}) - \eta'(\bar{p})(p-\bar{p})dx \leq d_3\|\tilde{p}\|^2. \tag{2.28}
$$

Hence there exist positive t-independent constants $C_{13}, ..., C_{16}$ such that

$$
\begin{aligned}
&C_{13}\left(\|\tilde{p}\|_{H^1}^2 + \|q\|_{H^1}^2\right) \leq V(t) \leq C_{14}\left(\|\tilde{p}\|_{H^1}^2 + \|q\|_{H^1}^2\right),\\
&C_{15}\left(\|\tilde{p}_x\|_{H^1}^2 + \|q_x\|^2 + \varepsilon\|q_x\|_{H^1}^2\right)\\
\leq& W(t) \leq C_{16}\left(\|\tilde{p}_x\|_{H^1}^2 + \int_0^1 \frac{(\tilde{p}_x)^2}{p}dx + \|q_x\|^2 + \varepsilon\|q_x\|_{H^1}^2\right).
\end{aligned} \tag{2.29}
$$

Due to the boundary conditions and Poincaré inequality: $\|f\| \leq \|f_x\|$ on $[0,1]$, we have

$$
\begin{aligned}
\frac{1}{2}\|\tilde{p}\|_{H^1}^2 \leq \|\tilde{p}_x\|^2 \leq \|\tilde{p}\|_{H^1}^2,\\
\frac{1}{2}\|q\|_{H^1}^2 \leq \|q_x\|^2 \leq \|q\|_{H^1}^2,
\end{aligned}
$$

which together with (2.26) and (2.29) imply the exponential decay of $V(t)$. Thus, we have

$$
\|\tilde{p}(t)\|_{H^1}^2 + \|q(t)\|_{H^1}^2 \leq C_{17}\big(\|\tilde{p}(0)\|_{H^1}^2 + \|q(0)\|_{H^1}^2\big)e^{-C_{18}t},
$$

for some positive constants C_{17} and C_{18}. We remark that the constants $C_8, ..., C_{18}$ are independent of t and ε.

Step 4. H^2 *estimate.* Differentiating system (2.8) twice in x, then taking inner product of the first equation of resulting system with $\tilde{p}_{xx}$ and second equation with $\bar{p}q_{xx}$, we obtain in a similar fashion as before

$$
\frac{d}{dt}\left(\frac{1}{2}\|\tilde{p}_{xx}\|^2 + \frac{\bar{p}}{2}\|q_{xx}\|^2\right) + \frac{1}{2}\|\tilde{p}_{xxx}\|^2 + \frac{\varepsilon\bar{p}}{2}\|q_{xxx}\|^2 \leq \frac{1}{2}\|(\tilde{p}q)_{xx}\|^2 + C_{19}\varepsilon\|q_{xx}\|^2. \tag{2.30}
$$

Similar to (2.15), we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|q_{xx}\|^2-\int_0^1\tilde{p}_x q_{xx}dx\right)+\frac{\bar{p}}{2}\|q_{xx}\|^2+\frac{\varepsilon}{2}\|q_{xxx}\|^2\\ \le&\frac{1}{2\bar{p}}\|(\tilde{p}q)_{xx}\|^2+\|\tilde{p}_{xx}\|^2+C_{20}\varepsilon\|q_{xx}\|^2.\end{aligned}\tag{2.31}$$

Multiplying (2.30) by 4 and combining the result with (2.31) we get

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|q_{xx}\|^2-\int_0^1\tilde{p}_x q_{xx}dx+2\|\tilde{p}_{xx}\|^2\right.\\ &\left.+2\bar{p}\|q_{xx}\|^2\right)+2\|\tilde{p}_{xxx}\|^2+\frac{\bar{p}}{2}\|q_{xx}\|^2+\varepsilon\left(2\bar{p}+\frac{1}{2}\right)\|q_{xxx}\|^2\\ \le&\left(2+\frac{1}{2\bar{p}}\right)\|(\tilde{p}q)_{xx}\|^2+\|\tilde{p}_{xx}\|^2+C_{21}\varepsilon\|q_{xx}\|^2.\end{aligned}$$

Using previously established estimates, we can show that

$$\begin{aligned}\|(\tilde{p}q)_{xx}\|^2&\le C_{22}\left(\|\tilde{p}_x\|_\infty^2\|q_x\|^2+\|\tilde{p}\|_\infty^2\|q_{xx}\|^2+\|q\|_\infty^2\|\tilde{p}_{xx}\|^2\right)\\ &\le C_{23}\left(\|\tilde{p}_x\|\|\tilde{p}_{xx}\|\|q_x\|^2+\|\tilde{p}\|\|\tilde{p}_x\|\|q_{xx}\|^2+\|q\|\|q_x\|\|\tilde{p}_{xx}\|^2\right)\\ &\le C_{24}\left(\|\tilde{p}_{xx}\|\|q_x\|^2+\|\tilde{p}_x\|\|q_{xx}\|^2+\|q_x\|\|\tilde{p}_{xx}\|^2\right)\\ &\le C_{25}\left(\|\tilde{p}_{xx}\|^2+\|q_x\|^2\right)+C_{26}(\delta)\|\tilde{p}_x\|^2\|q_{xx}\|^2+\delta\|q_{xx}\|^2.\end{aligned}\tag{2.32}$$

By choosing $\delta>0$ sufficiently small we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|q_{xx}\|^2-\int_0^1\tilde{p}_x q_{xx}dx\right.\\ &\left.+2\|\tilde{p}_{xx}\|^2+2\bar{p}\|q_{xx}\|^2\right)+2\|\tilde{p}_{xxx}\|^2+\frac{\bar{p}}{4}\|q_{xx}\|^2+\varepsilon\left(2\bar{p}+\frac{1}{2}\right)\|q_{xxx}\|^2\\ \le&C_{27}\left(\|\tilde{p}_{xx}\|^2+\|q_x\|^2+\varepsilon\|q_{xx}\|^2\right)+C_{28}\|\tilde{p}_x\|^2\|q_{xx}\|^2.\end{aligned}$$

We note that, by virtue of Poincaré inequality: $\|\tilde{p}_x\|^2\le\|\tilde{p}_{xx}\|^2$, it holds that

$$\frac{1}{2}\|q_{xx}\|^2-\int_0^1\tilde{p}_x q_{xx}dx+2\|\tilde{p}_{xx}\|^2\ge\frac{1}{4}\|q_{xx}\|^2+\int_0^1\left(\frac{1}{2}q_{xx}-\tilde{p}_x\right)^2dx+\|\tilde{p}_{xx}\|^2.\tag{2.33}$$

Substituting (2.33) into the previous estimate we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|q_{xx}\|^2-\int_0^1\tilde{p}_x q_{xx}dx\right.\\ &\left.+2\|\tilde{p}_{xx}\|^2+2\bar{p}\|q_{xx}\|^2\right)+2\|\tilde{p}_{xxx}\|^2+\frac{\bar{p}}{4}\|q_{xx}\|^2+\varepsilon\left(2\bar{p}+\frac{1}{2}\right)\|q_{xxx}\|^2\\ \le&C_{27}\left(\|\tilde{p}_{xx}\|^2+\|q_x\|^2+\varepsilon\|q_{xx}\|^2\right)\\ &+C_{29}\|\tilde{p}_x\|^2\left(\frac{1}{2}\|q_{xx}\|^2-\int_0^1\tilde{p}_x q_{xx}dx+2\|\tilde{p}_{xx}\|^2+2\bar{p}\|q_{xx}\|^2\right).\end{aligned}$$

Applying Gronwall inequality to the above estimate, using the uniform-in-time integrabilities of $\|\tilde{p}_{xx}\|^2$, $\|\tilde{p}_x\|^2$, $\|q_x\|^2$ and $\varepsilon\|q_{xx}\|^2$, and applying (2.33) we have

$$\|\tilde{p}_{xx}(t)\|^2+\|q_{xx}(t)\|^2+\int_0^t\left(\|\tilde{p}_{xxx}(\tau)\|^2+\|q_{xx}(\tau)\|^2+\varepsilon\|q_{xxx}\|^2\right)d\tau\le C_{30}.\tag{2.34}$$

Substituting the uniform estimate of $\|q_{xx}(t)\|^2$ into the estimate before (2.33) we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|q_{xx}\|^2-\int_0^1\tilde{p}_x q_{xx}dx\right.\\&\left.+2\|\tilde{p}_{xx}\|^2+2\bar{p}\|q_{xx}\|^2\right)+2\|\tilde{p}_{xxx}\|^2+\frac{\bar{p}}{4}\|q_{xx}\|^2+\varepsilon\left(2\bar{p}+\frac{1}{2}\right)\|q_{xxx}\|^2\\\le&C_{31}\left(\|\tilde{p}_{xx}\|^2+\|\tilde{p}_x\|^2+\|q_x\|^2+\varepsilon\|q_{xx}\|^2\right).\end{aligned}$$

We observe that the quantity $W(t)$ in (2.26) contains the four terms on the right hand side of above estimate. Hence, by combining (2.26) and above estimate one can show that

$$\frac{d}{dt}Y(t)+Z(t)\le 0, \tag{2.35}$$

where the quantities $Y(t)$ and $Z(t)$ satisfy

$$\begin{aligned}C_{32}\left(\|\tilde{p}\|_{H^2}^2+\|q\|_{H^2}^2\right)\le&Y(t)\le C_{33}\left(\|\tilde{p}\|_{H^2}^2+\|q\|_{H^2}^2\right),\\C_{34}\left(\|\tilde{p}\|_{H^3}^2+\|q\|_{H^2}^2+\varepsilon\|q\|_{H^3}^2\right)\le&Z(t)\le C_{35}\left(\|\tilde{p}\|_{H^3}^2+\int_0^1\frac{(\tilde{p}_x)^2}{p}dx+\|q\|_{H^2}^2+\varepsilon\|q\|_{H^3}^2\right).\end{aligned}\tag{2.36}$$

Hence, we have

$$\|\tilde{p}(t)\|_{H^2}^2+\|q(t)\|_{H^2}^2+\int_0^t\left(\|\tilde{p}(\tau)\|_{H^3}^2+\|q(\tau)\|_{H^2}^2+\varepsilon\|q(\tau)\|_{H^3}^2\right)d\tau\le C_{36}, \tag{2.37}$$

and

$$\|\tilde{p}(t)\|_{H^2}^2+\|q(t)\|_{H^2}^2\le C_{37}\left(\|\tilde{p}(0)\|_{H^2}^2+\|q(0)\|_{H^2}^2\right)e^{-C_{38}t}. \tag{2.38}$$

We remark again that the constants $C_{19},\dots,C_{38}$ are independent of t and ε.

Step 5. *Uniqueness.* Suppose that there are two solutions (p_1,q_1) and (p_2,q_2). Let $\check{p}=p_1-p_2$ and $\check{q}=q_1-q_2$. Then it is easy to see that $\check{p}$ and $\check{q}$ satisfy

$$\begin{cases}\check{p}_t-(p_1\check{q})_x-(\check{p}q_2)_x=\check{p}_{xx},\\\check{q}_t-\check{p}_x=\varepsilon\check{q}_{xx}+\varepsilon\check{q}_x(q_1+q_2)+\varepsilon\check{q}(q_1+q_2)_x;\\(\check{p},\check{q})(x,0)=(0,0);\\\check{p}_x|_{x=0,x=1}=\check{q}|_{x=0,x=1}=0.\end{cases}\tag{2.39}$$

Taking L^2 inner products of $(2.39)_1$ with $\check{p}$ and $(2.39)_2$ with $\check{q}$, and adding the results, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\check{p}(t)\|^2+\|\check{q}(t)\|^2\right)+\|\check{p}_x\|^2+\varepsilon\|\check{q}_x\|^2\\=&\int_0^1(\check{q}-p_1\check{q}-\check{p}q_2)\check{p}_xdx+\frac{\varepsilon}{2}\int_0^1(\check{q})^2(q_1+q_2)_xdx.\end{aligned}\tag{2.40}$$

Applying Cauchy-Schwarz inequality to the right-hand side of (2.40), we have

$$\begin{aligned}\int_0^1(\check{q}-p_1\check{q}-\check{p}q_2)\check{p}_xdx\le&\frac{1}{2}\|\check{p}_x\|^2+\frac{1}{2}\|\check{q}-p_1\check{q}-\check{p}q_2\|^2\\\le&\frac{1}{2}\|\check{p}_x\|^2+\frac{3}{2}\left(\|\check{q}\|^2+\|p_1\|_\infty^2\|\check{q}\|^2+\|q_2\|_\infty^2\|\check{p}\|^2\right)\\\le&\frac{1}{2}\|\check{p}_x\|^2+C\left(\|\check{q}\|^2+\|\check{p}\|^2\right),\end{aligned}\tag{2.41}$$

where we have applied the uniform estimates of $\|p_1\|_\infty$ and $\|q_2\|_\infty$. Substituting (2.41) into (2.40), we obtain

$$\frac{1}{2}\frac{d}{dt}\big(\|\check{p}(t)\|^2+\|\check{q}(t)\|^2\big)+\frac{1}{2}\|\check{p}_x\|^2+\varepsilon\|\check{q}_x\|^2\le C\big(\|\check{q}(t)\|^2+\|\check{p}(t)\|^2\big)$$

where we used the uniform estimate of $\|(q_1+q_2)_x\|_\infty$. In particular, it holds that

$$\frac{1}{2}\frac{d}{dt}\big(\|\check{p}(t)\|^2+\|\check{q}(t)\|^2\big)\le C\big(\|\check{q}(t)\|^2+\|\check{p}(t)\|^2\big). \tag{2.42}$$

Then solving (2.42) with the initial conditions in (2.39) yields that $\|\check{q}(t)\|^2+\|\check{p}(t)\|^2=0$ and hence the uniqueness of the solution follows. The proof of Theorem 1.1 is complete. □

3. **Diffusion limit and convergence rate (proof of Theorem 1.2).** In this section we study the diffusion limit and convergence rate of (1.5) toward the non-diffusive problem (i.e., (1.5) with $\varepsilon=0$) when $\varepsilon\to 0$. Let (p,q) and $(p^\varepsilon,q^\varepsilon)$ be the solutions to the non-diffusive problem and the diffusive problem, respectively, with the same initial data. Let $\tilde{p}=p-\bar{p}$, $\hat{p}=p-p^\varepsilon$ and $\hat{q}=q-q^\varepsilon$ where $\bar{p}=\int_0^1 p_0(x)dx$. Then we have the following IBVP:

$$\begin{cases}\hat{p}_t-(\tilde{p}\hat{q})_x-\bar{p}\hat{q}_x-(\hat{p}q^\varepsilon)_x=\hat{p}_{xx},\\ \hat{q}_t-\hat{p}_x=-\varepsilon q^\varepsilon_{xx}-\varepsilon[(q^\varepsilon)^2]_x;\\ (\hat{p},\hat{q})(x,0)=(0,0);\\ \hat{p}_x|_{x=0,1}=\hat{q}|_{x=0,1};\\ \tilde{p}_x|_{x=0,1}=q^\varepsilon|_{x=0,1}=0.\end{cases} \tag{3.1}$$

Taking L^2 inner products of $(3.1)_1$ with $\hat{p}$ and $(3.1)_2$ with $\bar{p}\hat{q}$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\hat{p}\|^2+\bar{p}\|\hat{q}\|^2)+\|\hat{p}_x\|^2&=\int_0^1[(\tilde{p}\hat{q})_x+(\hat{p}q^\varepsilon)_x]\hat{p}dx-\varepsilon\bar{p}\int_0^1\left\{q^\varepsilon_{xx}+[(q^\varepsilon)^2]_x\right\}\hat{q}dx\\&=-\int_0^1(\tilde{p}\hat{q}+\hat{p}q^\varepsilon)\hat{p}_xdx+\varepsilon\bar{p}\int_0^1[q^\varepsilon_x+(q^\varepsilon)^2]\hat{q}_xdx,\end{aligned}$$

which gives by Hölder's inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\hat{p}\|^2+\bar{p}\|\hat{q}\|^2)+\|\hat{p}_x\|^2&\le\big(\|\tilde{p}\|_\infty\|\hat{q}+\|q^\varepsilon\|_\infty\|\hat{p}\|\big)\|\hat{p}_x\|+\varepsilon\bar{p}\big(\|q^\varepsilon_x\|+\|q^\varepsilon\|_\infty\|q^\varepsilon\|\big)\|\hat{q}_x\|\\&\le\big(\|(\tilde{p},q^\varepsilon)\|_\infty\|(\hat{p},\hat{q})\|\big)\|\hat{p}_x\|+C\varepsilon\|q^\varepsilon\|_{H^1}\|\hat{q}_x\|.\end{aligned} \tag{3.2}$$

Using the first two equations of (3.1) we have

$$\hat{q}_{xt}+\bar{p}\hat{q}_x=-(\tilde{p}\hat{q})_x-(\hat{p}q^\varepsilon)_x-\varepsilon q^\varepsilon_{xxx}-\varepsilon[(q^\varepsilon)^2]_{xx}+\hat{p}_t. \tag{3.3}$$

Taking L^2 inner product of (3.3) with $\hat{q}_x$ we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\hat{q}_x\|^2+\bar{p}\|\hat{q}_x\|^2\\&=-\int_0^1[(\tilde{p}\hat{q})_x+(\hat{p}q^\varepsilon)_x]\hat{q}_xdx-\varepsilon\int_0^1\left\{q^\varepsilon_{xxx}+[(q^\varepsilon)^2]_{xx}\right\}\hat{q}_xdx+\int_0^1\hat{p}_t\hat{q}_xdx.\end{aligned} \tag{3.4}$$

For the last term on the right hand side of (3.4), by using $(3.1)_2$ we have

$$\int_0^1\hat{p}_t\hat{q}_xdx=\frac{d}{dt}\int_0^1\hat{p}\hat{q}_xdx+\|\hat{p}_x\|^2-\varepsilon\int_0^1\left\{q^\varepsilon_{xx}+[(q^\varepsilon)^2]_x\right\}\hat{p}_xdx. \tag{3.5}$$

Substituting (3.5) into (3.4) one has

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\hat{q}_x\|^2-\int_0^1\hat{p}\hat{q}_xdx\right)+\bar{p}\|\hat{q}_x\|^2\\
&=-\int_0^1[(\tilde{p}\hat{q})_x+(\hat{p}q^\varepsilon)_x]\hat{q}_xdx+\|\hat{p}_x\|^2\\
&\quad-\varepsilon\int_0^1\left\{q^\varepsilon_{xxx}+[(q^\varepsilon)^2]_{xx}\right\}\hat{q}_xdx-\varepsilon\int_0^1\left\{q^\varepsilon_{xx}+[(q^\varepsilon)^2]_x\right\}\hat{p}_xdx\\
&\le\big(\|(\tilde{p},\tilde{p}_x,q^\varepsilon,q^\varepsilon_x)\|_\infty\|(\hat{p},\hat{p}_x,\hat{q},\hat{q}_x)\|\big)\|\hat{q}_x\|+\|\hat{p}_x\|^2+C\varepsilon\|q^\varepsilon\|_{H^3}\big(\|\hat{p}_x\|+\|\hat{q}_x\|\big).
\end{aligned}\tag{3.6}$$

Taking spatial derivatives of the first two equations of (3.1) we get

$$\begin{cases}\hat{p}_{xt}-(\tilde{p}\hat{q})_{xx}-\bar{p}\hat{q}_{xx}-(\hat{p}q^\varepsilon)_{xx}=\hat{p}_{xxx},\\ \hat{q}_{xt}-\hat{p}_{xx}=-\varepsilon q^\varepsilon_{xxx}-\varepsilon[(q^\varepsilon)^2]_{xx}.\end{cases}\tag{3.7}$$

Taking L^2 inner products of $(3.7)_1$ with $\hat{p}_x$ and $(3.7)_2$ with $\bar{p}\hat{q}_x$ we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\hat{p}_x\|^2+\bar{p}\|\hat{q}_x\|^2)+\|\hat{p}_{xx}\|^2\\
&=-\int_0^1[(\tilde{p}\hat{q})_x+(\hat{p}q^\varepsilon)_x]\hat{p}_{xx}dx-\varepsilon\bar{p}\int_0^1\left\{q^\varepsilon_{xxx}+[(q^\varepsilon)^2]_{xx}\right\}\hat{q}_xdx\\
&\le\big(\|(\tilde{p},\tilde{p}_x,q^\varepsilon,q^\varepsilon_x)\|_\infty\|(\hat{p},\hat{p}_x,\hat{q},\hat{q}_x)\|\big)\|\hat{p}_{xx}\|+C\varepsilon\|q^\varepsilon\|_{H^3}\|\hat{q}_x\|
\end{aligned}\tag{3.8}$$

where the results of Theorem 1.1 and the Sobolev inequality $\|f\|^2_\infty\le C\|f\|^2_{H^1}$ have been used.

By multiplying (3.2) by 4 and then taking the sum of the result with (3.6) and (3.8) one has

$$\begin{aligned}
\frac{d}{dt}X(t)+Y(t)\le&C\big(\|(\tilde{p},\tilde{p}_x,q^\varepsilon,q^\varepsilon_x)\|_\infty\|(\hat{p},\hat{p}_x,\hat{q},\hat{q}_x)\|\big)\big(\|\hat{p}_x\|+\|\hat{p}_{xx}\|+\|\hat{q}_x\|\big)\\
&+C\varepsilon\|q^\varepsilon\|_{H^3}\big(\|\hat{p}_x\|+\|\hat{q}_x\|\big),
\end{aligned}$$

where

$$\begin{aligned}
&X(t)=2\|\hat{p}\|^2+2\bar{p}\|\hat{q}\|^2+\frac{1}{2}\|\hat{q}_x\|^2-\int_0^1\hat{p}\hat{q}_xdx+\frac{1}{2}\|\hat{p}_x\|^2+\frac{\bar{p}}{2}\|\hat{q}_x\|^2,\\
&Y(t)=3\|\hat{p}_x\|^2+\bar{p}\|\hat{q}_x\|^2+\|\hat{p}_{xx}\|^2.
\end{aligned}$$

Here we should remark that $X(t)$ is non-negative for all $t\ge0$. Indeed by Young's inequality $ab\le\frac{a^2}{4}+b^2$ for all $a,b\ge0$, we have $\int_0^1\hat{p}\hat{q}_xdx\le\frac{1}{4}\int_0^1\hat{q}_x^2dx+\int_0^1\hat{p}^2dx=\frac{1}{4}\|\hat{q}_x\|^2+\|\hat{p}\|^2$, and hence

$$X(t)\ge\|\hat{p}\|^2+2\bar{p}\|\hat{q}\|^2+\frac{1}{4}\|\hat{q}_x\|^2+\frac{1}{2}\|\hat{p}_x\|^2+\frac{\bar{p}}{2}\|\hat{q}_x\|^2.$$

Therefore, by Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\frac{d}{dt}X(t)+\frac{1}{2}Y(t)&\le C\big(\|\tilde{p}\|^2_\infty+\|\tilde{p}_x\|^2_\infty+\|q^\varepsilon\|^2_\infty+\|q^\varepsilon_x\|^2_\infty\big)X(t)+C\varepsilon^2\|q^\varepsilon\|^2_{H^3}\\
&\le C\big(\|\tilde{p}\|^2_{H^2}+\|q^\varepsilon\|^2_{H^2}\big)X(t)+C\varepsilon^2\|q^\varepsilon\|^2_{H^3}.
\end{aligned}$$

Gronwall's inequality then implies that

$$X(t)\le C\exp\left\{C\int_0^t\big(\|\tilde{p}\|^2_{H^2}+\|q^\varepsilon\|^2_{H^2}\big)d\tau\right\}\cdot\varepsilon\int_0^t\varepsilon\|q^\varepsilon\|^2_{H^3}d\tau.$$

Using the uniform temporal integrabilities of $\|\tilde{p}\|_{H^2}^2$, $\|q^\varepsilon\|_{H^2}^2$ and $\varepsilon\|q^\varepsilon\|_{H^3}^2$, see Theorem 1.1, we finally get

$$X(t) + \int_0^t Y(\tau)d\tau \le C\varepsilon$$

for some constant $C > 0$ which is independent of t. In particular, it holds that

$$\|p - p^\varepsilon\|_{H^1}^2 + \|q - q^\varepsilon\|_{H^1}^2 + \int_0^t \left(\|p - p^\varepsilon\|_{H^2}^2 + \|q - q^\varepsilon\|_{H^1}^2\right) d\tau \le C\varepsilon. \tag{3.9}$$

Moreover, by Sobolev embedding we have

$$\|p - p^\varepsilon\|_{L^\infty}^2 + \|q - q^\varepsilon\|_{L^\infty}^2 \le C\varepsilon.$$

This completes the proof of Theorem 1.2. □

4. **Results for the original chemotaxis model.** In this section we study long-time dynamics, diffusion limit and convergence rate of the original model (1.2) based on the results for the transformed system (1.5).

4.1. **Long-time dynamics (proof of Corollary 1).** Let (u, v) be the solution to (1.6). Since the dynamics of $u = p$ is clear from Theorem 1.1, we are only concerned with the dynamics of the original function v. We consider the following equation

$$[\ln(v)]_t = u - \mu + \varepsilon q_x + \varepsilon q^2$$

where $q = [\ln(v)]_x = [\ln(ve^{\mu t})]_x$. Integrating the above equation over $[0, 1] \times [0, t]$ we get

$$\int_0^1 \ln(v)dx = \int_0^1 \ln(v_0)dx + (\bar{u} - \mu)t + \varepsilon \int_0^t \|q\|^2 d\tau.$$

Define

$$\xi(x, t) = \ln(v) - \int_0^1 \ln(v_0)dx - (\bar{u} - \mu)t - \varepsilon \int_0^t \|q\|^2 d\tau. \tag{4.1}$$

It is straightforward to check that

$$\xi_x = q, \quad \text{and} \quad \int_0^1 \xi(x, t)dx = 0.$$

By Poincaré inequality we have $\|\xi\|^2 \le \|q\|^2$. From (2.38) we see that

$$\|\xi(t)\|_{H^3}^2 \le \alpha e^{-\beta t} \tag{4.2}$$

for some positive constants α and β which are independent of t.

Now from (4.1) we see that

$$v(x, t) = \exp\left\{\xi(x, t) + \int_0^1 \ln(v_0)dx + \varepsilon \int_0^t \|q\|^2 d\tau\right\} \exp\left\{(\bar{u} - \mu)t\right\}.$$

From (2.37) and (4.2) one sees that

$$\gamma_1 \exp\left\{(\bar{u} - \mu)t\right\} \le v(x, t) \le \gamma_2 \exp\left\{(\bar{u} - \mu)t\right\}$$

for some positive constants γ_1 and γ_2 which are independent of t. Thus

$$v(x, t) \to 0 \quad \text{as} \quad t \to \infty, \quad \text{when} \quad \bar{u} < \mu,$$
$$v(x, t) \to +\infty \quad \text{as} \quad t \to \infty, \quad \text{when} \quad \bar{u} > \mu.$$

This completes the proof of Corollary 1. □

4.2. **Diffusion limit and convergence rate (proof of Corollary 2).** Let $(u^\varepsilon, v^\varepsilon)$ be the solution to (1.6) and (u, v) be the solution to (1.6) with $\varepsilon = 0$. Noticing that $u^\varepsilon = p^\varepsilon$ and $u = p$, the first part of Corollary 2 is obtained directly from Theorem 1.2. It remains to prove the convergence for v^ε only. Note that

$$\begin{cases} [\ln(v^\varepsilon)]_t = u^\varepsilon - \mu + \varepsilon q_x^\varepsilon + \varepsilon (q^\varepsilon)^2, \\ [\ln(v)]_t = u - \mu, \end{cases} \tag{4.3}$$

where $q^\varepsilon = [\ln(v^\varepsilon)]_x$. We consider the difference of the two equations:

$$[\ln(v^\varepsilon) - \ln(v)]_t = (u^\varepsilon - u) + \varepsilon q_x^\varepsilon + \varepsilon (q^\varepsilon)^2. \tag{4.4}$$

Integrating the above inequality with respect to t one has

$$\frac{v^\varepsilon(x,t)}{v(x,t)} = \frac{v^\varepsilon(x,0)}{v(x,0)} \exp\left\{ \int_0^t \left[(u^\varepsilon - u) + \varepsilon q_x^\varepsilon + \varepsilon (q^\varepsilon)^2 \right] d\tau \right\}.$$

Since it is assumed that $v^\varepsilon(x,0) = v(x,0)$, it follows that

$$|v^\varepsilon(x,t) - v(x,t)| = |v(x,t)| \cdot \left| \exp\left\{ \sqrt{\varepsilon} \left(\int_0^t \left[\frac{(u^\varepsilon - u)}{\sqrt{\varepsilon}} + \sqrt{\varepsilon} q_x^\varepsilon + \sqrt{\varepsilon} (q^\varepsilon)^2 \right] d\tau \right) \right\} - 1 \right|.$$

Next, it follows from Hölder's inequality that

$$\begin{aligned} &\int_0^t \left[\frac{(u^\varepsilon - u)}{\sqrt{\varepsilon}} + \sqrt{\varepsilon} q_x^\varepsilon + \sqrt{\varepsilon} (q^\varepsilon)^2 \right] d\tau \\ \le& \frac{1}{\sqrt{\varepsilon}} \int_0^t \|u^\varepsilon - u\|_{L^\infty} d\tau + \sqrt{\varepsilon} \int_0^t \|q_x^\varepsilon\|_{L^\infty} d\tau + \sqrt{\varepsilon} \int_0^t \|q^\varepsilon\|_{L^\infty}^2 d\tau \\ \le& \frac{1}{\sqrt{\varepsilon}} \left(\int_0^t \|u^\varepsilon - u\|_{L^\infty}^2 d\tau \right)^{\frac{1}{2}} t^{\frac{1}{2}} + \sqrt{\varepsilon} \left(\int_0^t \|q_x^\varepsilon\|_{L^\infty}^2 d\tau \right)^{\frac{1}{2}} t^{\frac{1}{2}} + \sqrt{\varepsilon} \int_0^t \|q^\varepsilon\|_{L^\infty}^2 d\tau \\ \le& \frac{1}{\sqrt{\varepsilon}} \left(\int_0^t \|u^\varepsilon - u\|_{H^1}^2 d\tau \right)^{\frac{1}{2}} t^{\frac{1}{2}} + \sqrt{\varepsilon} \left(\int_0^t \|q_x^\varepsilon\|_{H^1}^2 d\tau \right)^{\frac{1}{2}} t^{\frac{1}{2}} + \sqrt{\varepsilon} \int_0^t \|q^\varepsilon\|_{H^1}^2 d\tau \\ \le& \frac{1}{\sqrt{\varepsilon}} (C\varepsilon)^{\frac{1}{2}} t^{\frac{1}{2}} + \sqrt{\varepsilon} (C_{36})^{\frac{1}{2}} t^{\frac{1}{2}} + \sqrt{\varepsilon} C_{36} \\ \le& C(t), \end{aligned}$$

where we have used the estimates (3.9) and (2.37), and the smallness of ε.

From [11] we know that the non-diffusive chemical concentration $v(x,t)$ satisfies

$$|v(x,t)| \le \begin{cases} C, & \text{if } \displaystyle\int_0^1 u_0(x)dx = \bar{u} \le \mu, \\ Ce^{(\bar{u}-\mu)t}, & \text{if } \bar{u} > \mu \end{cases}$$

where the constant C is independent of t.

From the above estimates we know that for any fixed $t > 0$ it holds that

$$\|v^\varepsilon - v\|_{L^\infty} = O(\sqrt{\varepsilon}).$$

Hence,

$$\|v^\varepsilon - v\|_{L^\infty}^2 = O(\varepsilon).$$

This completes the proof of Corollary 2. □

5. **Numerical illustrations.** Generally it is unfeasible to simulate the chemotaxis model (1.2) directly by the routine numerical scheme due to the singularity term $\nabla \ln(v) = \nabla v/v$. By noting that the transformed system (1.4) removes the singularity appearing in (1.2), the Hopf-Cole transformation (1.3) enables us to study the model not only analytically (as shown in the paper) but also numerically. From the original chemotaxis model (1.2) to the transformed parabolic system (1.4), the cell density $u = p$ remains the same. Therefore we can numerically solve system (1.4) to obtain the numerical value of u. Nevertheless it is mathematically interesting alone to numerically investigate system (1.4) as a newly derived system of conservations laws from biology.

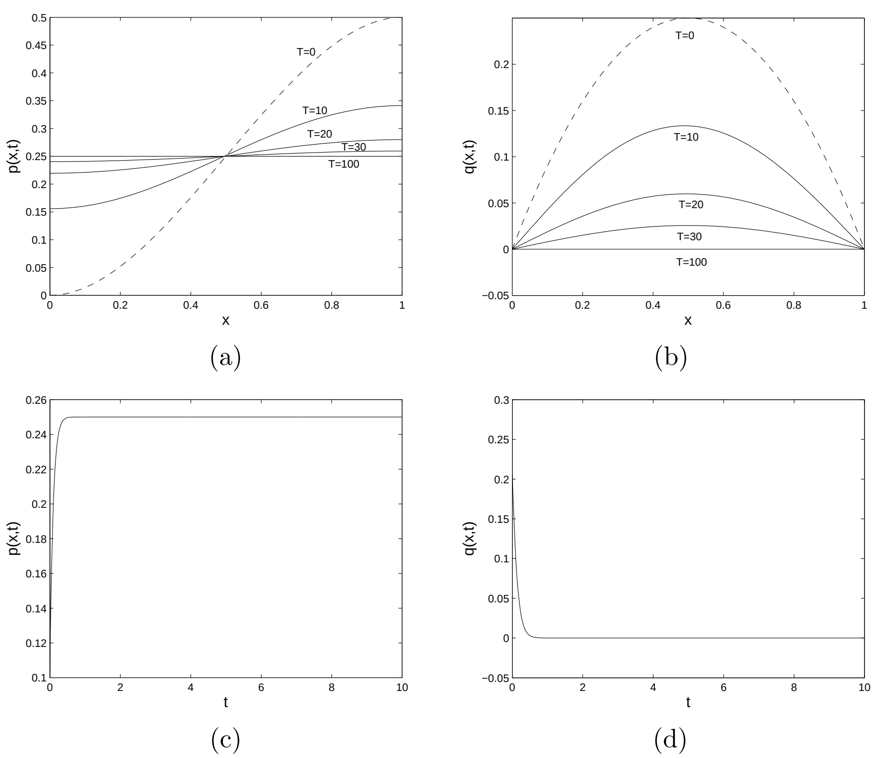

FIGURE 1. Numerical solutions to system (1.5) in $(0,1)$ with initial data $p_0(x) = 3(x^2/2 - x^3/3), q_0(x) = x(1-x), \varepsilon = 1$. (a)-(b) plot the snapshot of solutions at different time steps, and (c)-(d) plot time evolution of solutions at a fixed position $x = 0.1$.

In this section, we shall numerically illustrate the analytical results derived in this paper for the one-dimensional version of model (1.4), i.e. model (1.5). We are particularly interested in showing the asymptotical behavior and diffusion limits of solutions. The Matlab PDE solver will be implemented to solve system (1.5) in $(0,1)$, where the time step size $\Delta t = 0.01$ and spatial step size $\Delta x = 0.01$. Numerical results are presented in Fig. 1 and Fig. 2.

In the simulation, we choose initial data $p_0(x) = 3(x^2/2 - x^3/3), q_0(x) = x(1-x)$ such that the conditions of Theorem 1.1 are satisfied where $\bar{p} = \int_0^1 p_0(x)dx = 0.25$. By Theorem 1.1, the solutions p and q converge to 0.25 and 0, respectively. Fig. 1

(a)-(b) plot the snapshot of solutions at different time steps, which illustrate that the solution p asymptotically approaches 0.25 and q asymptotically approaches 0. Fig. 1 (c)-(d) give alternate visualization of the asymptotics of solutions where the time evolution of solutions at a specific location $x = 0.2$ was plotted.

Fig. 2 numerically illustrates the diffusion limit of solutions as $\varepsilon \to 0$ at a given time step $T = 20$. It shows that, as $\varepsilon \to 0$, the solution profiles are getting closer to the solution profiles with $\varepsilon = 0$. This is consistent with Theorem 1.2.

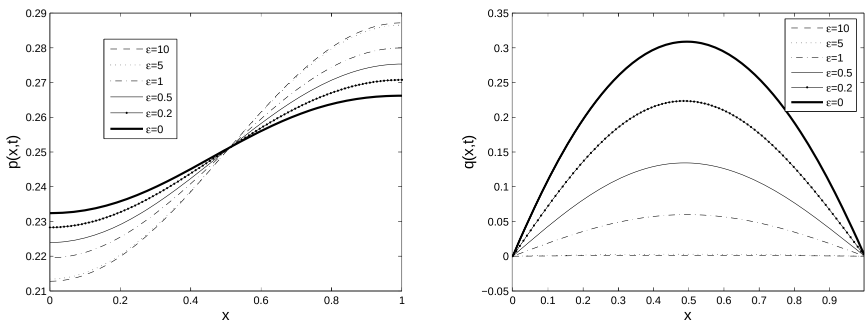

FIGURE 2. Numerical solutions to system (1.5) in $(0, 1)$ at time step $T = 20$ for different values of ε, where the initial data was the same as in Figure 1.

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