

GEOMETRY EFFECTS ON THE BOUNDARY-LAYER PROFILES OF THE KELLER-SEGEL SYSTEM

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ABSTRACT. We consider the boundary-layer problem of a nonlocal semilinear elliptic equation in a bounded smooth domain of all dimensions with the Dirichlet boundary condition, which arises as the stationary problem of the Keller-Segel system with physical boundary conditions describing the boundary-layer formation driven by chemotaxis. Using the Fermi coordinates and delicate analysis with subtle estimates, we rigorously derive the asymptotic expansion of the boundary-layer profile and thickness in terms of the small diffusion rate with coefficients explicitly expressed by the domain geometric properties including mean curvature, volume and surface area. By these expansions, one can explicitly find the joint impact of the mean curvature, surface area and volume of the spatial domain on the boundary-layer steepness and thickness. This seems to be the first result revealing how the boundary-layer profiles depend on the domain geometries for chemotaxis models.

1. INTRODUCTION

This paper is concerned with the stationary problem of the Keller-Segel chemotaxis system

$$(1.1) \quad \begin{cases} v_t = \Delta v - \nabla \cdot (\chi v \nabla \phi(u)) & \text{in } \Omega, \\ u_t = \varepsilon^2 \Delta u - uv & \text{in } \Omega, \end{cases}$$

subject to the physical boundary conditions

$$(1.2) \quad (\nabla v - \chi v \nabla \phi(u)) \cdot \nu = 0, \quad u = \bar{u} \quad \text{on } \partial\Omega,$$

where $v(x, t)$ and $u(x, t)$ denote the cell density and chemical (signal) concentration, respectively, ν is the normal vector of $\partial\Omega$, $\bar{u}$ is a positive constant, Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary, and $\varepsilon > 0$ represents the chemical diffusion rate. The function $\phi(u)$ is the chemotactic sensitivity function accounting for the signal transduction mechanism in response to the chemical signal.

Received by the editors March 21, 2025.

2020 *Mathematics Subject Classification.* Primary 35K57, 35Q92, 92D25.

Key words and phrases. Boundary layer, singularity, nonlocal problem, boundary curvature, nonlinear stability.

The first author was supported by the Taiwan Ministry of Science and Technology (Grant numbers: 110-2115-M-007-003-MY2 and 112-2115-M-007-008-MY2). The second author was supported by the National Research Foundation of Korea grant funded by the Korea government (MSIT) (RS-2022-NR072398). The third author was supported by Hong Kong RGC GRF grant No.15305824 and an internal grant No. 4-ZZPY from the Hong Kong Polytechnic University. The fourth author was supported by National Key R&D Program of China 2022YFA1006800, NSFC, China No. 12171456, NSFC, China No. 12271369, FDCT No. 0070/2024/RIA1, Start-up Research Grant No. SRG2023-00067-FST, Multi-Year Research Grant No. MYRG-GRG2024-00082-FST and UMDP-TISF/2025/006/FST.

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There are two prevailing prototypes: $\phi(u) = u$ (linear sensitivity) and $\phi(u) = \log u$ (logarithmic sensitivity). The model (1.1) was first proposed in [21] to explain the propagation of traveling bands driven by the bacterial chemotaxis observed in the celebrated experiment of Adler [1] where the oxygen is supplied at the end (boundary) of the capillary tube filled with water to attract the bacteria inside the capillary tube to move towards the oxygen, which corresponds to the boundary conditions given in (1.2). Later it was employed to describe many other important biological processes such as the initiation of angiogenesis [25, 26], boundary movement of chemotactic bacteria [35], reinforced random walks [24, 36], and so on. The mathematical derivation of (1.1) was given in [24, 36] based on the random-walk framework. In particular, the model (1.1) coupled to fluid dynamics was employed in [42] to explain the boundary accumulation layer formed on the drop edge (air-water interface) in a sessile drop mixed with aerobic bacterial *Bacillus subtilis* due to chemotaxis.

Literature review. Due to its strong biological relevance, the Keller-Segel system (1.1) has attracted extensive attention and various analytical results have been developed, such as the stability of traveling waves (see [7, 8, 10, 20, 30, 33]), global well-posedness of solutions (see [9, 28, 29, 32, 34, 38, 46] in one-dimensional bounded domain with various boundary conditions or in the whole real line $\mathbb{R}$, and [12, 17, 27, 32, 37, 39, 41, 43, 45] in multidimensional spaces), just to mention a few and more relevant works can be found in the above-mentioned references. In contrast, the boundary-layer problem of (1.1) pertinent to the experimental observation in [42] is less studied. Below we shall briefly recall these results in connection with our current study, based on two different cases: linear and logarithmic sensitivities.

- $\phi(u) = u$ (linear sensitivity). In this case, the first rigorous result was due to [23] showing that the existence of unique stationary solution of (1.1) and convergence of it as $\varepsilon \rightarrow 0$ in any dimension. Furthermore, the stability of stationary boundary-layer problem solution was proved in one dimension [19] and in multi-dimensions [31]. The existence of classical solutions in the two-dimensional radially symmetric domain and weak solutions or small-data classical solutions in higher dimensions were proved in [22, 44]. Recently the convergence of solutions to the time-dependent problem (1.1) as $\varepsilon \rightarrow 0$ was shown in [4] and in [18] for degenerate and non-degenerate initial data, respectively.
- $\phi(u) = \log u$ (logarithmic sensitivity). In this case, the analytical results of the boundary-layer problem of (1.1) seem to be less complete. The first result was obtained in [5] showing that (1.1) in one dimension admits a unique stationary boundary-layer solution which is locally asymptotically stable, succeeded by a work [40] further giving the stabilization rate. Similar results have been extended to more general consumption rate functions in [13]. In a two-dimensional smooth domain or a ball in three dimensions and higher, the existence of unique stationary solution of (1.1) was obtained in [2]. Recently it was shown in [6] that (1.1) with any $\varepsilon > 0$ admits a unique stationary solution in all dimensions which is a boundary-layer profile as $\varepsilon \rightarrow 0$. When the domain is radially symmetric, the asymptotic expansion of boundary-layer profile and thickness near the boundary in terms of small $\varepsilon > 0$ was further established in [6]. We remark that the analyses for (1.1)

with logarithmic sensitivity $\phi(u) = \log u$ are much harder than the linear sensitivity $\phi(u) = u$ since the former generates a singularity as $\varepsilon \rightarrow 0$ (see Section 2.3).

A remarkable feature of the boundary-layer phenomenon observed in both laboratory and numerical experiments shown in the paper [42] is that the boundary-layer thickness varies with the boundary curvature. So far, the above-mentioned works only established the existence of boundary-layer solutions of (1.1) in arbitrary dimensions for the stationary problem and in one dimension for the time-dependent problem, but none of them studied the impact of domain geometric properties, such as boundary curvature or domain surface area, on the boundary-layer profile such as the steepness and thickness. Since boundary-layer solutions describe the rapid change of solution values near the boundary, it is natural to ask how the domain geometry affects the boundary-layer shape localized near the boundary. The main goal of this paper is to answer this question by studying the stationary problem of (1.1). Specifically, by assuming the domain is smooth, we rigorously derive explicit asymptotic expansions of the boundary-layer profile, steepness and thickness up to higher-order terms in terms of small $\varepsilon > 0$, by which we can pinpoint the impact of domain geometry (curvature, volume and surface area of the domain) on the boundary-layer profile, steepness and thickness. Though boundary-layer problems have been widely studied for fluid dynamics (cf. [3, 14]) or some other biological models (cf. [15, 16]), it seems that the effect of domain geometric properties on the boundary-layer formation was rarely studied (if none). Our result seems to be the first one that can explicitly address this issue. We conclude this section by rigorously deriving that the stationary problem of (1.1) can be reduced to a scalar nonlocal Dirichlet elliptic problem.

Derivation of the stationary problem. First we note that the integration of the first equation of (1.1) over Ω immediately yields

$$(1.3) \quad \int_{\Omega} v(x, t) dx = \int_{\Omega} v_0(x) dx := m,$$

which entails that the mass of v is preserved, denoted by $m > 0$, where $v_0 \geq (\neq 0)$ denotes the initial value of v . This implies that the mass conservation (1.3) is an inherent constraint which should be prescribed in the analytical study. Therefore the stationary solutions of (1.1), still denoted by $(v, u)(x)$ without ambiguity, satisfy

$$(1.4) \quad \begin{cases} \Delta v - \nabla \cdot (\chi v \nabla \phi(u)) = 0 & \text{in } \Omega, \\ \varepsilon^2 \Delta u - vu = 0 & \text{in } \Omega, \\ (\nabla v - \chi v \nabla \phi(u)) \cdot \nu = 0, u = \bar{u} & \text{on } \partial\Omega, \\ \int_{\Omega} v(x) dx = m. \end{cases}$$

Multiplying the first equation of (1.4) by $\ln v - \chi \phi(u)$, and integrating the equation on Ω , we have

$$(1.5) \quad \int_{\Omega} v |\nabla (\log v - \chi \phi(u))|^2 dx = 0.$$

Since we are interested in the non-negative solutions, we have $v(x) \geq 0$ and $u(x) \geq 0$ for any $x \in \bar{\Omega}$. Applying the strong maximum principle to the second equation of

(1.4), we have $u(x) > 0$ for any $x \in \bar{\Omega}$. We next write the first equation of (1.4) as

$$-\Delta v + \chi \phi'(u) \nabla u \nabla v + \frac{\chi}{\varepsilon^2} u \phi'(u) v^2 = -\chi v \phi''(u) |\nabla u|^2 \geq 0.$$

Then by the strong maximum principle and Hopf's boundary point lemma along with the fact $\int_{\Omega} v dx = m$, one has $v(x) > 0$ for all $x \in \bar{\Omega}$. Thus, it follows from (1.5) that

$$\log v - \chi \phi(u) = c_0$$

for an arbitrary constant c_0 . Therefore, we get a constant $\lambda = e^{c_0} > 0$ such that

$$(1.6) \quad v = \lambda e^{\chi \phi(u)}, \quad \lambda = \frac{m}{\int_{\Omega} e^{\chi \phi(u)} dx},$$

where the constant $\lambda = \frac{m}{\int_{\Omega} e^{\chi \phi(u)} dx}$ is due to the mass constraint in (1.4). Then the second equation of (1.4) can be rewritten as a nonlocal problem as follows:

$$(1.7) \quad \begin{cases} \varepsilon^2 \Delta u = \frac{m}{\int_{\Omega} e^{\chi \phi(u)} dx} u e^{\chi \phi(u)} & \text{in } \Omega, \\ u = \bar{u} > 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore the stationary problem (1.4) is equivalent to the nonlocal problem (1.7) with (1.6). The existence of the unique positive solution of (1.7) with both linear and logarithmic sensitivity in any dimensions was shown in [23] and [6], respectively. Apart from the existence, the asymptotic expansion of the boundary-layer profile thickness as $\varepsilon \rightarrow 0$ was also derived for the radially symmetric domain up to the first order term in [23] and leading order term in [6]. It turns out the results are quite different between linear and logarithmic sensitivity, where the latter generates a singularity making the analysis much more difficult. As far as we know, the asymptotic expansion of the boundary-layer profile and thickness as $\varepsilon \rightarrow 0$ in a general multi-dimensional domain remains unknown. In a general domain, the domain geometry properties, such as (mean) curvature, volume or surface, will become important and hence affect the boundary-layer profile. In particular, one may expect that the boundary-layer solution will behave differently at different boundary points. The main goal of this paper is to develop new ideas to explore this question. Our results and analyses will be divided into two different cases: linear and logarithmic sensitivities, which will be treated by different technicalities.

2. STATEMENT OF MAIN RESULTS

In this section, we shall state our main results for the semi-linear nonlocal elliptic problem (1.7) with linear and logarithmic sensitivities. We first introduce some notation.

Notation.

- We write $a \sim b$ if there exist two positive constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$.
- By C , we denote a generic positive constant that may vary among different formulas and places. To highlight the dependence of C on some quantity say K , we shall write it as $C(K)$.

2.1. Laplacian operator in terms of Fermi coordinates. To analyze the behavior of solution near $\partial\Omega$, we will employ Fermi coordinates close to the boundary. Given a smooth bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$, for any small $\delta > 0$, we define

$$\Omega_\delta := \{x \in \Omega \mid 0 < \text{dist}(x, \partial\Omega) < \delta\}.$$

Then for any $x \in \Omega_\delta$, we can parameterize a point on $\partial\Omega$ along with the distance from this point to the boundary. This allows us to represent $x \in \Omega$ in the following manner:

$$X : (y, z) \in \partial\Omega \times \mathbb{R}^+ \rightarrow x = X(y(x), z(x)) = y(x) + z(x)\nu(y(x)) \in \Omega_\delta,$$

where $y(x) \in \partial\Omega$ is the point such that

$$|x - y(x)| = \text{dist}(x, \partial\Omega), \quad z(x) = |x - y(x)|$$

and $\nu(y(x))$ denotes the unit interior normal vector at $y(x) \in \partial\Omega$. There exists a constant $\delta_0 > 0$ small enough, such that

$$(2.1) \quad X : \partial\Omega \times (0, \delta) \rightarrow \Omega_\delta \text{ is a diffeomorphism for any } \delta \in (0, \delta_0).$$

Conventionally, letting $\kappa_i(y(x)), i = 1, 2, \dots, n-1$, be the principal curvature at $y(x) \in \partial\Omega$, then the mean curvature at $y(x)$ can be defined as

$$H_{\partial\Omega}(y(x)) = \frac{1}{n-1} \sum_{i=1}^{n-1} \kappa_i(y(x)).$$

While for $x \in \Omega_\delta$, we set

$$\Gamma_{z(x)} := \{p \in \Omega \mid \text{dist}(p, \partial\Omega) = z(x)\}.$$

The mean curvature of $\Gamma_{z(x)}$ at x is defined by

$$(2.2) \quad H_{\Gamma_{z(x)}}(y(x)) := \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{\kappa_i(y(x))}{1 - z(x)\kappa_i(y(x))}.$$

Using $H_{\Gamma_{z(x)}}(x)$, we can express Δ in terms of the Fermi coordinate system $(y, z)^1$, see [11, (2.14)],

$$(2.3) \quad \Delta \cdot = \frac{\partial^2}{\partial z^2} \cdot - (n-1)H_{\Gamma_{z(x)}}(y(x)) \frac{\partial}{\partial z} \cdot + \Delta_{\Gamma_{z(x)}} \cdot,$$

where $\Delta_{\Gamma_{z(x)}}$ stands for the Laplace-Beltrami operator on $\Gamma_{z(x)}$. In particular,

$$(2.4) \quad \Delta z(x) = -(n-1)H_{\Gamma_{z(x)}}(y(x)) \quad \text{in } \Omega_\delta$$

since $z(x)$ is a fixed constant on $\Gamma_{z(x)}$.

¹See Figure 1

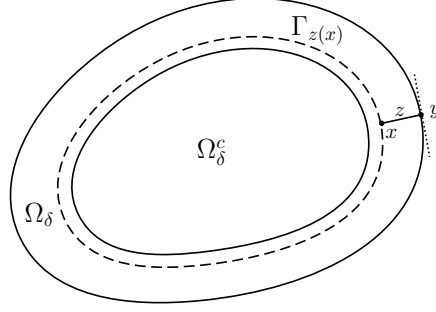


FIGURE 1. Illustration of domains and Fermi coordinates

By the useful Fermi coordinate, we can rewrite (1.7) in Ω_δ as

$$(2.5) \quad \varepsilon^2 \left(\frac{\partial^2}{\partial z^2} u_\varepsilon - (n-1) H_{\Gamma_{z(x)}}(y(x)) \frac{\partial}{\partial z} u_\varepsilon + \Delta_{\Gamma_{z(x)}} u_\varepsilon \right) = \frac{m}{\int_\Omega e^{\chi\phi(u_\varepsilon)} dx} u_\varepsilon e^{\chi\phi(u_\varepsilon)},$$

where we have used the fact that $u_\varepsilon(x) = u_\varepsilon(y(x) + z(x)\nu(y(x)))$ for $x \in \Omega_\delta$.

Before concluding this subsection, we prove a co-area formula which will be used later.

Lemma 2.1. *Let h be a smooth function on $\overline{\Omega}$. Then for each $\delta \in (0, \delta_0)$ it holds that*

$$(2.6) \quad \int_{\Omega_\delta} h(x) dx = \int_0^\delta \int_{\partial\Omega} h(y, z) (1 - (n-1)z H_{\partial\Omega}(y) + O(z^2)) d\sigma_y dz,$$

where $h(x) = h(y, z)$ for $x = y + z\nu(y) \in \Omega_\delta$.

Proof. We postpone the proof to the Appendix. $\square$

2.2. Linear sensitivity. When $\phi(u) = u$, the problem (1.7) reduces to the following nonlocal semilinear elliptic Dirichlet problem

$$(2.7) \quad \begin{cases} \varepsilon^2 \Delta u = \frac{m}{\int_\Omega e^u dx} u e^u & \text{in } \Omega, \\ u = \bar{u} & \text{on } \partial\Omega, \end{cases}$$

where we have assumed $\chi = 1$ without loss of generality, and the original system can be simply recovered by making the following change of variables: $u \rightarrow \chi u, \bar{u} \rightarrow \chi \bar{u}$. For convenience, we define

$$(2.8) \quad \begin{aligned} f(t) &:= te^t, \quad F(t) := \int_0^t f(s) ds = se^s - e^s + 1 > 0 \text{ for } s > 0, \\ \mathfrak{F}(t) &= \int_0^t \sqrt{2\rho F(s)} ds, \end{aligned}$$

where

$$\rho = \frac{m}{|\Omega|}$$

denotes the average mass. It was shown in [23] that (2.7) admits a unique classical solution which behaves like a boundary-layer profile as $\varepsilon \rightarrow 0$, and the asymptotic profile of the solution as $\varepsilon \rightarrow 0$ was further refined when Ω is radially symmetric.

In this paper, we shall step forward to investigate how the geometry properties of a general smooth domain Ω , such as the curvature, volume or surface area, collectively affect the boundary-layer profile and thickness near the boundary. Since now Ω is an arbitrary domain, the techniques used in [23] for the radially symmetric domain are inapplicable. We need to explore the problem (2.7) with new approaches. In particular, the effect of curvature will become significant, and the boundary-layer profile is expected to behave differently at each boundary point. Thus, we have to get a full understanding of the solution behavior near the boundary first. Let us recall the result of [23, Corollary 2.3] showing that the boundary-layer thickness $z(x)$ is of order ε , namely $\lim_{\varepsilon \rightarrow 0} \frac{z(x)}{\varepsilon} = L$ with $L \in (0, \infty)$ being a constant, denoted by

$$(2.9) \quad z(x) \sim \varepsilon, \text{ as } \varepsilon \rightarrow 0.$$

The difference between $\int_{\Omega} e^{u_{\varepsilon}} dx$ and $|\Omega|$ is also of order ε (see [23, (4.13)]), namely

$$(2.10) \quad 0 < \int_{\Omega} e^{u_{\varepsilon}} dx - |\Omega| \leq C\varepsilon$$

for some constant $C > 0$, where u_{ε} denotes the solution of (2.7). Consequently, the function $W\left(\frac{z}{\varepsilon}\right)$, which satisfies the following equation

$$(2.11) \quad \begin{cases} W''(z) = \rho W(z)e^{W(z)}, & z > 0, \\ W(0) = \bar{u}, & W(z) \rightarrow 0 \text{ as } z \rightarrow +\infty, \end{cases}$$

shall be the leading order approximation for u_{ε} . In Lemma 3.2 established in Section 3, we will prove that (2.11) admits a unique solution which is monotonically decreasing and decays exponentially at infinity. Next, we will derive the higher-order terms of the expansion of u_{ε} in terms of ε . It turns out that two factors will play key roles: one involves the term with mean curvature $H_{\Gamma_{z(x)}}(y(x))\frac{\partial u_{\varepsilon}}{\partial z}$, and the other arises from the difference between the nonlocal term $\int_{\Omega} e^{u_{\varepsilon}} dx$ and $|\Omega|$. Both factors will contribute to the first-order approximation of u_{ε} . Specifically, we will introduce two non-negative functions ϕ_1 and ϕ_2 satisfying the following second-order ODEs:

$$(2.12) \quad \begin{cases} \phi_1''(z) = \rho(1 + W(z))e^{W(z)}\phi_1(z) + W(z)e^{W(z)}, & z > 0, \\ \phi_1(0) = 0, & \phi_1(z) \rightarrow 0 \text{ as } z \rightarrow \infty, \end{cases}$$

and

$$(2.13) \quad \begin{cases} \phi_2''(z) = \rho(1 + W(z))e^{W(z)}\phi_2(z) + W'(z), & z > 0, \\ \phi_2(0) = 0, & \phi_2(z) \rightarrow 0 \text{ as } z \rightarrow \infty. \end{cases}$$

In Section 3, we will prove that both ODEs (2.12) and (2.13) (see Lemma 3.4) have a unique solution decaying exponentially at infinity. Utilizing ϕ_1 and ϕ_2 , we can obtain a more accurate approximation for u_{ε} . Simply speaking, we will derive a three-term expansion for $\int_{\Omega} e^{u_{\varepsilon}} dx$ in terms of the diffusion coefficient ε . To present this result, we introduce a function

$$(2.14) \quad Q_F(t) := \int_0^t \frac{e^s - 1}{\sqrt{2(se^s - e^s + 1)}} ds, \quad t \in [0, \infty),$$

and two constants:

$$\begin{aligned}
 (2.15) \quad I_1 &= \rho^{-\frac{1}{2}} |\partial\Omega| Q_F(\bar{u}), \\
 (2.16) \quad I_2 &= - (n-1) \rho^{-\frac{1}{2}} \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \int_0^\infty Q_F(W(t)) dt \\
 &\quad - \sqrt{\rho} \frac{|\partial\Omega|^2}{|\Omega|} Q_F(\bar{u}) \int_0^\infty e^{W(t)} \phi_1(t) dt \\
 &\quad + (n-1) \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \int_0^\infty e^{W(t)} \phi_2(t) dt.
 \end{aligned}$$

Then we first prove the following key result which gives more detailed information than (2.10).

Proposition 2.2. *Consider the problem (2.7). Let I_1 and I_2 be given in (2.15) and (2.16), respectively. Then as $\varepsilon \rightarrow 0$ we have*

$$(2.17) \quad \int_{\Omega} (e^{u_\varepsilon} - 1) dx = I_1 \varepsilon + I_2 \varepsilon^2 + o_\varepsilon(1) \varepsilon^2,$$

where $o_\varepsilon(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Using ϕ_1 , ϕ_2 , and Proposition 2.2, we can derive an explicit three-term expansion for u_ε in Ω_δ for any $\delta \in (0, \delta_0)$ (i.e. the expansion near the boundary). Additionally, we will derive an expansion for the normal derivative of u_ε on the boundary, by which we can further find an explicit expansion of boundary-layer thickness $z(x)$ in terms of ε up to the order ε^2 . By these results, we can identify the joint impact of domain geometry properties, including boundary curvature, surface area and volume, on the boundary-layer profile and thickness. Our results are stated below.

Theorem 2.3. *Let $W(z)$, $\phi_1(z)$, and $\phi_2(z)$ denote the solutions of (2.11), (2.12) and (2.13), respectively. Let $Q_F(t)$ be defined by (2.14) and $H_{\Gamma_{z(x)}}(y(x))$ by (2.2). Then, for any $x \in \Omega_\delta$, the solution u_ε of (2.7) satisfies*

$$\begin{aligned}
 (2.18) \quad u_\varepsilon(x) &= W\left(\frac{z(x)}{\varepsilon}\right) - \varepsilon \sqrt{\rho} \frac{|\partial\Omega|}{|\Omega|} Q_F(\bar{u}) \phi_1\left(\frac{z(x)}{\varepsilon}\right) \\
 &\quad + \varepsilon(n-1) H_{\Gamma_{z(x)}}(y(x)) \phi_2\left(\frac{z(x)}{\varepsilon}\right) + O(\varepsilon^2),
 \end{aligned}$$

where $z(x) = \text{dist}(x, \partial\Omega)$, $y(x)$ denotes the closest point to x on $\partial\Omega$ and $\rho = \frac{m}{|\Omega|}$. Moreover, as $\varepsilon \rightarrow 0$, we obtain

$$\begin{aligned}
 \partial_\nu u_\varepsilon(\xi) &= -\frac{1}{\varepsilon} \sqrt{2\rho F(\bar{u})} + \frac{|\partial\Omega|}{|\Omega|} Q_F(\bar{u}) \sqrt{\frac{F(\bar{u})}{2}} \\
 &\quad + (n-1) H_{\partial\Omega}(\xi) \sqrt{\frac{1}{2\rho F(\bar{u})}} \mathfrak{F}(\bar{u}) + o_\varepsilon(1), \quad \forall \xi \in \partial\Omega,
 \end{aligned}$$

where $\mathfrak{F}(t)$ is defined in (2.8). For any $a \in (0, \bar{u})$, we represent any point $x \in \{\xi \in \Omega \mid u_\varepsilon(\xi) = a\}$ by the Fermi coordinates, i.e.,

$$x = y(x) + z(x)\nu(y(x)).$$

Then the boundary-layer thickness, represented by $z(x)$, has the following expansion

$$(2.19) \quad z(x) = \varepsilon z_{1,a} + \varepsilon^2 \frac{1}{W'(z_{1,a})} \left(\sqrt{\rho} \frac{|\partial\Omega|}{|\Omega|} Q_F(\bar{u}) \phi_1(z_{1,a}) - (n-1) H_{\partial\Omega}(y(x)) \phi_2(z_{1,a}) \right) + o_\varepsilon(1) \varepsilon^2,$$

where $z_{1,a}$ is the unique point such that $W(z) = a$.

Remark 2.1. We have two remarks for the main results in Theorem 2.3.

- (1) Note that $W' < 0$, $\phi_1 < 0$, and $\phi_2 > 0$, from (2.19), we observe that the boundary-layer thickness increases with respect to the boundary curvature and surface area of the domain, but decreases with respect to the volume of the domain.
- (2) Let $(u_\varepsilon, v_\varepsilon)$ be a solution of (1.4). Then, using Theorem 2.3 and the relation (1.6) with $\phi(u) = u$ and $\chi = 1$ between u_ε and v_ε , we can also obtain the expansion of v_ε near the boundary: for any $x \in \Omega_\delta$,

$$v_\varepsilon(x) = \frac{m}{|\Omega|} e^{W(\frac{z(x)}{\varepsilon})} \left[1 - \varepsilon \rho^{-\frac{1}{2}} \frac{|\partial\Omega|}{|\Omega|} Q_F(\bar{u}) - \varepsilon \sqrt{\rho} \frac{|\partial\Omega|}{|\Omega|} Q_F(\bar{u}) \phi_1\left(\frac{z(x)}{\varepsilon}\right) + \varepsilon(n-1) H_{\Gamma_{z(x)}}(y(x)) \phi_2\left(\frac{z(x)}{\varepsilon}\right) + O(\varepsilon^2) \right]$$

and, for $\xi \in \partial\Omega$

$$\partial_\nu v_\varepsilon(\xi) = \frac{m e^{\bar{u}}}{|\Omega|} \left[-\frac{1}{\varepsilon} \sqrt{2\rho F(\bar{u})} + \frac{|\partial\Omega|}{|\Omega|} Q_F(\bar{u}) \sqrt{2F(\bar{u})} + \frac{|\partial\Omega|}{|\Omega|} Q_F(\bar{u}) \sqrt{\frac{F(\bar{u})}{2}} + (n-1) H_{\partial\Omega}(\xi) \sqrt{\frac{1}{2\rho F(\bar{u})}} \mathfrak{F}(\bar{u}) + o_\varepsilon(1) \right].$$

2.3. Logarithmic sensitivity. For the case $\phi(u) = \log u$, the nonlocal problem (1.7) can be written as

$$(2.20) \quad \begin{cases} \varepsilon^2 \Delta u = \frac{m}{\int_\Omega u^\chi dx} u^{\chi+1} & \text{in } \Omega, \\ u = \bar{u} & \text{on } \partial\Omega. \end{cases}$$

Compared with the case of linear sensitivity, it is more challenging to study the asymptotic behavior of solutions as $\varepsilon \rightarrow 0$ for the nonlocal problem (2.20) since the denominator $\int_\Omega u^\chi dx \rightarrow 0$ as $\varepsilon \rightarrow 0$ (see (2.23) and [6, Lemma 4.4] for the case $\chi > 0$), which generates a singularity. In [6], we have given a qualitative study on the nonlocal problem (2.20), including the existence and uniqueness of solutions for the general domain and asymptotic profile of the solution on the boundary as $\varepsilon \rightarrow 0$ for the radially symmetric domain. In particular, we found that if $\chi > 2$, the second-order term in the expansion of the normal derivative of the boundary-layer profile on the boundary becomes $\log \varepsilon$ in the radially symmetric domain. In this paper, we shall prove this is also true for the general smooth domain. To state our result, we introduce constants

$$(2.21) \quad A = \left(\frac{2(\chi+2)}{m\chi^2} \right)^{\frac{1}{\chi}}, \quad B = \left(\frac{A}{\bar{u}} \right)^{\frac{\chi}{2}}, \quad q^- = \frac{1 - \sqrt{1 + \frac{8(\chi+1)(\chi+2)}{\chi^2}}}{2} < -\frac{2}{\chi}$$

and functions

$$(2.22) \quad U(z) = \frac{A}{(B+z)^{\frac{2}{\chi}}}, \quad \phi_3(z) = \frac{A}{\chi+4}(B+z)^{\frac{\chi-2}{\chi}} - \frac{AB^{\frac{\chi-2}{\chi}-q^-}}{\chi+4}(B+z)^{q^-}.$$

Then we state the last result of this paper as follows.

Theorem 2.4. *Let u_ε be the solution of (2.20) with $\chi > 2$ and $H_{\partial\Omega}(x)$ denote the mean curvature of $x \in \partial\Omega$. Denote $\lambda_\varepsilon = \left(\int_\Omega u_\varepsilon^\chi\right)^{\frac{1}{2}}$. Then as $\varepsilon \rightarrow 0$, we have*

$$(2.23) \quad \lambda_\varepsilon = \frac{A^\chi}{B} \varepsilon |\partial\Omega| \left(1 - \frac{8(n-1)A^\chi}{\chi+4} \varepsilon^2 |\log \varepsilon| \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y\right) + O(\varepsilon^3),$$

and for any $x \in \Omega_\delta$, it holds that

$$\begin{aligned} & \left| u_\varepsilon(x) - U\left(\frac{z(x)}{\varepsilon\lambda_\varepsilon}\right) - \varepsilon\lambda_\varepsilon(n-1)H_{\Gamma_{z(x)}}(y(x))\phi_3\left(\frac{z(x)}{\varepsilon\lambda_\varepsilon}\right) \right| \\ & \leq C(\varepsilon\lambda_\varepsilon)^2 \left(1 + \frac{z(x)}{\varepsilon\lambda_\varepsilon}\right)^{2-\frac{2}{\chi}}. \end{aligned}$$

For any $\xi \in \partial\Omega$, we have

$$\partial_\nu u_\varepsilon(\xi) = -\frac{\chi m \bar{u}}{\varepsilon^2(\chi+2)|\partial\Omega|} - \frac{16(n-1)}{\chi(\chi+4)} \frac{\bar{u}}{|\partial\Omega|} |\log \varepsilon| \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y + O(1).$$

In addition, for any $a \in (0, \bar{u})$, we represent any point $x \in \{x \in \Omega \mid u_\varepsilon(x) = a\}$ by the Fermi-coordinate, i.e.,

$$x = y(x) + z(x)\nu(y(x)).$$

Then $z(x)$ admits the following expansion

$$(2.24) \quad z(x) = \left(\frac{A^\chi}{B} |\partial\Omega| \varepsilon^2 - \frac{A^\chi}{B} \frac{8(n-1)A^\chi}{\chi+4} |\partial\Omega| \varepsilon^4 |\log \varepsilon| \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y\right) z_{u,a} + O(\varepsilon^4)$$

where $z_{u,a} = \left(\frac{A}{a}\right)^{\frac{2}{\chi}} - B$ is the unique point such that $U(z) = a$.

Remark 2.2. Unlike the case of linear sensitivity, we are unable to identify the effect of mean curvature at each boundary point on the boundary-layer thickness in the case of logarithmic sensitivity as shown in (2.24) by which we see the total curvature will decrease the boundary-layer thickness. By the expansions given in Theorem 2.4, one easily observes that the expansion is irregular with respect to ε . This is perhaps caused by the logarithmic singularity. How to find the effect of mean curvature at each boundary point for the logarithmic sensitivity is still an open challenging question. Let $(u_\varepsilon, v_\varepsilon)$ be a solution of (1.4). Similar to Remark 2.1, using Theorem 2.4 and the relation (1.6), we can obtain the expansion of v_ε near the boundary: for any $x \in \Omega_\delta$,

$$\begin{aligned} v_\varepsilon(x) &= \frac{1}{\varepsilon^2} \frac{mB^2}{A^{2\chi}|\partial\Omega|^2} U^\chi\left(\frac{z(x)}{\varepsilon\lambda_\varepsilon}\right) \left[1 + \frac{16(n-1)A^\chi}{\chi+4} \varepsilon^2 |\log \varepsilon| \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y\right] \\ &+ O\left(\left(1 + \frac{z(x)}{\varepsilon\lambda_\varepsilon}\right)^{-1}\right) \end{aligned}$$

and, for $\xi \in \partial\Omega$

$$\partial_\nu v_\varepsilon(\xi)$$

$$= -\frac{1}{\varepsilon^2} \frac{m^2 \chi^2}{2(\chi+2)|\partial\Omega|^3} \left[\frac{\chi m}{\varepsilon^2(\chi+2)} + \frac{48(n-1)}{\chi(\chi+4)} |\log \varepsilon| \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y + O(1) \right].$$

The organization of this paper is as follows. In Section 3, we construct a refined approximation of the solution to (2.7) and prove Theorem 2.3 in Section 4. In Section 5, we shall derive the counterpart results for the case of logarithmic sensitivity and prove Theorem 2.4.

3. PROOF OF THEOREM 2.3 (CASE OF LINEAR SENSITIVITY)

In this section, we shall consider the case of linear sensitivity and prove related main theorems (Proposition 2.2 and Theorem 2.3). We start with an effective approximate solution.

3.1. Approximate solutions. In this subsection, we present some preliminary results and derive an effective approximate solution to the following nonlocal problem

$$(3.1) \quad \begin{cases} \varepsilon^2 \Delta u = \frac{m}{\int_{\Omega} e^u dx} u e^u & \text{in } \Omega, \\ u = \bar{u} & \text{on } \partial\Omega. \end{cases}$$

For the non-negative solutions of (3.1), we have the following estimate.

Lemma 3.1. *Let Ω be a smooth bounded domain and u_ε be a solution to (3.1). For $\varepsilon > 0$ sufficiently small, there exists a positive constant C independent of ε such that*

$$(3.2) \quad u_\varepsilon(x) \leq \bar{u} \exp\left(-\frac{C}{\varepsilon} \text{dist}(x, \partial\Omega)\right), \quad \forall x \in \Omega.$$

Proof. First we recall in the proof of [23, Corollary 2.3] (see [23, (4.18)]), it was shown that

$$(3.3) \quad u_\varepsilon(x) \leq \bar{u} \exp\left(-\frac{C_\delta}{\varepsilon} \text{dist}(x, \partial\Omega)\right), \quad \text{for all } x \in \Omega_\delta = \{x \in \Omega \mid 0 < \text{dist}(x, \partial\Omega) < \delta\}$$

for some constant $C_\delta > 0$ independent of ε . In the following, for convenience, we denote the interior of the complement of Ω_δ in Ω by Ω_δ^c , namely,

$$\Omega_\delta^c = \overline{\Omega} \setminus \overline{\Omega_\delta}.$$

For each $x \in \Omega_\delta^c$, it was proved in [23, (4.12)] that there is a constant $M(\overline{\Omega_\delta^c}) > 0$ independent of ε , such that

$$(3.4) \quad u_\varepsilon(x) \leq C(\overline{\Omega_\delta^c}) e^{-M(\overline{\Omega_\delta^c})/\varepsilon}.$$

Now we choose

$$C = \min \left\{ C_\delta, \frac{M(\overline{\Omega_\delta^c})}{2 \text{Diam}(\Omega)} \right\} \quad \text{and} \quad \varepsilon < \frac{M(\overline{\Omega_\delta^c})}{2 \log \frac{C(\overline{\Omega_\delta^c}) + \bar{u}}{\bar{u}}},$$

where $\text{Diam}(\Omega) := \max_{x_1, x_2 \in \partial\Omega} \text{dist}(x_1, x_2)$. Then we get (3.2) by using (3.3) and (3.4). $\square$

In order to get the first-order approximation of the solution to (3.1), we need to study the positive solution of the following ODE problem

$$(3.5) \quad \begin{cases} W''(z) = \rho W(z) e^{W(z)}, & z > 0, \\ W(0) = \bar{u}, & W(z) \rightarrow 0 \text{ as } z \rightarrow +\infty. \end{cases}$$

Consider the following function

$$W_1(z) := \bar{u} \exp\left(-\sqrt{\rho e^{\bar{u}}} z\right).$$

Obviously, it follows that

$$W_1(z) \text{ is a strictly decreasing function, } W_1(0) = \bar{u} \text{ and } \lim_{z \rightarrow \infty} W_1(z) = 0.$$

By a direct computation, one has

$$W_1''(z) = \rho W_1(z) e^{\bar{u}} \geq \rho W_1(z) e^{W_1(z)}.$$

Therefore, $W_1(z)$ is a sub-solution to (3.5). Similarly, we can show that the function $W_2(z) = \bar{u} \exp(-\sqrt{\rho} z)$ provides a super-solution of (3.5). Then by the method of sub-super solutions, we can get a unique positive solution of (3.5), denoted by $W(z)$, satisfying

$$(3.6) \quad \bar{u} \exp\left(-\sqrt{\rho e^{\bar{u}}} z\right) \leq W(z) \leq \bar{u} \exp(-\sqrt{\rho} z).$$

Using (3.5), we can see that $W''(z)$ does not change sign, which implies that $W(z)$ is a convex and decreasing function with exponential decay at infinity.

In addition, we can show that $W'(z)$ also decays exponentially at infinity. Indeed multiplying the first equation (3.5) by W' and integrating the result from z to ∞ , we derive that

$$\frac{1}{2}(W'(z))^2 = \rho(W(z)e^{W(z)} - e^{W(z)} + 1),$$

which yields

$$(3.7) \quad W'(z) = -\sqrt{2\rho F(W(z))} < 0, \quad \text{where } F(s) = se^s - e^s + 1.$$

It is straightforward to check that

$$(3.8) \quad 2e^{\bar{u}}x \geq xe^x - e^x + 1 \geq \frac{1}{2}x^2, \quad \forall x \in (0, \bar{u}).$$

Therefore, using (3.8) and (3.7), we have

$$(3.9) \quad -2\sqrt{\rho e^{\bar{u}} W(z)} \leq W'(z) \leq -\sqrt{\rho} W(z).$$

Summarizing the above results in (3.5)–(3.9), we get Lemma 3.2.

Lemma 3.2. *The problem (3.5) admits a unique solution W , which is a strictly convex and monotonically decreasing function. Moreover, for all $z > 0$, it holds that*

$$(3.10) \quad W(0) = \bar{u} > 0, \quad 0 < W(z) \leq \bar{u} \exp(-\sqrt{\rho} z),$$

and

$$(3.11) \quad W'(0) = -\sqrt{2\rho F(\bar{u})}, \quad W'(z) = -\sqrt{2\rho F(W(z))}, \quad |W'(z)| \leq 2\sqrt{\rho \bar{u} e^{\bar{u}}} \exp\left(-\frac{1}{2}\sqrt{\rho} z\right).$$

Lemma 3.3 asserts that $W\left(\frac{z(x)}{\varepsilon}\right)$ can be regarded as the leading order approximation of $u_\varepsilon(x)$.

Lemma 3.3. *Let u_ε and W be the solutions of (3.1) and (3.5) respectively. Then for $\varepsilon > 0$ sufficiently small, we have*

$$\max_{x \in \Omega} \left| u_\varepsilon(x) - W\left(\frac{z(x)}{\varepsilon}\right) \right| \leq C\varepsilon.$$

Proof. For $x \in \Omega \setminus \Omega_\delta$, by Lemma 3.1 and Lemma 3.2, we have

$$(3.12) \quad \left| u_\varepsilon(x) - W\left(\frac{z(x)}{\varepsilon}\right) \right| \leq u_\varepsilon(x) + W\left(\frac{z(x)}{\varepsilon}\right) \leq C \exp\left(-\frac{\delta}{\varepsilon}\right) \leq C\varepsilon$$

for $\varepsilon > 0$ sufficiently small. It remains to prove that the inequality holds for $x \in \Omega_\delta$ with $\delta > 0$ small. When $x \in \Omega_\delta$, we write $x = y(x) + z(x)\nu(y(x))$ by the Fermi coordinate, as introduced in Section 2.1. Then by the definition of W in (3.5), we have

$$(3.13) \quad \begin{aligned} \varepsilon^2 \Delta \left(u_\varepsilon(x) - W\left(\frac{z(x)}{\varepsilon}\right) \right) &= \frac{mf(u_\varepsilon(x))}{\int_\Omega e^{u_\varepsilon} dx} - \rho f\left(W\left(\frac{z(x)}{\varepsilon}\right)\right) - \varepsilon W'\left(\frac{z(x)}{\varepsilon}\right) \Delta z(x) \\ &= \rho \left(f(u_\varepsilon(x)) - f\left(W\left(\frac{z(x)}{\varepsilon}\right)\right) \right) + E_u(x), \end{aligned}$$

where f is defined in (2.8) and

$$(3.14) \quad E_u(x) = B_\varepsilon f(u_\varepsilon(x)) - \varepsilon W'\left(\frac{z(x)}{\varepsilon}\right) \Delta z(x) \quad \text{and} \quad B_\varepsilon = \frac{m}{\int_\Omega e^{u_\varepsilon} dx} - \rho.$$

From [23, (4.13)], see also (2.10), one can directly check that

$$(3.15) \quad |B_\varepsilon| \leq C\varepsilon.$$

As a consequence of (3.15) and (3.11), one can get that $|E_u| \leq C\varepsilon$. It is known that

$$(3.16) \quad u_\varepsilon(x) - W\left(\frac{z(x)}{\varepsilon}\right) = 0 \quad \text{for all } x \in \partial\Omega.$$

With (3.12) and (3.16) in hand, it remains to prove that Lemma 3.3 also holds for $x \in \Omega_\delta$. Suppose $p \in \Omega_\delta$ is the point where $|u_\varepsilon(x) - W\left(\frac{z(x)}{\varepsilon}\right)|$ obtains its maximum. Without loss of generality, we may assume that

$$u_\varepsilon(p) - W\left(\frac{z(p)}{\varepsilon}\right) \geq 0.$$

Then using the fact f is a strictly increasing function, we have

$$(3.17) \quad 0 \leq \rho f'(\theta) \left(u_\varepsilon(p) - W\left(\frac{z(p)}{\varepsilon}\right) \right) \leq |E_u|,$$

where $\theta \in \left(W\left(\frac{z(p)}{\varepsilon}\right), u_\varepsilon(p)\right)$. Using $f'(\theta) = e^\theta + \theta e^\theta \geq 1$, we get from (3.17) that

$$\left| u_\varepsilon(p) - W\left(\frac{z(p)}{\varepsilon}\right) \right| \leq C\rho^{-1}\varepsilon.$$

While if $u_\varepsilon(p) - W\left(\frac{z(p)}{\varepsilon}\right)$ is negative, we can still get (3.17). Combined (3.12) with (3.16), we finish the proof. $\square$

As mentioned in Section 2, we need to study the following two ODEs, which will help us pinpoint the effects of nonlocal term and mean curvature of the domain surface on the boundary-layer profiles, respectively,

$$(3.18) \quad \begin{cases} \phi_1''(z) = \rho (W(z)e^{W(z)} + e^{W(z)})\phi_1(z) + W(z)e^{W(z)}, & z > 0, \\ \phi_1(0) = 0, \quad \phi_1(z) \rightarrow 0 \text{ as } z \rightarrow \infty, \end{cases}$$

and

$$(3.19) \quad \begin{cases} \phi_2''(z) = \rho (W(z)e^{W(z)} + e^{W(z)})\phi_2(z) + W'(z), & z > 0, \\ \phi_2(0) = 0, \quad \phi_2(z) \rightarrow 0 \text{ as } z \rightarrow \infty. \end{cases}$$

Concerning equations (3.18)–(3.19), we have the following results.

Lemma 3.4. *Let $F(z)$ and $\mathfrak{F}(z)$ be given in (2.8). Then the following results hold.*

(i) *The problem (3.18) admits a unique solution $\phi_1(z)$, which satisfies*

$$(3.20) \quad \begin{aligned} \phi_1(z) &\leq 0, \quad \phi_1'(z) + \sqrt{\frac{\rho}{2F(W(z))}} f(W(z))\phi_1(z) = -\sqrt{\frac{F(W(z))}{2\rho}}, \\ \phi_1'(0) &= -\sqrt{\frac{F(\bar{u})}{2\rho}}, \end{aligned}$$

and there exist some constants C and M independent of ε such that

$$(3.21) \quad |\phi_1(z)| + |\phi_1'(z)| \leq Ce^{-Mz}, \quad z \in [0, \infty).$$

(ii) *The problem (3.19) admits a unique solution $\phi_2(z)$ satisfying*

$$(3.22) \quad \begin{aligned} \phi_2'(z) + \sqrt{\frac{\rho}{2F(W(z))}} f(W(z))\phi_2(z) &= \sqrt{\frac{1}{2\rho F(W(z))}} \mathfrak{F}(W(z)), \\ \phi_2(z) &\geq 0, \quad \phi_2'(0) = \sqrt{\frac{1}{2\rho F(\bar{u})}} \mathfrak{F}(\bar{u}), \end{aligned}$$

and there exist some constants C and M independent of ε such that

$$(3.23) \quad |\phi_2(z)| + |\phi_2'(z)| \leq Ce^{-Mz}, \quad z \in [0, \infty).$$

Proof. We first prove the assertion (i). It is clear that $\phi_{1,\text{sup}} = 0$ provides a super solution to (3.18). Define a function $\phi_{1,\text{sub}}(z) = -\frac{2\bar{u}e^{\bar{u}}}{\rho}e^{-\frac{1}{2}\sqrt{\rho}z}$. By a direct computation, we have

$$\begin{aligned} \phi_{1,\text{sub}}''(z) - \rho(W(z)e^{W(z)} + e^{W(z)})\phi_{1,\text{sub}}(z) &\geq \frac{\rho}{4}\phi_{1,\text{sub}} - \rho\phi_{1,\text{sub}} = -\frac{3}{4}\rho\phi_{1,\text{sub}} \\ &= \frac{3}{2}\bar{u}e^{\bar{u}}e^{-\frac{1}{2}\sqrt{\rho}z} > W(z)e^{W(z)}, \end{aligned}$$

where we have used $W(z) \leq \bar{u}$ (see Lemma 3.2) and (3.10). Therefore $\phi_{1,\text{sub}}$ is a sub-solution to (3.18). By the method of super-lower solutions, we get a negative solution ϕ_1 to (3.18) satisfying

$$(3.24) \quad -\frac{2\bar{u}e^{\bar{u}}}{\rho}e^{-\frac{1}{2}\sqrt{\rho}z} \leq \phi_1(z) \leq 0,$$

which also indicates that $\phi_1(z)$ decays exponentially to zero as $z \rightarrow \infty$. Using (3.18) and (3.10) we can see that $\phi_1''(z)$ also decays exponentially, which together with (3.24) implies that $\phi_1'(z)$ decays exponentially at infinity. Hence (3.21) holds.

It remains to show the second identity in (3.20) since the third equality of (3.20) follows easily from it. Multiplying equation (3.18) by ϕ_1' and integrating from z to ∞ , we have

$$(3.25) \quad -\frac{1}{2}(\phi_1'(z))^2 = \int_z^\infty \phi_1''(s)\phi_1'(s)ds = \int_z^\infty (\rho f'(W(s))\phi_1(s)\phi_1'(s) + f(W(s))\phi_1'(s))ds.$$

Using (3.5) and (3.18), we get that

$$(3.26) \quad \begin{aligned} -\frac{1}{2}\left(\phi_1'(z) - \frac{1}{\rho}W'(z)\right)^2 &= \int_z^\infty \left(\phi_1(s) - \frac{1}{\rho}W(s)\right)'' \left(\phi_1(s) - \frac{1}{\rho}W(s)\right)' ds \\ &= \int_z^\infty \rho f'(W(s))\phi_1(s) \left(\phi_1'(s) - \frac{1}{\rho}W'(s)\right) ds \\ &= f(W(z))\phi_1(z) \\ &\quad + \int_z^\infty (\rho f'(W(s))\phi_1(s)\phi_1'(s) + f(W(s))\phi_1'(s)) ds. \end{aligned}$$

Using (3.25) and (3.26), we obtain

$$\frac{1}{2}(\phi_1'(z))^2 - \frac{1}{2}\left(\phi_1'(z) - \frac{1}{\rho}W'(z)\right)^2 = f(W(z))\phi_1(z).$$

This gives

$$\phi_1'(z)W'(z) - \frac{1}{2\rho}(W'(z))^2 = \rho f(W(z))\phi_1(z).$$

By (3.11), we see that $W'(z) < 0$ for z positive. Hence from the above equation, we can further derive that

$$\phi_1'(z) - \frac{1}{2\rho}W'(z) = \frac{\rho f(W(z))}{W'(z)}\phi_1(z).$$

Together with the fact $W'(z) = -\sqrt{2\rho F(W(z))}$ (see (3.11)), we get the second identity in (3.20). This completes the proof of (3.20).

For the second assertion (ii) of the lemma, as we did for proving the existence of ϕ_1 , one can see that $\phi_{2,\text{sub}} = 0$ and $\phi_{2,\text{sup}} = \frac{8}{3}\sqrt{\frac{\bar{u}e^{\bar{u}}}{\rho}}\exp(-\frac{1}{2}\sqrt{\rho}x)$ provide the sub-solution and super-solution of (3.19), respectively. Then we can show both $\phi_2(z)$ and $\phi_2''(z)$ exponentially decay as done for ϕ_1 , which implies that $\phi_2'(z)$ also exponentially decays as $|z| \rightarrow \infty$. Thus (3.23) is obtained. To derive the second identity in (3.22), using (3.5), (3.10) and (3.19), we have

$$(3.27) \quad \begin{aligned} 0 &= \int_z^\infty (W''(s) - \rho f(W(s)))\phi_2'(s)ds \\ &= -W'(z)\phi_2'(z) - \int_z^\infty (W'(s)\phi_2''(s) + \rho f(W(s))\phi_2'(s))ds \\ &= -W'(z)\phi_2'(z) - \int_z^\infty \left[(\rho f(W(s))\phi_2(s))' - \sqrt{2\rho F(W(s))}W'(s)\right]ds \\ &= \sqrt{2\rho F(W(z))}\phi_2'(z) + \rho f(W(z))\phi_2(z) - \int_0^{W(z)} \sqrt{2\rho F(s)}ds. \end{aligned}$$

This gives the second equation in (3.22) and $\phi'_2(0) = \sqrt{\frac{1}{2\rho F(\bar{u})}} \mathfrak{F}(\bar{u})$ follows directly. Thus the proof is completed. $\square$

From the lemma concerning ϕ_1 and ϕ_2 above, we can find an improved approximation of $u_\varepsilon(x)$ in a small neighborhood Ω_δ of $\partial\Omega$. We recall the definition of δ_0 in (2.1) and fix $\delta \in (0, \delta_0)$.

Lemma 3.5. *Let u_ε and W be the solutions of (3.1) and (3.5) respectively. Then for $\varepsilon > 0$ sufficiently small, there exists a constant C independent of ε such that*

$$(3.28) \quad \max_{x \in \Omega_\delta} \left| u_\varepsilon(x) - W\left(\frac{z(x)}{\varepsilon}\right) - B_\varepsilon \phi_1\left(\frac{z(x)}{\varepsilon}\right) - \varepsilon(n-1)H_{\Gamma_{z(x)}}(y(x))\phi_2\left(\frac{z(x)}{\varepsilon}\right) \right| \leq C\varepsilon^2,$$

where $H_{\Gamma_{z(x)}}(y(x))$ is introduced in (2.2) and B_ε is defined in (3.14).

Proof. For $x \in \Omega_\delta$, we write it as $x = y(x) + z(x)\nu(y(x))$ with $y(x) \in \partial\Omega$ and $z(x) = \text{dist}(x, y(x))$. Using the form of Laplace operator in terms of the Fermi-coordinate (2.3), we rewrite (3.18) and (3.19) in Ω_δ as

$$\begin{aligned} \varepsilon^2 \Delta \phi_1\left(\frac{z(x)}{\varepsilon}\right) &= \rho f'\left(W\left(\frac{z(x)}{\varepsilon}\right)\right) \phi_1\left(\frac{z(x)}{\varepsilon}\right) + f\left(W\left(\frac{z(x)}{\varepsilon}\right)\right) \\ &\quad - \varepsilon(n-1)H_{\Gamma_{z(x)}}(y(x))\phi'_1\left(\frac{z(x)}{\varepsilon}\right), \\ \varepsilon^2 \Delta \phi_2\left(\frac{z(x)}{\varepsilon}\right) &= \rho f'\left(W\left(\frac{z(x)}{\varepsilon}\right)\right) \phi_2\left(\frac{z(x)}{\varepsilon}\right) + W'\left(\frac{z(x)}{\varepsilon}\right) \\ &\quad - \varepsilon(n-1)H_{\Gamma_{z(x)}}(y(x))\phi'_2\left(\frac{z(x)}{\varepsilon}\right), \end{aligned}$$

where we have used the fact that $\phi_1\left(\frac{z(x)}{\varepsilon}\right)$ and $\phi_2\left(\frac{z(x)}{\varepsilon}\right)$ are constants on $\Gamma_{z(x)}$.

Similarly, using (2.3) and (2.4), we can rewrite (3.13) as

$$(3.29) \quad \begin{aligned} &\varepsilon^2 \Delta \left(u_\varepsilon(x) - W\left(\frac{z(x)}{\varepsilon}\right) \right) \\ &= \frac{mf(u_\varepsilon(x))}{\int_\Omega e^{u_\varepsilon} dx} - \rho f\left(W\left(\frac{z(x)}{\varepsilon}\right)\right) + \varepsilon(n-1)H_{\Gamma_{z(x)}}(y(x))W'\left(\frac{z(x)}{\varepsilon}\right). \end{aligned}$$

Due to the presence of the nonlocal (integration) term and $W'\left(\frac{z(x)}{\varepsilon}\right)$ in (3.29), we define the approximate solution by

$$(3.30) \quad u_{\varepsilon, \text{app}} := W\left(\frac{z(x)}{\varepsilon}\right) + B_\varepsilon \phi_1\left(\frac{z(x)}{\varepsilon}\right) + \varepsilon(n-1)H_{\Gamma_{z(x)}}(y(x))\phi_2\left(\frac{z(x)}{\varepsilon}\right).$$

By direct computations, we have

$$(3.31) \quad \varepsilon^2 \Delta (u_\varepsilon(x) - u_{\varepsilon, \text{app}}(x)) = \rho f'\left(W\left(\frac{z(x)}{\varepsilon}\right)\right) (u_\varepsilon(x) - u_{\varepsilon, \text{app}}(x)) + \sum_{i=1}^4 E_{i, \varepsilon},$$

where

$$\begin{aligned} E_{1,\varepsilon} &:= \varepsilon(n-1)B_\varepsilon H_{\Gamma_{z(x)}}(y(x))\phi_1' \left(\frac{z(x)}{\varepsilon} \right) + \varepsilon^2(n-1)^2 H_{\Gamma_{z(x)}}^2(y(x))\phi_2' \left(\frac{z(x)}{\varepsilon} \right), \\ E_{2,\varepsilon} &:= \varepsilon^3(n-1) \left[H_{\Gamma_{z(x)}}(y(x))\Delta\phi_2 \left(\frac{z(x)}{\varepsilon} \right) - \Delta \left(H_{\Gamma_{z(x)}}(y(x))\phi_2 \left(\frac{z(x)}{\varepsilon} \right) \right) \right], \\ E_{3,\varepsilon} &:= B_\varepsilon \left(f(u_\varepsilon) - f \left(W \left(\frac{z(x)}{\varepsilon} \right) \right) \right), \\ E_{4,\varepsilon} &:= \rho \left[f(u_\varepsilon(x)) - f \left(W \left(\frac{z(x)}{\varepsilon} \right) \right) - f' \left(W \left(\frac{z(x)}{\varepsilon} \right) \right) \left(u_\varepsilon(x) - W \left(\frac{z(x)}{\varepsilon} \right) \right) \right]. \end{aligned}$$

Below we shall estimate the error terms $E_{i,\varepsilon}$ ($i = 1, 2, 3, 4$). Using the fact that Ω is a smooth domain and hence the mean curvature $H_{\Gamma_{z(x)}}(y(x))$ is uniformly bounded, (3.15) and Lemma 3.4, we have

$$(3.32) \quad \max_{x \in \Omega_\delta} |E_{1,\varepsilon}(x)| \leq C\varepsilon|B_\varepsilon| + C\varepsilon^2 \leq C\varepsilon^2.$$

By Lemma 3.4 (ii), we get that

$$\begin{aligned} (3.33) \quad & \max_{x \in \Omega_\delta} |E_{2,\varepsilon}(x)| \\ &= \varepsilon^3(n-1) \max_{x \in \Omega_\delta} \left| \phi_2 \left(\frac{z(x)}{\varepsilon} \right) \Delta H_{\Gamma_{z(x)}}(y(x)) + \frac{2}{\varepsilon} \phi_2' \left(\frac{z(x)}{\varepsilon} \right) \nabla H_{\Gamma_{z(x)}}(y(x)) \nabla z(x) \right| \\ &\leq C\varepsilon^2, \end{aligned}$$

where we have used the fact that $\nabla H_{\Gamma_{z(x)}}(y(x))$ is uniformly bounded due to the smoothness of the domain Ω and $|\nabla z(x)|$ is bounded by its definition. For the third one, using (3.15) and Lemma 3.3, we obtain that

$$(3.34) \quad \max_{x \in \Omega_\delta} |E_{3,\varepsilon}(x)| \leq (e^{\bar{u}} + \bar{u}e^{\bar{u}})|B_\varepsilon| \max_{x \in \Omega_\delta} \left| u(x) - W \left(\frac{z(x)}{\varepsilon} \right) \right| \leq C\varepsilon^2.$$

Concerning the last one, by Lemma 3.3, we have

$$(3.35) \quad \max_{x \in \Omega_\delta} |E_{4,\varepsilon}(x)| \leq \rho(2e^{\bar{u}} + \bar{u}e^{\bar{u}}) \max_{x \in \Omega_\delta} \left(u_\varepsilon(x) - W \left(\frac{z(x)}{\varepsilon} \right) \right)^2 \leq C\varepsilon^2.$$

As a consequence of (3.32)–(3.35), we have

$$(3.36) \quad \max_{x \in \Omega_\delta} \left| \sum_{i=1}^4 E_{i,\varepsilon} \right| \leq C\varepsilon^2.$$

We set

$$\psi_\varepsilon(x) = u_\varepsilon(x) - u_{\varepsilon,\text{app}}(x).$$

Using (3.31), we have

$$(3.37) \quad \varepsilon^2 \Delta \psi_\varepsilon = \rho f' \left(W \left(\frac{z(x)}{\varepsilon} \right) \right) \psi_\varepsilon + \sum_{i=1}^4 E_{i,\varepsilon} \quad \text{in } \Omega_\delta.$$

On $\partial\Omega$ we have

$$(3.38) \quad \max_{x \in \partial\Omega} \psi_\varepsilon = 0,$$

while on $\partial\Omega \setminus \partial\Omega_\delta$, using Lemma 3.1, Lemma 3.2 and Lemma 3.4, we have

$$(3.39) \quad \max_{x \in \partial\Omega_\delta \setminus \partial\Omega} |\psi_\varepsilon| \leq C e^{-C \frac{\delta}{\varepsilon}} \leq C \varepsilon^2,$$

provided $\varepsilon > 0$ is sufficiently small. In the interior of Ω_δ , using (3.37) we have

$$\min_{\Omega_\delta} \psi_\varepsilon \geq - \frac{\left| \sum_{i=1}^4 E_{i,\varepsilon} \right|}{\rho \min_{\Omega_\delta} f' \left(W \left(\frac{z(x)}{\varepsilon} \right) \right)} \quad \text{and} \quad \max_{\Omega_\delta} \psi_\varepsilon \leq \frac{\left| \sum_{i=1}^4 E_{i,\varepsilon} \right|}{\rho \min_{\Omega_\delta} f' \left(W \left(\frac{z(x)}{\varepsilon} \right) \right)},$$

which together with (3.36) implies

$$(3.40) \quad |\psi_\varepsilon(x)| \leq \frac{\left| \sum_{i=1}^4 E_{i,\varepsilon} \right|}{2\rho} \leq C \varepsilon^2.$$

Here we have used the simple fact $f'(s) = 2se^s + e^s \geq 2$ for $s \geq 0$. Combining (3.38), (3.39) and (3.40), we finally obtain that

$$|\psi_\varepsilon(x)| \leq C \varepsilon^2 \quad \text{for } x \in \Omega_\delta.$$

It implies the desired conclusion (3.28). $\square$

Remark 3.1. For $x := x_\varepsilon \in \overline{\Omega_\delta}$ satisfying that $\lim_{\varepsilon \rightarrow 0} \frac{z(x_\varepsilon)}{\varepsilon} < \infty$, we have by the smoothness of the domain

$$|H_{\partial\Omega}(y(x)) - H_{\Gamma_{z(x)}}(y(x))| \leq C z(x) \leq C \varepsilon.$$

Then from Lemma 3.5, we get that

$$\begin{aligned} u_\varepsilon(x) &= W \left(\frac{z(x)}{\varepsilon} \right) + \left(\frac{m}{\int_\Omega e^{u_\varepsilon} dx} - \rho \right) \phi_1 \left(\frac{z(x)}{\varepsilon} \right) \\ &\quad + \varepsilon(n-1) H_{\partial\Omega}(z(x)) \phi_2 \left(\frac{z(x)}{\varepsilon} \right) + O(\varepsilon^2). \end{aligned}$$

3.2. Expansions of boundary-layer profile and thickness. In this section, we will prove Proposition 2.2 and Theorem 2.3. A crucial step is to derive the expansion of the nonlocal integration, which highlights the primary difference from the local problem.

We start with Lemma 3.6.

Lemma 3.6. *Let $W(z), \phi_1(z), \phi_2(z)$ be the solutions of equations (3.5), (3.18), (3.19) respectively, and $\delta(\varepsilon)$ be a sequence of numbers such that*

$$\delta(\varepsilon) \rightarrow 0 \quad \text{and} \quad \frac{\delta(\varepsilon)}{\varepsilon} \rightarrow \infty \quad \text{as } \varepsilon \rightarrow 0.$$

Then it follows that

$$(3.41) \quad \int_0^{\frac{\delta(\varepsilon)}{\varepsilon}} (e^{W(t)} - 1) dt = \rho^{-\frac{1}{2}} Q_F(\bar{u}) + O \left(e^{-C \frac{\delta(\varepsilon)}{\varepsilon}} \right),$$

$$(3.42) \quad \int_0^{\frac{\delta(\varepsilon)}{\varepsilon}} t(e^{W(t)} - 1) dt = \rho^{-\frac{1}{2}} \int_0^\infty Q_F(W(t)) dt + O \left(\frac{\delta(\varepsilon)}{\varepsilon} e^{-C \frac{\delta(\varepsilon)}{\varepsilon}} \right),$$

and

(3.43)

$$\int_0^{\frac{\delta(\varepsilon)}{\varepsilon}} t^k e^{W(t)} \phi_1(t) dt = \int_0^\infty t^k e^{W(t)} \phi_1(t) dt + O\left(\frac{(\delta(\varepsilon))^k}{\varepsilon^k} e^{-C\frac{\delta(\varepsilon)}{\varepsilon}}\right), \quad k = 0, 1, 2,$$

(3.44)

$$\int_0^{\frac{\delta(\varepsilon)}{\varepsilon}} t^k e^{W(t)} \phi_2(t) dt = \int_0^\infty t^k e^{W(t)} \phi_2(t) dt + O\left(\frac{(\delta(\varepsilon))^k}{\varepsilon^k} e^{-C\frac{\delta(\varepsilon)}{\varepsilon}}\right), \quad k = 0, 1, 2,$$

where $C > 0$ is a generic constant independent of ε , and $Q_F(t)$ is given in (2.14).

Proof. By Lemma 3.2, one can check that

$$\begin{aligned} \int_0^{\frac{\delta(\varepsilon)}{\varepsilon}} (e^{W(t)} - 1) dt &= - \int_0^{\frac{\delta(\varepsilon)}{\varepsilon}} (e^{W(t)} - 1) \sqrt{\frac{1}{2\rho(W(t)e^{W(t)} - e^{W(t)} + 1)}} W'(t) dt \\ &= \int_{W(\frac{\delta(\varepsilon)}{\varepsilon})}^{\bar{u}} (e^t - 1) \sqrt{\frac{1}{2\rho(te^t - e^t + 1)}} dt \\ &= \rho^{-\frac{1}{2}} \left(Q_F(\bar{u}) - \int_0^{W(\frac{\delta(\varepsilon)}{\varepsilon})} \frac{e^t - 1}{\sqrt{2(te^t - e^t + 1)}} dt \right) \\ &= \rho^{-\frac{1}{2}} Q_F(\bar{u}) + O\left(e^{-C\frac{\delta(\varepsilon)}{\varepsilon}}\right), \end{aligned}$$

which leads to (3.41). Here we have used the fact that $W(t) \in (0, \bar{u}]$ and

$\sup_{t \in (0, \bar{u}]} \frac{e^t - 1}{\sqrt{2(te^t - e^t + 1)}}$ is finite.

Similarly, we have

$$\begin{aligned} \int_0^{\frac{\delta(\varepsilon)}{\varepsilon}} t(e^{W(t)} - 1) dt &= - \int_0^{\frac{\delta(\varepsilon)}{\varepsilon}} t(e^{W(t)} - 1) \sqrt{\frac{1}{2\rho(W(t)e^{W(t)} - e^{W(t)} + 1)}} W'(t) dt \\ &= -\rho^{-\frac{1}{2}} \int_0^{\frac{\delta(\varepsilon)}{\varepsilon}} t(Q_F(W(t)))' dt \\ &= \rho^{-\frac{1}{2}} \int_0^{\frac{\delta(\varepsilon)}{\varepsilon}} Q_F(W(t)) dt + O\left(\frac{\delta(\varepsilon)}{\varepsilon} e^{-C\frac{\delta(\varepsilon)}{\varepsilon}}\right) \\ &= \rho^{-\frac{1}{2}} \int_0^\infty Q_F(W(t)) dt + O\left(\frac{\delta(\varepsilon)}{\varepsilon} e^{-C\frac{\delta(\varepsilon)}{\varepsilon}}\right), \end{aligned}$$

which implies (3.42), where we have used the following inequalities from Lemma 3.2

$$Q_F(W(t)) \leq \bar{u} \left(\sup_{t \in (0, \bar{u}]} \left| \frac{e^t - 1}{\sqrt{2(te^t - e^t + 1)}} \right| \right) e^{-\sqrt{\rho}t} \leq C e^{-\sqrt{\rho}t},$$

and

$$\left| \int_{\frac{\delta(\varepsilon)}{\varepsilon}}^\infty Q_F(W(t)) dt \right| \leq C \left| \int_{\frac{\delta(\varepsilon)}{\varepsilon}}^\infty e^{-\sqrt{\rho}t} dt \right| = O\left(e^{-C\frac{\delta(\varepsilon)}{\varepsilon}}\right).$$

By Lemma 3.2 and Lemma 3.5 we have

$$\left| \int_{\frac{\delta(\varepsilon)}{\varepsilon}}^{\infty} t^k e^{W(t)} \phi_1(t) dt \right| \leq C e^{\bar{u}} \left| \int_{\frac{\delta(\varepsilon)}{\varepsilon}}^{\infty} t^k e^{-Ct} dt \right| = O \left(\frac{(\delta(\varepsilon))^k}{\varepsilon^k} e^{-C \frac{\delta(\varepsilon)}{\varepsilon}} \right),$$

which implies (3.43). Similarly one can get (3.44). The proof is completed. $\square$

Now we are ready to give the proof for Proposition 2.2.

Proof of Proposition 2.2. We shall use Lemma 2.1 and Lemma 3.5 to compute the expansion of $\int_{\Omega} (e^{u_{\varepsilon}} - 1) dx$. Recall that (see (3.2) and (3.6))

$$u(x) \in (0, \bar{u}] \text{ and } W(z(x)) \in (0, \bar{u}] \text{ for } x \in \Omega.$$

For $\delta \in (0, \delta_0)$, we get from Lemma 3.5 that

$$(3.45) \quad \max_{x \in \Omega_{\delta}} \left| e^{u_{\varepsilon}(x)} - e^{W\left(\frac{z(x)}{\varepsilon}\right)} - e^{W\left(\frac{z(x)}{\varepsilon}\right)} \left[B_{\varepsilon} \phi_1 \left(\frac{z(x)}{\varepsilon} \right) + \varepsilon(n-1) H_{\Gamma_{z(x)}}(y(x)) \phi_2 \left(\frac{z(x)}{\varepsilon} \right) \right] \right| \leq C \varepsilon^2.$$

On the other hand, in $\overline{\Omega \setminus \Omega_{\delta}}$, by (3.2) we have

$$(3.46) \quad \sup_{\overline{\Omega \setminus \Omega_{\delta}}} |e^{u_{\varepsilon}} - 1| \leq C e^{-C \frac{\delta}{\varepsilon}}.$$

Now we shall choose $\delta(\varepsilon) = C\varepsilon^{\frac{1}{2}}$. Then $\delta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. By (3.45)–(3.46) we have

$$(3.47) \quad \begin{aligned} \int_{\Omega} (e^{u_{\varepsilon}(x)} - 1) dx &= \int_{\Omega_{\delta(\varepsilon)}} (e^{u_{\varepsilon}(x)} - 1) dx + O \left(e^{-C \frac{\delta(\varepsilon)}{\varepsilon}} \right) \\ &= \int_{\Omega_{\delta(\varepsilon)}} \left(e^{W\left(\frac{z(x)}{\varepsilon}\right)} - 1 \right) dx + B_{\varepsilon} \int_{\Omega_{\delta(\varepsilon)}} e^{W\left(\frac{z(x)}{\varepsilon}\right)} \phi_1 \left(\frac{z(x)}{\varepsilon} \right) dx \\ &\quad + \varepsilon(n-1) \int_{\Omega_{\delta(\varepsilon)}} e^{W\left(\frac{z(x)}{\varepsilon}\right)} H_{\Gamma_{z(x)}}(y(x)) \phi_2 \left(\frac{z(x)}{\varepsilon} \right) dx \\ &\quad + O(\varepsilon^2) |\Omega_{\delta(\varepsilon)}| + O \left(e^{-C \frac{\delta(\varepsilon)}{\varepsilon}} \right). \end{aligned}$$

For the first term on the right hand side of (3.47), using Lemma 2.1 and Lemma 3.6 we have

$$(3.48) \quad \begin{aligned} &\int_{\Omega_{\delta(\varepsilon)}} \left(e^{W\left(\frac{z(x)}{\varepsilon}\right)} - 1 \right) dx \\ &= \int_0^{\delta(\varepsilon)} \int_{\partial\Omega} \left(e^{W\left(\frac{z}{\varepsilon}\right)} - 1 \right) (1 - (n-1)z H_{\partial\Omega}(y) + O(z^2)) d\sigma_y dz \\ &= |\partial\Omega| \int_0^{\delta(\varepsilon)} \left(e^{W\left(\frac{z}{\varepsilon}\right)} - 1 \right) (1 + O(z^2)) dz \\ &\quad - (n-1) \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \int_0^{\delta(\varepsilon)} z \left(e^{W\left(\frac{z}{\varepsilon}\right)} - 1 \right) dz \\ &= \varepsilon |\partial\Omega| \int_0^{\frac{\delta(\varepsilon)}{\varepsilon}} \left(e^{W(t)} - 1 \right) (1 + O(\varepsilon^2) t^2) dt \end{aligned}$$

$$\begin{aligned}
& -\varepsilon^2(n-1) \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \int_0^{\frac{\delta(\varepsilon)}{\varepsilon}} t(e^{W(t)} - 1) dt \\
& = \varepsilon \rho^{-\frac{1}{2}} \left(|\partial\Omega| Q_F(\bar{u}) - \varepsilon(n-1) \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \int_0^\infty Q_F(W(t)) dt \right) \\
& \quad + O(\varepsilon^3).
\end{aligned}$$

Similarly, for the second and third terms, we can obtain the following identities

$$\begin{aligned}
& \int_{\Omega_{\delta(\varepsilon)}} e^{W(\frac{z(x)}{\varepsilon})} \phi_1 \left(\frac{z(x)}{\varepsilon} \right) dx \\
& = \varepsilon |\partial\Omega| \int_0^{\frac{\delta(\varepsilon)}{\varepsilon}} e^{W(t)} \phi_1(t) (1 + O(\varepsilon^2)t^2) dt \\
(3.49) \quad & - \varepsilon^2(n-1) \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \int_0^{\frac{\delta(\varepsilon)}{\varepsilon}} t e^{W(t)} \phi_1(t) dt \\
& = \varepsilon |\partial\Omega| \int_0^\infty e^{W(t)} \phi_1(t) (1 + O(\varepsilon^2)t^2) dt \\
& \quad - \varepsilon^2(n-1) \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \int_0^\infty t e^{W(t)} \phi_1(t) dt + O(\varepsilon^3)
\end{aligned}$$

and

$$\begin{aligned}
(3.50) \quad & \int_{\Omega_{\delta(\varepsilon)}} e^{W(\frac{z(x)}{\varepsilon})} \phi_2 \left(\frac{z(x)}{\varepsilon} \right) H_{\Gamma_{z(x)}}(y(x)) dx \\
& = \int_{\Omega_{\delta(\varepsilon)}} e^{W(\frac{z(x)}{\varepsilon})} \phi_2 \left(\frac{z(x)}{\varepsilon} \right) (H_{\partial\Omega}(y(x)) + O(\delta(\varepsilon))) dx \\
& = \varepsilon \left(\int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y + O(\delta(\varepsilon)) \right) \int_0^\infty e^{W(t)} \phi_2(t) (1 + O(\varepsilon^2)t^2) dt \\
& \quad - \varepsilon^2(n-1) \left(\int_{\partial\Omega} H_{\partial\Omega}^2(y) d\sigma_y + O(\delta(\varepsilon)) \right) \int_0^\infty t e^{W(t)} \phi_2(t) dt + O(\varepsilon^2\delta(\varepsilon)),
\end{aligned}$$

where we have used

$$\max_{x \in \Omega_{\delta(\varepsilon)}} |H_{\Gamma_{z(x)}}(y(x)) - H_{\partial\Omega}(y(x))| = O(\delta(\varepsilon)).$$

Note that $\delta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Combining (3.48), (3.49), (3.50) and (3.15), we carefully examine the asymptotic expansion of each term up to ε^2 order and obtain

$$\begin{aligned}
(3.51) \quad & \int_{\Omega} (e^{u_\varepsilon} - 1) dx \\
& = \int_{\Omega_{\delta(\varepsilon)}} \left(e^{W(\frac{z(x)}{\varepsilon})} - 1 \right) dx + B_\varepsilon \int_{\Omega_{\delta(\varepsilon)}} e^{W(\frac{z(x)}{\varepsilon})} \phi_1 \left(\frac{z(x)}{\varepsilon} \right) dx \\
& \quad + \varepsilon(n-1) \int_{\Omega_{\delta(\varepsilon)}} e^{W(\frac{z(x)}{\varepsilon})} H_{\Gamma_{z(x)}}(y(x)) \phi_2 \left(\frac{z(x)}{\varepsilon} \right) dx + O(\varepsilon^2) |\Omega_{\delta(\varepsilon)}| \\
& = \varepsilon \rho^{-\frac{1}{2}} \left(|\partial\Omega| Q_F(\bar{u}) - \varepsilon(n-1) \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \int_0^\infty Q_F(W(t)) dt \right)
\end{aligned}$$

$$\begin{aligned}
& + \varepsilon B_\varepsilon \left[|\partial\Omega| \int_0^\infty e^{W(t)} \phi_1(t) dt - \varepsilon(n-1) \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \int_0^\infty t e^{W(t)} \phi_1(t) dt \right] \\
& + \varepsilon^2(n-1) \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \int_0^\infty e^{W(t)} \phi_2(t) dt + O(\varepsilon^2 \delta(\varepsilon)).
\end{aligned}$$

Then, by $|B_\varepsilon| \leq C\varepsilon$, we can formally rewrite the above equation as

$$(3.52) \quad \int_\Omega (\varepsilon^{u_\varepsilon} - 1) dx = D_1 \varepsilon + D_2 \varepsilon^2 + o_\varepsilon(1) \varepsilon^2, \quad 0 < \varepsilon \ll 1,$$

for some positive constants D_1 and D_2 . It remains to show that $D_i = I_i, i = 1, 2$ (I_1 and I_2 are given in (2.15) and (2.16) respectively). From (3.51) one can see that

$$(3.53) \quad D_1 = \rho^{-\frac{1}{2}} |\partial\Omega| Q_F(\bar{u}) = I_1.$$

As a consequence, we get that

$$(3.54) \quad B_\varepsilon = \frac{m}{\int_\Omega e^{u_\varepsilon} dx} - \rho = -\varepsilon \sqrt{\rho} \frac{|\partial\Omega|}{|\Omega|} Q_F(\bar{u}) + O(\varepsilon^2).$$

Substituting (3.54) into (3.51) we can derive that

$$\begin{aligned}
(3.55) \quad D_2 &= -(n-1) \rho^{-\frac{1}{2}} \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \int_0^\infty Q_F(W(t)) dt \\
&\quad - \sqrt{\rho} \frac{|\partial\Omega|^2}{|\Omega|} Q_F(\bar{u}) \int_0^\infty e^{W(t)} \phi_1(t) dt \\
&\quad + (n-1) \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \int_0^\infty e^{W(t)} \phi_2(t) dt \\
&= I_2.
\end{aligned}$$

Therefore, by (3.53), (3.55) and (3.52), we get (2.17) and finish the proof. $\square$

Based on Proposition 2.2, we can derive the expansion of the normal derivative of the solution near the boundary and hence prove Theorem 2.3.

Proof of Theorem 2.3. From the proof of Lemma 3.5, we have

$$(3.56) \quad u_\varepsilon(x) = W\left(\frac{z(x)}{\varepsilon}\right) + B_\varepsilon \phi_1\left(\frac{z(x)}{\varepsilon}\right) + \varepsilon(n-1) H_{\Gamma_{z(x)}}(y(x)) \phi_2\left(\frac{z(x)}{\varepsilon}\right) + \psi_\varepsilon(x).$$

Recall that $\psi_\varepsilon(x)$ (see (3.37)) satisfies

$$\begin{cases} \varepsilon^2 \Delta \psi_\varepsilon(x) = \rho f' \left(W\left(\frac{z(x)}{\varepsilon}\right) \right) \psi_\varepsilon(x) + \sum_{i=1}^4 E_{i,\varepsilon} & \text{in } \Omega_\delta, \\ \psi_\varepsilon(x) = 0 & \text{on } \partial\Omega, \\ \psi_\varepsilon(x) = O(\varepsilon^2) & \text{on } \partial\Omega_\delta \setminus \partial\Omega. \end{cases}$$

We set

$$\tilde{\psi}_\varepsilon(\tilde{x}) = \psi_\varepsilon(\varepsilon \tilde{x}), \quad \text{for } \tilde{x} \in \Omega_\delta^\varepsilon.$$

Then $\tilde{\psi}_\varepsilon(\tilde{x})$ satisfies

$$(3.57) \quad \begin{cases} \Delta_{\tilde{x}} \tilde{\psi}_\varepsilon(\tilde{x}) = \rho f' \left(W\left(\frac{z(\varepsilon \tilde{x})}{\varepsilon}\right) \right) \tilde{\psi}_\varepsilon(\tilde{x}) + \sum_{i=1}^4 E_{i,\varepsilon}(\varepsilon \tilde{x}) & \text{in } \tilde{x} \in \Omega_\delta^\varepsilon, \\ \tilde{\psi}_\varepsilon(\tilde{x}) = 0 & \text{on } \partial\Omega^\varepsilon, \\ \tilde{\psi}_\varepsilon(\tilde{x}) = O(\varepsilon^2) & \text{on } \partial\Omega^\varepsilon \setminus \partial\Omega_\delta^\varepsilon, \end{cases}$$

where

$$\Omega_\delta^\varepsilon := \{\tilde{x} \mid \varepsilon\tilde{x} \in \Omega_\delta\} \quad \text{and} \quad \Omega^\varepsilon = \{\tilde{x} \mid \varepsilon\tilde{x} \in \Omega\}.$$

By classical elliptic regularity for equation (3.57), we have (for some $q > 1$)

$$\begin{aligned} \|\tilde{\psi}_\varepsilon\|_{W^{2,q}(B_R(\tilde{x}_0) \cap \Omega_\delta^\varepsilon)} &\leq C \left(\|\tilde{\psi}_\varepsilon\|_{L^q(B_{2R}(\tilde{x}_0) \cap \Omega_\delta^\varepsilon)} + \left\| \sum_{i=1}^4 E_{i,\varepsilon}(\varepsilon y) \right\|_{L^q(B_{2R}(\tilde{x}_0) \cap \Omega_\delta^\varepsilon)} \right) \\ &\leq C\varepsilon^2, \end{aligned}$$

which together with the Sobolev embedding implies that

$$|\tilde{\psi}_\varepsilon(\tilde{x})| + |\nabla_{\tilde{x}} \tilde{\psi}_\varepsilon(\tilde{x})| = O(\varepsilon^2) \quad \text{for } \tilde{x} \in \Omega_\delta^\varepsilon.$$

Equivalently, we have

$$|\psi_\varepsilon(x)| = O(\varepsilon^2), \quad |\nabla_x \psi_\varepsilon(x)| = O(\varepsilon) \quad \text{for } x \in \Omega_\delta.$$

From the above estimate, (3.54) and (3.56), we get (2.18). In addition, for any $\xi \in \partial\Omega$, we can compute its normal derivative as

$$\begin{aligned} \partial_\nu u_\varepsilon(\xi) &= \frac{1}{\varepsilon} W'(0) - \sqrt{\rho} \frac{|\partial\Omega|}{|\Omega|} Q_F(\bar{u}) \phi_1'(0) + (n-1) H_{\partial\Omega}(x) \phi_2'(0) + o_\varepsilon(1) \\ &= -\frac{1}{\varepsilon} \sqrt{2\rho F(\bar{u})} + \frac{|\partial\Omega|}{|\Omega|} Q_F(\bar{u}) \sqrt{\frac{F(\bar{u})}{2}} \\ &\quad + (n-1) H_{\partial\Omega}(\xi) \sqrt{\frac{1}{2\rho F(\bar{u})}} \mathfrak{F}(\bar{u}) + o_\varepsilon(1) \end{aligned}$$

where we have used (3.7), (3.20), (3.22) and (3.56). It remains to study the representation of the level set. For any $a \in (0, \bar{u})$, we consider the following equation

$$\begin{aligned} (3.58) \quad a = u_\varepsilon(x) &= W\left(\frac{z(x)}{\varepsilon}\right) - \varepsilon \sqrt{\rho} \frac{|\partial\Omega|}{|\Omega|} Q_F(\bar{u}) \phi_1\left(\frac{z(x)}{\varepsilon}\right) \\ &\quad + \varepsilon(n-1) H_{\partial\Omega}(y(x)) \phi_2\left(\frac{z(x)}{\varepsilon}\right) + O(\varepsilon^2), \end{aligned}$$

where we have used the fact that $z(x) = O(\varepsilon)$ for $x \in \{p \in \Omega \mid u_\varepsilon(p) = a\}$ with $a \in (0, \bar{u})$, see (2.9). As a consequence, together with the smoothness of the domain, we have

$$|H_{\partial\Omega}(y(x)) - H_{\Gamma_{z(x)}}(y(x))| = O(\varepsilon) \quad \text{for } x \in \{p \in \Omega \mid u_\varepsilon(p) = a\}.$$

Since $W(z)$ is a strictly decreasing function, we can find a unique $z_{1,a}$ such that $W(z_{1,a}) = a$. Using (3.58), we can set

$$z(x) = \varepsilon z_{1,a} + \varepsilon^2 z_{2,a}(x) + o_\varepsilon(1)\varepsilon^2.$$

Substituting this expansion into (3.58) gives

$$\begin{aligned} a &= W(z_{1,a}) + \varepsilon W'(z_{1,a}) z_{2,a}(x) - \varepsilon \sqrt{\rho} \frac{|\partial\Omega|}{|\Omega|} Q_F(\bar{u}) \phi_1(z_{1,a}) \\ &\quad + \varepsilon(n-1) H_{\partial\Omega}(y(x)) \phi_2(z_{1,a}) + o_\varepsilon(1)\varepsilon. \end{aligned}$$

We can get from the above equation that

$$z_{2,a}(x) = \frac{1}{W'(z_{1,a})} \left(\sqrt{\rho} \frac{|\partial\Omega|}{|\Omega|} Q_F(\bar{u}) \phi_1(z_{1,a}) - (n-1) H_{\partial\Omega}(y(x)) \phi_2(z_{1,a}) \right) + o_\varepsilon(1).$$

Therefore, for any $x_0 \in \partial\Omega$, if $x = x_0 + z(x)\nu(x_0) \in \partial\Omega_\delta$ is the point such that $u_\varepsilon(x) = a \in (0, \bar{u})$. Then it follows that

$$z(x) = \varepsilon z_{1,a} + \varepsilon^2 \frac{1}{W'(z_{1,a})} \left(\sqrt{\rho} \frac{|\partial\Omega|}{|\Omega|} Q_F(\bar{u}) \phi_1(z_{1,a}) - (n-1) H_{\partial\Omega}(x_0) \phi_2(z_{1,a}) \right) + o_\varepsilon(1) \varepsilon^2.$$

□

4. PROOF OF THEOREM 2.4 (CASE OF LOGARITHMIC SENSITIVITY)

In this section, we shall study the logarithmic sensitivity model and obtain the counterpart results. We consider the following equation:

$$(4.1) \quad \begin{cases} \varepsilon^2 \Delta u = \frac{m}{\int_\Omega u^\chi dx} u^{\chi+1} & \text{in } \Omega, \\ u = \bar{u} & \text{on } \partial\Omega, \end{cases}$$

where $\chi > 2$. For the non-negative solutions of (4.1), we have the following assertions.

Lemma 4.1. *Let Ω be a smooth bounded domain and u_ε be the solution to (4.1). For ε sufficiently small, there exist positive constants C_1, C_2, c_1, c_2 independent of ε such that*

$$(4.2) \quad C_2 \left(1 + c_2 \frac{\text{dist}(x, \partial\Omega)}{\varepsilon^2} \right)^{-\frac{2}{\chi}} \leq u_\varepsilon(x) \leq C_1 \left(1 + c_1 \frac{\text{dist}(x, \partial\Omega)}{\varepsilon^2} \right)^{-\frac{2}{\chi}}, \quad \forall x \in \Omega_\delta.$$

Furthermore, there exists a positive constant C independent of ε such that

$$(4.3) \quad u_\varepsilon(x) \leq \bar{u} \left(1 + C \frac{\text{dist}(x, \partial\Omega)}{\varepsilon^2} \right)^{-\frac{2}{\chi}}, \quad \forall x \in \Omega.$$

Proof. Estimate (4.2) follows from [6, Lemma 4.4]. It remains to show estimate (4.3). In the proof of [6, (4.16)], we have already shown that

$$(4.4) \quad u_\varepsilon(x) \leq \bar{u} \left(1 + C_\delta \frac{\text{dist}(x, \partial\Omega)}{\varepsilon^2} \right)^{-\frac{2}{\chi}}, \quad \text{for all } x \in \Omega_\delta.$$

While for $x \in \Omega_\delta^c$, it was proved in [23, (4.18)] that

$$(4.5) \quad u_\varepsilon(x) \leq C(\bar{\Omega}_\delta^\varepsilon) \varepsilon^{\frac{4}{\chi}}.$$

Now we choose

$$C = \min \left\{ C_\delta, \frac{\bar{u}^{\frac{\chi}{2}}}{\text{Diam}(\Omega) C(\bar{\Omega}_\delta^\varepsilon)^{\frac{\chi}{2}}} \right\} \quad \text{and} \quad \varepsilon < \left(\frac{\bar{u}}{C(\bar{\Omega}_\delta^\varepsilon)} \right)^{\frac{\chi}{4}},$$

where $\text{Diam}(\Omega) := \max_{x_1, x_2 \in \partial\Omega} \text{dist}(x_1, x_2)$. Then using (4.4) and (4.5) we get (4.3). □

Recall from (2.21)–(2.22) that:

$$(4.6) \quad U(z) = \frac{A}{(B+z)^{\frac{2}{\chi}}}, \quad \text{where } A = \left(\frac{2(\chi+2)}{m\chi^2} \right)^{\frac{1}{\chi}} \quad \text{and } B = \left(\frac{A}{\bar{u}} \right)^{\frac{\chi}{2}}.$$

It is straightforward to check that $U(z)$ satisfies the following equation

$$(4.7) \quad \begin{cases} U''(z) = mU^{\chi+1}(z), & z > 0, \\ U(0) = \bar{u}, & U(z) \rightarrow 0 \text{ as } z \rightarrow +\infty, \end{cases}$$

which can be considered as the leading order equation of (4.1). Let

$$\lambda_\varepsilon = \left(\int_{\Omega} u_\varepsilon^\chi dx \right)^{\frac{1}{2}}.$$

We note it follows from [6, Lemma 4.4] that $\lambda_\varepsilon \sim \varepsilon$. That is there are two constants $d_1, d_2 > 0$ such that

$$d_1 \varepsilon \leq \lambda_\varepsilon \leq d_2 \varepsilon.$$

We set $\sigma_\varepsilon = \varepsilon \lambda_\varepsilon$ for convenience. In the following, we shall see that $U\left(\frac{z(x)}{\sigma_\varepsilon}\right)$ can be regarded as the leading order approximation of $u_\varepsilon(x)$.

Lemma 4.2. *Let u_ε and U be the solution of (4.1) and (4.7), respectively. Then for $\varepsilon > 0$ sufficiently small, we have*

$$\max_{x \in \Omega} \left| u_\varepsilon(x) - U\left(\frac{z(x)}{\sigma_\varepsilon}\right) \right| \leq C \varepsilon^{\frac{4}{\chi}}.$$

Proof. For $x \in \Omega \setminus \Omega_\delta$, by Lemma 4.1 and the fact $\lambda_\varepsilon \sim \varepsilon$, we have

$$(4.8) \quad \left| u_\varepsilon(x) - U\left(\frac{z(x)}{\sigma_\varepsilon}\right) \right| \leq u_\varepsilon(x) + U\left(\frac{z(x)}{\sigma_\varepsilon}\right) \leq C \left(\frac{\delta}{\varepsilon^2}\right)^{-\frac{2}{\chi}} \leq C \varepsilon^{\frac{4}{\chi}}$$

for $\varepsilon > 0$ sufficiently small and some constant $C > 0$. Thus, it only remains to prove that the inequality holds for $x \in \Omega_\delta$ with $\delta > 0$ small.

For $x \in \Omega_\delta$, we write $x = y(x) + z(x)\nu(y(x))$ by the Fermi coordinate, as introduced in Section 2. Noticing that $U\left(\frac{z(x)}{\sigma_\varepsilon}\right)$ satisfies (4.7), we have

$$(4.9) \quad \begin{aligned} \sigma_\varepsilon^2 \Delta \left(u_\varepsilon(x) - U\left(\frac{z(x)}{\sigma_\varepsilon}\right) \right) &= mg(u_\varepsilon(x)) - mg\left(U\left(\frac{z(x)}{\sigma_\varepsilon}\right)\right) - \sigma_\varepsilon U' \left(\frac{z(x)}{\sigma_\varepsilon} \right) \Delta z(x) \\ &= m \left(g(u_\varepsilon(x)) - g\left(U\left(\frac{z(x)}{\sigma_\varepsilon}\right)\right) \right) + \mathcal{E}_u(x) \\ &= m(\chi + 1) \tilde{w}_\varepsilon^\chi(x) \left(u_\varepsilon(x) - U\left(\frac{z(x)}{\sigma_\varepsilon}\right) \right) + \mathcal{E}_u(x), \end{aligned}$$

where $g(u) = u^{\chi+1}$, $\tilde{w}_\varepsilon(x) \in \left(\min \left\{ U\left(\frac{z(x)}{\sigma_\varepsilon}\right), u_\varepsilon(x) \right\}, \max \left\{ U\left(\frac{z(x)}{\sigma_\varepsilon}\right), u_\varepsilon(x) \right\} \right)$ and

$$\mathcal{E}_u(x) = -\sigma_\varepsilon U' \left(\frac{z(x)}{\sigma_\varepsilon} \right) \Delta z(x) = \sigma_\varepsilon U' \left(\frac{z(x)}{\sigma_\varepsilon} \right) (n-1) H_{\Gamma_{z(x)}}(y(x)) \quad \text{in } \Omega_\delta,$$

here (2.4) was used. We note that, by Lemma 4.1 and $\lambda_\varepsilon \sim \varepsilon$, we have

$$(4.10) \quad \min \left\{ U\left(\frac{z(x)}{\sigma_\varepsilon}\right), u_\varepsilon(x) \right\} \geq C_3 \left(1 + c_3 \frac{z(x)}{\sigma_\varepsilon} \right)^{-\frac{2}{\chi}} \quad \text{in } \Omega_\delta.$$

Note $H_{\Gamma_{z(x)}}$ is uniformly bounded due to the smoothness of Ω . Then using the definition of U given in (4.6) and the fact that $\sigma_\varepsilon \sim \varepsilon^2$, we have

$$(4.11) \quad \mathcal{E}_u(x) \geq -c_0 \varepsilon^2 \left(B + \frac{z(x)}{\varepsilon^2} \right)^{-\frac{2}{\chi}-1}$$

for some constant $c_0 > 0$. In order to estimate $u_\varepsilon(x) - U(\frac{z(x)}{\sigma_\varepsilon})$, let us define a comparison function

$$w_\varepsilon(x) = \left(B + \frac{z(x)}{\sigma_\varepsilon}\right)^{1-\frac{2}{\chi}}.$$

Then, by (2.3) and $|H_{\Gamma_{z(x)}}(y(x))| \leq C$ in Ω_δ , we have that

$$\begin{aligned} & \sigma_\varepsilon^2 \Delta w_\varepsilon(x) - m(\chi + 1) \tilde{w}_\varepsilon^\chi(x) w_\varepsilon(x) \\ & \leq \frac{-2(\chi - 2)}{\chi^2} \left(B + \frac{z(x)}{\sigma_\varepsilon}\right)^{-1-\frac{2}{\chi}} - m(\chi + 1) \tilde{w}_\varepsilon^\chi(x) w_\varepsilon(x) \\ & \quad - \sigma_\varepsilon(n - 1) H_{\Gamma_{z(x)}}(y(x)) \frac{\chi - 2}{\chi} \left(B + \frac{z(x)}{\sigma_\varepsilon}\right)^{-\frac{2}{\chi}} \\ (4.12) \quad & \leq \sigma_\varepsilon \left(B + \frac{z(x)}{\sigma_\varepsilon}\right)^{-\frac{2}{\chi}} \left[C - \frac{m(\chi + 1) C_3^\chi}{\sigma_\varepsilon + c_3 z(x)} \frac{\left(B + \frac{z(x)}{\sigma_\varepsilon}\right)}{\left(1 + c_3 \frac{z(x)}{\sigma_\varepsilon}\right)} \right] \\ & \quad - \frac{2(\chi - 2)}{\chi^2} \left(B + \frac{z(x)}{\sigma_\varepsilon}\right)^{-1-\frac{2}{\chi}} \\ & \leq -\frac{2(\chi - 2)}{\chi^2} \left(B + \frac{z(x)}{\sigma_\varepsilon}\right)^{-1-\frac{2}{\chi}} \end{aligned}$$

for $x \in \Omega_{\delta_1}$ with $\delta_1 \in (0, \delta_0)$ being sufficiently small and independent of small ε . We note that, by Lemma 4.1 and the definition of U and w_ε , it holds that

$$(4.13) \quad w_\varepsilon(x) \sim \varepsilon^{\frac{4}{\chi}-2} \quad \text{and} \quad \left| u_\varepsilon(x) - U\left(\frac{z(x)}{\sigma_\varepsilon}\right) \right| \leq C \varepsilon^{\frac{4}{\chi}} \quad \text{on } \Gamma_{\delta_1}$$

and

$$(4.14) \quad u_\varepsilon(x) - U\left(\frac{z(x)}{\sigma_\varepsilon}\right) = 0 \quad \text{and} \quad w_\varepsilon(x) > 0 \quad \text{for } x \in \partial\Omega,$$

where $\Gamma_{\delta_1} := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) = \delta_1\}$, as introduced in Section 2.

Let $W_\varepsilon(x) = C_0 \varepsilon^2 w_\varepsilon(x)$, where $C_0 > 0$ is a constant so that $C_0 \frac{2(\chi-2)}{\chi^2} > c_0$ for c_0 given in (4.11). Then it follows from (4.12) that

$$(4.15) \quad \sigma_\varepsilon^2 \Delta W_\varepsilon(x) - m(\chi + 1) \tilde{w}_\varepsilon^\chi(x) W_\varepsilon(x) \leq -C_0 \varepsilon^2 \frac{2(\chi - 2)}{\chi^2} \left(B + \frac{z(x)}{\sigma_\varepsilon}\right)^{-1-\frac{2}{\chi}}.$$

Now comparing (4.15) to (4.9) with (4.11) and using the boundary conditions in (4.13) and (4.14), we get by the comparison principle that

$$(4.16) \quad \left| u_\varepsilon(x) - U\left(\frac{z(x)}{\sigma_\varepsilon}\right) \right| \leq W_\varepsilon(x) = C_0 \varepsilon^2 w_\varepsilon(x), \quad x \in \Omega_{\delta_1}.$$

Since $w_\varepsilon(z(x))$ is increasing in z due to $\chi > 2$, we obtain that

$$\left| u_\varepsilon(x) - U\left(\frac{z(x)}{\sigma_\varepsilon}\right) \right| \leq C_0 \varepsilon^2 w_\varepsilon(x) \leq C \varepsilon^2 \left(B + \frac{\delta_1}{\sigma_\varepsilon}\right)^{1-\frac{2}{\chi}} \leq C_0 \varepsilon^{\frac{4}{\chi}}, \quad x \in \Omega_{\delta_1}.$$

Combining the above inequality with (4.8) and setting $\delta = \delta_1$, we complete the proof. $\square$

Recall that in (2.21)–(2.22), we introduced the following function

$$(4.17) \quad \begin{aligned} \phi_3(z) &= \frac{A}{\chi+4}(B+z)^{\frac{\chi-2}{\chi}} - \frac{AB^{\frac{\chi-2}{\chi}-q^-}}{\chi+4}(B+z)^{q^-}, \\ q^- &= \frac{1 - \sqrt{1 + \frac{8(\chi+1)(\chi+2)}{\chi^2}}}{2} < -\frac{2}{\chi}. \end{aligned}$$

With $U(z)$ given in (4.6), one can readily verify that $\phi_3(z)$ satisfies

$$(4.18) \quad \begin{cases} \phi_3''(z) = m(\chi+1)U^\chi(z)\phi_3(z) + U'(z), & z > 0, \\ \phi_3(0) = 0. \end{cases}$$

Then with ϕ_3 in hand, we can give a refined approximation of $u_\varepsilon(x)$ as given below. Recalling the definition of δ_0 in (2.1), we fix $\delta \in (0, \delta_0)$.

Lemma 4.3. *Let u_ε and U be the solution of (4.1) and (4.7), respectively. As $\varepsilon > 0$ is sufficiently small, there exists a constant C independent of ε such that*

$$(4.19) \quad \begin{aligned} \max_{x \in \Omega} \left| u_\varepsilon(x) - U\left(\frac{z(x)}{\sigma_\varepsilon}\right) - \sigma_\varepsilon(n-1)H_{\Gamma_{z(x)}}(y(x))\phi_3\left(\frac{z(x)}{\sigma_\varepsilon}\right) \right| \\ \leq C\varepsilon^4 \left(1 + \frac{z(x)}{\sigma_\varepsilon}\right)^{2-\frac{2}{\chi}}, \end{aligned}$$

where $H_{\Gamma_{z(x)}}(y(x))$ is defined in (2.2).

Proof. For $x \in \Omega_\delta$, we write it as $x = y(x) + z(x)\nu(y(x))$ with $y(x) \in \partial\Omega$ and $z(x) = \text{dist}(x, y(x))$. Using the form of Laplace operator in terms of Fermi-coordinate in (2.3) and (2.4), we can rewrite (4.9) as

$$(4.20) \quad \begin{aligned} \sigma_\varepsilon^2 \Delta \left(u_\varepsilon(x) - U\left(\frac{z(x)}{\sigma_\varepsilon}\right) \right) &= m u_\varepsilon^{\chi+1}(x) - m U^{\chi+1}\left(\frac{z(x)}{\sigma_\varepsilon}\right) \\ &\quad + \sigma_\varepsilon(n-1)H_{\Gamma_{z(x)}}(y(x))U'\left(\frac{z(x)}{\sigma_\varepsilon}\right). \end{aligned}$$

Using (2.3), we rewrite (4.18) in Ω_δ as

$$\begin{aligned} \sigma_\varepsilon^2 \Delta \phi_3\left(\frac{z(x)}{\sigma_\varepsilon}\right) &= m(\chi+1)U^\chi\left(\frac{z(x)}{\sigma_\varepsilon}\right)\phi_3\left(\frac{z(x)}{\sigma_\varepsilon}\right) + U'\left(\frac{z(x)}{\sigma_\varepsilon}\right) \\ &\quad - \sigma_\varepsilon(n-1)H_{\Gamma_{z(x)}}(y(x))\phi_3'\left(\frac{z(x)}{\sigma_\varepsilon}\right), \end{aligned}$$

where we have used the fact that $\phi_3\left(\frac{z(x)}{\sigma_\varepsilon}\right)$ is constant on $\Gamma_{z(x)}$.

Due to the appearance of $U'\left(\frac{z(x)}{\sigma_\varepsilon}\right)$ in (4.20), we define an approximate solution by

$$(4.21) \quad u_{\varepsilon, \text{app}} := U\left(\frac{z(x)}{\sigma_\varepsilon}\right) + \sigma_\varepsilon(n-1)H_{\Gamma_{z(x)}}(y(x))\phi_3\left(\frac{z(x)}{\sigma_\varepsilon}\right).$$

By a direct computation, we have

$$(4.22) \quad \sigma_\varepsilon^2 \Delta (u_\varepsilon(x) - u_{\varepsilon, \text{app}}(x)) = m(\chi+1)U^\chi\left(\frac{z(x)}{\sigma_\varepsilon}\right)(u_\varepsilon(x) - u_{\varepsilon, \text{app}}(x)) + \sum_{i=1}^3 \mathbb{E}_{i, \varepsilon},$$

where

$$\begin{aligned}\mathbb{E}_{1,\varepsilon} &:= \sigma_\varepsilon^2 (n-1)^2 H_{\Gamma_{z(x)}}^2(y(x)) \phi_3' \left(\frac{z(x)}{\sigma_\varepsilon} \right), \\ \mathbb{E}_{2,\varepsilon} &:= \varepsilon^3 \lambda_\varepsilon^3 (n-1) \left[H_{\Gamma_{z(x)}}(y(x)) \Delta \phi_3 \left(\frac{z(x)}{\sigma_\varepsilon} \right) - \Delta \left(H_{\Gamma_{z(x)}}(y(x)) \phi_3 \left(\frac{z(x)}{\sigma_\varepsilon} \right) \right) \right], \\ \mathbb{E}_{3,\varepsilon} &:= m \left[u_\varepsilon^{\chi+1}(x) - U^{\chi+1} \left(\frac{z(x)}{\sigma_\varepsilon} \right) - (\chi+1) U^\chi \left(\frac{z(x)}{\sigma_\varepsilon} \right) \left(u_\varepsilon(x) - U \left(\frac{z(x)}{\sigma_\varepsilon} \right) \right) \right].\end{aligned}$$

We will provide the estimates for the error terms $\mathbb{E}_{i,\varepsilon}$ ($i = 1, 2, 3$) below. Using the fact that the mean curvature $H_{\Gamma_{z(x)}}(y(x))$ are uniformly bounded (since Ω is smooth), we have

$$(4.23) \quad |\mathbb{E}_{1,\varepsilon}(x)| \leq C \sigma_\varepsilon^2 \left(B + \frac{z(x)}{\sigma_\varepsilon} \right)^{-\frac{2}{\chi}} \leq C \varepsilon^4 \left(B + \frac{z(x)}{\sigma_\varepsilon} \right)^{-\frac{2}{\chi}}.$$

From (4.17), we can see that

$$\begin{aligned}(4.24) \quad & |\mathbb{E}_{2,\varepsilon}(x)| \\ &= \varepsilon^3 \lambda_\varepsilon^3 (n-1) \left| \phi_3 \left(\frac{z(x)}{\sigma_\varepsilon} \right) \Delta H_{\Gamma_{z(x)}}(y(x)) + \frac{2}{\sigma_\varepsilon} \phi_3' \left(\frac{z(x)}{\sigma_\varepsilon} \right) \nabla H_{\Gamma_{z(x)}}(y(x)) \nabla z(x) \right| \\ &\leq C \varepsilon^3 \lambda_\varepsilon^3 \left(B + \frac{z(x)}{\sigma_\varepsilon} \right)^{\frac{\chi-2}{\chi}} + C \sigma_\varepsilon^2 \left(B + \frac{z(x)}{\sigma_\varepsilon} \right)^{-\frac{2}{\chi}} \leq C \varepsilon^4 \left(B + \frac{z(x)}{\sigma_\varepsilon} \right)^{-\frac{2}{\chi}}.\end{aligned}$$

Concerning the last one, by (4.16) in the proof of Lemma 4.2, we obtain that (4.25)

$$|\mathbb{E}_{3,\varepsilon}(x)| \leq C U^{\chi-1} \left(\frac{z(x)}{\sigma_\varepsilon} \right) \left(u_\varepsilon(x) - U \left(\frac{z(x)}{\sigma_\varepsilon} \right) \right)^2 \leq C \varepsilon^4 \left(B + \frac{z(x)}{\sigma_\varepsilon} \right)^{-\frac{2}{\chi}}.$$

We set

$$\psi_\varepsilon(x) = u_\varepsilon(x) - u_{\varepsilon,\text{app}}(x).$$

It follows from (4.22) that

$$(4.26) \quad \sigma_\varepsilon^2 \Delta \psi_\varepsilon = m(\chi+1) U^\chi \left(\frac{z(x)}{\sigma_\varepsilon} \right) \psi_\varepsilon + \sum_{i=1}^3 \mathbb{E}_{i,\varepsilon} \quad \text{in } \Omega_\delta.$$

To estimate ψ_ε , we define a comparison function

$$w_{1,\varepsilon}(x) = \left(B + \frac{z(x)}{\sigma_\varepsilon} \right)^{2-\frac{2}{\chi}}.$$

Then, by (2.3) and the fact that $|H_{\Gamma_{z(x)}}(y(x))| \leq C$ in Ω_δ , we have that

$$\begin{aligned}& \sigma_\varepsilon^2 \Delta w_{1,\varepsilon}(x) - m(\chi+1) U^\chi \left(\frac{z(x)}{\sigma_\varepsilon} \right) w_{1,\varepsilon}(x) \\ &\leq \frac{(2\chi-2)(\chi-2)}{\chi^2} \left(B + \frac{z(x)}{\sigma_\varepsilon} \right)^{-\frac{2}{\chi}} - m(\chi+1) A^\chi \left(B + \frac{z(x)}{\sigma_\varepsilon} \right)^{-2} \left(B + \frac{z(x)}{\sigma_\varepsilon} \right)^{2-\frac{2}{\chi}} \\ &\quad - \sigma_\varepsilon (n-1) H_{\Gamma_{z(x)}}(y(x)) \frac{2\chi-2}{\chi} \left(B + \frac{z(x)}{\sigma_\varepsilon} \right)^{1-\frac{2}{\chi}} \\ &= \frac{(2\chi-2)(\chi-2)}{\chi^2} \left(B + \frac{z(x)}{\sigma_\varepsilon} \right)^{-\frac{2}{\chi}} - \frac{2(\chi+2)(\chi+1)}{\chi^2} \left(B + \frac{z(x)}{\sigma_\varepsilon} \right)^{-\frac{2}{\chi}}\end{aligned}$$

$$\begin{aligned}
& -\sigma_\varepsilon(n-1)H_{\Gamma_{z(x)}}(y(x))\frac{2\chi-2}{\chi}\left(B+\frac{z(x)}{\sigma_\varepsilon}\right)^{1-\frac{2}{\chi}} \\
& \leq C\sigma_\varepsilon\left(B+\frac{z(x)}{\sigma_\varepsilon}\right)^{1-\frac{2}{\chi}} - \frac{12}{\chi}\left(B+\frac{z(x)}{\sigma_\varepsilon}\right)^{-\frac{2}{\chi}} \\
& \leq \sigma_\varepsilon\left(B+\frac{z(x)}{\sigma_\varepsilon}\right)^{1-\frac{2}{\chi}}\left[C-\frac{6}{\chi}(B\sigma_\varepsilon+z(x))^{-1}\right] - \frac{6}{\chi}\left(B+\frac{z(x)}{\sigma_\varepsilon}\right)^{-\frac{2}{\chi}} \\
& \leq -\frac{6}{\chi}\left(B+\frac{z(x)}{\sigma_\varepsilon}\right)^{-\frac{2}{\chi}}
\end{aligned}$$

for $x \in \Omega_{\delta_1}$, where $\delta_1 \in (0, \delta_0)$ is sufficiently small and independent of ε . Since

$$w_{1,\varepsilon}(x) \sim \varepsilon^{\frac{4}{\chi}-4} \quad \text{and} \quad |\psi_\varepsilon(x)| \leq C\varepsilon^{\frac{4}{\chi}} \quad \text{on} \quad \Gamma_{\delta_1}$$

and

$$(4.27) \quad \max_{x \in \partial\Omega} |\psi_\varepsilon| = 0, \quad \max_{x \in \partial\Omega_\delta \setminus \partial\Omega} |\psi_\varepsilon| = O(\varepsilon^{\frac{4}{\chi}}) \quad \text{on} \quad \partial\Omega_{\delta_1},$$

by the maximum principle, we have $|\psi_\varepsilon(x)| \leq C\varepsilon^4 w_{1,\varepsilon}$ in Ω_{δ_1} . Thus, it remains to prove (4.19) holds in $\Omega \setminus \Omega_{\delta_1}$. We note that there is a constant $C > 0$ independent of ε such that

$$C\varepsilon^4 \left(1 + \frac{z(x)}{\sigma_\varepsilon}\right)^{2-\frac{2}{\chi}} \geq \varepsilon^{\frac{4}{\chi}}, \quad \forall x \in \Omega \setminus \Omega_{\delta_1}.$$

Combining this and Lemma 4.2, we complete the proof of the inequality (4.19). $\square$

Next, we shall calculate $\int_\Omega u_\varepsilon^\chi dx$ for the case $\chi > 2$. By Lemma 4.1 and simple calculations, we have $\int_\Omega u_\varepsilon^\chi dx = \int_{\Omega_\delta} u_\varepsilon^\chi dx + O(\varepsilon^4)$. From Lemma 4.3, we get

$$\begin{aligned}
\int_{\Omega_\delta} u_\varepsilon^\chi dx &= \int_{\Omega_\delta} \left[U\left(\frac{z(x)}{\sigma_\varepsilon}\right) + \sigma_\varepsilon(n-1)H_{\Gamma_{z(x)}}(y(x))\phi_3\left(\frac{z(x)}{\sigma_\varepsilon}\right) \right. \\
&\quad \left. + O\left(\varepsilon^4\left(1 + \frac{z(x)}{\sigma_\varepsilon}\right)^{2-\frac{2}{\chi}}\right) \right]^\chi dx \\
&= \int_{\Omega_\delta} U^\chi\left(\frac{z(x)}{\sigma_\varepsilon}\right) dx \\
&\quad + \chi \int_{\Omega_\delta} U^{\chi-1}\left(\frac{z(x)}{\sigma_\varepsilon}\right) \left(\sigma_\varepsilon(n-1)H_{\Gamma_{z(x)}}(y(x))\phi_3\left(\frac{z(x)}{\sigma_\varepsilon}\right) \right) dx \\
&\quad + \int_{\Omega_\delta} U^{\chi-1}\left(\frac{z(x)}{\sigma_\varepsilon}\right) O\left(\varepsilon^4\left(1 + \frac{z(x)}{\sigma_\varepsilon}\right)^{2-\frac{2}{\chi}}\right) dx \\
&\quad + \int_{\Omega_\delta} U^{\chi-2}\left(\frac{z(x)}{\sigma_\varepsilon}\right) O\left(\varepsilon^4\phi_3^2\left(\frac{z(x)}{\sigma_\varepsilon}\right) + \varepsilon^8\left(1 + \frac{z(x)}{\sigma_\varepsilon}\right)^{4-\frac{4}{\chi}}\right) dx,
\end{aligned}$$

where we have used the facts $\sigma_\varepsilon \sim \varepsilon^2$ and

$$U\left(\frac{z(x)}{\sigma_\varepsilon}\right) \geq C\varepsilon^2\phi_3\left(\frac{z(x)}{\sigma_\varepsilon}\right) \geq C\varepsilon^4\left(1 + \frac{z(x)}{\sigma_\varepsilon}\right)^{2-\frac{2}{\chi}} \quad \text{in} \quad \Omega_\delta.$$

Using the co-area formula in Lemma 2.1, we have

$$\begin{aligned}
\int_{\Omega_\delta} U^\chi \left(\frac{z(x)}{\sigma_\varepsilon} \right) dx &= \int_0^\delta \int_{\partial\Omega} U^\chi \left(\frac{z}{\sigma_\varepsilon} \right) (1 - (n-1)zH_{\partial\Omega}(y) + O(z^2)) d\sigma_y dz \\
&= \sigma_\varepsilon |\partial\Omega| \int_0^{\delta/(\sigma_\varepsilon)} U^\chi(z) (1 + O(\varepsilon^4 z^2)) dz \\
&\quad - \sigma_\varepsilon^2 (n-1) \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \int_0^{\delta/(\sigma_\varepsilon)} z U^\chi(z) dz \\
&= \frac{A^\chi}{B} \sigma_\varepsilon |\partial\Omega| - (n-1) A^\chi \sigma_\varepsilon^2 |\log(\sigma_\varepsilon)| \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y + O(\varepsilon^4),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Omega_\delta} (U^{\chi-1} \phi_3) \left(\frac{z(x)}{\sigma_\varepsilon} \right) H_{\Gamma_{z(x)}}(y(x)) dx \\
&= \int_{\Omega_\delta} (U^{\chi-1} \phi_3) \left(\frac{z(x)}{\sigma_\varepsilon} \right) (H_{\partial\Omega}(y(x)) + O(z(x))) dx \\
&= \sigma_\varepsilon \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \int_0^{\delta/(\sigma_\varepsilon)} (U^{\chi-1} \phi_3)(z) (1 + O(\varepsilon^2 z)) dz \\
&= \frac{A^\chi}{\chi+4} \sigma_\varepsilon |\log(\sigma_\varepsilon)| \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y + O(\varepsilon^2).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\lambda_\varepsilon^2 &= \frac{A^\chi}{B} \sigma_\varepsilon |\partial\Omega| - (n-1) A^\chi \sigma_\varepsilon^2 |\log(\sigma_\varepsilon)| \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \\
&\quad + \frac{(n-1)\chi A^\chi}{\chi+4} \sigma_\varepsilon^2 |\log(\sigma_\varepsilon)| \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y + O(\varepsilon^4) \\
&= \frac{A^\chi}{B} \sigma_\varepsilon |\partial\Omega| - \frac{8(n-1)A^\chi}{\chi+4} \sigma_\varepsilon^2 |\log \varepsilon| \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y + O(\varepsilon^4).
\end{aligned}$$

With $\sigma_\varepsilon = \varepsilon \lambda_\varepsilon$, solving the above equation, we can get the following key result.

Lemma 4.4. *Consider equation (4.1) with Ω being a smooth domain. Let $\lambda_\varepsilon = (\int_\Omega u_\varepsilon^\chi dx)^{\frac{1}{\chi}}$. Then*

$$\begin{aligned}
(4.28) \quad \lambda_\varepsilon &= \frac{\frac{A^\chi}{B} \varepsilon |\partial\Omega| + O(\varepsilon^3)}{1 + \frac{8(n-1)A^\chi}{\chi+4} \varepsilon^2 |\log \varepsilon| \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y} \\
&= \frac{A^\chi}{B} \varepsilon |\partial\Omega| \left(1 - \frac{8(n-1)A^\chi}{\chi+4} \varepsilon^2 |\log \varepsilon| \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \right) + O(\varepsilon^3).
\end{aligned}$$

Based on Lemmas 4.3 and 4.4, we are able to derive a three-term expansion for the normal derivative of the solution on the boundary.

Proof of Theorem 2.4. Let

$$\psi_\varepsilon(x) = u_\varepsilon(x) - U \left(\frac{z(x)}{\sigma_\varepsilon} \right) - \sigma_\varepsilon (n-1) H_{\Gamma_{z(x)}}(y(x)) \phi_3 \left(\frac{z(x)}{\sigma_\varepsilon} \right).$$

By Lemma 4.3 and the fact $\sigma_\varepsilon = \varepsilon \lambda_\varepsilon \sim \varepsilon^2$, we have

$$(4.29) \quad |\psi_\varepsilon(x)| \leq C\varepsilon^4 \left(1 + \frac{z(x)}{\sigma_\varepsilon}\right)^{2-\frac{2}{\chi}} \leq C\varepsilon^{\frac{4}{\chi}} \quad \text{for } x \in \Omega_\delta.$$

Set

$$\hat{\psi}_\varepsilon(\hat{x}) = \psi_\varepsilon(\sigma_\varepsilon \hat{x}), \quad \text{for } \hat{x} \in \Omega_\delta^{\sigma_\varepsilon}.$$

Then $\hat{\psi}_\varepsilon(\hat{x})$ satisfies

$$(4.30) \quad \begin{cases} \Delta_{\hat{x}} \hat{\psi}_\varepsilon(\hat{x}) = m(\chi+1)U^\chi \left(\frac{z(\sigma_\varepsilon \hat{x})}{\sigma_\varepsilon}\right) \hat{\psi}_\varepsilon(\hat{x}) + \sum_{i=1}^3 E_{i,\varepsilon}(\sigma_\varepsilon \hat{x}) & \text{in } \hat{x} \in \Omega_\delta^{\sigma_\varepsilon}, \\ \hat{\psi}_\varepsilon(\hat{x}) = 0 & \text{on } \partial\Omega^{\sigma_\varepsilon}, \\ \hat{\psi}_\varepsilon(\hat{x}) = O(\varepsilon^{\frac{4}{\chi}}) & \text{on } \partial\Omega^{\sigma_\varepsilon} \setminus \partial\Omega_\delta^{\sigma_\varepsilon}, \end{cases}$$

where

$$\Omega_\delta^{\sigma_\varepsilon} := \{\hat{x} \mid \sigma_\varepsilon \hat{x} \in \Omega_\delta\} \quad \text{and} \quad \Omega^{\sigma_\varepsilon} = \{\hat{x} \mid \sigma_\varepsilon \hat{x} \in \Omega\}.$$

On the other hand, by Lemma 4.3, we have

$$\hat{\psi}_\varepsilon(\hat{x}) \leq C\varepsilon^4 \left(1 + \frac{z(\sigma_\varepsilon \hat{x})}{\sigma_\varepsilon}\right)^{2-\frac{2}{\chi}}.$$

As a consequence, if $z(\sigma_\varepsilon \hat{x}) \sim \sigma_\varepsilon$, i.e., for $\text{dist}(x, \partial\Omega) \sim \sigma_\varepsilon$, we can show that

$$|\hat{\psi}_\varepsilon(\hat{x})| = |\psi_\varepsilon(x)| \leq C\varepsilon^4.$$

Then applying the classical elliptic regularity theory for equation (4.30) in the region $\Omega_{3\sigma_\varepsilon}^{\sigma_\varepsilon}$, we get that

$$\begin{aligned} \|\hat{\psi}_\varepsilon\|_{W^{2,q}(B_1(\hat{x}_0) \cap \Omega_\delta^{\sigma_\varepsilon})} &\leq C \left(\|\hat{\psi}_\varepsilon\|_{L^q(B_2(\hat{x}_0) \cap \Omega_\delta^{\sigma_\varepsilon})} + \left\| \sum_{i=1}^3 \mathbb{E}_{i,\varepsilon}(\sigma_\varepsilon \hat{x}) \right\|_{L^q(B_2(\hat{x}_0) \cap \Omega_\delta^{\sigma_\varepsilon})} \right) \\ &\leq C\varepsilon^4, \end{aligned}$$

where $\text{dist}(\hat{x}_0, \partial\Omega^{\sigma_\varepsilon}) \leq 2$. Thanks to the Sobolev embedding theorem we get that

$$|\hat{\psi}_\varepsilon(\hat{x})| + |\nabla_{\hat{x}} \hat{\psi}_\varepsilon(\hat{x})| \leq C\varepsilon^4 \quad \text{for } \sigma_\varepsilon \hat{x} \in \Omega_{2\sigma_\varepsilon}.$$

Equivalently, we have

$$|\psi_\varepsilon(x)| = O(\varepsilon^4), \quad |\nabla_x \psi_\varepsilon(x)| = O(\varepsilon^2) \quad \text{for } x \in \Omega_{2\sigma_\varepsilon}.$$

As a consequence, for any $x \in \partial\Omega$, $|\nabla \psi_\varepsilon(x)| = O(\varepsilon^2)$. Then we can compute the normal derivative of the solution u_ε on the boundary by U , ϕ_3 and Lemma 4.4, and obtain

$$(4.31) \quad \begin{aligned} \partial_\nu u_\varepsilon(x) &= \frac{1}{\sigma_\varepsilon} U'(0) + (n-1)H_{\partial\Omega}(x)\phi_3'(0) + O(\varepsilon^2) = -\frac{1}{\sigma_\varepsilon} \frac{2}{\chi} \frac{A}{B^{\frac{2+\chi}{\chi}}} + O(1) \\ &= -\frac{1}{\varepsilon^2} \frac{2}{\chi} \frac{1}{B^{\frac{2}{\chi}} A^{\chi-1} |\partial\Omega|} - \frac{16(n-1)}{\chi(\chi+4)} \frac{A}{B^{\frac{2}{\chi}} |\partial\Omega|} |\log \varepsilon| \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y + O(1) \\ &= -\frac{\chi m \bar{u}}{\varepsilon^2 (\chi+2) |\partial\Omega|} - \frac{16(n-1)}{\chi(\chi+4)} \frac{\bar{u}}{|\partial\Omega|} |\log \varepsilon| \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y + O(1). \end{aligned}$$

For any $a \in (0, \bar{u})$, we consider the following equation

$$(4.32) \quad a = u_\varepsilon(x) = U\left(\frac{z(x)}{\sigma_\varepsilon}\right) + \sigma_\varepsilon(n-1)H_{\Gamma_{z(x)}}(y(x))\phi_3\left(\frac{z(x)}{\sigma_\varepsilon}\right) + \psi_\varepsilon(x).$$

Obviously, we get that $z(x) \sim \sigma_\varepsilon$. As a consequence of above analysis, we have $\psi_\varepsilon(x) = O(\varepsilon^4)$. Furthermore by the smoothness of the domain, we have

$$|H_{\partial\Omega}(y(x)) - H_{\Gamma_{z(x)}}(y(x))| = O(\sigma_\varepsilon)$$

for any $x \in \{\xi \in \Omega \mid u_\varepsilon(\xi) = a\}$ with $a \in (0, \bar{u})$. Then solving (4.32) is equivalent to solving

$$(4.33) \quad a = u_\varepsilon(x) = U\left(\frac{z(x)}{\sigma_\varepsilon}\right) + \sigma_\varepsilon(n-1)H_{\partial\Omega}(y(x))\phi_3\left(\frac{z(x)}{\sigma_\varepsilon}\right) + O(\varepsilon^4).$$

Since $U(z)$ is a strictly decreasing function, we can find a unique $z_{u,a}$ such that $U(z_{u,a}) = a$. By the expression of $U(z)$ we have $z_{u,a} = \left(\frac{A}{a}\right)^{\frac{2}{\chi}} - B$. Using (4.28), we end up with

$$\begin{aligned} z(x) &= \sigma_\varepsilon z_{u,a} + O(\sigma_\varepsilon^2) \\ &= \left(\frac{A^\chi}{B} |\partial\Omega| \varepsilon^2 - \frac{A^\chi}{B} \frac{8(n-1)A^\chi}{\chi+4} |\partial\Omega| |\log \varepsilon| \varepsilon^4 \int_{\partial\Omega} H_{\partial\Omega}(y) d\sigma_y \right) z_{u,a} + O(\varepsilon^4), \end{aligned}$$

which yield (2.24) and hence completes the proof. $\square$

5. APPENDIX (PROOF OF LEMMA 2.1)

Proof of Lemma 2.1. For any point $y^0 \in \partial\Omega$, there is a small neighborhood of y^0 , denoted by $\mathcal{U}_{y^0}$, such that the n -th coordinate of any point y in $\mathcal{U}_{y^0}$ can be represented as a function depending on the previous $n-1$ coordinates, i.e., we can find a smooth function φ such that $y = (y_1, \dots, y_{n-1}, \varphi(y_1, \dots, y_{n-1}))$ for any $y \in \mathcal{U}_{y^0}$. Based on this setting, the unit inner normal vector is given by

$$(5.1) \quad \nu(y) = \frac{(-D\varphi(y'), 1)}{\sqrt{1 + |D\varphi(y')|^2}},$$

where

$$D\varphi(y') = \left(\frac{\partial\varphi(y')}{\partial y_1}, \dots, \frac{\partial\varphi(y')}{\partial y_{n-1}} \right) \quad \text{and} \quad y' = (y_1, \dots, y_{n-1}).$$

Then the mean curvature at $y \in \mathcal{U}_{y^0}$ is

$$(5.2) \quad H_{\partial\Omega}(y) := -\frac{1}{n-1} \sum_{i=1}^{n-1} \partial_{y_i} \nu_i = \frac{1}{n-1} \sum_{i=1}^{n-1} \partial_{y_i} \left(\frac{\partial_{y_i} \varphi(y')}{\sqrt{1 + |D\varphi(y')|^2}} \right).$$

According to the definition of the map $X : \partial\Omega \times (0, \delta) \rightarrow \Omega_\delta$, we have

$$(5.3) \quad x = X(y, z) = \tilde{X}(y', z) = (y', \varphi(y')) + z \frac{(-D\varphi(y'), 1)}{\sqrt{1 + |D\varphi(y')|^2}}.$$

By a direct computation, we have

$$\begin{aligned}
 \frac{\partial \tilde{X}(y', z)}{\partial(y', z)} &= \left[\begin{array}{c|c} \text{Id}_{(n-1) \times (n-1)} & \begin{matrix} \nu_1 \\ \vdots \\ \nu_{n-1} \end{matrix} \\ \hline \partial_{y_1} \varphi & \cdots & \partial_{y_{n-1}} \varphi & \nu_n \end{array} \right] \\
 (5.4) \quad &+ z \left[\begin{array}{c|c} \partial_{y_1} \nu_1 & \cdots & \partial_{y_{n-1}} \nu_1 \\ \vdots & \ddots & \vdots \\ \partial_{y_1} \nu_{n-1} & \cdots & \partial_{y_{n-1}} \nu_{n-1} \\ \partial_{y_1} \nu_n & \cdots & \partial_{y_{n-1}} \nu_n \end{array} \middle| \begin{matrix} 0 \\ \vdots \\ 0 \\ 0 \end{matrix} \right] \\
 &= \mathcal{M}_1(y') + z\mathcal{M}_2(y').
 \end{aligned}$$

By (5.1)–(5.4), we get from direct computation that

$$(5.5) \quad \det \mathcal{M}_1(y') = \nu_n(y) - \sum_{i=1}^{n-1} \nu_i(y) \partial_{y_i} \varphi(y') = \sqrt{1 + |D\varphi(y')|^2},$$

and

$$\begin{aligned}
 (5.6) \quad \det \left(\frac{\partial \tilde{X}(y', z)}{\partial(y', z)} \right) &= \det \mathcal{M}_1(y') \det (\text{Id}_{n \times n} + z\mathcal{M}_1^{-1}(y')\mathcal{M}_2(y')) \\
 &= \sqrt{1 + |D\varphi(y')|^2} \left(1 + z \text{Trace}(\mathcal{M}_1^{-1}(y')\mathcal{M}_2(y')) + O(z^2) \right).
 \end{aligned}$$

In order to compute $\text{Trace}(\mathcal{M}_1^{-1}(y')\mathcal{M}_2(y'))$, we first compute the inverse matrix of $\mathcal{M}_1(y')$ and derive that

$$\begin{aligned}
 &\mathcal{M}_1^{-1}(y') \\
 &= \left[\begin{array}{c|c} \text{Id}_{(n-1) \times (n-1)} + (\det \mathcal{M}_1(y'))^{-1} \nu_i \partial_{y_j} \varphi & \begin{matrix} -(\det \mathcal{M}_1(y'))^{-1} \nu_1 \\ \vdots \\ -(\det \mathcal{M}_1(y'))^{-1} \nu_{n-1} \end{matrix} \\ \hline -(\det \mathcal{M}_1(y'))^{-1} \partial_{y_1} \varphi & \cdots & -(\det \mathcal{M}_1(y'))^{-1} \partial_{y_{n-1}} \varphi & (\det \mathcal{M}_1(y'))^{-1} \end{array} \right].
 \end{aligned}$$

As a consequence, using (5.2) and elementary matrix computation, one gets

$$\begin{aligned}
 &\text{Trace}(\mathcal{M}_1^{-1}(y')\mathcal{M}_2(y')) \\
 (5.7) \quad &= \sum_{i=1}^{n-1} \left(\partial_{y_i} \nu_i + \sum_{j=1}^{n-1} \frac{\nu_i \partial_{y_j} \varphi \partial_{y_i} \nu_j}{\det \mathcal{M}_1(y')} - \frac{\nu_i \partial_{y_i} \nu_n}{\det \mathcal{M}_1(y')} \right) \\
 &= -(n-1)\mathcal{H}_{\partial\Omega}(y) + \frac{1}{\det \mathcal{M}_1(y')} \sum_{i=1}^{n-1} \left(\sum_{j=1}^{n-1} \nu_i \partial_{y_j} \varphi \partial_{y_i} \nu_j - \nu_i \partial_{y_i} \nu_n \right) \\
 &= -(n-1)\mathcal{H}_{\partial\Omega}(y),
 \end{aligned}$$

where we have used that

$$\begin{aligned}
& \sum_{i=1}^{n-1} \left(\sum_{j=1}^{n-1} \nu_i \partial_{y_j} \varphi \partial_{y_i} \nu_j - \nu_i \partial_{y_i} \nu_n \right) \\
&= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left(\frac{\partial_{y_i} \varphi \partial_{y_j} \varphi \partial_{y_i y_j} \varphi}{1 + |D\varphi|^2} - \frac{\partial_{y_i} \varphi (\partial_{y_j} \varphi)^2}{(1 + |D\varphi|^2)^2} \sum_{k=1}^{n-1} \partial_{y_i y_k} \varphi \partial_{y_k} \varphi \right) \\
&\quad - \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \frac{\partial_{y_i} \varphi}{(1 + |D\varphi|^2)^2} \sum_{k=1}^{n-1} \partial_{y_i y_k} \varphi \partial_{y_k} \varphi \\
&= 0.
\end{aligned}$$

Therefore, by (5.6) and (5.7) we obtain that

$$\det(\mathcal{M}_1(y') + z\mathcal{M}_2(y')) = \sqrt{1 + |D\varphi(y')|^2} (1 - (n-1)\mathcal{H}_{\partial\Omega}(y)z + O(z^2)).$$

On the other hand, using $d\sigma_y = \sqrt{1 + |D\varphi(y')|^2} dy'$, we have

$$\begin{aligned}
dx &= \sqrt{1 + |D\varphi(y')|^2} (1 - (n-1)z\mathcal{H}_{\partial\Omega}(y) + O(z^2)) dy' dz \\
&= (1 - (n-1)z\mathcal{H}_{\partial\Omega}(y) + O(z^2)) d\sigma_y dz.
\end{aligned}$$

Then we get (2.6) for a part of Ω_δ , where the n -th coordinate can be represented by the other $n-1$ ones. The other parts can be treated similarly, where the only difference is that we may represent the i -th coordinate (other than the n -th one) by the other $n-1$ coordinates. $\square$

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