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On the Lotka–Volterra competition system with dynamical resources and density-dependent diffusion

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Abstract

In this paper, we consider the following Lotka–Volterra competition system with dynamical resources and density-dependent diffusion

$$\begin{cases} u_t = \Delta(d_1(w)u) + u(a_1 w - b_1 u - c_1 v), & x \in \Omega, \quad t > 0, \\ v_t = \Delta(d_2(w)v) + v(a_2 w - b_2 u - c_2 v), & x \in \Omega, \quad t > 0, \\ w_t = \Delta w - w(u+v) + \mu w(m(x) - w), & x \in \Omega, \quad t > 0, \end{cases} \qquad (*)$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^2$ with homogeneous Neumann boundary conditions, where the parameters μ, a_i, b_i, c_i $(i = 1, 2)$ are positive constants, $m(x)$ is the prey's resource, and the dispersal rate function $d_i(w)$ satisfies the the following hypothesis:

- $d_i(w) \in C^2([0, \infty))$, $d_i'(w) \leq 0$ on $[0, \infty)$ and $d(w) > 0$.

When $m(x)$ is constant, we show that the system (*) with has a unique global classical solution when the initial datum is in functional space $W^{1,p}(\Omega)$ with $p > 2$. By constructing appropriate Lyapunov functionals and using LaSalle's invariant principle, we further prove that the solution of (*) converges to the co-existence steady state exponentially or competitive exclusion steady state algebraically as time tends to infinity in different parameter regimes. Our results reveal that once the resource w has temporal dynamics, two competitors may coexist in the case of weak competition regardless of their dispersal rates and initial values no matter whether there is explicit dependence in dispersal or not. When the prey's resource is spatially heterogeneous (i.e. $m(x)$ is non-constant), we use numerical simulations to demonstrate that the striking phenomenon "slower diffuser always prevails" (cf. Dockery et al. in J Math Biol 37(1):61–83, 1998; Lou in J Differ Equ 223(2):400–426, 2006) fails to appear if the non-random dispersal strategy is employed by competing species (i.e. either $d_1(w)$ or $d_2(w)$ is non-constant) while it still holds true if both $d(w)$ and $d_2(w)$ are constant.

Extended author information available on the last page of the article

Keywords Lotka–Volterra competition · Dynamical resources · Density-dependent diffusion · Homogeneous and heterogenous resource · Asymptotic dynamics

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1 Introduction

1.1 Background, motivation and main results

The evolution of dispersal (either random or non-random) is one of the most interesting topics in theoretical studies of population dynamics and various mathematical models have been studied to understand the process of dispersal and its ecological effect and evolution [e.g., see the survey papers (Cosner 2014; Lou 2008) or book (Cantrell and Cosner 2004)]. Among other things, we consider the following Lotka–Volterra diffusion-competition model

$$\begin{cases} u_t = d_1\Delta u + u(a_1 - b_1 u - c_1 v), & \text{in } \Omega \times \mathbb{R}^+, \\ v_t = d_2\Delta v + v(a_2 - b_2 u - c_2 v), & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ (u, v)(x, 0) = (u_0, v_0)(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $u(x, t)$ and $v(x, t)$ represent the population densities of two competing species at location $x \in \Omega$ and at time $t > 0$, and the habitat Ω is a bounded smooth domain in $\mathbb{R}^n (n \geq 2)$; $d_1, d_2 > 0$ are the dispersal rates of u and v, respectively. $a_i, b_i, c_i (i = 1, 2)$ are all positive constants, where a_i represent the intrinsic growth rates of species, b_1 and c_2 are the death rates due to intra-specific competition and c_1 and b_2 are the death rates due to inter-specific competition; $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the usual Laplace operator and $\partial_\nu = \frac{\partial}{\partial \nu}$, where ν denotes the outward unit normal vector on $\partial\Omega$, is the normal derivative on the boundary. The zero-flux boundary condition is prescribed to warrant that no individual crosses the boundary of the habitat. The initial data u_0 and v_0 are nonnegative and nontrivial (i.e., not identically zero). The system (1.1) has been extensively studied in the literature (cf. Lou and Ni 1996; Brown 1980; Jüngel 2010 and references therein) among which the main concern was under what conditions competition exclusion or co-existence will be achieved asymptotically. It turned out that the asymptotic behavior of solutions to (1.1) essentially depends on the value of the ecological reaction coefficients $a_i, b_i, c_i (i = 1, 2)$. For simplicity, the following changes of variables and parameters:

$$\tilde{u} = b_1 u, \ \tilde{v} = c_2 v, \ b = \frac{b_2}{b_1}, \ c = \frac{c_1}{c_2}$$

have been often used to simplify the system (1.1) to

$$\begin{cases} u_t = d_1 \Delta u + u(a_1 - u - cv), & \text{in } \Omega \times \mathbb{R}^+, \\ v_t = d_2 \Delta v + v(a_2 - bu - v), & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ (u, v)(x, 0) = (u_0, v_0)(x), & \text{in } \Omega, \end{cases} \tag{1.2}$$

where tildes on u and v have been suppressed for convenience. Set

$$A = a_1/a_2, \ B = 1/b, \ C = c. \tag{1.3}$$

Then the following results are well known (cf. Lou and Ni 1996):

- **Weak competition** $C < A < B$. The solution (u, v) of (1.2) converges to $(u_*, v_*) = ((a_1 - a_2 c)/(1 - bc), (a_2 - ba_1)/(1 - bc))$ uniformly (i.e., regardless of initial values) as $t \to \infty$, namely the coexistence steady state (u_*, v_*) is globally asymptotically stable.
- **Competitive exclusion** $A < \min\{B, C\}$ (reps. $A > \max\{B, C\}$). The solution (u, v) of (1.2) converges to $(0, a_2)$ (reps. $(a_1, 0)$) uniformly as $t \to \infty$; that is, one species dominates and the other becomes extinct (one species wipes out the other).
- **Strong competition** $B < A < C$. The steady states $(a_1, 0)$ and $(0, a_2)$ are locally stable, but (u_*, v_*) is unstable. If the domain is convex, no stable positive steady states exist.

When the spatial heterogeneity of resource (or environment) is considered, say $a_i = m(x)$ with $m(x)$ being a non-constant function representing the local carrying capacity of species, then the Lotka–Volterra competition-diffusion system in (1.1) can be extended to the following one:

$$\begin{cases} u_t = d_1 \Delta u + u(m(x) - u - cv), & \text{in } \Omega \times \mathbb{R}^+, \\ v_t = d_2 \Delta v + v(m(x) - bu - v), & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ (u, v)(x, 0) = (u_0, v_0)(x), & \text{in } \Omega. \end{cases} \tag{1.4}$$

The most prominent feature of (1.4), in contrast to (1.1), is perhaps the so-called "slower diffuser wins" phenomenon.

With $g(x) \in C^\alpha(\overline{\Omega})(0 < \alpha < 1)$ with $\int_\Omega g dx \geq 0$ and $g \not\equiv 0$, we denote by $\theta_{d,g}$ the unique positive solution of

$$d\Delta\theta + \theta(g(x) - \theta) = 0 \text{ in } \Omega, \quad \partial_\nu \theta = 0 \text{ on } \partial\Omega \tag{1.5}$$

where the proof of existence and uniqueness of solutions to (1.5) was given in Cantrell and Cosner (2004). The result of Dockery et al. (1998) asserts that if $m(x) \in C^\alpha(\overline{\Omega})(0 < \alpha < 1)$, then every solution (u, v) of (1.4) with $b = c = 1$

converges to $(\theta_{d_1,m}, 0)$ as $t \to \infty$ when $d_1 < d_2$. Simply speaking, *the slower diffuser wipes out its fast competitor regardless of the initial value*. When $0 < b, c < 1$ (weak competition), it was further proved by Lou (2006) that for any $b \in (1/E(m), 1)$ there exists a $\bar{c} > 0$ such that if $c \in (\bar{c}, 1)$, then $(\theta_{d_1,m}, 0)$ is globally asymptotically stable for some $0 < d_1 < d_2$, where $E(m) = \sup_{d>0} \overline{\theta_{d,m}}/\overline{m}$ with $\bar{f} = \frac{1}{|\Omega|}\int_\Omega f dx$. This remarkable result implies that co-existence may be no longer possible even in the case of weak competition $0 < b, c < 1$, which is very different from constant $m(x)$. The results are further completed in Lam and Ni (2012). It was also conjectured that the same results should for any $c \in (0, 1)$. This conjecture is confirmed in an important work of He and Ni (2017). In fact, a complete dynamics for $bc \leq 1$ including the case $b = c = 1$ with different heterogeneity of resources for different competing species was obtained in a series of important papers by He and Ni (2016a, b) and He and Ni (2017).

Both the system (1.1) and (1.4) do not take into account the non-random dispersion of species towards the resource (like food, light). Recently the following reaction-diffusion-advection model

$$\begin{cases} u_t = \nabla \cdot (d_1 \nabla u - \chi_1 u \nabla m) + u(m(x) - u - cv), & \text{in } \Omega \times \mathbb{R}^+, \\ v_t = \nabla \cdot (d_2 \nabla v - \chi_2 v \nabla m) + v(m(x) - bu - v), & \text{in } \Omega \times \mathbb{R}^+, \\ d_1 \partial_\nu u - \chi_1 u \partial_\nu m = d_2 \partial_\nu v - \chi_2 u \partial_\nu m = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ (u, v)(x, 0) = (u_0, v_0)(x), & \text{in } \Omega, \end{cases} \tag{1.6}$$

has been considered by adding the advection (directed movement) of species along the gradient of the resource. Due to its complexity, the model (1.6) still remains poorly understood and not many results are available. We refer to Cantrell et al. (2006), Cantrell et al. (2007), Chen et al. (2008), Averill et al. (2017) for some interesting results obtained on (1.6) and Cosner (2014), Lou (2008) for open questions imposed.

We note that all these relevant works mentioned above have assumed the (environmental) carrying capacity/recource $m(x)$ is either constant or spatially variable. However the resource often changes in time (like seasonal changes or temporarily varying population in the predator-prey system), and hence it would be of interest to consider the dynamics in a spatio-temporal heterogenous environment. It seems few research projects have been conducted in this direction. Given heterogeneous time-periodic resource $m(x, t)$, it is shown in Hutson et al. (2001) that the system (1.4) with $b = c = 1$ exhibits quite different dynamics, where the competing species may coexist at different dispersal rates and even faster diffuser may prevail under suitable choices of d_1, d_2 and m. The traveling wave solutions (see Zhao and Ruan 2011; Bao and Wang 2013) and the free boundary problem (e.g., see Chen et al. 2016; Wang and Zhang 2016 and references therein) of (1.4) with heterogenous time-periodic environment have been investigated. Though time-dependent resources are considered in these works, they are still given as functions of time without temporal dynamics. By considering the feedback of resources from exploitations by consumers, Zhang et al. (2017) extend, based on their experimental findings, the scalar logistic models to consumer-resource reaction-diffusion models to include exploitable renewed resources with temporal dynamics. The experiment of Zhang et al. (2017) not only verifies some hypotheses on the single logistic model but also finds that homogeneous resources can support larger population than heterogeneous resources—a surprising result. The model in Zhang et al. (2017) was further analytically studied in a recent work (He et al. 2019).

In this paper, we shall consider another scenario where the resource has temporal dynamics. To be specific, we consider the competition of two species in a predator-prey system, where the prey as a resource has spatial movement, intrinsic birth-death kinetics and loss due to predation. Furthermore in the realistic predator-prey system, the dispersal rates of predators should depend on the distribution of the prey (see Kareiva and Odell 1987). Taking into account these two important effects in the competition system, we consider the following Lotka–Volterra competition model with dynamical resource and density-dependent diffusion (i.e. non-random dispersion):

$$\begin{cases} u_t = \Delta(d_1(w)u) + u(a_1 w - u - cv), & x \in \Omega, \quad t > 0, \\ v_t = \Delta(d_2(w)v) + v(a_2 w - bu - v), & x \in \Omega, \quad t > 0, \\ w_t = \Delta w - w(u+v) + \mu w(1-w), & x \in \Omega, \quad t > 0, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x \in \partial\Omega, \quad t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega, \end{cases} \tag{1.7}$$

where $u(x,t)$ and $v(x,t)$ denote the densities of two competing species (e.g. predators), and $w(x,t)$ denotes the density of predators' resources (e.g. the prey). The third equation of (1.7) describes the dynamics of the resource, which for instance can be regarded as the prey in the predator-prey system. $d_i(w)(i = 1, 2)$ denotes the resource-dependent dispersal rate of species with a monotone property: $d_i'(w) \le 0$, to comply with the fact that the predators will reduce its motility for exploitation when encountering the prey observed in the field experiment of Kareiva and Odell (1987). This dispersal mechanism is called "density-suppressed motility" and was also found in the bacterial movement (cf. Fu et al. 2012; Jin et al. 2018). Common examples include $d_i(w) = 1/(1+w)^{\lambda_i}$ (algebraic decay) or $d_i(w) = e^{-\lambda_i w}$ (exponential decay) with $\lambda_i > 0$. By expanding the Laplace operator, it is easy to see that the nonlinear diffusion in (1.7) actually consists of both diffusive and advective flux

$$\begin{aligned} \Delta(d_1(w)u) &= \nabla \cdot (d_1(w)\nabla u - u\chi_1(w)\nabla w), \\ \Delta(d_2(w)v) &= \nabla \cdot (d_2(w)\nabla v - v\chi_2(w)\nabla w), \end{aligned} \tag{1.8}$$

with $\chi_i(w) = -d_i'(w) \ge 0 (i = 1, 2)$. Hence the system (1.7) can be regarded as a generalization of the reaction-diffusion-advection model (1.6).

Throughout the paper, we shall assume the motility function $d_i(w)$ $(i = 1; 2)$ satisfies the following hypothesis:

(H1): $d_i(w) \in C^2([0,\infty))$, $d_i'(w) \le 0$ on $[0, \infty)$ and $d(w) > 0$.

Due to the presence of the density dependent diffusion coefficient, the system (1.7) is a cross diffusion system and the maximum principle is no longer applicable. Thus the boundedness of solutions (prevention of overcrowding of population) is not an obvious result and needs to be justified. Hence the first goal of this paper is to prove that the system (1.7) has a unique global classical solution uniformly bounded in time. We shall prove our results by the method of energy estimates and Moser iteration. The second goal of this paper is to identify conditions under which coexistence or exclusion steady state will be asymptotically achieved. Then we finally give some biological interpretations for our results. We shall prove our results based on a parabolic approach—constructing Lyapunoval functions, which is different from elliptic approaches used in the literature (cf. He and Ni 2016a; Lam and Ni 2012; Lou 2006). Our first result is the global existence of solutions with uniform-in-time bound.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and the hypothesis (H1) hold. Assume $(u_0, v_0, w_0) \in [W^{1,p}(\Omega)]^3$ with $p > 2$ and $u_0, v_0, w_0 \geq 0 (\not\equiv 0)$. Then there exists a global classical solution $(u, v, w) \in [C^0([0,\infty) \times \bar{\Omega}) \cap C^{2,1}((0,\infty) \times \bar{\Omega})]^3$ solving the system (1.7). Moreover, the solution satisfies $u, v, w > 0$ for all $t > 0$ and*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t > 0, \tag{1.9}$$

where $C > 0$ is a constant independent of t. In particular, we have $0 < w \leq K$, where

$$K := \max\{1, \|w_0\|_{L^\infty}\}. \tag{1.10}$$

Our second main result is concerned with the asymptotic behavior of solutions to (1.7), which is connected to the homogeneous steady state (u_s, v_s, w_s) of (1.7) satisfying the following equations

$$\begin{cases} u(a_1 w - u - cv) = 0, \\ v(a_2 w - bu - v) = 0, \\ w(\mu w - \mu + u + v) = 0. \end{cases} \tag{1.11}$$

Beyond two trivial solutions (extinction steady state (0, 0, 0) and resource-only steady state (0, 0, 1)), we see that (1.11) has some other non-trivial solutions depending on the value of parameters a_i, b, c. They can be classified into the following three categories similar to the classical Lotka–Volterra competition system

$$\begin{aligned} &\text{Case 1}: c < \frac{a_1}{a_2} < 1/b (\text{weak competition}); \\ &\text{Case 2}: \frac{a_1}{a_2} < \min\{1/b, c\} (v \text{ has advantage over } u); \\ &\text{Case 3}: \frac{a_1}{a_2} > \max\{1/b, c\} (u \text{ has advantage over } v). \end{aligned} \tag{1.12}$$

For convenience, we denote

$$L := \mu(bc - 1) + a_1(b - 1) + a_2(c - 1).$$

One can check that $L < 0$ in Case 1 ($c < \frac{a_1}{a_2} < \frac{1}{b}$). Then the corresponding homogeneous steady state (u_s, v_s, w_s) can be solved as follows:

$$(u_s, v_s, w_s) = \begin{cases} (u_1^*, v_1^*, w_1^*) \text{ or } (0, v_2^*, w_2^*) \text{ or } (u_3^*, 0, w_3^*), & \text{in Case 1,} \\ (0, v_2^*, w_2^*) \text{ or } (u_3^*, 0, w_3^*), & \text{in Case 2 and Case 3,} \end{cases}$$

where

$$(u_1^*, v_1^*, w_1^*) := \frac{\mu}{L}(a_2 c - a_1, a_1 b - a_2, bc - 1) \tag{1.13}$$

and

$$(v_2^*, w_2^*) := \left(\frac{\mu a_2}{a_2 + \mu}, \frac{\mu}{a_2 + \mu}\right), \quad (u_3^*, w_3^*) := \left(\frac{\mu a_1}{a_1 + \mu}, \frac{\mu}{a_1 + \mu}\right). \tag{1.14}$$

To state our main results on the large time behavior of solutions, we introduce some further notations. Denote

$$\mathcal{K}_i := \max_{0 \le w \le K} \frac{|d_i'(w)|^2}{d_i(w)}, \quad i = 1, 2, \tag{1.15}$$

where K is defined in (1.10). Let

$$\begin{cases} \delta_1 = (a_1 b + a_2 c)^2 - 4a_1 a_2, & \text{in Case 1,} \\ \delta_2 = a_1(b+1)^2 - 4a_2, & \text{in Case 2,} \\ \delta_3 = a_2(c+1)^2 - 4a_1, & \text{in Case 3,} \end{cases} \tag{1.16}$$

and

$$\begin{cases} \mu_1^* = \frac{c(a_1+a_2-a_1 b)^2 + b(a_1+a_2-a_2 c)^2}{4bc(a_1+a_2)(1-bc)}, & \text{in Case 1,} \\ \mu_2^* = \frac{(a_1(b+1)-2a_2)^2}{4(a_2-a_1 b)}, & \text{in Case 2,} \\ \mu_3^* = \frac{(a_2(c+1)-2a_1)^2}{4(a_1-a_2 c)}, & \text{in Case 3.} \end{cases} \tag{1.17}$$

Then our second result is stated in the follow theorem.

Theorem 1.2 *Let the assumptions in Theorem 1.1 hold. Then the solution (u, v, w) of (1.7) obtained in Theorem 1.1 has the following convergence properties.*

(1) *Assume $c < \frac{a_1}{a_2} < \frac{1}{b}$ (Case 1) and $\mathcal{K}_1 + \mathcal{K}_2 \le 4$ (“=” holds if $\|w_0\|_{L^\infty} \le 1$). If $\delta_1 < 0$ or $\delta_1 \ge 0$ and $\mu > \mu_1^*$, then*

$$(u, v, w) \to (u_1^*, v_1^*, w_1^*) \text{ exponentially as } t \to \infty.$$

(2) *Suppose $\frac{a_1}{a_2} < \min\{\frac{1}{b}, c\}$ (Case 2) and $\mathcal{K}_2 \le 4$ (“=” holds if $\|w_0\|_{L^\infty} \le 1$). If $\delta_2 < 0$ or $\delta_2 \ge 0$ and $\mu > \mu_2^*$, then*

$$(u, v, w) \to (0, v_2^*, w_2^*) \text{ algebraically as } t \to \infty.$$

(3) *Assume* $\frac{a_1}{a_2} > \max\{\frac{1}{b}, c\}$ *(Case 3) and* $\mathcal{K}_1 \le 4$ *("=" holds if* $\|w_0\|_{L^\infty} \le 1$ *). If* $\delta_3 < 0$ *or* $\delta_3 \ge 0$ *and* $\mu > \mu_3^*$*, then*

$$(u, v, w) \to (u_3^*, 0, w_3^*) \text{ algebraically as } t \to \infty.$$

1.2 Implications of results

This paper investigates the asymptotic dynamics of a diffusive Lotka–Volterra competition system, where the resource has spatio-temporal dynamics and diffusion of competitors depends on the resource distribution. Below we shall discuss the applications of our results and compare them with existing findings. It is well known that the model (1.4) will exhibit the striking phenomenon "slower diffuser always prevails" when the resource is spatially heterogeneous but given. Therefore the first issue we are concerned with is whether the same phenomenon still exists when the resource w has temporal dynamics such as in the system (1.7). To this end, we consider (1.7) with a special case $d_i(w) = D_i =$ constant $(i = 1, 2)$ and $a_1 = a_2 = a$:

$$\begin{cases} u_t = D_1\Delta u + u(aw - u - cv), & x \in \Omega, \quad t > 0, \\ v_t = D_2\Delta v + v(aw - bu - v), & x \in \Omega, \quad t > 0, \\ w_t = \Delta w - w(u + v) + \mu w(1 - w), & x \in \Omega, \quad t > 0, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x \in \partial\Omega, \quad t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega. \end{cases} \tag{1.18}$$

The system (1.18) is directly comparable with the Lotka–Volterra competition-diffusion system (1.4) where the spatially heterogenous resource $m(x)$ is given without temporal dynamics. It can be easily checked that $\delta_i < 0$ $(i = 1, 2, 3)$ in each case of (1.12) if $a_1 = a_2$. Hence Theorem 1.2 applied to (1.18) yields the following results.

Corollary 1.3 *Let* (u, v, w) *be the unique classical solution of* (1.18) *with* $a_1 = a_2$ *obtained in Theorem* 1.1. *Then the following results hold.*

(1) *If* $c < 1 < \frac{1}{b}$ *(Case 1), then* $(u, v, w) \to (u_1^*, v_1^*, w_1^*)$ *exponentially as* $t \to \infty$.

(2) *If* $1 < \min\{\frac{1}{b}, c\}$ *(Case 2), then*
$$(u, v, w) \to (0, v_2^*, w_2^*) \text{ algebraically as } t \to \infty.$$

(3) *If* $1 > \max\{\frac{1}{b}, c\}$ *(Case 3), then*
$$(u, v, w) \to (u_3^*, 0, w_3^*) \text{ algebraically as } t \to \infty.$$

The prominent phenomenon derived from the system (1.4) is that "slower diffuser always wipes out faster diffuser" even for the weak competition (Case 1). However the results in Corollary 1.3 show that this phenomenon no longer exists if the resource has temporal dynamics and co-existence may be achieved in the case of weak competition regardless of the size of D_i $(i = 1, 2)$ and initial values. In this situation, the asymptotic dynamics of (1.18) is more like the one for the classical Lotka–Volterra diffusion-competition model (1.1) with spatially homogeneous resources.

Next we consider the density-dependent motility function $d_i(w)$ and interpret the meaning of $\mathcal{K}_i$ defined in (1.15). To see this, we consider following examples under the hypothesis (H1):

$$d_i(w)=\frac{1}{(1+w)^{\lambda_i}}\quad \text{or}\quad d_i(w)=\exp(-\lambda_i w),\quad \lambda_i>0. \tag{1.19}$$

Then it is easy to verify that $\mathcal{K}_i=\max_{0\le w\le K}\frac{|d_i'(w)|^2}{d_i(w)}=|\lambda_i|^2$. This means that $\mathcal{K}_i$ is a measurement of the decay rates of $d_i(w)$ with respect to w. In terms of the decay rate λ_i, we have the following results as a consequence of Theorem 1.2.

Corollary 1.4 *Let $d_i(w)$ $(i=1,2)$ be given in (1.19), and (u,v,w) be the unique classical solution of (1.7) with $a_1=a_2$ obtained in Theorem 1.1. Then the following results hold.*

(1) *If $c<1<\frac{1}{b}$ (Case 1) and $\lambda_1^2+\lambda_2^2<4$, then*

$$(u,v,w)\to(u_1^*,v_1^*,w_1^*)\ \textit{exponentially as}\ t\to\infty.$$

(2) *If $1<\min\{\frac{1}{b},c\}$ (Case 2) and $\lambda_2<2$, then*

$$(u,v,w)\to(0,v_2^*,w_2^*)\ \textit{algebraically as}\ t\to\infty.$$

(3) *If $1>\max\{\frac{1}{b},c\}$ (Case 3) and $\lambda_1<2$, then*

$$(u,v,w)\to(u_3^*,0,w_3^*)\ \textit{algebraically as}\ t\to\infty.$$

The results in Corollary 1.4 indicate that even two non-randomly dispersing competitors have different dispersal rates (i.e. $\lambda_1\neq\lambda_2$), which is comparable with the case $d_1\neq d_2$ in the system (1.4), the co-existence steady state can be achieved in the case of weak competition if the dispersion decay rates of both competitors are not large (i.e. $\lambda_1^2+\lambda_2^2<4$) no matter whether they are equal or not. This again shows that the phenomenon "slower diffuser always prevails" does not exist any more. In the original model (1.4) deriving the prominent phenomenon "slower diffuser always prevails", the given resource is spatially heterogenous. Hence a natural question is what the asymptotic dynamics will be if the prey's resource (resource supplied to the prey) is spatially heterogenous given that the prey has temporal dynamics, namely replacing the third equation of (1.7) by

$$w_t=\Delta w-w(u+v)+\mu w(m(x)-w)$$

where $m(x)$ represents the prey's resource which is non-constant. The analytical study of this question will be very delicate and has gone beyond the scope of this paper, but we shall numerically explore it in the last section. It turns out the phenomenon "slower diffuser always prevails" will fail to appear when the non-random dispersal strategy is employed by the competing species even if the prey's resource $m(x)$ is spatially heterogeneous, whereas it still holds if the species use the random dispersion only (see simulations and discussions in Sect. 4).

The rest of this paper is organized as follows. In Sect. 2, we shall address the global boundedness of solutions to (1.7) and show Theorem 1.1. In Sect. 3, we construct

Lyapunov functionals to prove the asymptotic behavior of solutions asserted in Theorem 1.2. In Sect. 4, we shall make a summary of our results, show numerical results on the heterogenous prey's resources and discuss their biological implications.

2 Boundedness of solutions (Proof of Theorem 1.1)

Theorem 1.1 is a consequence of the local existence theorem (see Lemma 2.1) and the a priori estimates (see Lemma 2.6 and Lemma 2.7), which shall be detailed in the subsequent subsections.

2.1 Preliminaries

Before proceeding, we introduce some notations used throughout the paper.

Notation. For simplicity, we replace $\int_0^t \int_\Omega f(\cdot, s)dxds$ and $\int_\Omega f(\cdot, t)dx$ by $\int_0^t \int_\Omega f$ and $\int_\Omega f$, respectively. In addition, we denote $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$ for short, and C_i ($i = 1, 2, 3, \cdots$) stand for generic constants which may alter from line to line.

First, we establish the local existence of solutions to the system (1.7) by the abstract theory of quasilinear parabolic systems established in Amann (1993).

Lemma 2.1 (Local existence) *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume that the parameters μ, a_1, a_2, b, c are positive constants and the hypothesis (H1) holds. Suppose that $(u_0, v_0, w_0) \in [W^{1,p}(\Omega)]^3$ with $u_0, v_0, w_0 \geq 0 (\not\equiv 0)$ and $p > 2$. Then there exists a constant $T_{max} \in (0, \infty]$ such that system (1.7) has a unique classical solution (u, v, w) fulfilling $u, v, w > 0$ for all $t > 0$ and*

$$(u, v, w) \in [C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))]^3.$$

Moreover if $T_{max} < \infty$, then

$$\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty} + \|w(\cdot, t)\|_{W^{1,\infty}} \to \infty \ \text{ as } \ t \nearrow T_{max}.$$

Proof Denote $z = (u, v, w)$. Then the system (1.7) can be written as

$$\begin{cases} z_t = \nabla \cdot (P(z)\nabla z) + Q(z), & x \in \Omega, t > 0, \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ z(\cdot, 0) = (u_0, v_0, w_0), & x \in \Omega, \end{cases} \tag{2.1}$$

where

$$P(z) = \begin{pmatrix} d_1(w) & 0 & ud_1'(w) \\ 0 & d_2(w) & vd_2'(w) \\ 0 & 0 & 1 \end{pmatrix}, \quad Q(z) = \begin{pmatrix} u(a_1 w - u - cv) \\ v(a_2 w - bu - v) \\ -w(u + v) + \mu w(1 - w) \end{pmatrix}.$$

Since $d_i(w) > 0$ ($i = 1, 2$), the matrix $P(z)$ is positive definite for the given initial data, which asserts that the system (2.1) is normally parabolic. Then the application of

(Amann 1990, Theorem 7.3) yields a $T_{max} > 0$ such that system (2.1) admits a unique solution $(u, v, w) \in [C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))]^3$. The nonnegativity of (u, v, w) follows from the maximum principle. To be precise, we rewrite the first equation of system (1.7) as follows

$$\begin{cases} u_t - d_1(w)\Delta u + q_1(x,t)\nabla w \cdot \nabla u + q_2(x,t)u = 0, & x \in \Omega, t \in (0, T_{max}), \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0, T_{max}), \\ u(x, 0) = u_0 \geq 0, & x \in \Omega, \end{cases} \tag{2.2}$$

where $q_1(x,t) = -2d_1'(w)$ and $q_2(x,t) = -d_1''(w)|\nabla w|^2 - d_1'(w)\Delta w - (a_1 w - u - cv)$. Then one applies the maximum principle to system (2.2) and gets that $u(x,t) \geq 0$ for all $(x,t) \in \Omega \times (0, T_{max})$. Since $u_0 \not\equiv 0$, then $u > 0$ holds by strong maximum principle. Similarly, we can derive that $v, w > 0$ for all $(x,t) \in \Omega \times (0, T_{max})$. Moreover, we see that $P(z)$ is an upper triangular matrix, which along with (Amann 1989, Theorem 5.2) yields the blowup criterion as claimed. Consequently, the proof is finished. □

Lemma 2.2 (Kowalczyk and Szymańska 2008) *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and $T \in (0, \infty]$. Suppose that $y(x,t) \in C^0(\bar{\Omega} \times [0,T)) \cap C^{2,1}(\bar{\Omega} \times (0,T))$ satisfies*

$$\begin{cases} y_t = \Delta y - y + h(x,t), & x \in \Omega, t \in (0,T), \\ \frac{\partial y}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,T), \end{cases}$$

where $h(x,t) \in L^\infty((0,T); L^p(\Omega))$. Then there exists a constant $C > 0$ such that

$$\|y(\cdot,t)\|_{W^{1,q}} \leq C$$

with

$$q \in \begin{cases} [1, \frac{np}{n-p}), & \text{if } p \leq n, \\ [1, \infty], & \text{if } p > n. \end{cases}$$

Lemma 2.3 (Jin and Wang 2017) *Let the assumptions in Lemma 2.1 hold. Then the solution (u, v, w) of system (1.7) satisfies that*

$$\|w(\cdot,t)\|_{L^\infty} \leq K \tag{2.3}$$

for all $t > 0$, where K is defined by (1.10). Moreover

$$\limsup_{t\to\infty} w(\cdot,t) \leq 1 \quad \text{for all } x \in \bar{\Omega}. \tag{2.4}$$

Lemma 2.4 *Let (u, v, w) be a solution of (1.7) under the assumptions in Lemma 2.1. Then it follows that*

$$\int_\Omega |\nabla w(\cdot,t)|^2 \leq C \tag{2.5}$$

and

$$\int_t^{t+\tau}\int_\Omega (u^2+v^2)(\cdot,s) \le C \quad \textit{and} \quad \int_t^{t+\tau}\int_\Omega |\Delta w(\cdot,s)|^2 \le C, \tag{2.6}$$

where $\tau = \min\{1, \frac{T_{max}}{2}\}$ *and* $C > 0$ *is a constant independent of* t.

Proof Integrating the first equation of (1.7) over Ω and using Young's inequality with (2.3), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u + \int_\Omega u &= a_1\int_\Omega uw - \int_\Omega u^2 - c\int_\Omega uv + \int_\Omega u \\ &\le -\frac{1}{2}\int_\Omega u^2 + \frac{(a_1K+1)^2}{2}|\Omega|\end{aligned}$$

which gives

$$\frac{d}{dt}\int_\Omega u + \int_\Omega u + \frac{1}{2}\int_\Omega u^2 \le C_1, \tag{2.7}$$

where $C_1 = \frac{(a_1K+1)^2}{2}|\Omega|$. Similarly from the second equation of system (1.7), we derive that

$$\frac{d}{dt}\int_\Omega v + \int_\Omega v + \frac{1}{2}\int_\Omega v^2 \le C_2 \tag{2.8}$$

with $C_2 = \frac{(a_2K+1)^2}{2}|\Omega|$.

Multiplying the third equation of (1.7) by $-\Delta w$, using (2.3) and Young's inequality, one derives that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 + \frac{1}{2}\int_\Omega |\nabla w|^2 + \int_\Omega |\Delta w|^2 \\ &\le K\int_\Omega (u+v)|\Delta w| + \mu K(K+1)\int_\Omega |\Delta w| - \frac{1}{2}\int_\Omega w\Delta w \\ &\le \frac{3}{4}\int_\Omega |\Delta w|^2 + K^2\int_\Omega u^2 + K^2\int_\Omega v^2 + C_3\end{aligned}$$

with $C_3 = K^2\left(\mu(1+K)+\frac{1}{2}\right)^2|\Omega|$. This yields that

$$\frac{d}{dt}\int_\Omega |\nabla w|^2 + \int_\Omega |\nabla w|^2 + \frac{1}{2}\int_\Omega |\Delta w|^2 \le 2K^2\int_\Omega (u^2+v^2) + 2C_3. \tag{2.9}$$

Multiplying (2.7) and (2.8) by $6K^2$ and combining them with (2.9), we end up with

$$\phi' + \phi + K^2\int_\Omega (u^2+v^2) + \frac{1}{2}\int_\Omega |\Delta w|^2 \le C_4, \tag{2.10}$$

where $\phi(t) = 6K^2\int_\Omega u + 6K^2\int_\Omega v + \int_\Omega |\nabla w|^2$ and $C_4 = 6C_1K^2 + 6C_2K^2 + 2C_3$. Then the application of Grönwall inequality to (2.10) gives

$$\phi(t) \le \phi(0) + C_4, \tag{2.11}$$

which yields (2.5). Furthermore, integrating (2.10) over $(t, t+\tau)$ with $\tau = \min\{1, \frac{T_{max}}{2}\}$ and using (2.11), we have that

$$K^2\int_t^{t+\tau}\int_\Omega (u^2+v^2) + \frac{1}{2}\int_t^{t+\tau}\int_\Omega |\Delta w|^2 \le \phi(t) + C_4\tau \\ \le \phi(0) + C_4(1+\tau),$$

which gives (2.6). Hence the proof of Lemma 2.4 is completed. □

2.2 A priori estimates

Motivated by an idea of Jin et al. (2018), we will derive the boundedness of $\|u(\cdot,t)\|_{L^2}$ and $\|v(\cdot,t)\|_{L^2}$ with the help of (2.6). Furthermore, we derive the uniform boundedness of the solution.

Lemma 2.5 *Assuming the conditions of Lemma 2.1 hold, the solution of (1.7) satisfies*

$$\|u(\cdot,t)\|_{L^2} + \|v(\cdot,t)\|_{L^2} + \|w(\cdot,t)\|_{W^{1,4}} \le C \text{ for all } t\in(0,T_{max}) \tag{2.12}$$

where $C > 0$ is a constant of independent of t.

Proof Multiplying the first equation of system (1.7) by u and applying Young's inequality, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \int_\Omega \frac{d_1(w)}{2}|\nabla u|^2 \le \int_\Omega \frac{|d_1'(w)|^2}{2d_1(w)}u^2|\nabla w|^2 + a_1K\int_\Omega u^2 - \int_\Omega u^3 \\ \le \frac{\mathcal{K}_1}{2}\left(\int_\Omega u^4\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla w|^4\right)^{\frac{1}{2}} - \frac{1}{2}\int_\Omega u^3 + C_1, \tag{2.13}$$

where $C_1 = \frac{16a_1^3K^3}{27}|\Omega|$ and $\mathcal{K}_1$ is defined in (1.15). The Gagliardo-Nirenberg inequality in two dimensions ($n=2$) yields that

$$\|u\|_{L^4}^2 \le C_2(\|\nabla u\|_{L^2}\|u\|_{L^2} + \|u\|_{L^2}^2). \tag{2.14}$$

On the other hand, when $n=2$, applying (Jin et al. 2018, Lemma 2.5), we have

$$\|\nabla w\|_{L^4}^2 \le C_3(\|\Delta w\|_{L^2}\|\nabla w\|_{L^2} + \|\nabla w\|_{L^2}^2) \le C_4(\|\Delta w\|_{L^2} + 1) \tag{2.15}$$

where (2.5) has been used. Then the combination of (2.14) and (2.15) gives that

$$\begin{aligned}
&\frac{\mathcal{K}_1}{2}\left(\int_\Omega u^4\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla w|^4\right)^{\frac{1}{2}}\\
&\quad\le \frac{\mathcal{K}_1 C_2 C_4}{2}(\|\nabla u\|_{L^2}\|u\|_{L^2}\|\Delta w\|_{L^2}+\|u\|_{L^2}^2\|\Delta w\|_{L^2}+\|\nabla u\|_{L^2}\|u\|_{L^2}+\|u\|_{L^2}^2)\\
&\quad\le \frac{d_1(K)}{2}\|\nabla u\|_{L^2}^2+C_5\|u\|_{L^2}^2\|\Delta w\|_{L^2}^2+C_5\|u\|_{L^2}^2
\end{aligned} \tag{2.16}$$

with $C_5 = \frac{1}{d_1(K)}[\mathcal{K}_1 C_2 C_4 + d_1(K)]^2$. Substituting (2.16) into (2.13) and using Young's inequality give that

$$\frac{d}{dt}\|u\|_{L^2}^2 - 2C_5\|u\|_{L^2}^2\|\Delta w\|_{L^2}^2 \le C_6 \tag{2.17}$$

for all $t \in (0, T_{max})$, where $C_6 = 2(C_1 + \frac{64C_5^3}{27}|\Omega|)$. Recalling (2.6), one can find $t_0 = t_0(t) \in ((t-\tau)_+, t)$ for any $t \in (0, T_{max})$ such that

$$\|u(\cdot, t_0)\|_{L^2}^2 \le C_7 \tag{2.18}$$

in both cases $t \in (0, \tau)$ and $t \ge \tau$, where τ is defined in Lemma 2.4. By (2.6), there exists a constant $C_8 > 0$ such that

$$\int_{t_0}^{t_0+\tau}\int_\Omega |\Delta w(\cdot, s)|^2 \le C_8. \tag{2.19}$$

With a notice of $t_0 < t \le t_0 + \tau \le t_0 + 1$, we integrate (2.17) over (t_0, t) and use (2.18)–(2.19) to derive that

$$\begin{aligned}
\|u(\cdot, t)\|_{L^2}^2 &\le \|u(\cdot, t_0)\|_{L^2}^2 e^{2C_5\int_{t_0}^t \|\Delta w(\cdot,s)\|_{L^2}^2 ds} + C_6\int_{t_0}^t e^{2C_5\int_s^t \|\Delta w(\cdot,\rho)\|_{L^2}^2 d\rho} ds\\
&\le (C_7 + C_6\tau)e^{2C_5C_8}
\end{aligned} \tag{2.20}$$

for all $t \in (0, T_{max})$. Treating v in the same way, we have

$$\|v(\cdot, t)\|_{L^2}^2 \le C_9 \tag{2.21}$$

for all $t \in (0, T_{max})$. Furthermore, we apply the parabolic regularity (see Lemma 2.2) to the third equation of system (1.7) and obtain that $\|w(\cdot, t)\|_{W^{1,4}} \le C_9$ which, together with (2.20)–(2.21), yields (2.12). □

With the boundedness of $\|u(\cdot, t)\|_{L^2}$ and $\|v(\cdot, t)\|_{L^2}$ in hand, we next derive the uniform-in-time boundedness of the solution (u, v, w).

Lemma 2.6 *Let the assumptions in Lemma 2.1 hold. Then the solution of (1.7) satisfies for all $t \in (0, T_{max})$ that*

$$\|w(\cdot, t)\|_{W^{1,\infty}} \le C, \tag{2.22}$$

where $C > 0$ is a constant independent of t.

Proof First with $0 < w \le K$ in (2.3) and hypothesis (H1), we have $0 < d_1(K) \le d_1(w)$ and $\frac{|d_1'(w)|^2}{d_1(w)} \le \mathcal{K}_1$. Then one multiplies the first equation of system (1.7) by u^2 and integrates the result over Ω to derive that

$$\begin{aligned}
&\frac{1}{3}\frac{d}{dt}\int_\Omega u^3 + 2\int_\Omega d_1(w)u|\nabla u|^2 + \int_\Omega u^4 \\
&\le -2\int_\Omega d_1'(w)u^2\nabla u\cdot\nabla w + a_1\int_\Omega u^3 w - c\int_\Omega u^3 v \\
&\le \int_\Omega d_1(w)u|\nabla u|^2 + \mathcal{K}_1\left(\int_\Omega u^6\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla w|^4\right)^{\frac{1}{2}} + a_1K\int_\Omega u^3. \qquad (2.23)
\end{aligned}$$

From Lemma 2.5, we have $\|\nabla w\|_{L^4} \le C_2$ and $\|u^{\frac{3}{2}}\|^2_{L^{\frac{4}{3}}} = \|u\|^3_{L^2} \le C_3$. Then one applies the Gagliardo-Nirenberg inequality and Young's inequality to obtain that

$$\begin{aligned}
4\mathcal{K}_1\left(\int_\Omega u^6\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla w|^4\right)^{\frac{1}{2}} &\le 4\mathcal{K}_1C_2^2\|u^{\frac{3}{2}}\|^2_{L^4} \\
&\le C_4(\|\nabla u^{\frac{3}{2}}\|^{\frac{4}{3}}_{L^2}\|u^{\frac{3}{2}}\|^{\frac{2}{3}}_{L^{\frac{4}{3}}} + \|u^{\frac{3}{2}}\|^2_{L^{\frac{4}{3}}}) \\
&\le \frac{2d_1(K)}{9}\|\nabla u^{\frac{3}{2}}\|^2_{L^2} + C_5, \qquad (2.24)
\end{aligned}$$

where $C_5 = C_3C_4(1 + \frac{3C_4^2}{d_1^2(K)})$. Since $\frac{2d_1(K)}{9}\|\nabla u^{\frac{3}{2}}\|^2_{L^2} = \frac{d_1(K)}{2}\|u^{\frac{1}{2}}\nabla u\|^2_{L^2}$, we substitute (2.24) into (2.23) and employ Young's inequality again to show that

$$\frac{1}{3}\frac{d}{dt}\int_\Omega u^3 + \frac{1}{3}\int_\Omega u^3 + \frac{d_1(K)}{2}\int_\Omega u|\nabla u|^2 + \frac{1}{2}\int_\Omega u^4 \le C_6$$

with $C_6 = \frac{|\Omega|}{4}(27a_1^4K^4 + \frac{1}{3}) + C_5$, that is

$$\frac{d}{dt}\int_\Omega u^3 + \int_\Omega u^3 \le 3C_6.$$

Therefore, the application of Grönwall inequality gives that

$$\int_\Omega u^3 \le \int_\Omega u_0^3 + 3C_6. \qquad (2.25)$$

We conclude similarly that $\|v\|_{L^3} \le C_7$, which, together with (2.25) and (2.3) gives (2.22) directly by the parabolic regularity (cf. Lemma 2.2). □

Lemma 2.7 *Let the assumptions in Lemma* 2.1 *hold and assume that* (u, v, w) *is a solution of* (1.7). *Then there exists a positive constant* C *independent of* t *such that*

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} \le C$$

for all $t \in (0, T_{max})$.

Proof Since $w \in L^\infty(\bar{\Omega} \times [0, T_{max}))$, there is a constant $M > 0$ such that $|w| \leq M$ for all $t \in [0, T_{max})$. Multiplying the first equation of (1.7) by u^{p-1} with $p \geq 2$ and integrating the result by parts, we arrive at

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)\int_\Omega d_1(w)u^{p-2}|\nabla u|^2 \\
&\quad \leq (p-1)\int_\Omega |d_1'(w)|u^{p-1}|\nabla u||\nabla w| + a_1\int_\Omega wu^p - \int_\Omega u^p(u+cv) \\
&\quad \leq \frac{(p-1)}{2}\int_\Omega d_1(w)u^{p-2}|\nabla u|^2 + \frac{p-1}{2}\int_\Omega \frac{|d_1'(w)|^2}{d_1(w)}u^p|\nabla w|^2 + a_1M\int_\Omega u^p.
\end{aligned} \tag{2.26}$$

Since $0 < w \in W^{1,\infty}(\bar{\Omega} \times [0, T_{max}])$, it follows from the hypothesis (H1) that there exist constants $C_i(i = 1, 2, 3) > 0$ such that

$$C_1 \leq d_1(w) \leq C_2, \quad \text{and} \quad \frac{|d_1'(w)|^2}{d_1(w)} \leq C_3. \tag{2.27}$$

Then we have from (2.26) that

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p + \frac{C_1p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 \leq C_4p(p-1)\int_\Omega u^p \tag{2.28}$$

with $C_4 = (\frac{C_3}{2}\|\nabla w\|^2_{L^\infty} + a_1M + 1)$. By the Gagliardo-Nirenberg inequality to $\int_\Omega u^p$, we get

$$\begin{aligned}
C_4p(p-1)\int_\Omega u^p &= C_4p(p-1)\|u^{\frac{p}{2}}\|^2_{L^2} \\
&\leq C_5p(p-1)(\|\nabla u^{\frac{p}{2}}\|^{\frac{2n}{n+2}}_{L^2}\|u^{\frac{p}{2}}\|^{\frac{4}{n+2}}_{L^1} + \|u^{\frac{p}{2}}\|^2_{L^1}) \\
&\leq \frac{2C_1(p-1)}{p}\|\nabla u^{\frac{p}{2}}\|^2_{L^2} + C_6p(p-1)(p^n+1)\|u^{\frac{p}{2}}\|^2_{L^1},
\end{aligned}$$

where $C_6 = C_5[(\frac{C_5}{2C_1})^{\frac{n}{2}} + 1]$. Noting that $\int_\Omega u^{p-2}|\nabla u|^2 = \frac{4}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2$ and that $p^n + 1 \leq (p+1)^n$, one derives that

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p \leq C_6p(p-1)(p+1)^n\left(\int_\Omega u^{\frac{p}{2}}\right)^2 \tag{2.29}$$

from (2.28). Furthermore, it follows from (2.29) that

$$\int_\Omega u^p \leq \int_\Omega u_0^p + C_6(p+1)^n \sup_{0\leq t\leq T_{max}}\left(\int_\Omega u^{\frac{p}{2}}\right)^2$$

$$\leq \|u_0\|_{L^\infty}^p|\Omega| + C_6(p+1)^n \sup_{0\leq t\leq T_{max}} \left(\int_\Omega u^{\frac{p}{2}}\right)^2. \tag{2.30}$$

If $\sup_{0\leq t\leq T_{max}} \left(\int_\Omega u^{\frac{p}{2}}\right)^{\frac{2}{p}} \leq \|u_0\|_{L^\infty}$, we have from (2.30) that

$$\int_\Omega u^p \leq (C_6 + |\Omega|)(p+1)^n \|u_0\|_{L^\infty}^p.$$

While if $\sup_{0\leq t\leq T_{max}} \left(\int_\Omega u^{\frac{p}{2}}\right)^{\frac{2}{p}} > \|u_0\|_{L^\infty}$, it follows from (2.30) that

$$\int_\Omega u^p \leq (C_6 + |\Omega|)(p+1)^n \sup_{0\leq t\leq T_{max}} \left(\int_\Omega u^{\frac{p}{2}}\right)^2.$$

Denote

$$N(p) = \max\left\{\|u_0\|_{L^\infty}, \sup_{0\leq t\leq T_{max}} \left(\int_\Omega u^p\right)^{\frac{1}{p}}\right\}$$

and let $C_7 = C_6+|\Omega|$. Now if $\sup_{0\leq t\leq T_{max}} \left(\int_\Omega u^p\right)^{\frac{1}{p}} < \|u_0\|_{L^\infty}$, we immediately have that $\|u\|_{L^\infty} < C_8$ for some constant $C_8 > 0$, which completes the proof. Otherwise it follows from the inequality (2.30) that

$$N(p) \leq C_7^{\frac{1}{p}}(p+1)^{\frac{n}{p}} N\left(\frac{p}{2}\right).$$

where w.l.o.g we have assumed $C_7 > 1$. Taking $p = 2^j$, $j = 1, 2, \ldots$, we have that

$$\begin{aligned}
N(2^j) &\leq C_7^{2^{-j}}(1+2^j)^{n2^{-j}} N(2^{j-1}) \\
&\leq \prod_{k=1}^{j} C_7^{2^{-k}}(1+2^k)^{n2^{-k}} N(1) \\
&\leq \prod_{k=1}^{j}(1+2^{-k})^{n2^{-k}} \left(C_7^{\sum_{k=1}^{j} 2^{-k}}\right)\left(2^{\sum_{k=1}^{j} kn2^{-k}}\right) N(1) \\
&\leq 2^{3n} C_7 N(1).
\end{aligned} \tag{2.31}$$

Since $u \in L^1(\Omega \times [0, T_{max}])$, we get $N(1) \leq C_9$. Sending $j \to \infty$ in (2.31), one has

$$\|u\|_{L^\infty} \leq 2^{3n} C_7 N(1) =: C_{10} \text{ for all } t \in (0, T_{max}).$$

Performing the same procedure to v, we can get a constant $C_{11} > 0$ such that $\|u\|_{L^\infty} \leq C_{11}$ for all $t \in (0, T_{max})$. This completes the proof. □

3 Stabilization and convergence rate

In this section, we will investigate the asymptotic behavior of solutions solving system (1.7) and prove Theorem 1.2 based on Lyapunov functional method along with LaSalle's invariant principle. Though the ideas are in the same spirit as a previous work Jin and Wang (2017) which deals with a two-component prey-taxis system with constant diffusion, the analyses in our present work are much more technical and complex since (1.7) is a three-component system with competition and density-dependent diffusion. In particular, the technique of choosing appropriate coefficients to appear in the Lyapunov functionals is quite different.

3.1 Stabilization

We aligned out analysis into three distinct scenarios.

Case 1 (weak competition): $c < \frac{a_1}{a_2} < \frac{1}{b}$. In this case, we can easily check that u_1^*, v_1^* and w_1^* defined by (1.13) are all positive. Then we consider the energy functional

$$\mathscr{E}_1(t) := \mathscr{E}_1[u(t), v(t), w(t)] = \xi_1 \int_\Omega \left(u - u_1^* - u_1^* \ln \frac{u}{u_1^*} \right) + \eta_1 \int_\Omega \left(v - v_1^* - v_1^* \ln \frac{v}{v_1^*} \right) + \int_\Omega \left(w - w_1^* - w_1^* \ln \frac{w}{w_1^*} \right), \tag{3.1}$$

where $\xi_1, \eta_1 > 0$ are constants defined by

$$\xi_1 = \begin{cases} \frac{1}{a_1}, & \delta_1 < 0, \\ \frac{b}{a_1+a_2}, & \delta_1 \geq 0, \end{cases} \quad \text{and} \quad \eta_1 = \begin{cases} \frac{1}{a_2}, & \delta_1 < 0, \\ \frac{c}{a_1+a_2}, & \delta_1 \geq 0, \end{cases} \tag{3.2}$$

where δ_1 is defined in (1.16).

Lemma 3.1 *Suppose that $c < \frac{a_1}{a_2} < \frac{1}{b}$ and δ_1, μ_1^* are defined by (1.16) and (1.17), respectively. Let $\mathscr{E}_1(t)$ be the energy functional defined in (3.1) with the solution (u, v, w) obtained in Theorem 1.1. Then the following results hold.*

(1) *$\mathscr{E}_1(t) \geq 0$ for all $t > 0$.*

(2) *Assume that*

$$\mathcal{K}_1 + \mathcal{K}_2 \leq 4 \quad (\text{“} = \text{” holds if } \|w_0\|_{L^\infty} \leq 1), \tag{3.3}$$

where $\mathcal{K}_1, \mathcal{K}_2$ are defined by (1.15). There exists two constants $\alpha_1 > 0$ and $T_1 > 0$ such that

$$\frac{d}{dt}\mathscr{E}_1(t) \leq -\alpha_1 \mathcal{F}_1(t) \tag{3.4}$$

holds for all $t > T_1$ if either $\delta_1 < 0$ or $\delta_1 \geq 0$ and $\mu > \mu_1^*$, where

$$\mathcal{F}_1(t) = \int_\Omega (u - u_1^*)^2 + \int_\Omega (v - v_1^*)^2 + \int_\Omega (w - w_1^*)^2. \tag{3.5}$$

Proof First, we show that $\mathscr{E}_1(t) \geq 0$ for all $t > 0$. For convenience, we rewrite (3.1) as

$$\mathscr{E}_1(t) = \xi_1 I_1(t) + \eta_1 I_2(t) + I_3(t), \tag{3.6}$$

where

$$\begin{cases} I_1(t) = \int_\Omega \left(u - u_1^* - u_1^* \ln \frac{u}{u_1^*}\right), \\ I_2(t) = \int_\Omega \left(v - v_1^* - v_1^* \ln \frac{v}{v_1^*}\right), \\ I_3(t) = \int_\Omega \left(w - w_1^* - w_1^* \ln \frac{w}{w_1^*}\right). \end{cases}$$

Let $\varphi(z) := z - u_1^* \ln z$ for $z > 0$. Then it holds $\varphi'(z) = 1 - \frac{u_1^*}{z}$ and $\varphi''(z) = \frac{u_1^*}{z^2}$. By the Taylor's expansion, we can find a constant $\xi > 0$ between u and u_1^* such that

$$u - u_1^* - u_1^* \ln \frac{u}{u_1^*} = \varphi(u) - \varphi(u_1^*) = \frac{\varphi''(\xi)}{2}(u - u_1^*)^2 = \frac{u_1^*}{2\xi^2}(u - u_1^*)^2 \geq 0,$$

which implies $I_1(t) \geq 0$. Similarly, we have that $I_2(t) \geq 0$ and $I_3(t) \geq 0$. Therefore, by (3.6), $\mathscr{E}_1(t) \geq 0$ for all $t > 0$ since $\xi_1, \eta_1 > 0$.

Next we show $\mathscr{E}_1(t)$ satisfies (3.4) under certain conditions. To this aim, we use the fact that $a_1 w_1^* - u_1^* - c v_1^* = 0$ to estimate $I_1(t)$ as follows

$$\begin{aligned} \frac{d}{dt} I_1(t) &= \int_\Omega \left(1 - \frac{u_1^*}{u}\right) u_t \\ &= -u_1^* \int_\Omega \frac{d_1(w)|\nabla u|^2}{u^2} - u_1^* \int_\Omega \frac{d_1'(w)\nabla u \cdot \nabla w}{u} + \int_\Omega (u - u_1^*)(a_1 w - u - cv) \\ &= -u_1^* \int_\Omega \frac{d_1(w)|\nabla u|^2}{u^2} - u_1^* \int_\Omega \frac{d_1'(w)\nabla u \cdot \nabla w}{u} - c \int_\Omega (u - u_1^*)(v - v_1^*) \\ &\quad - \int_\Omega (u - u_1^*)^2 + a_1 \int_\Omega (u - u_1^*)(w - w_1^*). \end{aligned} \tag{3.7}$$

Similarly, from the second and third equations of (1.7), we have

$$\begin{aligned} \frac{d}{dt} I_2(t) &= -v_1^* \int_\Omega \frac{d_2(w)|\nabla v|^2}{v^2} - v_1^* \int_\Omega \frac{d_2'(w)\nabla v \cdot \nabla w}{v} - b \int_\Omega (u - u_1^*)(v - v_1^*) \\ &\quad - \int_\Omega (v - v_1^*)^2 + a_2 \int_\Omega (v - v_1^*)(w - w_1^*) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \frac{d}{dt} I_3(t) &= -w_1^* \int_\Omega \frac{|\nabla w|^2}{w^2} - \int_\Omega (u - u_1^*)(w - w_1^*) - \int_\Omega (v - v_1^*)(w - w_1^*) \\ &\quad - \mu \int_\Omega (w - w_1^*)^2, \end{aligned} \tag{3.9}$$

where we have used identities $a_2 w_1^* = bu_1^* + v_1^*$ and $u_1^* + v_1^* = \mu(1 - w_1^*)$. Combining (3.7)–(3.9) with (3.6) gives that

$$\frac{d}{dt}\mathscr{E}_1(t) = -\int_\Omega X_1 A_1 X_1^T - \int_\Omega Y_1 B_1 Y_1^T, \tag{3.10}$$

where $X_1 = (u - u_1^*, v - v_1^*, w - w_1^*)$ and $Y_1 = \left(\frac{\nabla u}{u}, \frac{\nabla v}{v}, \nabla w\right)$ and A_1, B_1 are matrices denoted by

$$A_1 := \begin{pmatrix} \xi_1 & \frac{c\xi_1 + b\eta_1}{2} & \frac{1 - a_1\xi_1}{2} \\ \frac{c\xi_1 + b\eta_1}{2} & \eta_1 & \frac{1 - a_2\eta_1}{2} \\ \frac{1 - a_1\xi_1}{2} & \frac{1 - a_2\eta_1}{2} & \mu \end{pmatrix}, \quad B_1 := \begin{pmatrix} \xi_1 u_1^* d_1(w) & 0 & \frac{\xi_1 u_1^* d_1'(w)}{2} \\ 0 & \eta_1 v_1^* d_2(w) & \frac{\eta_1 v_1^* d_2'(w)}{2} \\ \frac{\xi_1 u_1^* d_1'(w)}{2} & \frac{\eta_1 v_1^* d_2'(w)}{2} & \frac{w_1^*}{w^2} \end{pmatrix}.$$

Next, we shall show the nonnegativity of the matrices A_1 and B_1. When $\delta_1 < 0$, we let $\xi_1 = \frac{1}{a_1}$ and $\eta_1 = \frac{1}{a_2}$, which implies $1 - a_1\xi_1 = 0$ and $1 - a_2\eta_1 = 0$. This leads to

$$|A_{11}| := \begin{vmatrix} \xi_1 & \frac{c\xi_1 + b\eta_1}{2} \\ \frac{c\xi_1 + b\eta_1}{2} & \eta_1 \end{vmatrix} = \frac{-\delta_1}{4a_1^2 a_2^2} > 0 \quad \text{and} \quad |A_1| = \mu |A_{11}| > 0.$$

When $\delta_1 \geq 0$, we choose $\xi_1 = \frac{b}{a_1 + a_2}$ and $\eta_1 = \frac{c}{a_1 + a_2}$. Then one can derive that

$$|A_{11}| = \frac{bc(1 - bc)}{(a_1 + a_2)^2}$$

and

$$\begin{aligned} |A_1| &= \mu|A_{11}| + \frac{1}{4(a_1 + a_2)^3}\Big[2bc(a_1 + a_2 - a_1 b)(a_1 - a_2 c + a_2) \\ &\quad - c(a_1 + a_2 - a_1 b)^2 - b(a_1 - a_2 c + a_2)^2\Big] \\ &> |A_{11}|\left(\mu - \frac{c(a_1 + a_2 - a_1 b)^2 + b(a_1 + a_2 - a_2 c)^2}{4bc(a_1 + a_2)(1 - bc)}\right). \end{aligned}$$

Therefore under the conditions $c < \frac{a_1}{a_2} < \frac{1}{b}$ and $\mu > \mu_1^*$ defined in (1.17), one has that $|A_{11}| > 0$ and $|A_1| > 0$. Based on the Sylvester's criterion, the matrix A_1 is positive definite and we can find a constant $\alpha_1 > 0$ such that

$$X_1(x,t) A_1 X_1^T(x,t) \geq \alpha_1 |X_1|^2, \quad \text{if } \delta_1 < 0 \text{ or } \delta_1 \geq 0 \text{ and } \mu > \mu_1^*. \tag{3.11}$$

For B_1, first we see $\xi_1 u_1^* d_1(w) > 0$ and hence

$$\begin{vmatrix} \xi_1 u_1^* d_1(w) & 0 \\ 0 & \eta_1 v_1^* d_2(w) \end{vmatrix} = \xi_1 \eta_1 u_1^* v_1^* d_1(w) d_2(w) > 0.$$

To proceed, we claim that

$$\frac{\xi_1 u_1^*}{w_1^*} < 1 \quad \text{and} \quad \frac{\eta_1 v_1^*}{w_1^*} < 1. \tag{3.12}$$

Let $\delta_1 < 0$. In fact, since $\frac{1}{b} > \frac{a_1}{a_2}$, we have $a_2 > a_1 b$, which implies that

$$\frac{a_1 - a_2 c}{a_1(1-bc)} < 1 \quad \Leftrightarrow \quad \frac{u_1^*}{a_1 w_1^*} < 1 \tag{3.13}$$

by recalling the definition of u_1^* and w_1^*. On the other hand, when $\delta_1 \geq 0$, thanks to $a_2 > a_1 b$ and (3.13), one has

$$\frac{b u_1^*}{(a_1 + a_2) w_1^*} < \frac{b u_1^*}{a_2 w_1^*} < \frac{u_1^*}{a_1 w_1^*} < 1. \tag{3.14}$$

The combination of (3.13) and (3.14) gives that $\frac{\xi_1 u_1^*}{w_1^*} < 1$ in either case. Similarly, we can derive that $\frac{\eta_1 v_1^*}{w_1^*} < 1$. Hence, (3.12) holds in case 1.

Next, we claim that there is a $T_1 > 0$ such that for all $t > T_1$, it holds

$$\frac{\xi_1 u_1^* w^2 |d_1'(w)|^2}{4 w_1^* d_1(w)} + \frac{\eta_1 v_1^* w^2 |d_2'(w)|^2}{4 w_1^* d_2(w)} < 1. \tag{3.15}$$

In fact if $\|w_0\|_{L^\infty} \leq 1$ by (2.3), we see $\|w(\cdot, t)\|_{L^\infty} \leq 1$. Then it follows from (3.3) and (3.12) that

$$\frac{\xi_1 u_1^* w^2 |d_1'(w)|^2}{4 w_1^* d_1(w)} + \frac{\eta_1 v_1^* w^2 |d_2'(w)|^2}{4 w_1^* d_2(w)} < \frac{w^2}{4}(\mathcal{K}_1 + \mathcal{K}_2) \leq \frac{1}{4}(\mathcal{K}_1 + \mathcal{K}_2) \leq 1.$$

If $\|w_0\|_{L^\infty} > 1$, we suppose that $\frac{1}{4}(\mathcal{K}_1 + \mathcal{K}_2) < 1$ holds (see (3.3)), then there exists a constant $\varepsilon_0 > 0$ such that

$$\frac{1}{4}(\mathcal{K}_1 + \mathcal{K}_2) + \varepsilon_0 \leq 1. \tag{3.16}$$

Since $w \in C^{2,1}(\bar{\Omega} \times (0, \infty))$, it follows from (2.4) that

$$\limsup_{t \to \infty} \frac{w^2}{4}(\mathcal{K}_1 + \mathcal{K}_2) \leq \frac{1}{4}(\mathcal{K}_1 + \mathcal{K}_2),$$

which allows us to find a constant $T_1 > 0$ such that

$$\frac{w^2}{4}(\mathcal{K}_1 + \mathcal{K}_2) \leq \frac{1}{4}(\mathcal{K}_1 + \mathcal{K}_2) + \varepsilon_0 \tag{3.17}$$

for all $t > T_1$. The combination of (3.12) and (3.16)–(3.17) guarantees (3.15) in the case $\|w_0\|_{L^\infty} > 1$. This proves the claim (3.15).

Now from (3.15), we obtain directly that

$$|B_1| = \frac{\xi_1\eta_1 u_1^* v_1^* w_1^* d_1(w) d_2(w)}{w^2}\left(1 - \frac{\xi_1 u_1^* w^2 |d_1'(w)|^2}{4w_1^* d_1(w)} - \frac{\eta_1 v_1^* w^2 |d_2'(w)|^2}{4w_1^* d_2(w)}\right) > 0$$

for all $t > T_1$. Then the Sylvester's criterion enables us to get

$$Y_1(x,t) B_1 Y_1^T(x,t) \geq 0. \tag{3.18}$$

Hence, the combination of (3.5), (3.10), (3.11) and (3.18) implies for all $t > T_1$ that

$$\frac{d}{dt}\mathscr{E}_1(t) \leq -\alpha_1 \mathcal{F}_1(t) \quad \text{either } \delta_1 < 0 \text{ or } \delta_1 \geq 0 \text{ and } \mu > \mu_1^*,$$

which yields (3.4) and hence completes the proof. □

Lemma 3.2 *Suppose that the conditions of Lemma 3.1 hold. Then we have*

$$\|u(\cdot,t) - u_1^*\|_{L^\infty} + \|v(\cdot,t) - v_1^*\|_{L^\infty} + \|w(\cdot,t) - w_1^*\|_{L^\infty} \to 0 \quad as\ t \to \infty. \tag{3.19}$$

Proof Let $\phi(t) := \phi(t; u_0, v_0, w_0) = (u, v, w)(t)$ denote the unique global classical solution of system (1.7) with initial data (u_0, v_0, w_0). This defines a semi-flow (or trajectory) on $L^\infty(\bar{\Omega})$ (see Amann 1989) due to Theorem 1.1. By (3.4), we know that $\mathscr{E}_1(\phi) \leq \mathscr{E}_1(\phi_0) =: c$ where $\phi_0 = \phi(0)$. Then clearly $\Omega_c = \{\phi \in \mathbb{R}^3 : \mathscr{E}_1(\phi) \leq c\}$ is a positively invariant compact set by Theorem 1.1.

From Lemma 3.1 and (3.1), we know that the function $\mathscr{E}_1(\phi)$ is continuously differentiable and enjoys the following properties:

(1) $\mathscr{E}_1(\phi) > 0$ for all $\phi \neq (u_1^*, v_1^*, w_1^*)$;
(2) $\frac{d}{dt}\mathscr{E}_1(\phi) \leq 0$ for all $\phi > 0$,

where $\frac{d}{dt}\mathscr{E}_1(\phi) = 0$ if and only if $\phi = (u_1^*, v_1^*, w_1^*)$. Then the LaSalle's invariance principle (e.g. see LaSalle 1960, Theorem 3 or Sastry 2013, pp. 198–199, Theorem 5.23) asserts that (u_1^*, v_1^*, w_1^*) is globally asymptotic stable, namely (3.19) holds. □

Case 2: $\frac{a_1}{a_2} < \min\{\frac{1}{b}, c\}$. In this case, we employ the following energy functional

$$\begin{aligned}\mathscr{E}_2(t) := \mathscr{E}_2[u(t), v(t), w(t)] = \xi_2 \int_\Omega u + \frac{1}{a_2}\int_\Omega \left(v - v_2^* - v_2^* \ln \frac{v}{v_2^*}\right) \\ + \int_\Omega \left(w - w_2^* - w_2^* \ln \frac{w}{w_2^*}\right)\end{aligned} \tag{3.20}$$

to study the asymptotic behavior of the solution (u, v, w) solving system (1.7), where (v_2^*, w_2^*) is given in (1.14) and

$$\xi_2 = \begin{cases} \frac{1}{a_1}, & \delta_2 < 0, \\ \frac{2a_2 - a_1 b}{a_1^2}, & \delta_2 \geq 0. \end{cases}$$

Note that $\xi_2 > 0$ in the case 2. Then we have the following results.

Lemma 3.3 *Let $\mathscr{E}_2(t)$ be the functional defined by (3.20). Then for all $t > 0$, we have $\mathscr{E}_2(t) \geq 0$. Moreover, under the condition $\frac{a_1}{a_2} < \min\{\frac{1}{b}, c\}$ and*

$$\mathcal{K}_2 \leq 4 \quad (\text{“} = \text{” holds if } \|w_0\|_{L^\infty} \leq 1), \tag{3.21}$$

there exists two constants $\alpha_2 > 0$ and $T_2 > 0$ such that if $\delta_2 < 0$ or $\delta_2 \geq 0$ and $\mu > \mu_2^*$, then

$$\frac{d}{dt}\mathscr{E}_2(t) \leq -\alpha_2 \mathcal{F}_2(t), \tag{3.22}$$

where

$$\mathcal{F}_2(t) = \int_\Omega u^2 + \int_\Omega (v - v_2^*)^2 + \int_\Omega (w - w_2^*)^2$$

and μ_2^* is defined by (1.17).

Proof With

$$J_1(t) = \int_\Omega u,\ J_2(t) = \int_\Omega \left(v - v_2^* - v_2^* \ln \frac{v}{v_2^*}\right),\ J_3(t) = \int_\Omega \left(w - w_2^* - w_2^* \ln \frac{w}{w_2^*}\right),$$

we rewrite the energy functional $\mathscr{E}_2(t)$ as

$$\mathscr{E}_2(t) = \xi_2 J_1(t) + \frac{1}{a_2} J_2(t) + J_3(t). \tag{3.23}$$

By the similar arguments as in Lemma 3.1, we apply the Taylor formula to obtain that $J_2 \geq 0$ and $J_3 \geq 0$, which immediately indicae $\mathscr{E}_2(t) \geq 0$ thanks to the positivity of u.

We proceed to show (3.22). Note that v_2^* and w_2^* satisfy $v_2^* = a_2 w_2^*$ and $v_2^* = \mu(1 - w_2^*)$, which along with the equations of (1.7) and the fact $\frac{a_1}{a_2} < c$ gives

$$\begin{aligned} \frac{d}{dt} J_1(t) &= a_1 \int_\Omega uw - \int_\Omega u^2 - c \int_\Omega uv \\ &\leq a_1 \int_\Omega u(w - w_2^*) - \int_\Omega u^2 - \frac{a_1}{a_2} \int_\Omega u(v - v_2^*). \end{aligned} \tag{3.24}$$

Similarly we have

$$\frac{d}{dt} J_2(t) = a_2 \int_\Omega (v - v_2^*)(w - w_2^*) - b \int_\Omega u(v - v_2^*) - \int_\Omega (v - v_2^*)^2$$

$$-v_2^*\int_\Omega \frac{d_2(w)|\nabla v|^2}{v^2} - v_2^*\int_\Omega \frac{d_2'(w)\nabla v\cdot\nabla w}{v} \tag{3.25}$$

and

$$\frac{d}{dt}J_3(t) = -\int_\Omega u(w-w_2^*) - \int_\Omega (v-v_2^*)(w-w_2^*) - \mu\int_\Omega (w-w_2^*)^2 - w_2^*\int_\Omega \frac{|\nabla w|^2}{w^2}. \tag{3.26}$$

Then the combination of (3.23)–(3.26) leads to

$$\frac{d}{dt}\mathscr{E}_2(t) \le -\int_\Omega X_2 A_2 X_2^T - \int_\Omega Y_2 B_2 Y_2^T, \tag{3.27}$$

where $X_2 = (u, v-v_2^*, w-w_2^*)$, $Y_2 = \left(\frac{\nabla v}{v}, \nabla w\right)$, and matrices A_2, B_2 are defined as follows

$$A_2 := \begin{pmatrix} \xi_2 & \frac{b+\xi_2 a_1}{2a_2} & \frac{1-\xi_2 a_1}{2} \\ \frac{b+\xi_2 a_1}{2a_2} & \frac{1}{a_2} & 0 \\ \frac{1-\xi_2 a_1}{2} & 0 & \mu \end{pmatrix}, \qquad B_2 := \begin{pmatrix} \frac{v_2^* d_2(w)}{a_2} & \frac{v_2^* d_2'(w)}{2a_2} \\ \frac{v_2^* d_2'(w)}{2a_2} & \frac{w_2^*}{w^2} \end{pmatrix}.$$

If $\delta_2 < 0$, we have $\xi_2 = \frac{1}{a_1}$ from the definition of ξ_2, and hence

$$|A_{21}| := \begin{vmatrix} \xi_2 & \frac{b+\xi_2 a_1}{2a_2} \\ \frac{b+\xi_2 a_1}{2a_2} & \frac{1}{a_2} \end{vmatrix} = \frac{-\delta_2}{4a_1 a_2^2} > 0, \qquad |A_2| = -\frac{\mu\delta_2}{4a_1 a_2^2} > 0.$$

On the other hand, if $\delta_2 \ge 0$ we choose $\xi_2 = \frac{2a_2 - a_1 b}{a_1^2}$ and derive that

$$|A_{21}| = \frac{a_2 - a_1 b}{a_1^2 a_2} > 0$$

and

$$|A_2| = \mu|A_{21}| - \frac{(a_1(b+1)-2a_2)^2}{4a_1^2 a_2} = |A_{21}|\left(\mu - \frac{(a_1(b+1)-2a_2)^2}{4(a_2 - a_1 b)}\right) > 0$$

when $\mu > \mu_2^*$. Hence, there exists a constant $\alpha_2 > 0$ such that

$$X_2(x,t)A_2X_2^T(x,t) \ge \alpha_2|X_2|^2, \quad \text{if } \delta_2 < 0 \text{ or } \delta_2 \ge 0 \text{ and } \mu > \mu_2^* \tag{3.28}$$

based on the Sylvester's criterion. Under the condition (3.21), we can use the similar arguments as in Lemma 3.1 to find $T_2 > 0$ such that

$$\frac{w^2|d_2'(w)|^2}{4d_2(w)} < \frac{\mathcal{K}_2}{4} \le 1 \quad \text{for all } t > T_2$$

(" $=$ " holds if $\|w_0\|_{L^\infty} \le 1$), which implies that $\frac{d_2(w)}{w^2} > \frac{d_2'^2(w)}{4}$. Recalling the definition of v_2^* and w_2^*, one has

$$\frac{v_2^* d_2(w)}{a_2} > 0$$

and

$$|B_2| = \frac{v_2^*}{a_2^2}\left(\frac{a_2 w_2^* d_2(w)}{w^2} - \frac{v_2^* d_2'^2(w)}{4}\right) = \frac{\mu^2}{(a_2+\mu)^2}\left(\frac{d_2(w)}{w^2} - \frac{|d_2'(w)|^2}{4}\right) > 0$$

for all $t > T_2$. Therefore, the matrix B_2 is positive definite and then

$$Y_2(x,t) B_2 Y_2^T(x,t) \ge 0,$$

which together with (3.27) and (3.28), gives that

$$\frac{d}{dt}\mathscr{E}_2(t) \le -\alpha_2 \mathcal{F}_2(t), \quad \text{if } \delta_2 < 0 \text{ or } \delta_2 \ge 0 \text{ and } \mu > \mu_2^*$$

for all $t > T_2$. The proof is complete. □

Lemma 3.4 *Let (u, v, w) be the solution of system (1.7) and (v_2^*, w_2^*) be defined by (1.14). Assume that the conditions in Lemma 3.3 hold. Then it follows that*

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t) - v_2^*\|_{L^\infty} + \|w(\cdot,t) - w_2^*\|_{L^\infty} \to 0 \quad \text{as } t \to \infty. \tag{3.29}$$

Proof From Lemma 3.3, we see that the non-negative functional $\mathscr{E}_2(t)$ satisfies $\frac{d}{dt}\mathscr{E}_2(t) \le 0$ and $\frac{d}{dt}\mathscr{E}_2(t) = 0$ if and only if $(u, v, w) = (0, v_2^*, w_2^*)$. Using LaSalle's invariance principle again as in Lemma 3.2, we obtain (3.29). □

Case 3: $\frac{a_1}{a_2} > \max\{\frac{1}{b}, c\}$. This case is essentially the same as the Case 2. By simply swap $d_1(w), a_1, c$ with $d_2(w), a_2, b$ and u with v in the proof of Case 2, we get the following results directly.

Lemma 3.5 *Assume $\frac{a_1}{a_2} > \max\{\frac{1}{b}, c\}$ and $\mathcal{K}_1 \le 4$ (" $=$ " holds if $\|w_0\|_{L^\infty} \le 1$). Then the following convergence holds:*

$$\|u(\cdot,t) - u_3^*\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} + \|w(\cdot,t) - w_3^*\|_{L^\infty} \to 0 \quad \text{as } t \to \infty.$$

3.2 Convergence rate

In this subsection, we shall investigate the convergence rates of solutions. To this end, we first need to improve the regularity of solutions of the system (1.7).

Lemma 3.6 *Let* (u, v, w) *be the solution of system* (1.7) *obtained in Theorem* 1.1*, then there exists a constant* $C > 0$ *independent of* t *such that*

$$\|\nabla u(\cdot, t)\|_{L^4} + \|\nabla v(\cdot, t)\|_{L^4} \le C. \tag{3.30}$$

Proof First, we claim that there exists $\beta \in (0, 1)$ such that for all $t > 1$

$$\|w(\cdot, t)\|_{C^{2+\beta, 1+\frac{\beta}{2}}(\bar{\Omega}\times[t,t+1])} \le C_1, \tag{3.31}$$

where $C_1 > 0$ is a constant independent of t. Indeed since (u, v, w) is the classical solution of system (1.7), we get that

$$0 < u \le C_2 \quad \text{and} \quad \|w\|_{W^{1,\infty}} \le C_2.$$

Let $g_1(x, t, \nabla u) = d_1(w)\nabla u + d_1'(w)u\nabla w$ and $g_2(x, t) = u(a_1 w - u - cv)$. Then we rewrite the first equation of system (1.7) as

$$u_t = \nabla \cdot g_1(x, t, \nabla u) + g_2(x, t)$$

for all $x \in \Omega$ and $t > 0$. By the similar arguments in (Jin and Wang 2020, Lemma 4.1), we derive that

$$\nabla u \cdot g_1(x, t, \nabla u) \ge \frac{d_1(C_2)}{2}|\nabla u|^2 - C_3, \qquad |g_1(x, t, \nabla u)| \le d_1(0)|\nabla u| + C_4$$

and

$$|g_2(x, t)| \le C_5,$$

which, together with Hölder regularity, yields that

$$\|u(\cdot, t)\|_{C^{\beta, \frac{\beta}{2}}(\bar{\Omega}\times[t,t+1])} \le C_6 \tag{3.32}$$

for all $t > 1$. Similarly, we have

$$\|v(\cdot, t)\|_{C^{\beta, \frac{\beta}{2}}(\bar{\Omega}\times[t,t+1])} \le C_7 \tag{3.33}$$

for all $t > 1$. From the third equation of (1.7), the combination of (3.32)–(3.33) with the standard parabolic Schauder theory (cf. Ladyženskaja et al. 1968) yields (3.31).

On the other hand, it follows from the first equation of system (1.7) that

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla u|^4 &= \int_\Omega |\nabla u|^2 \nabla u\cdot\nabla u_t\\ &= \int_\Omega |\nabla u|^2\nabla u\cdot\nabla(\nabla\cdot(d_1(w)\nabla u)) + \int_\Omega |\nabla u|^2\nabla u\cdot\nabla(\nabla\cdot(d_1'(w)u\nabla w))\\ &\quad+\int_\Omega \nabla(a_1uw-u^2-cuv)\cdot\nabla u|\nabla u|^2.\end{aligned}\tag{3.34}$$

The first two terms in the right hand side of (3.34) can be estimated in the same way of Jin and Wang (2020, Lemma 4.2) or Jin et al. (2019, Lemma 3.6) and we have

$$\begin{aligned}&\int_\Omega |\nabla u|^2\nabla u\cdot\nabla(\nabla\cdot(d_1(w)\nabla u)) + \int_\Omega |\nabla u|^2\nabla u\cdot\nabla(\nabla\cdot(d_1'(w)u\nabla w))\\ &\quad\le -\frac{d_1(C_2)}{12}\int_\Omega |\nabla|\nabla u|^2|^2 - \frac{d_1(C_2)}{2}\int_\Omega |\nabla u|^2|D^2u|^2 + C_8\int_\Omega |\nabla u|^4 + C_8\end{aligned}\tag{3.35}$$

thanks to (3.31). We next estimate the last term on the right hand side of (3.34) as follows

$$\begin{aligned}&\int_\Omega \nabla(a_1uw-u^2-cuv)\cdot\nabla u|\nabla u|^2\\ &\quad\le a_1\int_\Omega w|\nabla u|^4 + a_1\int_\Omega u|\nabla u|^2\nabla u\cdot\nabla w - c\int_\Omega u|\nabla u|^2\nabla u\cdot\nabla v\\ &\quad\le C_9\int_\Omega |\nabla u|^4 + \frac{1}{4}\int_\Omega |\nabla v|^4 + C_9.\end{aligned}\tag{3.36}$$

Substituting (3.35) and (3.36) into (3.34) gives that

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla u|^4 + \frac{1}{2}\int_\Omega |\nabla u|^4 + \frac{d_1(C_2)}{12}\int_\Omega |\nabla|\nabla u|^2|^2\\ &\quad\le -\frac{d_1(C_2)}{2}\int_\Omega |\nabla u|^2|D^2u|^2 + C_{10}\int_\Omega |\nabla u|^4 + \frac{1}{4}\int_\Omega |\nabla v|^4 + C_{10}.\end{aligned}\tag{3.37}$$

Using integration by parts and Young's inequality one obtains

$$\begin{aligned}C_{10}\int_\Omega |\nabla u|^4 &= -C_{10}\int_\Omega u\nabla|\nabla u|^2\cdot\nabla u - C_{10}\int_\Omega u\Delta u|\nabla u|^2\\ &\le \frac{d_1(C_2)}{24}\int_\Omega |\nabla|\nabla u|^2|^2 + \frac{d_1(C_2)}{4}\int_\Omega |\nabla u|^2|D^2u|^2 + \frac{C_{10}}{2}\int_\Omega |\nabla u|^4 + C_{11},\end{aligned}$$

that is,

$$C_{10}\int_\Omega |\nabla u|^4 \le \frac{d_1(C_2)}{12}\int_\Omega |\nabla|\nabla u|^2|^2$$

$$+\frac{d_1(C_2)}{2}\int_\Omega |\nabla u|^2|D^2u|^2+2C_{11}. \tag{3.38}$$

From (3.37) and (3.38), we arrive at

$$\frac{d}{dt}\int_\Omega |\nabla u|^4+2\int_\Omega |\nabla u|^4\le \int_\Omega |\nabla v|^4+4(C_{10}+2C_{11}). \tag{3.39}$$

In the same manner, we get

$$\frac{d}{dt}\int_\Omega |\nabla v|^4+2\int_\Omega |\nabla v|^4\le \int_\Omega |\nabla u|^4+C_{12}. \tag{3.40}$$

Combining (3.39) with (3.40) gives that

$$\frac{d}{dt}\int_\Omega \left(|\nabla u|^4+|\nabla v|^4\right)+\int_\Omega \left(|\nabla u|^4+|\nabla v|^4\right)\le 4(C_{10}+2C_{11})+C_{12},$$

which, together with Grönwall inequality, yields (3.30). Therefore, the proof is completed. □

Lemma 3.7 *Let the assumptions in Lemma 3.1 hold. Then there exist two constants $\sigma>0$ and $C>0$ independent of t such that*

$$\|u-u_1^*\|_{L^\infty}+\|v-v_1^*\|_{L^\infty}+\|w-w_1^*\|_{L^\infty}\le Ce^{-\sigma t} \tag{3.41}$$

holds for all $t>T_0$ with some $T_0>0$, where u_1^*, v_1^* and w_1^* are defined in (1.13).

Proof Since $\|u-u_1^*\|_{L^\infty}\to 0$ as $t\to\infty$ (see Lemma 3.2), we apply the L'Hôpital's rule to derive that

$$\lim_{u\to u_1^*}\frac{u-u_1^*-u_1^*\ln\frac{u}{u_1^*}}{(u-u_1^*)^2}=\lim_{u\to u_1^*}\frac{1-\frac{u_1^*}{u}}{2(u-u_1^*)}=\lim_{u\to u_1^*}\frac{1}{2u}=\frac{1}{2u_1^*},$$

which by the continuity yields a constant $t_1>0$ such that

$$\frac{1}{4u_1^*}\int_\Omega (u-u_1^*)^2\le\int_\Omega\left(u-u_1^*-u_1^*\ln\frac{u}{u_1^*}\right)\le\frac{1}{u_1^*}\int_\Omega (u-u_1^*)^2 \tag{3.42}$$

for all $t>t_1$. Similarly, we can find a constant $t_2>0$ such that

$$\frac{1}{4v_1^*}\int_\Omega (v-v_1^*)^2\le\int_\Omega\left(v-v_1^*-v_1^*\ln\frac{v}{v_1^*}\right)\le\frac{1}{v_1^*}\int_\Omega (v-v_1^*)^2 \tag{3.43}$$

and

$$\frac{1}{4w_1^*}\int_\Omega (w-w_1^*)^2\le\int_\Omega\left(w-w_1^*-w_1^*\ln\frac{w}{w_1^*}\right)\le\frac{1}{w_1^*}\int_\Omega (w-w_1^*)^2 \tag{3.44}$$

hold for all $t > t_2$. Let $t_3 = \max\{t_1, t_2\}$. Then, it follows from the definition of $\mathscr{E}_1(t)$ and $\mathcal{F}_1(t)$ that

$$\mathscr{E}_1(t) \leq C_1\mathcal{F}_1(t) \quad \text{for all } t > t_3,$$

which, together with (3.4), implies that there exists a constant $t_4 > 0$ such that

$$\frac{d}{dt}\mathscr{E}_1(t) + \frac{\alpha_1}{C_1}\mathscr{E}_1(t) \leq 0 \quad \text{for all } t > t_4. \tag{3.45}$$

Applying the Grönwall's inequality to (3.45), we get

$$\mathscr{E}_1(t) \leq \mathscr{E}_1(0)e^{-\frac{\alpha_1}{C_1}t} \quad \text{for all } t > t_4. \tag{3.46}$$

On the other hand, one can find a constant $C_2 > 0$ such that $\mathcal{F}_1(t) \leq C_2\mathscr{E}_1(t)$ from the left inequalities of (3.42)–(3.44). Then, it follows from (3.46) that

$$\|u - u_1^*\|_{L^2} + \|v - v_1^*\|_{L^2} + \|w - w_1^*\|_{L^2} \leq C_3e^{-\frac{\alpha_1}{2C_1}t} \quad \text{for all } t > t_4. \tag{3.47}$$

To finish the proof, we need the higher-order estimates of solutions. With (1.9) and (3.30), we use the Gagliardo-Nirenberg inequality to obtain that

$$\|u - u_1^*\|_{L^\infty} \leq C_4(\|\nabla u\|_{L^4}^{\frac{2}{3}}\|u - u_1^*\|_{L^2}^{\frac{1}{3}} + \|u - u_1^*\|_{L^2}) \leq C_5\|u - u_1^*\|_{L^2}^{\frac{1}{3}}. \tag{3.48}$$

Similarly we have

$$\|v - v_1^*\|_{L^\infty} \leq C_6\|v - v_1^*\|_{L^2}^{\frac{1}{3}} \quad \text{and} \quad \|w - w_1^*\|_{L^\infty} \leq C_7\|w - w_1^*\|_{L^2}^{\frac{1}{3}},$$

which, together with (3.47) and (3.48), gives (3.41) by choosing C large enough and $\sigma = \frac{\alpha_1}{6C_1}$ by taking $T_0 = t_4$. □

Lemma 3.8 *Suppose that the conditions in Lemma 3.3 hold. Then there is a* $T_1 > 0$ *such that the solution* (u, v, w) *of system* (1.7) *satisfies*

$$\|u\|_{L^\infty} + \|v - v_2^*\|_{L^\infty} + \|w - w_2^*\|_{L^\infty} \leq \frac{C}{1+t} \tag{3.49}$$

for all $t > T_1$*, where* C *is a positive constant independent of* t.

Proof As in the proof of Lemma 3.7, we find a $t_1 > 0$ such that

$$\frac{1}{4v_2^*}\int_\Omega \left(v - v_2^*\right)^2 \leq \int_\Omega \left(v - v_2^* - v_2^* \ln\frac{v}{v_2^*}\right) \leq \frac{1}{v_2^*}\int_\Omega (v - v_2^*)^2 \tag{3.50}$$

and

$$\frac{1}{4w_2^*}\int_\Omega \left(w - w_2^*\right)^2 \leq \int_\Omega \left(w - w_2^* - w_2^* \ln\frac{w}{w_2^*}\right) \leq \frac{1}{w_2^*}\int_\Omega (w - w_2^*)^2 \tag{3.51}$$

for all $t > t_1$ thanks to Lemma 3.4. Recalling the definition of $\mathscr{E}_2(t)$ and $\mathcal{F}_2(t)$ and using the result of Theorem 1.1, it follows from (3.50) and (3.51) that

$$\begin{aligned}\mathscr{E}_2(t) &\leq C_1\left(\int_\Omega u + \int_\Omega (v-v_2^*)^2 + \int_\Omega (w-w_2^*)^2\right)\\ &\leq C_2\left(\left(\int_\Omega u^2\right)^{\frac{1}{2}} + \left(\int_\Omega (v-v_2^*)^2\right)^{\frac{1}{2}} + \left(\int_\Omega (w-w_2^*)^2\right)^{\frac{1}{2}}\right)\\ &\leq C_3\mathcal{F}_2^{\frac{1}{2}}(t),\end{aligned}$$

which, combining with (3.22), gives that

$$\frac{d}{dt}\mathscr{E}_2(t) + \frac{\alpha_2}{C_3^2}\mathscr{E}_2^2(t) \leq 0. \tag{3.52}$$

Solving this ordinary differential inequality (3.52), we arrive at

$$\mathscr{E}_2(t) \leq C_4(t+1)^{-1} \qquad \text{for all } t \geq t_1.$$

Using the same argument as in the proof of Lemma 3.7, we readily get (3.49) and complete the proof. □

With the same arguments for Lemma 3.8, we get the following conclusion with the omit of proof for simplicity.

Lemma 3.9 *Let (u, v, w) be the solution of system (1.7) and (u_3^*, w_3^*) be defined by (1.14). Suppose that the conditions in Lemma 3.5 hold. Then there exists a T_2 such that*

$$\|u-u_3^*\|_{L^\infty} + \|v\|_{L^\infty} + \|w-w_3^*\|_{L^\infty} \leq \frac{C}{1+t}$$

holds for all $t > T_2$, where $C > 0$ is a constant independent of t.

Proof of Theorem 1.2. Theorem 1.2 is a direct consequence of Lemmas 3.7–3.9.

4 Summary, simulations and discussions

4.1 Heterogenous prey's resources

Diffusive Lotka–Volterra competition systems with given spatially homogenous or heterogeneous resources have been widely studied in the past few decades and many interesting results/phenomena have been found. The most prominent result (cf. Dockery et al. 1998; Lou 2006) is perhaps the phenomenon "slower diffuser always prevails (species with slower dispersion rate will wipe out the one with faster dispersion rate in the competition)" if the resource is spatially heterogeneous without temporal dynamics, which has stimulated much interesting work to investigate its universality (cf. He and Ni 2016b; Hutson et al. 2001; He et al. 2019; Zhang et al. 2017). However whether this distinctive phenomenon exists when the spatially heterogeneous resource is not given but has temporal dynamics remains unknown. Toward this question, in this paper, we consider a diffusive Lotka–Volterra competition system (1.7) where the predators' resource has temporal dynamics and the diffusion rates of competing species depend on the prey density. Interestingly our analytical results show that phenomenon "slower diffuser always prevails" no longer appears and the co-existence steady state will be achieved asymptotically in the case of weak competition regardless of the size of dispersal rates of two competing species (see Corollaries 1.3 and 1.4). The results in the present paper, along with those in Hutson et al. (2001), show that when the resources are temporarily varying no matter whether they are given functions of time as in Hutson et al. (2001) or have the temporal dynamics as in the present paper, the dynamics of the competition system will be quite different from the case where the resources are only spatially varying. This finding may not be that surprising since no heterogeneity is present in the system (1.7), where in particular the prey's resource (i.e. the resource supplied to the prey) is spatially homogeneous. Since the phenomenon "slower diffuser always prevails" arises when the resource is heterogenous, a relevant question keen to be elucidated is what will happen if the prey's resource is heterogeneous. This motivates us to further consider the following system

$$\begin{cases} u_t = \Delta(d_1(w)u) + u(a_1 w - u - cv), & x \in \Omega, \quad t > 0, \\ v_t = \Delta(d_2(w)v) + v(a_2 w - bu - v), & x \in \Omega, \quad t > 0, \\ w_t = \Delta w - w(u+v) + \mu w(m(x) - w), & x \in \Omega, \quad t > 0, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x \in \partial\Omega, \quad t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega, \end{cases} \tag{4.1}$$

where $m(x)$ represents the prey's resource. When $m(x)$ is bounded, the global boundedness of solutions to (4.1) can be established with similar arguments as in Sect. 2 with direct modifications. However, the asymptotical behavior of solutions is very hard to obtain due to the heterogeneity of $m(x)$. Below we shall use numerical simulations to explore the asymptotic dynamics of solutions to (4.1) and examine the effect of heterogeneity of the prey's resource and non-random dispersion (i.e. $d_1(w)$ or $d_2(w)$ is non-constant) on the competition outcomes.

4.2 Numerical simulations

To investigate the effects of non-random dispersion and the heterogeneity of the prey's resource $m(x)$ on the competition outcomes, we set other competition conditions to be the same, and hence w.l.o.g we assume in the following that

$$a_1 = a_2 = b = c = \mu = 1.$$

We shall implement the numerical simulations in an interval $\Omega = [0, 10]$ by the Matlab Pdepe solver based on the finite difference scheme, with the following initial value

$$(u_0, v_0, w_0)(x) = (1 + \cos(\pi x), 1 + \cos(\pi x), 1 + \cos(\pi x)). \tag{4.2}$$

We divide our numerical simulations into three cases: random dispersion (constant diffusion), non-random dispersion (density-dependent diffusion) and mixed dispersal strategies, for both the spatially homogeneous (i.e. constant) and heterogenous (i.e. non-constant) prey's resource. For the spatially homogeneous prey resource, we shall simply set $m(x) = 1$ and for the spatially heterogenous one we set without loss of generality

$$m(x) = 2 + 2\cos(\pi x/2). \tag{4.3}$$

Case 1: random dispersion (constant diffusion). We first numerically explore the asymptotic dynamics of (4.1) with random dispersion only (i.e. both $d_1(w)$ and $d_2(w)$ are constant). Without loss of generality, we choose $d_1(w) = 1$ and $d_2(w) = 5$ so that u is the slower diffuser. When the prey's resource $m(x) = 1$ is spatially homogeneous, the numerical simulations of spatial-temporal patterns of the two competing species and their steady spatial profiles are plotted in Fig. 1, where we see that the spatially homogeneous co-existence steady state is asymptotically achieved. For the spatially heterogenous prey's resource given in (4.3), the corresponding numerical simulation results are plotted in Fig. 2 where we do observe that the faster diffuser v is asymptotically wiped out and the slower diffuser u wins—namely the slower diffuser prevails. The numerical simulations shown in Figs. 1 and 2 demonstrate that when both competing species employ the random dispersal strategies, the effect of heterogeneity of

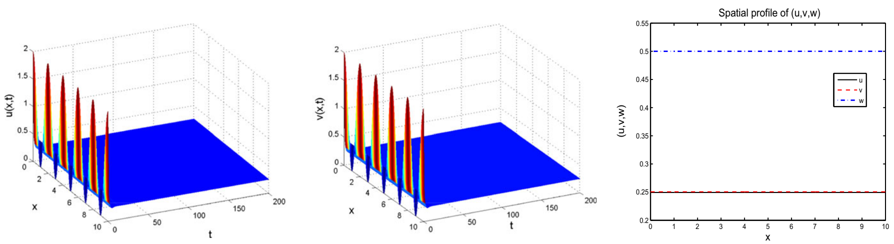


Fig. 1 Numerical simulations of the spatially homogeneous (constant) co-existence state to the system (4.1) with random dispersion (constant diffusion) and spatially homogenous prey's resource $m(x) = 1$, where $d_1(w) = 1, d_2(w) = 5$

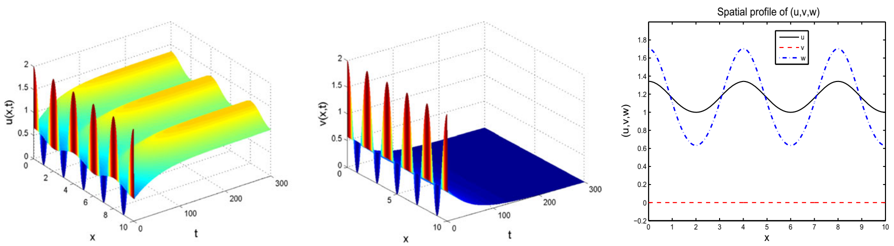


Fig. 2 Numerical simulations of the phenomenon "slower diffuser prevails" to the system (4.1) with random dispersion (constant diffusion) and spatially heterogeneous prey's resource $m(x) = 2 + 2\cos(\frac{\pi}{2}x)$, where $d_1(w) = 1, d_2(w) = 5$

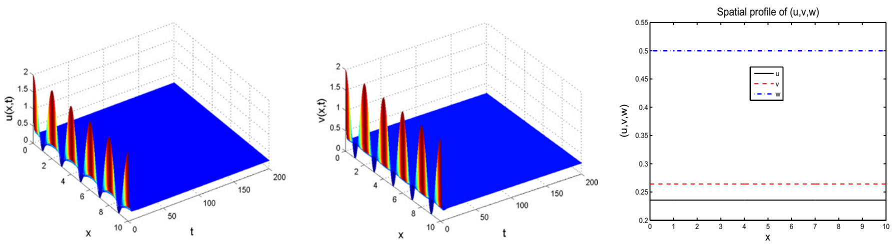


Fig. 3 Numerical simulations of the spatially homogeneous co-existence state to the system (4.1) with non-random dispersion and spatially homogeneous prey's resource $m(x) = 1$, where $d_1(w)$ and $d_2(w)$ are given in (4.4)

the prey's resource $m(x)$ on the competition outcomes is similar to the two-component competition system (1.4).

Case 2: non-random dispersion (density-dependent diffusion). We turn to numerically explore the asymptotic dynamics of (4.1) with non-random dispersion (i.e. both $d_1(w)$ and $d_2(w)$ are non-constant) to investigate the impact of the heterogeneity of the prey's resource $m(x)$ on the competition outcomes. To this end, we set

$$d_1(w) = e^{-10w},\ d_2(w) = e^{-5w} \tag{4.4}$$

so that $d_1(w) < d_2(w)$, that is u is the slower diffuser in the competition. We first look at the spatially homogeneous prey's resource $m(x) = 1$ for which the numerical simulations of spatial-temporal patterns of the two competing species and their steady spatial profiles are plotted in Fig. 3. Clearly we see that spatially homogeneous co-existence steady state is asymptotically achieved and the slower diffuser is slightly disadvantaged indeed in contrast to the numerical results for the random dispersion shown in Fig. 1. For the spatially heterogeneous prey's resource $m(x)$ given in (4.3), the corresponding numerical results are shown in Fig. 4 where we unexpectedly find that the two competing species u and v reach a spatially heterogeneous co-existence steady state, that is "slower diffuser always prevails" phenomenon no longer arises. Apart from this, we find a (weak) segregation phenomenon between the two competing species u and v in Fig. 4. These numerical observations indicate that the non-random dispersion will lead to competition outcomes different from the random dispersion .

Case 3: mixed random and non-random dispersions. From the numerical results shown in the above two cases, one finds that the choice of dispersal strategies is very important to the competition outcomes. Now we explore what kind of competition outcomes will be achieved if two species employ mixed dispersal strategies (i.e. one species uses the random dispersion and the other uses the non-random dispersion). To this end, we set

$$d_1(w) = e^{-5w},\ d_2(w) \text{ is constant},$$

that is the dispersion of the species u is non-random (density-dependent) while the dispersion of v is random. The initial condition is given by (4.2). For the spatially homogeneous prey's resource $m(x)$, the resulting numerical results are plotted in Fig.

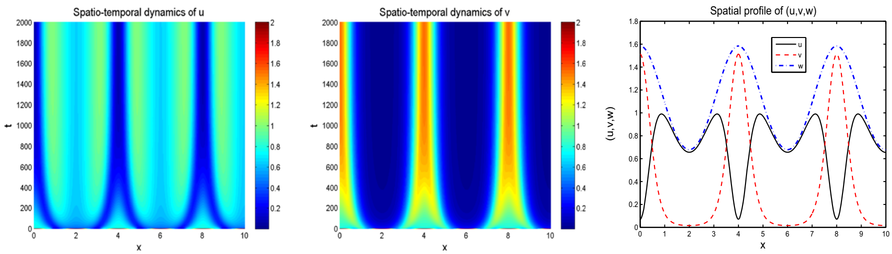


Fig. 4 Numerical simulations of the spatially heterogeneous (non-constant) co-existence state to the system (4.1) with non-random dispersion and spatially heterogeneous $m(x) = 2 + 2\cos(\frac{\pi}{2}x)$, where $d_i(w)(i = 1, 2)$ are given in (4.4)

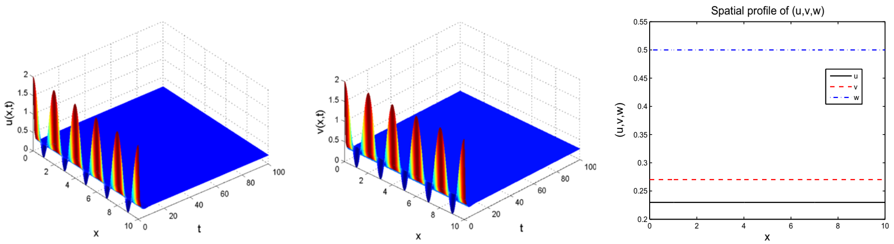


Fig. 5 Numerical simulations of the spatially homogeneous co-existence state to the system (4.1) with mixed random and non-random dispersions and the spatially homogeneous prey's resource $m(x) = 1$, where $d_1(w) = e^{-5w}$, $d_2(w) = 5$

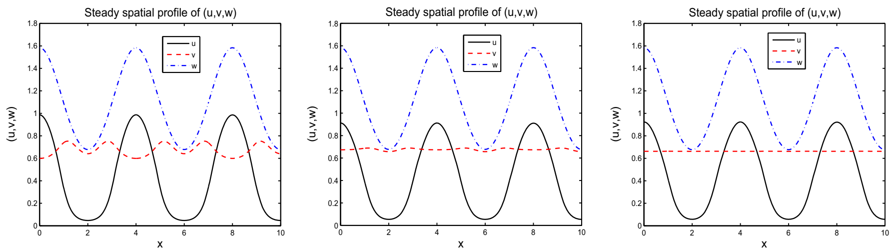
5 which illustrates that the homogeneous co-existence steady state is asymptotically achieved. When the prey's resource $m(x)$ is spatially heterogeneous, the steady spatial profiles of numerical solutions for different constant dispersal rates $d_2(w)$ are shown in Fig. 6 from which we find that the co-existence steady state is achieved, where the profile for the species u with non-random dispersion is heterogeneous while the one for the species v with random dispersion may be heterogeneous (resp. homogeneous) if its dispersal rate is small (resp. large). In particular, when $d_2(w) = 5$, the diffusion rate of u is less than the one of v, that is u is the slower diffuser with non-random dispersion . In this case, the simulation in Fig. 6c shows that the slower diffuser u does not wipe out the faster one v, instead they co-exist. This implies again that the non-random dispersion is a factor rendering the failure of the phenomenon "slower diffuser always prevails" in the competition system.

4.3 Biological implications

Based on the above numerical results obtained for the system (4.1), we may speculate the following biological implications.

First, regarding the prominent phenomenon "slower diffuser always prevails", the following facts are observed.

(a) When the two competing species employ the random dispersal strategies (i.e. both $d_1(w)$ and $d_2(w)$ are constant), the prominent phenomenon "slower diffuser



(a) $d_2(w) = 0.01$ **(b)** $d_2(w) = 0.1$ **(c)** $d_2(w) = 5$

Fig. 6 Numerical simulations of spatially heterogeneous co-existence state to the system (4.1) with mixed random and non-random dispersions and the spatially heterogeneous prey's resource $m(x) = 2+2\cos(\frac{\pi}{2}x)$, where $d_1(w) = e^{-5w}$ and $d_2(w)$ is a constant as shown

always prevails" holds true when the prey's resource $m(x)$ is spatially heterogeneous (see Fig. 2) while the co-existence will be achieved otherwise (see Fig. 1);

(b) Once there is a non-random dispersion employed amongst competing species (i.e. at least one of diffusion rates $d_i(w)$, $i = 1, 2$, is non-constant), the phenomenon "slower diffuser always prevails" will fail and instead coexistence will be achieved regardless of spatial homogeneity or heterogeneity of the prey's resource $m(x)$ (see Figs. 3, 4, 5, 6).

Second, the effects of dispersal strategy and resource heterogeneity on the slower diffusers are the following:

(c) Whether the prey's resource is spatially homogenous or heterogenous, non-random dispersion seems to be disadvantageous for the slower diffuser in terms of the total population supported in the space (see Figs. 1, 3 and 5 for the spatially homogeneous prey source, and see Figs. 2, 4 and 6 for the spatially heterogeneous prey source, where we find that the slower diffuser with random dispersion has a larger total population supported than the one with non-random dispersion);

(d) Given that two competing species employ the same dispersal strategies (either random or non-random dispersion), the resource heterogeneity is advantageous to the slower diffuser (compare Fig. 1 vs. Fig. 2 and Fig.3 vs. Fig. 4); but this seems not the case for mixed dispersal strategies (see Figs. 5 and 6).

The observation (a) agrees well with the prominent phenomenon "slower diffuser always prevails" when the competing species employ the random dispersion strategies. However the observation (b) suggests this phenomenon may fail with the non-random dispersion in the competition system. The observations (c) and (d) give possible effects of the non-random dispersion and heterogeneity of resources on the slower diffuser. All these numerical observations on the system (4.1) have not been justified analytically in this paper in particular for the heterogenous prey's resource $m(x)$, and hence raise interesting questions to pursue in the future. Finally we remark in the above simulations, we do not take into account the effect of the total amount of the resource $m(x)$ on the global dynamics of the system (4.1). It appears from the simulations (not shown here) that the size of total amount of the resource $m(x)$ will have an impact on the asymptotic profiles of (u, v, w), but this has gone beyond the interest of this paper and hence we do not discuss it here.

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