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Radial spiky steady states of a flux-limited Keller–Segel model: Existence, asymptotics, and stability

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Abstract

This paper is concerned with the radial stationary problem of a flux-limited Keller–Segel model derived in a multidimensional bounded domain with Neumann boundary conditions. With the global bifurcation theory and Helly compactness theorem by treating the chemotactic coefficient as a bifurcation parameter, we establish the existence of nonconstant monotone radial stationary solutions and further show that the radial stationary solution will tend to a Dirac delta mass as the chemotactic coefficient tends to infinity. By using the stability criterion of Crandall and Rabinnowitz, we prove the linearized stability of bifurcating stationary solutions near the bifurcation points.

KEYWORDS

flux-limited Keller–Segel model, global bifurcation theory, Helly compactness theorem, linearized stability, stationary solutions

1 | INTRODUCTION

Chemotaxis, the directed movement of species in response to chemical stimuli, is a fundamental cellular process in many important biological phenomena such as embryonic development,[1] wound healing,[2] blood vessel formation,[3] pattern formation,[4,5] and so on. Mathematical models of chemotaxis from both microscopic (individual) and macroscopic (population) scales have been developed and widely studied as one of the most popular topics in modern mathematical biology. The canonical macroscopic chemotaxis model was proposed by Keller–Segel in Ref. 6 to describe the aggregation of cellular slime molds *Dictyostelium discoideum* and in Ref. 7 to describe the wave propagation of bacterial chemotaxis. Because of the presence of the critical mass and spatial dimensions in determining whether solutions undergo smooth dispersion or blow-up, the Keller–Segel system and its variants have attracted enormous attentions from researcher (cf. Refs. 8–13).

In this paper, we are interested in the flux-limited Keller–Segel (FLKS) system, where some special forms of such system have already been introduced in Refs. 9, 14. Based on the assumption that the chemotactic flux is bounded in modeling velocity saturation in large gradient environment, the FLKS system reads

$$\begin{cases} \partial_t \rho = \Delta \rho - \nabla \cdot (\chi \rho \phi(|\nabla S|) \nabla S), \\ \partial_t S - \Delta S + \alpha S = \rho, \\ (\rho, S)(0, x) = (\rho_0, S_0)(x), \end{cases} \tag{1}$$

where $\rho(t, x)$ denotes the cell density and $S(t, x)$ the chemical signal concentration at time $t > 0$ and position x. $\phi \in C^3(\mathbb{R}; \mathbb{R}^+)$ is referred to as the chemotactic sensitivity function describing the signal response mechanism and $\chi > 0$ is the chemotactic coefficient. Here limited flux means there is a positive constant A_∞ such that

$$\max_{r \in \mathbb{R}^+} |r\phi(r)| = A_\infty \text{ and } \phi(0) > 0. \tag{2}$$

The motivation to study the FLKS system (1)–(2) comes from its derivation from kinetic chemotaxis model. Patlak[15] first used the kinetic theory to express the chemotactic velocity in terms of the average of velocities and run times of individual cells. Subsequently this approach was essentially boosted by Alt[16] and developed by Othmer, Dunber, and Alt[17] using a velocity-jump process, which assumes that cells run with some velocity and at random instants of time they changes velocities (directions) according to a Poisson process. Denoting by $f(t, x, v)$ the cell number density, at time t and position x, moving with a velocity $v \in V$ (compact set of $\mathbb{R}^n$ with rotational symmetry), the governing evolution equation of this process is described by a kinetic equation reading as:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \int_V \big(T[S](v, v') f(t, x, v') - T[S](v', v) f(t, x, v) \big) dv'. \tag{3}$$

The tumbling kernel $T[S](v, v')$ describes the frequency of changing trajectories from velocity v' (anterior) to v (posterior) depending on the chemical concentration S or its gradient. The advantage of kinetic models over macroscopic models is that details of the run-and-tumble motion at individual scales can be explicitly incorporated into the tumbling kernel and then passed to macroscopic quantities through bottom-up scaling (cf. Refs. 18–23), where the rigorous justification of upscaling limits have been studied in many works (see Refs. 24–28 and reference therein). As such the flux-limited term in the system (1) was derived from the so-called smoothed stiffness response in the signaling response of cells considered in the kinetic equation (3). When cells (like bacteria) respond to temporal gradients along their pathways, the tumbling kernel may depend on the pathway (directional) derivative in the form of (cf. Refs. 29, 22, 23):

$$T[S](v, v') = \lambda_0 + \sigma \Psi(D_t S / \varepsilon), \qquad D_t S = \partial_t S + v' \nabla S, \tag{4}$$

where λ_0 denotes a basal meaning tumbling frequency, σ accounts for the variation of tumble frequency modulation, Ψ denotes the signal response (sensing) function, and the parameter $\varepsilon > 0$ represents the stiffness of response. The stiffness of signal response has shown to be important to describe the traveling pulses of bacterial chemotaxis observed in the experiment[29,30,31] and is related to instabilities for both of the FLKS system and the kinetic equation.[32,33] With the parabolic scaling $t \to \varepsilon^2 t, x \to \varepsilon x$, the macroscopic limits of (3) with tumbling kernel (4) leads to the flux-limited equation in (1) satisfying the condition (2) when Ψ is a decreasing function expressing the fact that cells are less likely to tumble when the chemical concentration increases (see details in Refs. 34, 29).

The boundedness assumption (2) on the chemotactic flux ensures that solutions of (1) exist globally in time (cf. Refs. 14, 35), unlike the Keller–Segel system for which finite time blow-up may occur. In $\mathbb{R}^n (n = 2, 3)$, the large-time behavior of solutions (1) with (2) was studied in Ref. 34 for any initial mass $\int_{\mathbb{R}^n} \rho_0(x)dx$ if $\alpha > 0$ and for small initial mass if $\alpha = 0$. When $\alpha = 0$, the nonconstant radially symmetric steady states were found by the shooting method with some tricky transformation in Ref. 34. It was shown that the cell total mass, denoted by m, is an important parameter. In dimension $n = 2$, radial symmetric solutions exist if and only if $m > \frac{8\pi}{\phi(0)}$ with $\max_{r\in\mathbb{R}^+} \phi(r) = \phi(0)$. While in dimension $n > 2$, there is no positive nonconstant radial steady state for any finite mass. The main purpose of this paper is to study the stationary problem of (1) with (2) in a bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$ with Neumman boundary conditions for $\alpha > 0$. With the Neumann boundary conditions, the integration of the first equation of (1) yields the cell mass conservation:

$$\int_\Omega \rho(x,t)dx = \int_\Omega \rho_0(x)dx := m,$$

where $m > 0$ is a constant denoting the cell total mass. Therefore, the relevant stationary problem of (1) reads as

$$\begin{cases} \Delta\rho - \nabla\cdot(\chi\rho\phi(|\nabla S|)\nabla S) = 0, & x \in \Omega, \\ \Delta S - \alpha S + \rho = 0, & x \in \Omega, \\ \frac{\partial\rho}{\partial\nu}(x) = \frac{\partial S}{\partial\nu}(x) = 0, & x \in \partial\Omega, \\ \int_\Omega \rho(x)\mathrm{d}x = m, \end{cases} \tag{5}$$

where ν denotes the outer unit normal vector of $\partial\Omega$. It is quite difficult to establish the existence of nonconstant solutions to (5) for a general domain $\Omega \subset \mathbb{R}^n (n \geq 2)$ due to the cross-diffusion nature and nonlinearity of ϕ. As in Ref. 34, we will focus on the existence of nonconstant radial solutions to (5) subject to the condition (2). It was shown in Ref. 34 that (1)–(2) has no nonconstant stationary solutions in $\mathbb{R}^n (n = 2, 3)$ if $\alpha > 0$, while if $\alpha = 0$, the nonconstant radially symmetric stationary solutions exist only if $n = 2$ and $m > \frac{8\pi}{\phi(0)}$. The result of this paper will show that when (1)–(2) with $\alpha > 0$ is considered in a bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$ with Neumman boundary conditions, it admits nonconstant radial stationary solutions, which gives a very different result from the whole space $\mathbb{R}^n (n \geq 2)$ shown in Ref. 34. Unlike the classical Keller–Segel system where ϕ is constant, ρ cannot be solved in terms of S from the first equation of (5). When $\alpha = 0$, a tricky variable transformation was used in Ref. 34 to convert the radial form of (5) into a first-order ordinary differential equation so that shooting method can be applied. This idea, however, is inapplicable to the case $\alpha > 0$. In this paper, we consider the radially symmetric solutions of (5) in multidimensions and employ the global bifurcation theory by treating the chemotactic coefficient $\chi > 0$ as a bifurcation parameter to establish the existence of nonconstant radial solutions to (5) (see Theorem 1), find the asymptotic profile of radial stationary solutions as $\chi \to \infty$ (see Theorem 2) by Helly compactness theorem and establish the linearized stability of bifurcating stationary solutions near bifurcation points (see Theorem 3) by the Crandall and Rabinnowitz stability criterion. For the one-dimensional case, similar results can be directly obtained by the same approaches with much less endeavors, see Ref. [10] for details. We remark that the global bifurcation theory has been successfully used to establish the existence of nonconstant steady states of various one-dimensional chemotaxis systems in Ref. 36 first and then in Refs. 37–39. Another way to establish the existence of stationary solutions of chemotaxis models is the Degree theory as in Refs. 40, 41.

The rest of this paper is organized as follows. In Section 2, we consider (1)–(2) with $\alpha > 0$ in $\Omega \subset \mathbb{R}^n (n \geq 2)$ and establish the existence of radially symmetric nonconstant stationary solutions by the global bifurcation theory. In Section 3, we derive asymptotic profiles of radially symmetric stationary solutions as $\chi \to \infty$. In the final Section 4, we explore the linearized stability of bifurcating stationary solutions near bifurcation points.

2 | RADIAL STEADY STATES IN MULTIDIMENSIONS

In this section, we explore the existence of radially symmetric solutions of (5) on $\Omega = B_R(0)$, where $B_R(0)$ is the ball in $\mathbb{R}^n$ $(n \geq 2)$ with radius R centered at origin. In the sequel, when we say a solution of (5), it always means a nonconstant solution unless otherwise stated. For convenience, we denote $\psi(|\nabla S|^2) = \phi(|\nabla S|)$, where $\phi(\cdot) \in C^3(\mathbb{R}; \mathbb{R}^+)$. Then (5) becomes

$$\begin{cases} \Delta\rho - \nabla \cdot (\chi\rho\psi(|\nabla S|^2)\nabla S) = 0, & x \in B_R(0), \\ \Delta S - \alpha S + \rho = 0, & x \in B_R(0), \\ \frac{\partial\rho}{\partial\nu}(x) = \frac{\partial S}{\partial\nu}(x) = 0, & x \in \partial B_R(0), \\ \int_{B_R(0)} \rho(x)\mathrm{d}x = m, \end{cases} \tag{6}$$

where ψ satisfies the following condition from (2):

$$\max_{r\in\mathbb{R}} |r\psi(r^2)| = A_\infty \quad \text{and} \quad \psi(0) > 0. \tag{7}$$

Next we focus on the radially symmetric solution of (6)–(7), that is, $(\rho, S)(x) = (\rho, S)(|x|) =: \rho(r)$. To settle the possible singularity, we assume that $\rho'(0) = S'(0)$ naturally. Then the radial solution

$(\rho, S)(r)$ satisfies

$$\begin{cases} \rho''(r) + \frac{n-1}{r}\rho'(r) - \chi[\rho(r)\psi(|S'(r)|^2)(S''(r) + \frac{n-1}{r}S'(r)) \\ \quad +S'(r)\rho'(r)\psi(|S'(r)|^2) + 2(S'(r))^2 S''(r)\rho(r)\dot{\psi}(|S'(r)|^2)] = 0, & r \in (0,R), \\ S''(r) + \frac{n-1}{r}S'(r) + \rho(r) - \alpha S(r) = 0, & r \in (0,R), \\ \rho'(0) = \rho'(R) = S'(0) = S'(R) = 0, \\ \int_0^R r^{n-1}\rho(r)\mathrm{d}r = \frac{m}{\omega_{n-1}}, \end{cases} \tag{8}$$

where ω_{n-1} is the surface area of unit sphere and by $\dot{\psi}(s)$ we denote the derivative of ψ with respect to s. In the following, for brevity, we denote

$$M = \frac{nm}{\omega_{n-1}R^n}. \tag{9}$$

Then $\int_0^R r^{n-1}\rho(r)\mathrm{d}r = M\frac{R^n}{n}$ and $(M, \frac{M}{\alpha})$ is a constant solution of (8) and the system (8) can be equivalently written as

$$\begin{cases} [r^{n-1}(\rho'(r) - \chi\rho(r)S'(r)\psi(|S'(r)|^2))]' = 0 & r \in (0,R), \\ (r^{n-1}S'(r))' - \alpha r^{n-1}S(r) + r^{n-1}\rho(r) = 0, & r \in (0,R), \\ \rho'(0) = \rho'(R) = S'(0) = S'(R) = 0, \\ \int_0^R r^{n-1}\rho(r)\mathrm{d}r = M\frac{R^n}{n}. \end{cases} \tag{10}$$

Below we employ the global bifurcation theory to establish the existence of solutions of (8) by using $\chi > 0$ as the bifurcation parameter. Sometimes we use (8), while sometimes we use equivalent equations (10) for the sake of convenience.

2.1 | A priori bound

We first derive a priori bound for solutions of (8) needed for the global bifurcation theory.

Lemma 1. *Suppose ρ and S are the positive classical solution of (8). Then it follows that*

$$M\exp[-\chi RA_\infty] \le \rho(r) \le M\exp[\chi RA_\infty],\ \forall\, r \in [0,R] \tag{11}$$

and

$$\frac{M}{\alpha}\exp[-\chi RA_\infty] \le S(r) \le \frac{M}{\alpha}\exp[\chi RA_\infty],\ \forall\, r \in [0,R], \tag{12}$$

where A_∞ is given in (7).

Proof. Integrating the first equation in (10) over $[0, r]$, we have

$$\rho'(r) = \chi\rho(r)S'(r)\psi(|S'(r)|^2). \tag{13}$$

This along with (7) gives that

$$|(\ln\rho(r))'| = \left|\frac{\rho'(r)}{\rho(r)}\right| = \chi|S'(r)\psi(|S'(r)|^2)| \le \chi A_\infty. \tag{14}$$

Because $\int_0^R r^{n-1}\rho dr = \frac{R^n M}{n}$, there exists $r_0 \in (0, R)$ such that $\rho(r_0) = M$ by the mean value theorem of integrals. So (14) implies that

$$|\ln(\rho(r)) - \ln M| \le \chi R A_\infty, \tag{15}$$

which gives (11). We proceed to derive the bounds of $S(r)$. First if S attains its maximum at an interior point $r^* \in (0, R)$, that $\max_{r\in[0,R]} S(r) = S(r^*)$. Then applying the strong maximum principle on the second equation of (8), we obtain that

$$S''(r^*) = -\rho(r^*) + \alpha S(r^*) \le 0, \tag{16}$$

which is

$$S(r^*) \le \frac{\rho(r^*)}{\alpha} \le \frac{M}{\alpha}\exp[\chi R A_\infty]. \tag{17}$$

If S attains its maximum at R with $S(R) > \frac{M}{\alpha}\exp[\chi R A_\infty]$, which along with (11) implies that $\alpha S - \rho > 0$, then it follows from the Hopf boundary lemma that $S'(R) > 0$. This contradicts boundary conditions. Hence if S attains its maximum at R, then $S(R) \le \frac{M}{\alpha}\exp[\chi R A_\infty]$. By similar arguments, if S attains its maximum at 0, then $S(0) \le \frac{M}{\alpha}\exp[\chi R A_\infty]$. Collecting above results, we conclude that $S(r) \le \frac{M}{\alpha}\exp[\chi R A_\infty]$ for $r \in [0, R]$. By similar arguments, we can show that $S(r) \ge \frac{M}{\alpha}\exp[-\chi R A_\infty]$. This gives (12) and completes the proof. ∎

2.2 | Existence of solutions

In this subsection, we use the global bifurcation theory to establish the existence of solutions to (8). To this end, we denote

$$X = \{f \in H^2(B_R(0)) | f \text{ is a radial symmetric function with } f'(0) = f'(R) = 0\},$$

$$Y = \{f \in L^2(B_R(0)) | f \text{ is a radial symmetric function}\},$$

$$Y_0 = \left\{f \in Y : \int_0^R r^{n-1} f(r) dr = 0\right\}, \tag{18}$$

where $f = f(r)$. Now we define $\mathcal{F} : X \times X \times \mathbb{R} \to Y_0 \times Y \times \mathbb{R}$ and

$$\mathcal{F}(\rho, S, \chi) = -\begin{pmatrix} \mathcal{G}_1(r) \\ S''(r) + \frac{n-1}{r}S'(r) + \rho(r) - \alpha S(r) \\ \int_0^R r^{n-1}\rho(r)\mathrm{d}r - M\frac{R^n}{n} \end{pmatrix}, \tag{19}$$

where

$$\begin{aligned} \mathcal{G}_1(r) = \rho''(r) + \frac{n-1}{r}\rho'(r) - \chi[\rho(r)\psi(|S'(r)|^2)(S''(r) + \frac{n-1}{r}S'(r)) \\ + S'(r)\rho'(r) + 2(S'(r))^2S''(r)\rho(r)\dot{\psi}(|S'(r)|^2)]. \end{aligned} \tag{20}$$

It is obvious that all solutions (ρ, S) of $\mathcal{F}(\rho, S, \chi) = 0$ with fixed $\chi > 0$ are the solutions of (8) and vice versa. Because $\mathcal{F} : X \times X \times \mathbb{R} \to Y_0 \times Y \times \mathbb{R}$ is C^1-smooth and at the fixed point $(\rho_0, S_0) \in X \times X$, the Frechet derivative of $\mathcal{F}$ is

$$D_{(\rho,S)}\mathcal{F}(\rho_0, S_0, \chi)(\rho, S) = -\begin{pmatrix} \mathcal{G}_2(r) \\ -S'' - \frac{n-1}{r}S'(r) + \alpha S - \rho \\ \int_0^R r^{n-1}\rho(r)\mathrm{d}r \end{pmatrix} \tag{21}$$

for any $(\rho, S) \in X \times X$, where

$$\begin{aligned} \mathcal{G}_2(r) = \rho''(r) + \frac{n-1}{r}\rho'(r) - \chi[S_0'\psi\left(|S_0'(r)|^2\right)\rho' + \rho_0'\psi\left(|S_0'(r)|^2\right)S' \\ + 2\dot{\psi}\left(|S_0'(r)|^2\right)\rho_0'(S_0')^2S' + \rho(S_0')^2S_0''\psi\left(|S_0'|^2\right) + 2\rho_0S_0'S'S_0''\psi\left(|S_0'|^2\right) \\ + \rho_0S_0'S''\psi\left(|S_0'|^2\right) + 2\rho_0(S_0')^2S_0''\dot{\psi}\left(|S_0'|^2\right)S' + \rho\left(S_0'' + \frac{n-1}{r}S_0'\right)\psi\left(|S_0'|^2\right) \\ + \rho_0\left(S'' + \frac{n-1}{r}S'\right)\psi\left(|S_0'|^2\right) + 2\rho_0\left(S_0'' + \frac{n-1}{r}S_0'\right)\psi\left(|S_0'|^2\right)S_0'S']. \end{aligned} \tag{22}$$

Then we have the following results.

Lemma 2. *For any fixed point $(\rho_0, S_0) \in X \times X$, the Frechet derivative in (21), $D_{(\rho,S)}\mathcal{F}(\rho_0, S_0, \chi) : X \times X \to Y_0 \times Y \times \mathbb{R}$ is a Fredholm operator with zero index.*

Proof. Denote

$$D_{(\rho,S)}\mathcal{F}(\rho_0, S_0, \chi)(\rho, S) = \mathcal{F}_1(\rho, S) + \mathcal{F}_2(\rho, S), \tag{23}$$

where

$$\mathcal{F}_1(\rho,S)=\begin{pmatrix}\mathcal{G}_2\\ -S''-\frac{n-1}{r}S'(r)+\alpha S-\rho\\ 0\end{pmatrix},\quad \text{and}\quad \mathcal{F}_2(\rho,S)=\begin{pmatrix}0\\ 0\\ \int_0^R r^{n-1}\rho(r)\mathrm{d}r\end{pmatrix},\tag{24}$$

where $\mathcal{G}_2$ is defined in (22). Because $\mathcal{F}_2$ is a compact operator, by the well-known fact that the perturbation of a compact operator does not change the Fredholmness and the index of the original operator (see chapter 2 in Ref. 42), we only need to show that $\mathcal{F}_1$ is a Fredholm operator with index zero.

Observe that the term containing the second-order derivatives of ρ and S in the first and second components of $\mathcal{F}_1$ can be written as

$$\begin{pmatrix}-1 & \chi\psi(|S_0'|^2)[\rho_0 S_0'+\rho_0]\\ 0 & -1\end{pmatrix}\begin{pmatrix}\rho''\\ S''\end{pmatrix}.\tag{25}$$

By Remark 2.5 of case 3 in Ref. 43, $\mathcal{F}_1$ is elliptic and satisfies Agmon's condition (see Ref. 44). Using Theorem 3.3 and Remark 3.4 in Ref. 43, we know $\mathcal{F}_1$ and hence $D_{(\rho,S)}\mathcal{F}(\rho_0,S_0,)\,:\,X\times X\to Y\times Y$ is a Fredholm operator with zero index. With a similar argument as in the proof of Lemma 2.3 in Ref. 39, we finish the proof. ■

Now we take $\rho_0=\alpha S_0=M$. Then all possible bifurcation values of χ with the bifurcation points $(M,\frac{M}{\alpha},\chi)$ should satisfy

$$N\left(D_{(\rho,S)}\mathcal{F}\left(M,\frac{M}{\alpha},\chi\right)\right)\neq\{0\}.$$

The null space of $D_{(\rho,S)}\mathcal{F}(M,\frac{M}{\alpha},\chi)$ is the space of solutions (ρ,S) satisfying

$$\begin{cases}-[\rho''+\frac{n-1}{r}\rho']+\chi M\psi(0)[S''+\frac{n-1}{r}S']=0, & r\in(0,R)\\ -[S''+\frac{n-1}{r}S']+\alpha S-\rho=0, & r\in(0,R)\\ \int_0^R r^{n-1}\rho(r)\mathrm{d}r=0.\end{cases}\tag{26}$$

Now we consider the following eigenvalue problem for the radial function $u(r)$, which is

$$\begin{cases}u''(r)+\frac{n-1}{r}u'(r)+\lambda u(r)=0, & r\in(0,R),\\ u'(0)=u'(R)=0.\end{cases}\tag{27}$$

It is well known that all the solutions of (27) are

$$u(r)=r^{1-\frac{n}{2}}J_\gamma(\sqrt{\lambda}r),\tag{28}$$

where $\gamma=\frac{n}{2}-1$ and $J_\gamma(r)$ denotes the Bessel function of the first kind of order γ (see Ref. 45). Multiplying r^{n-1} by the equation of u in (27) and integrating the results over $[0,R]$, if $\lambda\neq 0$, we can easy obtain that

$$\int_0^R r^{n-1}u(r)\mathrm{d}r = 0. \tag{29}$$

Because $u'(R) = 0$, that is $(\frac{n}{2}-1)J_\gamma(\sqrt{\lambda}R) = R\sqrt{\lambda}J'_\gamma(\sqrt{\lambda}R)$. By q_i we denote the i-th positive root of the equation

$$\left(\frac{n}{2}-1\right)J_\gamma(q) = qJ'_\gamma(q), \tag{30}$$

with $q_i < q_{i+1}$ for all $i = 1, 2, \ldots$. Then the eigenfunctions of (27) corresponding to q_i are

$$u_i(r) = r^{1-\frac{n}{2}}J_\gamma\left(\frac{q_i}{R}r\right), \quad i = 1, 2, \ldots \tag{31}$$

in which we take $\lambda := \lambda_i = \frac{q_i^2}{R^2} > 0$. Denote $u_0(r) = 1$ for $\lambda := \lambda_0 = 0$.

Because the system (26) is linear, its solution is of the form $(\rho, S)(r) = \sum_{j=0}^{\infty}(t_j, s_j)u_j(r)$. Observe that $t_0 = s_0 = 0$ by $\int_0^R r^{n-1}\rho(r)\mathrm{d}r = 0$. Inserting (ρ, S) into (26), we have

$$\begin{cases} t_j - \chi M\psi(0)s_j = 0, \\ \frac{q_j^2}{R^2}s_j + \alpha s_j - t_j = 0, \end{cases} \tag{32}$$

whose nontrivial solutions exists if and only if $\chi = \bar{\chi}_j$ satisfying

$$\bar{\chi}_j = \frac{\alpha + \frac{q_j^2}{R^2}}{M\psi(0)} \tag{33}$$

for $m = 1, 2, \ldots$. Hence the null space $N(D_{(\rho,S)}\mathcal{F}(M, \frac{M}{\alpha}, \bar{\chi}_j))$ is one-dimensional and

$$N\left(D_{(\rho,S)}\mathcal{F}\left(M, \frac{M}{\alpha}, \bar{\chi}_j\right)\right) = \mathrm{span}\{(\bar{\rho}_j, \bar{S}_j)\}, \tag{34}$$

where

$$\bar{\rho}_j(r) = \left(\alpha + \frac{q_j^2}{R^2}\right)u_j(r), \quad \bar{S}_j(r) = u_j(r). \tag{35}$$

The transversality condition

$$D_{(\rho,S)\chi}\mathcal{F}\left(M, \frac{M}{\alpha}, \bar{\chi}_j\right)(\bar{\rho}_j, \bar{S}_j) \notin R\left(D_{(\rho,S)}\mathcal{F}\left(M, \frac{M}{\alpha}, \bar{\chi}_j\right)\right) \tag{36}$$

can be easily checked by

$$D_{(\rho,S)\chi}\mathcal{F}\left(M,\frac{M}{\alpha},\bar{\chi}_j\right)(\bar{\rho}_j,\bar{S}_j)=\begin{pmatrix}M\psi(0)\frac{q_j^2}{R^2}\\0\\0\end{pmatrix}u_j(r), \tag{37}$$

and $M\psi(0)\frac{q_j^2}{R^2}\neq 0$.

By Theorem 4.3 in Ref. 43, for any positive integer j, $\bar{\chi}_j$ is a bifurcation value; there exists $(-\delta,\delta)$ with $\delta>0$ and C^3- smooth mapping: $\tau\in(-\delta,\delta)\mapsto(\rho_j(\tau),S_j(\tau))\in X\times X$ and $\tau\in(-\delta,\delta)\mapsto\chi_j(\tau)\in\mathbb{R}$ such that

$$\chi_j(0)=\bar{\chi}_j \tag{38}$$

and

$$(\rho_j(\tau,r),S_j(\tau,r))=\left(M,\frac{M}{\alpha}\right)+\tau(\bar{\rho}_j(r),\bar{S}_j(r))+o(\tau) \tag{39}$$

is a solution of (8). In addition, all nonconstant solutions of (8) near the bifurcation point $(M,\frac{M}{\alpha},\bar{\chi}_j)$ live on the curve

$$C_j=(\rho_j(\tau,r),S_j(\tau,r),\chi_j(\tau)),\ \tau\in(-\delta,\delta),\ j\geq 1. \tag{40}$$

Denote the component of nontrivial solutions that contains C_1 by C.

Theorem 1. *Let $\psi\in C^2(\mathbb{R};\mathbb{R}^+)$ satisfy (7). Then for any fixed constants $M>0$ defined in (9) and*

$$\chi>\bar{\chi}_1=\frac{\alpha+\frac{q_1^2}{R^2}}{M\psi(0)}, \tag{41}$$

there exists a positive solution (ρ,S) of (8) satisfying $\rho',S'<0$ on $(0,R)$, where $(M,\frac{M}{\alpha})$ is the constant solution of (8).

Proof. The proof consists of several steps shown below.

Step 1.

We will prove that $\chi>0$ on C. Denote $\mathcal{A}_1=\{(\rho,S,\chi)\in C|\chi>0\}$. Then $\mathcal{A}_1$ is nonempty because $(M,\frac{M}{\alpha},\bar{\chi}_1)\in\mathcal{A}_1$. Observing that $\mathcal{A}_1$ is open in C, we only need to show $\mathcal{A}_1$ is closed in C to finish the proof of Step 1.

Take $(\rho_k,S_k,\chi_k)\in\mathcal{A}_1\to$ some $(\rho,S,\chi)\in C$ as $k\to\infty$ in the topology in $X\times X\times\mathbb{R}$ and hence in $C^2(\bar{B}_R(0))\times C^2(\bar{B}_R(0))\times\mathbb{R}$ by the elliptic regularity. Because $\chi_k\geq 0$ for all k, then $\chi\geq 0$. If

$\chi = 0$, we know

$$\begin{cases} \rho'' + \frac{n-1}{r}\rho' = 0, \\ S'' + \frac{n-1}{r}S' - \alpha S + \rho = 0, \\ \int_0^R r^{n-1}\rho(r)\mathrm{d}r = M\frac{R^n}{n}, \end{cases} \tag{42}$$

because $(\rho, S, \chi) \in \mathcal{C}$. Solving the first equation of (42) with $\rho'(R) = 0$, which is $(r^{n-1}\rho(r)')' = 0$, we have $\rho' \equiv 0$ and hence $(\rho, S) \equiv (M, \frac{M}{\alpha})$. Then $(\rho, S, \chi) = (M, \frac{M}{\alpha}, 0)$ is a new bifurcation point, which is impossible because we have shown all possible bifurcation values of χ are given by (33). Thus, we have $\chi > 0$, which implies $\mathcal{A}_1$ is closed in $\mathcal{C}$ and hence $\mathcal{A}_1 = \mathcal{C}$.

Step 2. We will prove that on $\mathcal{C}$, $\rho(r)$, $S(r) > 0$ for $r \in [0, R]$. Denote $\mathcal{A}_2 = \{(\rho, S, \chi) \in \mathcal{C} | \rho > 0, S > 0 \text{ for } r \in [0, R]\}$. Clearly $\mathcal{A}_2$ is nonempty because $(M, \frac{M}{\alpha}, \bar{\chi}_1) \in \mathcal{A}_2$. Obviously, $\mathcal{A}_2$ is open in $\mathcal{C}$. Next, we prove $\mathcal{A}_2$ is closed in $\mathcal{C}$.

Take $(\rho_k, S_k, \chi_k) \in \mathcal{A}_2 \to$ some $(\rho, S, \chi) \in \mathcal{C}$ as $k \to \infty$ in the topology in $X \times X \times \mathbb{R}$ and hence in $C^2(\bar{B}_R(0)) \times C^2(\bar{B}_R(0)) \times \mathbb{R}$ by the elliptic regularity. Because ρ_k, $S_k > 0$, we know ρ, $S \geq 0$. Assume there exists $r_0 \in [0, R]$ such that $S(r_0) = 0$. Applying the strong maximum principle and Hopf boundary point lemma to the following Neumann problem:

$$\begin{cases} rS'' + (n-1)S' - \alpha rS + r\rho = 0 \ \text{ on } r \in (0, R) \\ S'(0) = S'(R), \end{cases} \tag{43}$$

we get $S \equiv 0$, which implies that $\rho \equiv 0$. This is impossible because $\int_0^R \rho(r)\mathrm{d}r = MR > 0$. Thus, $S > 0$ on $[0, R]$. Similarly, if there exists $r_0 \in [0, R]$ such that $\rho(r_0) = 0$, applying strong maximum principle and Hopf boundary point lemma on

$$\begin{cases} r\rho'' + (n-1)\rho' - \chi[\rho\psi(rS'' + (n-1)S') + S'\rho' + 2\dot{\psi}r(S')^2S''\rho] = 0 \ \text{ on } r \in (0, R) \\ \rho'(0) = \rho'(L), \end{cases} \tag{44}$$

where the coefficients are bounded because $\psi \in C^2(\mathbb{R}, \mathbb{R}^+)$ and $S \in C^2(\bar{B}_R(0))$, we have $\rho \equiv 0$, which is impossible. Thus, $\rho > 0$ on $[0, R]$. So $\mathcal{A}_2$ is closed in $\mathcal{C}$ and then $\mathcal{A}_2 = \mathcal{C}$.

Step 3. Consider the "positive part" of curve $\mathcal{C}$ (the part of $\tau \in (0, \delta)$), denoted as $\mathcal{C}^+$ in the sequel. Then we will prove that $\rho'(r)$, $S'(r) < 0$ for $r \in (0, R)$ on $\mathcal{C}^+ \backslash \{(M, \frac{M}{\alpha}, \bar{\chi}_1)\}$.

Denote $\mathcal{A}_3 = \{(\rho, S, \chi) \in \mathcal{C}^+ \backslash \{(M, \frac{M}{\alpha}, \bar{\chi}_1)\} | \rho' < 0, S' < 0 \text{ for } r \in [\epsilon, R)\}$, where $0 < \epsilon < R$ is arbitrarily small.

Because all the points on $\mathcal{C}^+$ near the bifurcation point $(M, \frac{M}{\alpha}, \bar{\chi}_1)$ can be written as

$$(\rho(\tau, r), S(\tau, r)) = \left(M, \frac{M}{\alpha}\right) + \tau(\bar{\rho}_1(r), \bar{S}_1(r)) + o(\tau), \tag{45}$$

for some small $\tau \in (0, \delta)$, where $\gamma = \frac{n}{2} - 1$ and

$$\bar{\rho}_1(r) = \left(\alpha + \frac{q_1^2}{R^2}\right) r^{1-\frac{n}{2}} J_\gamma\left(\frac{q_1 r}{R}\right), \quad \bar{S}_1(r) = r^{1-\frac{n}{2}} J_\gamma\left(\frac{q_1 r}{R}\right). \tag{46}$$

It is easy to see $\bar{\rho}_1'(r), \bar{S}_1'(r) < 0$ on $r \in (0, R)$. Then $(\rho(\tau, r), S(\tau, r)) \in \mathcal{A}_3$ for τ small enough, which implies that $\mathcal{A}_3$ is nonempty.

Now we show $\mathcal{A}_3$ is open in $\mathcal{C}^+\backslash\{(M, \frac{M}{\alpha}, \bar{\chi}_1)\}$. Assume that there exists a sequence $(\rho_k, S_k, \chi_k) \in \mathcal{C}^+\backslash\{(M, \frac{M}{\alpha}, \bar{\chi}_1)\}$ converging to $(\rho, S, \chi) \in \mathcal{A}_3$ as $k \to \infty$ in the topology of $X \times X \times \mathbb{R}$ and hence $C^2(\bar{B}_R(0)) \times C^2(\bar{B}_R(0)) \times \mathbb{R}$. Because $\rho', S' < 0$, for large k, we have $\rho_k', S_k' \leq 0$ on $[\epsilon, R)$ and hence $\rho_k', S_k' < 0$ on any compact subset of $[\epsilon, R)$. So we can show the nondegeneracy of the second order of ρ and S at $r = R$ to ensure $\rho_k', S_k' < 0$ on $[\epsilon, R)$, which is $\rho''(r) \neq 0, S''(r) \neq 0$ at $r = R$. Observe that

$$\begin{cases} (S')'' + \frac{n-1}{r}(S')' - \left(\frac{n-1}{r^2} + \alpha\right) S' = -\rho' > 0, & r \in [\epsilon, R), \\ S'(0) = S'(R) = 0. \end{cases} \tag{47}$$

By Hopf boundary point lemma, we have $S''(R) > 0$. Because

$$r\rho'' + (n-1)\rho' - \chi[\rho\psi(rS'' + (n-1)S') + S'\rho' + 2\dot{\psi}r(S')^2 S''\rho] = 0, \tag{48}$$

we have $\rho''(R) = \chi\psi(0)\rho(R)S''(R)$, which implies that $\rho''(R) > 0$. Then we have proved that ρ_k', $S_k' < 0$ on $[\epsilon, R)$ for large k, which immediately gives the openness of $\mathcal{A}_3$.

We will prove $\mathcal{A}_3$ is closed in $\mathcal{C}^+\backslash\{(M, \frac{M}{\alpha}, \bar{\chi}_1)\}$ to finish this step. Assume there exists a sequence $(\rho_k, S_k, \chi_k) \in \mathcal{A}_3$ converging to $(\rho, S, \chi) \in \mathcal{C}^+\backslash\{(M, \frac{M}{\alpha}, \bar{\chi}_1)\}$ as $k \to \infty$. Obviously, ρ', $S' \leq 0$. If there exists $r_0 \in [\epsilon, R)$ such that $S'(r_0) = 0$, applying the strong maximum principle to (47), we have $S' \equiv 0$ and hence $S \equiv \frac{M}{\alpha}$ and $\rho \equiv M$. So χ is a bifurcation value and χ should be equal to $\bar{\chi}_j$ for some $j = 1, 2, \ldots$, where $j = 1$ is impossible because $(\rho, S, \chi) \in \mathcal{C}^+\backslash\{(M, \frac{M}{\alpha}, \bar{\chi}_1)\}$. $j \geq 2$ is impossible because ρ_k, S_k are monotone while all the points near $(M, \frac{M}{\alpha}, \bar{\chi}_j)$ for $j \geq 2$ are not possible either. Thus, $S' < 0$ on $[\epsilon, R)$. Integrating the first equation of (10), by Neumann boundary conditions, we know

$$\rho'(r) = \chi\rho(r)\psi(|S'(r)|^2)S'(r), \text{ for } r \in (0, R). \tag{49}$$

Because $\rho > 0$ on $[0, R]$, $\psi > 0$ on $\mathbb{R}^+$, we have $\rho' < 0$ on $[\epsilon, R)$. Thus, $\mathcal{A}_3$ is closed. Because ϵ is arbitrary, taking $\epsilon \to 0$ gives us that ρ', $S' < 0$ on $(0, R)$ for all points (ρ, S, χ) belonging to the curve $\mathcal{C}^+/\{(M, \frac{M}{\alpha}, \bar{\chi}_1)\}$.

Step 4. From Theorem 4.4 in Ref. 43, it follows that $\mathcal{C}^+$ satisfies at least one of the following alternatives:

(a) it is not compact in $X \times X \times \mathbb{R}$;
(b) it contains a point $(M, \frac{M}{\alpha}, \chi^*)$, $\chi^* \neq \bar{\chi}_1$;
(c) it contains a point $(M + \hat{\rho}, \frac{M}{\alpha} + \hat{S}, \chi)$ with $0 \neq (\hat{\rho}, \hat{S}) \in Z$, where Z denotes a closed linear subspace of $X \times X$ and is a complement of $N(D_{(\rho,S)}F(M, \frac{M}{\alpha}, \bar{\chi}_1)) = \text{span}\{(\bar{\rho}_1, \bar{S}_1)\})$, namely,

$$Z = \left\{ (\rho, S) \in X \times X \middle| \int_0^R (\bar{\rho}_1(r)\rho(r) + \bar{S}_1(r)S(r))\mathrm{d}x = 0 \right\}. \tag{50}$$

We will show that the situation (a) must occur.

If (b) happens, then χ^* must be some bifurcation value $\bar{\chi}_j$ and $j \geq 1$. By the arguments in in Step 3, this situation is impossible. If (c) happens, then $(M + \hat{\rho}, \frac{M}{\alpha} + \hat{S}, \chi) \in C^+ \backslash \{(M, \frac{M}{\alpha}, \bar{\chi}_1)\}$ and by Step 3, $\hat{\rho}', \hat{S}' < 0$ in $(0, R)$. Because $(\hat{\rho}, \hat{S}) \in Z$, then

$$0 = \int_0^R \left[\left(\alpha + \frac{q_1^2}{R^2} \right) \hat{\rho}(r) + \hat{S}(r) \right] u_1(r) \mathrm{d}r \tag{51}$$

$$= -\int_0^R \left[\left(\alpha + \frac{q_1^2}{R^2} \right) \hat{\rho}'(r) + \hat{S}'(r) \right] U_1(r) \mathrm{d}r > 0, \tag{52}$$

where $U_1(r)$ is the primitive of $u_1(r)$ and $U_1(r) > 0$ on $(0, R)$. This is a contradiction. So the situation (a) must occur.

Step 5. We prove below that the monotone positive solution of (8) exists for all $\chi > \bar{\chi}_1$.

By the estimate of ρ and S in Lemma 1, we know if χ is bounded from above by a positive constant, ρ and S are bounded in $H^3(\bar{B}_R(0))$ by elliptic regularity theory, which contradicts the fact that C^+ is not compact. So the coordinate of χ should cover $(\bar{\chi}_1, \infty)$ and we finish our proof. ■

3 | ASYMPTOTIC PROFILE AS $\chi \to \infty$

In this section, we are devoted to proving the following results, which state the asymptotic profile of solutions as $\chi \to \infty$.

Theorem 2. *Let (ρ, S) be the positive and monotone decreasing solution of (8) given in Theorem 1. Then we have*

$$\rho(r) \to \frac{MR^n}{n} \frac{\delta(r)}{r^{n-1}} \quad as \quad \chi \to \infty \tag{53}$$

in the sense of distribution, where $\delta(r)$ is the Dirac delta function, and $S(r) \to S_\infty(r)$ pointwise on $(0, R]$ as $\chi \to \infty$, where

$$S_\infty(r) = \begin{cases} C_2 S_\infty(r; 2) \\ C_n S_\infty(r; n) \end{cases} = \begin{cases} C_2 \sqrt{\alpha}[K_1(\sqrt{\alpha}R) I_0(\sqrt{\alpha}r) + I_1(\sqrt{\alpha}R) K_0(\sqrt{\alpha}r)], & n = 2, \\ C_n [c_1 r^{1-\frac{n}{2}} I_\gamma(\sqrt{\alpha}r) + c_2 r^{1-\frac{n}{2}} K_\gamma(\sqrt{\alpha}r)], & n \geq 3, \end{cases} \tag{54}$$

where $I_\gamma(s)$ and $K_\gamma(s)$ are the modified Bessel functions of the first and the second kinds with $\gamma = \frac{n}{2} - 1$,

$$c_1 = R^{-\frac{n}{2}} \left[\left(1 - \frac{n}{2} \right) K_\gamma(\sqrt{\alpha}R) + \sqrt{\alpha}R K_\gamma'(\sqrt{\alpha}R) \right],$$

$$c_2 = -R^{-\frac{n}{2}} \left[\left(1 - \frac{n}{2} \right) I_\gamma(\sqrt{\alpha}R) + \sqrt{\alpha}R I_\gamma'(\sqrt{\alpha}R) \right], \tag{55}$$

and C_n satisfies

$$C_n = \frac{MR^n}{n\int_0^R r^{n-1}S_\infty(r;n)\mathrm{d}r}, \qquad n = 2, 3, \ldots \tag{56}$$

Moreover, for $n = 2$, we have $C_2 = \frac{\sqrt{\alpha}MR^2}{2I_1(\sqrt{\alpha}R)}$.

Proof. Recall $\int_0^R r^{n-1}\rho(r)\mathrm{d}r = \frac{MR^n}{n}$. For any fixed $\varepsilon \in (0, R]$, we have

$$\varepsilon^{n-1}\int_\varepsilon^R \rho(r)\mathrm{d}r \le \int_0^R r^{n-1}\rho(r)\mathrm{d}r,$$

that is, $\int_\varepsilon^R \rho(r)\mathrm{d}r \le \frac{MR^n}{n\varepsilon^{n-1}}$. Then by the Helly's compactness theorem (see Ref. 46), we know that $\rho(r)$ converges to some $\rho_\infty(r)$ pointwise on $r \in [\varepsilon, R]$ after passing to a subsequence of $\chi \to \infty$. By the diagonal argument of compactness, $\rho(r) \to \rho_\infty(r)$ pointwise on $(0, R]$ after passing to a subsequence of $\chi \to \infty$.

To show (53), we only need to show $\rho_\infty(r) = 0$ on $(0, R]$. Arguing by contradiction, we suppose that $\rho_\infty(r_0) > 0$ for some $r_0 \in (0, R]$. Observe that $\rho(r)$ is decreasing on $(0, R)$ and hence $\rho_\infty(r)$ is nonincreasing on $(0, R)$. By the definition of $\rho_\infty(r)$, we know that for any positive constant $\tilde{c}$, there exists a positive constant M_{r_0} such that $|\rho_\infty(r_0) - \rho(r_0)| < \tilde{c}$, whenever $\chi > M_{r_0}$. Taking $\tilde{c} = \rho(r_0)$, we get $\rho_\infty(r_0) < 2\rho(r_0)$ for large χ. Then with our assumption $\rho_\infty(r_0) > 0$, we have

$$0 < \frac{\rho_\infty(r_0)}{2} < \rho(r_0) \le \rho\left(\frac{r_0}{2}\right) < 2\rho_\infty\left(\frac{r_0}{2}\right) \tag{57}$$

for large $\chi > 0$, where the third inequality comes from that $\rho_\infty(r)$ is nonincreasing on $(0, R)$. So for $r \in [\frac{r_0}{2}, r_0]$, we have

$$0 < \frac{\rho_\infty(r_0)}{2} < \rho(r_0) \le \rho(r) \le \rho\left(\frac{r_0}{2}\right) < 2\rho_\infty\left(\frac{r_0}{2}\right). \tag{58}$$

Recall the differentiation of second equation in (8)

$$(S'')' + \frac{n-1}{r}(S')' - \left(\frac{n-1}{r^2} + \alpha\right)S' + \rho' = 0 \tag{59}$$

and the equation $\rho' = \chi\rho\psi(|S'|^2)S'$, which comes form the integration of the first equation in (10). Combining these two equations, we have $(S')'' + \frac{n-1}{r}(S')' + (\chi\rho\psi(|S'|^2) - \alpha - \frac{n-1}{r^2})S' = 0$. Observing that $S''' + \frac{n-1}{r}(S')' = (r^{\frac{n-1}{2}}S')''r^{\frac{1-n}{2}} - \frac{(n-1)(n-3)}{4r^2}S'$, we can write this equation as

$$\begin{cases} \left(r^{\frac{n-1}{2}}S'\right)'' + \left(\chi\rho\psi(|S'|^2) - \alpha - \frac{n-1}{r^2} - \frac{(n-1)(n-3)}{4r^2}\right)\left(r^{\frac{n-1}{2}}S'\right) = 0, \\ S'(0) = S'(R) = 0. \end{cases} \tag{60}$$

Recall $\max_{y\in\mathbb{R}} |y\psi(y^2)| = A_\infty > 0$. By (7) and ψ is C^2 smooth, we know that $\chi\rho\psi(|S'|^2) - \alpha - \frac{n^2-1}{4r^2} \to \infty$ as $\chi \to \infty$ uniformly on $[\frac{r_0}{2}, r_0]$ because $r_0 > 0$. So the Sturm oscillation theorem implies that the sign of $r^{\frac{n-1}{2}} S'$, and hence S', changes many times on $[\frac{r_0}{2}, r_0]$ for large χ, which contradicts the fact that $S' < 0$ on $(0, R)$. Thus, $\rho_\infty(r) = 0$ on $(0, R]$.

Now we consider the asymptotic behavior of S as $\chi \to \infty$. Similarly, we know $S(r) \to$ some $S_\infty(r)$ on $(0, R]$ after passing to a subsequence of $\chi \to \infty$ and S_∞ should satisfy

$$S''_\infty(r) + \frac{n-1}{r}S'_\infty(r) - \alpha S_\infty(r) = 0,\ r \in (0, R],\ S'_\infty(R) = 0. \tag{61}$$

Taking $S_\infty(r) = r^{1-\frac{n}{2}}U(r)$, then we have

$$U''(r) + \frac{1}{r}U'(r) - \left(\alpha + \frac{(n-2)^2}{4r^2}\right)U(r) = 0. \tag{62}$$

Taking $s = \sqrt{\alpha}r$, we can see

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}U + \frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s}U - \left(1 + \frac{(n-2)^2}{4s^2}\right)U = 0, \tag{63}$$

whose solutions are the modified Bessel functions of the first kind, $I_\gamma(s)$, and the second kind $K_\gamma(s)$ with $\gamma = \frac{n-2}{2}$. So

$$S_\infty(r) = C_n\left[c_1 r^{1-\frac{n}{2}} I_\gamma(\sqrt{\alpha}r) + c_2 r^{1-\frac{n}{2}} K_\gamma(\sqrt{\alpha}r)\right], \tag{64}$$

where C_n is a constant depending on n and c_1, c_2 can be written explicitly below. For $n = 2$, which means $\gamma = 0$, we have

$$S_\infty(r) = C_2\left[c_1 I_0(\sqrt{\alpha}r) + c_2 K_0(\sqrt{\alpha}r)\right], \tag{65}$$

and $S'_\infty(R) = 0$ implies that $c_1 = \sqrt{\alpha}K_1(\sqrt{\alpha}R)$ and $c_2 = \sqrt{\alpha}I_1(\sqrt{\alpha}R)$. For $n \geq 3$, $c_1 = R^{-\frac{n}{2}}[(1 - \frac{n}{2})K_\gamma(\sqrt{\alpha}R) + \sqrt{\alpha}RK'_\gamma(\sqrt{\alpha}R)]$ and $c_2 = -R^{-\frac{n}{2}}[(1 - \frac{n}{2})I_\gamma(\sqrt{\alpha}R) + \sqrt{\alpha}RI'_\gamma(\sqrt{\alpha}R)]$ to ensure the boundary condition $S'(R) = 0$. The uniqueness of limits implies (54) and the mass constraint gives us (56) immediately. For $n = 2$, thanks to the recurrence relations

$$rI'_1(r) + I_1(r) = rI_0(r), \quad rK'_1(r) + K_1(r) = -rK_0(r), \tag{66}$$

and the fact that $\lim_{r\to 0} rK_1(r) = 1$, we directly calculate that

$$C_2 = \frac{\sqrt{\alpha}MR^2}{2I_1(\sqrt{\alpha}R)}. \tag{67}$$

Indeed from $\int_0^R rS(r)\mathrm{d}r = \frac{MR^2}{2}$, we can formally write

$$C_2 = \frac{MR^2}{2\int_0^R rS_\infty(r;2)\mathrm{d}r}. \tag{68}$$

Note that

$$\begin{aligned}
&\int_0^R rI_0(r)\mathrm{d}r = \int_0^R [rI_1'(r) + I_1(r)]\mathrm{d}r = RI_1(R) - \int_0^R I_1(r)\mathrm{d}r + \int_0^R I_1(r)\mathrm{d}r = RI_1(R),\\
&\int_0^R rK_0(r)\mathrm{d}r = -\int_0^R [rK_1'(r) + K_1(r)]\mathrm{d}r\\
&\qquad = 1 - RK_1(R) + \int_0^R K_1(r)\mathrm{d}r - \int_0^R K_1(r)\mathrm{d}r = 1 - RK_1(R).
\end{aligned} \tag{69}$$

Then

$$\begin{aligned}
&\int_0^R rS_\infty(r;2)\mathrm{d}r\\
&= \int_0^R \sqrt{\alpha}K_1\left(\sqrt{\alpha}R\right) rI_0\left(\sqrt{\alpha}r\right)\mathrm{d}r + \int_0^R \sqrt{\alpha}I_1\left(\sqrt{\alpha}R\right) rK_0\left(\sqrt{\alpha}r\right)\mathrm{d}r\\
&= \alpha^{-\frac{1}{2}}[K_1\left(\sqrt{\alpha}R\right)\int_0^{\sqrt{\alpha}R} rI_0(r)\,\mathrm{d}r + I_1\left(\sqrt{\alpha}R\right)\int_0^{\sqrt{\alpha}R} rK_0(r)\,\mathrm{d}r]\\
&= RK_1\left(\sqrt{\alpha}R\right)I_1\left(\sqrt{\alpha}R\right) - RI_1\left(\sqrt{\alpha}R\right)K_1\left(\sqrt{\alpha}R\right) + \alpha^{-\frac{1}{2}}I_1\left(\sqrt{\alpha}R\right)\\
&= \frac{I_1\left(\sqrt{\alpha}R\right)}{\sqrt{\alpha}}.
\end{aligned} \tag{70}$$

Substituting this into (68) yields (67) and hence completes the proof. ■

4 | STABILITY OF BIFURCATING SOLUTIONS

In this section, we show that any bifurcating solution near the bifurcation point $(M, \frac{M}{\alpha}, \bar{\chi}_j)$ is asymptotically stable. Without loss of generality, we only consider the monotone solution near the first bifurcation point $(M, \frac{M}{\alpha}, \bar{\chi}_1)$ as obtained in Theorem 1, where the solution, denoted by $(\rho_1(\tau, r), S_1(\tau, r))$, can be written as (see (46))

$$(\rho_1(\tau, r), S_1(\tau, r)) = \left(M, \frac{M}{\alpha}\right) + \tau(\bar{\rho}_1(r), \bar{S}_1(r)) + o(\tau),\ \tau \in (-\delta, \delta) \tag{71}$$

for some $\delta > 0$, where $(\bar{\rho}_1(r), \bar{S}_1(r)) = (\alpha + \frac{q_1^2}{R^2}, 1)u_1(r)$ is a solution of (8) and all nonconstant solutions of (8) near $(M, \frac{M}{\alpha}, \bar{\chi}_1)$ live on the curve $C_1 = (\chi_1(\tau), \rho_1(\tau, r), S_1(\tau, r))$ for $\tau \in (-\delta, \delta)$.

Let $\mathcal{Z}$ be a closed complement of $N(D_{(\rho,S)}\mathcal{F}(M, \frac{M}{\alpha}, \bar{\chi}_1)) = \text{span}\{(\bar{\rho}_1(r), \bar{S}_1(r))\}$ in $X \times X$ as following

$$\mathcal{Z} = \{(\rho, S) \in X \times X | \int_0^R r^{n-1}\bar{\rho}_1(r)\rho(r) + r^{n-1}\bar{S}_1(r)S(r)\mathrm{d}r = 0\}. \tag{72}$$

By Theorem 1.7 in Ref. 47, we have

$$(\rho_1(\tau, r), S_1(\tau, r)) - \left(M, \frac{M}{\alpha}\right) - \tau\left(\bar{\rho}_1(r), \bar{S}_1(r)\right) \in \mathcal{Z}, \quad \forall \tau \in (-\delta, \delta). \tag{73}$$

Recall that (ρ_1, S_1, χ_1) are C^3 smooth functions of τ and we have the following expansions:

$$\begin{cases} \rho_1(\tau, r) = M + \tau\left(\alpha + \frac{q_1^2}{R^2}\right)u_1(r) + \tau^2\phi_1(r) + \tau^3\eta_1(r) + o(\tau^3) \\ S_1(\tau, r) = \frac{M}{\alpha} + \tau u_1(r) + \tau^2\phi_2(r) + \tau^3\eta_2(r) + o(\tau^3) \\ \chi_1(\tau) = \bar{\chi}_1 + \tau\tilde{\chi}_2 + \tau^2\tilde{\chi}_3 + o(\tau^2), \end{cases} \tag{74}$$

where $(\phi_1, \phi_2), (\eta_1, \eta_2) \in \mathcal{Z}$, $\tilde{\chi}_2$ and $\tilde{\chi}_3$ are constants. By the mass constraint of ρ and S, we know

$$\int_0^R r^{n-1}\phi_i(r)\mathrm{d}r = \int_0^R r^{n-1}\eta_i(r)\mathrm{d}r = 0, \quad i = 1, 2. \tag{75}$$

Now inserting (74) into (10) with the Taylor expansion of ψ, we have

$$\begin{aligned} &\{r^{n-1}[\tau\bar{\rho}_1' + \tau^2\phi_1' + \tau^3\eta_1' + o(\tau^3) - (\bar{\chi}_1 + \tau\tilde{\chi}_2 + \tau^2\tilde{\chi}_3 + o(\tau^2))(M + \tau\bar{\rho}_1 + \tau^2\phi_1 + \tau^3\eta_1 + o(\tau^3)) \\ &\cdot(\tau\bar{S}_1' + \tau^2\phi_2' + \tau^3\eta_2' + o(\tau^3))(\psi(0) + \psi'(0)(\tau\bar{S}_1' + \tau^2\phi_2' + \tau^3\eta_2')^2 + o(\tau^3))]\}' = 0 \end{aligned} \tag{76}$$

and

$$\begin{aligned} &[r^{n-1}(\tau\bar{S}_1' + \tau^2\phi_2' + \tau^3\eta_2' + o(\tau^3))]' - \alpha r^{n-1}\left(\frac{M}{\alpha} + \tau\bar{S}_1 + \tau^2\phi_2 + \tau^3\eta_2 + o(\tau^3)\right) \\ &+ r^{n-1}(M + \tau\bar{\rho}_1 + \tau^2\phi_1 + \tau^3\eta_1 + o(\tau^3)) = 0. \end{aligned} \tag{77}$$

Collecting the $O(\tau^2)$ terms in (77) gives

$$(r^{n-1}\phi_2')' - \alpha r^{n-1}\phi_2 + r^{n-1}\phi_1 = 0. \tag{78}$$

Multiplying (78) by $u_1(r)$ and integrating over $[0, R]$, we have

$$\int_0^R r^{n-1}\left[u_1(r)\phi_1(r) - \left(\alpha + \frac{q_1^2}{R^2}\right)u_1(r)\phi_2(r)\right]\mathrm{d}r = 0. \tag{79}$$

Recall that $(\phi_1, \phi_2) \in \mathcal{Z}$, which implies that

$$\int_0^R r^{n-1}\left[\left(\alpha + \frac{q_1^2}{R^2}\right)u_1(r)\phi_1(r) + u_1(r)\phi_2(r)\right]\mathrm{d}r = 0. \tag{80}$$

By (79) and (80), it is easy to see that

$$\int_0^R r^{n-1}u_1(r)\phi_1(r)\mathrm{d}r = \int_0^R r^{n-1}u_1(r)\phi_2(r)\mathrm{d}r = 0. \tag{81}$$

Now we consider the $O(\tau^2)$ terms in (76)

$$\{r^{n-1}[\phi_1' - \bar{\chi}_1 M\phi_2'\psi(0) - \bar{\chi}_1\bar{\rho}_1\bar{S}_1'\psi(0) - \tilde{\chi}_2 M\bar{S}_1'\psi(0)]\}' = 0. \tag{82}$$

Multiplying (82) by $u_1(r)$ and integrating the result by parts along with (81) we have

$$\tilde{\chi}_2 = \frac{1}{2M^2\psi(0)}\left(\alpha + \frac{q_1^2}{R^2}\right)^2 \frac{\int_0^R u_1^3(r)r^{n-1}\mathrm{d}r}{\int_0^R u_1^2(r)r^{n-1}\mathrm{d}r}. \tag{83}$$

Lemma 3. *Consider the eigenvalue problem of (10) as $\tau \to 0$ (i.e., linearization of (10) at $(M, \frac{M}{\alpha})$)*

$$\begin{cases} [r^{n-1}(\rho' - \bar{\chi}_1 M\psi(0)S')]' = \eta r^{n-1}\rho, & r \in (0, R), \\ (r^{n-1}S')' - \alpha r^{n-1}S + r^{n-1}\rho = \eta r^{n-1}S, & r \in (0, R), \\ \rho'(0) = \rho'(R) = S'(0) = S'(R) = 0, \\ \int_0^R r^{n-1}\rho(r)\mathrm{d}r = 0, \end{cases} \tag{84}$$

where η is the eigenvalue of (84) with corresponding eigenfunction $(r^{n-1}\rho, r^{n-1}S)$. Then 0 is the only eigenvalue of (84) when Re $\eta \geq -\hat{\delta}$ with $\hat{\delta} > 0$ small.

Proof. The solution of the linear system (84) has the form of

$$\rho = \sum_{k=0}^{\infty} a_k u_k, \quad S = \sum_{k=0}^{\infty} b_k u_k. \tag{85}$$

Inserting (85) into (84), we have

$$\begin{cases} \frac{q_k^2}{R^2}a_k - \bar{\chi}_1 M\psi(0)\frac{q_k^2}{R^2}b_k = -\eta a_k, \\ \left(\frac{q_k^2}{R^2}+\alpha\right)b_k - a_k = -\eta b_k, \end{cases} \tag{86}$$

where $a_0 = b_0 = 0$ by $\int_0^R r^{n-1}\rho(r)\mathrm{d}r = 0$. Then η is an eigenvalue of (84) if and only if there exists $k > 0$ with $(a_k, b_k) \neq (0,0)$ such that

$$\eta^2 + \left(\frac{2q_k^2}{R^2}+\alpha\right)\eta + \frac{q_k^2}{R^2}\left(\frac{q_k^2}{R^2}+\alpha-\bar{\chi}_1 M\psi(0)\right) = 0. \tag{87}$$

Because the discriminant of the above equation is $\Delta = \alpha^2 + \frac{4\bar{\chi}_1 M\psi(0)q_k^2}{R^2} > 0$, then the equation has two roots η_1 and η_2 satisfying

$$\eta_1+\eta_2 = -\left(\frac{2q_k^2}{R^2}+\alpha\right) < 0, \quad \eta_1\eta_2 = \frac{q_k^2}{R^2}\left(\frac{q_k^2}{R^2}+\alpha-\bar{\chi}_1 M\psi(0)\right) = \frac{q_k^2(q_k^2-q_1^2)}{R^4}. \tag{88}$$

So $\eta_1\eta_2 = 0$ as $k = 1$ and $\eta_1\eta_2 > 0$ as $k > 1$. Therefore 0 is the only eigenvalue fulfilling $\mathrm{Re}\,\eta \geq -\hat{\delta}$ for arbitrarily small $\hat{\delta} > 0$. ■

Next we study the asymptotic stability of $(\rho_1, S_1)(\tau, r)$, which is determined by the sign of the real part of eigenvalue λ of the following eigenvalue problem:

$$\begin{cases} [r^{n-1}(\rho' - \chi_1(\tau)(S_1'\psi(|S_1'|^2)\rho + \rho_1\psi(|S_1'|^2)S' + 2\rho_1(S_1')^2\dot{\psi}(|S_1'|^2)S))]' = \lambda r^{n-1}\rho, & r\in(0,R) \\ (r^{n-1}S')' - \alpha r^{n-1}S + r^{n-1}\rho = \lambda r^{n-1}S, & r\in(0,R) \\ \rho'(0) = \rho'(R) = S'(0) = S'(R) = 0, \\ \int_0^R r^{n-1}\rho(r)\mathrm{d}r = 0. \end{cases} \tag{89}$$

It is easy to see when $\tau = 0$, 0 is an eigenvalue. When $\tau \neq 0$, define $\mathcal{H} : X\times X \to Y_0 \times Y \times \mathbb{R}$ by

$$\mathcal{H}(\rho,S) = \begin{pmatrix} r^{n-1}\rho - \frac{1}{R}\int_0^R r^{n-1}\rho(r)\mathrm{d}r \\ r^{n-1}S \\ 0 \end{pmatrix}. \tag{90}$$

Then the following eigenvalue problem:

$$r^{n-1}D_{(\rho,S)}\mathcal{F}(\rho_\tau, S_\tau, \chi_1(\tau))(\rho,S) = \lambda\mathcal{H}(\rho,S), \quad (\rho,S)\in X\times X, \tag{91}$$

is equivalent to (89). Then we can easily see that $\lambda = 0$ is a simple eigenvalue of the pair $(r^{n-1}D_{(\rho,S)}\mathcal{F}(M, \frac{M}{\alpha}, \bar{\chi}_1), \mathcal{H})$ by the Fredholmness of $r^{n-1}D_{(\rho,S)}\mathcal{F}(M, \frac{M}{\alpha}, \bar{\chi}_1)$ and the fact that $\mathcal{H}(\bar{\rho}_1, \bar{S}_1) \notin R(r^{n-1}D_{(\rho,S)}\mathcal{F}(M, \frac{M}{\alpha}, \bar{\chi}_1))$.

Theorem 3. *Let $\tilde{\chi}_2$ be defined in (83) and assume that $\tilde{\chi}_2 > 0$. For $\tau \in (-\delta, \delta)$ and $\tau \neq 0$, the steady-state $(\rho_1(\tau, r), S_1(\tau, r))$ is asymptotically stable if $\tau > 0$ and unstable if $\tau < 0$.*

Proof. Recall that $\lambda = 0$ is a simple eigenvalue of the pair $(r^{n-1}D_{(\rho,S)}\mathcal{F}(M, \frac{M}{\alpha}, \bar{\chi}_1), \mathcal{H})$. Using Corollary 1.13 in Ref. 48 with "K" in this corollary being identity, there exists C^1-smooth mapping $\lambda_1(\chi)$: $N \to \mathbb{R}$ and $\lambda_2(\tau) : (-\delta, \delta) \to \mathbb{R}$, where N is a neighborhood of $\bar{\chi}_1$, such that $\lambda_1(\bar{\chi}_1) = 0$, $\lambda_2(0) = 0$, and $\lambda_1(\chi)$ is a real eigenvalue of

$$r^{n-1}D_{(\rho,S)}\mathcal{F}\left(M, \frac{M}{\alpha}, \chi\right)(\rho, S) = \lambda\mathcal{H}(\rho, S), \quad (\rho, S) \in X \times X, \tag{92}$$

and $\lambda_2(\tau)$ is an eigenvalue of (91) and hence (89). In the complex plane, in any small neighborhood of the origin, $\lambda_1(\chi)$ is the only eigenvalue of (92) and $\lambda_2(\tau)$ is the only eigenvalue of (89), namely, $\lambda = \lambda_2(\tau)$. Hence it remains only to determine the sign of $\lambda_2(\tau)$.

Write eigenfunction of (92) as $(\rho(\chi, r), S(\chi, r))$. By Corollary 1.13 in Ref. 48, this eigenfunction depends on χ smoothly and is uniquely determined by

$$(\rho(\bar{\chi}_1, r), S(\bar{\chi}_1, r))) = (\bar{\rho}_1(r), \bar{S}_1(r)), \quad (\rho(\chi, r), S(\chi, r)) - (\bar{\rho}_1(r), \bar{S}_1(r)) \in \mathcal{Z}. \tag{93}$$

Differentiating (92) with respect to χ and then setting $\chi = \bar{\chi}_1$, we have

$$\begin{cases} [r^{n-1}(\dot{\rho}' - M\psi(0)S' - \bar{\chi}_1 M\psi(0)\dot{S}')]' = \dot{\lambda}_1(\bar{\chi}_1)\bar{\rho}_1 r^{n-1}, & r \in (0, R), \\ (r^{n-1}\dot{S}')' - \alpha r^{n-1}\dot{S} + r^{n-1}\dot{\rho} = \dot{\lambda}_1(\bar{\chi}_1)r^{n-1}\bar{S}_1, & r \in (0, R), \\ \rho'(0) = \rho'(R) = S'(0) = S'(R) = 0. \end{cases} \tag{94}$$

Multiplying the $\dot{\rho}$-equation by u_1 and the $\dot{S}$-equation by $\frac{q_1^2}{R^2}u_1$, adding the resulting equations, integrating with respect to r over $[0, R]$, we get

$$\dot{\lambda}_1(\bar{\chi}_1) = \frac{q_1^2}{\alpha R^2 + 2q_1^2} M\psi(0) > 0. \tag{95}$$

From Theorem 1.16 in Ref. 48, we know whenever $\lambda_2(\tau) \neq 0$,

$$\lim_{\tau \to 0} \frac{-\tau\chi_1'(\tau)\dot{\lambda}_1(\bar{\chi}_1)}{\lambda_2(\tau)} = 1, \tag{96}$$

where $\chi_1(\tau) = \bar{\chi}_1 + \tau\tilde{\chi}_2 + o(\tau^2)$ and hence $\chi_1'(\tau) = \tilde{\chi}_2 + o(\tau)$. Therefore $\tilde{\chi}_2 > 0$ implies that $\operatorname{sgn}\lambda_2(\tau) = -\operatorname{sgn}\tau$ for small $\tau \neq 0$. To conclude our results, it suffices to show that $\lambda_2(\tau)$ is not a complex eigenvalue with positive real part. This would follow from the standard eigenvalue perturbation theory if we can show that the eigenvalue problem (84) (namely, the limit of (89) as

$\tau \to 0$) has no nonzero eigenvalues with nonnegative real parts. This, however, has been shown in Lemma 3. Therefore, we conclude our results and complete the proof. ∎

Remark 1. Recalling the explicit expression $\tilde{\chi}_2$ given in (83), we find the sign of $\tilde{\chi}_2$ is the same as that of $\int_0^R u_1^3(r)r^{n-1}\mathrm{d}r$. If $n = 3$, $\tilde{\chi}_2 > 0$ will follow automatically. To see this, we first notice that $u_1(r) = r^{-\frac{1}{2}}J_{\frac{1}{2}}(\frac{q_1}{R}r) = \sqrt{\frac{2R}{\pi q_1}}\frac{\sin\frac{q_1}{R}r}{r}$ for $n = 3$, by the fact that $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$, where $q_1 \in [\pi, \frac{3}{2}\pi]$ is the first positive root of the equation $\tan x = x$. By direct calculations, we have

$$\int_0^R u_1^3(r)r^{n-1}\mathrm{d}r = \left(\frac{2R}{\pi q_1}\right)^{\frac{3}{2}}\int_0^R \frac{(\sin\frac{q_1}{R}r)^3}{r}\mathrm{d}r = \left(\frac{2R}{\pi q_1}\right)^{\frac{3}{2}}\int_0^{q_1}\frac{(\sin r)^3}{r}\mathrm{d}r, \tag{97}$$

where $\int_0^{q_1}\frac{(\sin r)^3}{r}\mathrm{d}r > \int_0^{\frac{3}{2}\pi}\frac{(\sin r)^3}{r}\mathrm{d}r > 0$, because $\sin r < 0$ for $r \in (\pi, \frac{3}{2}\pi)$, and

$$\int_0^{\frac{3}{2}\pi}\frac{(\sin r)^3}{r}\mathrm{d}r = \int_0^{\frac{1}{2}\pi}\frac{(\sin r)^3}{r}\mathrm{d}r + \int_{\frac{1}{2}\pi}^{\pi}\frac{(\sin r)^3}{r}\mathrm{d}r + \int_{\pi}^{\frac{3}{2}\pi}\frac{(\sin r)^3}{r}\mathrm{d}r \tag{98}$$

$$= \int_0^{\frac{1}{2}\pi}\frac{(\sin r)^3}{r}\mathrm{d}r + \int_{\frac{1}{2}\pi}^{\pi}\frac{(\sin r)^3}{r}\mathrm{d}r - \int_{\frac{1}{2}\pi}^{\pi}\frac{(\sin r)^3}{2\pi - r}\mathrm{d}r \tag{99}$$

$$= \int_0^{\frac{1}{2}\pi}\frac{(\sin r)^3}{r}\mathrm{d}r + \int_{\frac{1}{2}\pi}^{\pi}\left[\frac{(\sin r)^3}{r} - \frac{(\sin r)^3}{2\pi - r}\right]\mathrm{d}r \tag{100}$$

$$> 0, \tag{101}$$

by the fact that $\frac{(\sin r)^3}{r} > 0$ on $[0, \frac{1}{2}]$ and $\frac{(\sin r)^3}{r} - \frac{(\sin r)^3}{2\pi - r} > 0$ on $[\frac{1}{2}\pi, \pi)$.

When $n = 2$, using Mathematica, we can find $\int_0^R J_0^3(\frac{q_1}{R}r)r\mathrm{d}r = \frac{R^2}{q_1^2}\int_0^{q_1} J_0^3(r)\mathrm{d}r > 0$. Because J_0 is a sign-changing function without explicit representation, we are unable to verify this rigorously.

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