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Asymptotic dynamics on a singular chemotaxis system modeling onset of tumor angiogenesis

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Abstract

The asymptotic behavior of solutions to a singular chemotaxis system modeling the onset of tumor angiogenesis in two and three dimensional whole spaces is investigated in the paper. By a Cole–Hopf type transformation, the singular chemotaxis is converted into a non-singular hyperbolic system. Then we study the transformed system and establish the global existence, asymptotic decay rates and diffusion convergence rate of solutions by the method of energy estimates. The main novelty of our results is the finding of a hidden interactive dissipation structure in the system by which the energy dissipation is established.
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1. Introduction

It is widely recognized that tumor angiogenesis plays a central role in spreading cancer cells to other tissues in cancer metastasis, and hence making cancer a potentially life-threatening disease. Therefore it is of great importance and interest to understand the underlying mechanism of

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tumor angiogenesis which starts with cancerous tumor cells releasing signaling molecules vascular endothelial growth factor (VEGF) to surrounding normal host tissue and activate the motion of vascular endothelial cells. To capture the main interaction between VEGF and vascular endothelial cells, the following PDE model was proposed in [12]

$$\begin{cases} u_t = \nabla \cdot \big(D\nabla u - \chi u \nabla \ln c\big), \\ c_t = \varepsilon \Delta c - \mu u c \end{cases} \tag{1.1}$$

where $u(x,t)$ and $c(x,t)$ denote the density of vascular endothelial cells and concentration of VEGF, respectively. The parameter $D > 0$ is the diffusivity of endothelial cells, $\chi > 0$ is referred to as the chemotactic coefficient measuring the intensity of chemotaxis and μ denotes the degradation rate of the chemical (VEGF) c. The parameter $\varepsilon \geq 0$ denotes the chemical diffusion rate and could be small or negligible since the chemical diffusion is far less important than its interaction with endothelial cells as treated in [12]. For more information on the cancer modeling, we refer to a review paper [4] and the references therein. Except the afore-mentioned applications, the model (1.1) was also previously considered in [22] to examine the boundary movement of bacterial population chemotaxis, and a specialized case investigated in [21,27] for traveling wave solutions.

The striking feature of model (1.1) is that the first equation contains a logarithmic sensitivity function $\ln c$ which is singular at $c = 0$. This singular logarithmic sensitivity was first used by Keller and Segel in their seminal paper [10] to describe the propagation of traveling wave band formed by bacterial chemotaxis observed in the experiment of Adler [1]. Its mathematical derivation was later given in [23] and biological basis was provided in [9] by both experimental measurements and model simulations. Therefore the logarithmic sensitivity is meaningful both mathematically and biologically though it causes great difficulties in its mathematical analysis and numerical computations. Among other things, the foremost mathematical question is therefore how to resolve the singularity. Toward this end, a Cole–Hopf type transformation as follows was used in [11,31]

$$\mathbf{v} = -\nabla \ln c = -\frac{\nabla c}{c} \tag{1.2}$$

which, together with scalings $\tilde{t} = \frac{\chi\mu}{D} t$, $\tilde{x} = \frac{\sqrt{\chi\mu}}{D} x$, $\tilde{\mathbf{v}} = \sqrt{\frac{\chi}{\mu}}\mathbf{v}$, transforms the system (1.1) into a hyperbolic system:

$$\begin{cases} u_t - \Delta u = \nabla \cdot (u\mathbf{v}), & x \in \Omega,\ t > 0, \\ \mathbf{v}_t - \varepsilon \Delta \mathbf{v} = \nabla(-\varepsilon |\mathbf{v}|^2 + u), & x \in \Omega,\ t > 0, \\ (u, \mathbf{v})(x, 0) = (u_0, \mathbf{v}_0)(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where tildes have been dropped for convenience and Ω is either the whole space or a bounded domain with smooth boundary. Compared to the original model (1.1), the transformed system (1.3) is much more manipulable mathematically since the singularity vanishes. There was an amount of interesting works carried out for the transformed system (1.3) and hence for the original model (1.1) by reverting the Cole–Hopf transformation (1.2). We briefly review these results below by the nature of domain. First in the one dimensional bounded domain $\Omega \subset \mathbb{R}$, the global existence of solutions of (1.3) with $\varepsilon = 0$ subject to Neumann–Dirichlet boundary condition was first established in [32] for small data, and later in [29] for large data with any $\varepsilon \geq 0$. Recently the initial–boundary value problem (1.3) with Dirichlet boundary condition in one dimension was extensively studied in [20]. In the multidimensional bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, the global existence and exponential decay rates of solutions under Neumann boundary conditions were obtained in [15] for $\varepsilon = 0$ for small initial data. In the one dimensional whole space $\Omega = \mathbb{R}$, the traveling wave solution of (1.3) was explicitly solved in [31] and its nonlinear stability with large wave amplitude was established for $\varepsilon > 0$ in [8,17,18] and for $\varepsilon = 0$ in [19,16] by the first author with his collaborators. The stability of composite waves was proved in [14] with $\varepsilon = 0$. Furthermore the one-dimensional Cauchy problem of (1.3) with $\varepsilon = 0$ was established in [5] for large data under the condition that v_0 has a positive lower bound. For the multidimensional unbounded domain $\Omega = \mathbb{R}^d$ $(d \geq 2)$, when the initial data is close to the constant ground state $(\bar{u}, 0)$ with $\bar{u} > 0$, there are a few studies on the system (1.3). First in [13], Li, Li and Zhao obtained the global well-posedness, regularity criterion and large time behavior of classical solutions of the Cauchy problem (1.3) with $\varepsilon = 0$ if $(u_0, \mathbf{v}_0) \in H^s(\mathbb{R}^d)$ for $s > \frac{d}{2} + 1$ and $\|(u_0 - \bar{u}, \mathbf{v}_0)\|_{H^s}$ is small. Later Hao [7] established the global existence of mild solutions in the critical Besov space $\dot{B}^{-\frac{1}{2}}_{2,1} \times (\dot{B}^{-\frac{1}{2}}_{2,1})^d$ with minimal regularity in the Chemin–Lerner space framework. The global well-posedness of strong solutions of (1.3) in $\mathbb{R}^3$ was recently established in [2] via the Fourier analysis if $\|(u_0 - \bar{u}, \mathbf{v}_0)\|_{L^2 \times H^1}$ is small. If the initial data has a higher regularity such that $\|(u_0 - \bar{u}, \mathbf{v}_0)\|_{H^2 \times H^1}$ is small, the algebraic decay of solutions in $\mathbb{R}^3$ was further derived in [2].

The afore-mentioned results on the whole space $\mathbb{R}^d$ are obtained only for the case $\varepsilon = 0$. A similar problem (i.e., replacing $\nabla(-\varepsilon|\mathbf{v}|^2)$ by $\nabla(\varepsilon|\mathbf{v}|^2)$ in (1.3)) modeling repulsive chemotaxis was studied in [24], where the global existence of solutions in $\mathbb{R}^3$ was established if $(u_0 - \bar{u}, \mathbf{v}_0) \in H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$ and $\|(u_0 - \bar{u}, \mathbf{v}_0)\|_{L^2(\mathbb{R}^3)}$ is small. As far as we known, the result for the model (1.3) with $\varepsilon > 0$ in multi-dimensions remains entirely open. The purpose of this paper is to establish the asymptotic behavior (global existence and time decay rates) of solutions of (1.3) for any $\varepsilon \geq 0$ in $\mathbb{R}^d$ for $d = 2, 3$ and ε-convergence of solutions by the method of energy estimates. Precisely we first establish the global existence of solutions of (1.3) with initial data near a constant ground state $(\bar{u}, 0)$ with $\bar{u} > 0$ and furthermore derive the explicit time decay rates of solutions. Then we study the solution behavior as $\varepsilon \to 0$. Finally we transfer the results back to the angiogenesis chemotaxis model (1.1). We should stress that the mathematical study of system (1.3) with $\varepsilon > 0$ is not a simple extension of the case $\varepsilon = 0$. Indeed it is much harder and involved since the parameter ε in the transformed system (1.3) plays a dual role: coefficient of diffusion and convection. The former is a smoothing factor and the later is opposite in general. For example, in the case $\varepsilon = 0$, the system (1.3) has a Lyapunov functional $F(p, \mathbf{v}) = \int_{\mathbb{R}^d} u \ln u + \frac{|\mathbf{v}|^2}{2} dx$, which is invalid for $\varepsilon > 0$ due to the nonlinear convection term $-\varepsilon|\mathbf{v}|^2$. Moreover in the study of stability of traveling waves of (1.3) for $\varepsilon > 0$, it was found if $\varepsilon > 0$ is large, the diffusion cannot compensate the convection effect and hence the stability was established only for $\varepsilon > 0$ small. Hence in our analysis, on one hand we need to perform delicate coupling estimates to balance the dissipation and convection for $\varepsilon > 0$. On the other hand we cannot use the dissipation provided by $\varepsilon \Delta \mathbf{v}$ since otherwise our results are invalid for $\varepsilon = 0$. Thus we need to develop new dissipation mechanisms hidden in the system (1.3), which is the key in our energy estimates (see Section 2.3).

The theorem on global existence of solutions is as follows.

Theorem 1.1 *(Global well-posedness). Let* $(u_0 - \bar{u}, \mathbf{v}_0) \in H^k(\mathbb{R}^d) \times H^k(\mathbb{R}^d)$ $(d = 2, 3)$ *with some integer* $k \geq 2$ *for some constant background state* $\bar{u} > 0$*. Then for any constant* $M_0 > 0$

with $\|\nabla^2 u_0\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^2 \mathbf{v}_0\|^2_{L^2(\mathbb{R}^d)} \le M_0^2$, *there exists a positive constant* η *depending on* M_0 *such that if*

$$\|u_0 - \bar{u}\|^2_{H^1(\mathbb{R}^d)} + \|\mathbf{v}_0\|^2_{H^1(\mathbb{R}^d)} \le \eta^2,$$

the system (1.3) *with* $\varepsilon \ge 0$ *admits a unique global solution* $(u, \mathbf{v}) \in C\big([0, +\infty), H^k(\mathbb{R}^d)\big)$ *satisfying:*

$$\begin{aligned} &\|u(t) - \bar{u}\|^2_{H^k(\mathbb{R}^d)} + \|\mathbf{v}(t)\|^2_{H^k(\mathbb{R}^d)} \\ &\qquad + \int_0^t \Big(\|\nabla u(\tau)\|^2_{H^k(\mathbb{R}^d)} + \|\nabla \mathbf{v}(\tau)\|^2_{H^{k-1}(\mathbb{R}^d)} + \varepsilon \|\nabla^{k+1}\mathbf{v}(\tau)\|^2_{L^2(\mathbb{R}^d)}\Big) d\tau \\ &\le C\big(\|u_0 - \bar{u}\|^2_{H^k(\mathbb{R}^d)} + \|\mathbf{v}_0\|^2_{H^k(\mathbb{R}^d)}\big) \end{aligned} \tag{1.4}$$

for all $t > 0$, *where* C *is a positive constant independent of* η *and* t.

Remark 1.1. In Theorem 1.1, the minimal regularity of initial data for the existence of global solutions is required to be in the class of $H^2(\mathbb{R}^d)$, which improves the results of [13,24] where the initial data is in $H^3(\mathbb{R}^d)$ and $\varepsilon = 0$.

Our second result concerns the asymptotic decay rates of solutions. As mentioned before, for system (1.3) with $\varepsilon = 0$, the algebraic decay of solutions in $\mathbb{R}^3$ was derived in [2] via the Fourier analysis under the assumption that the initial perturbation is small. Here, we shall further investigate the decay rates of solutions for the system (1.3) with any $\varepsilon \ge 0$ in both $\mathbb{R}^3$ and $\mathbb{R}^2$ by using the energy analysis. To this end, we introduce the homogeneous negative index Sobolev space $\dot{H}^{-s}(\mathbb{R}^d)$:

$$\dot{H}^{-s}(\mathbb{R}^d) := \big\{ f \in L^2(\mathbb{R}^d) \,:\, \big\| |\xi|^{-s} \hat{f}(\xi) \big\|_{L^2(\mathbb{R}^d)} < \infty \big\}$$

endowed with the norm $\|f\|_{\dot{H}^{-s}(\mathbb{R}^d)} := \big\| |\xi|^{-s} \hat{f}(\xi) \big\|_{L^2(\mathbb{R}^d)}$. Notice that the classical Littlewood–Payley decomposition implies that $f \in \dot{H}^{-s}(\mathbb{R}^d)$ for any $s \in (0, \frac{d}{2})$ if $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Thanks for the mass conservation, we see that $\dot{H}^{-s}(\mathbb{R}^d)$ is a natural function space for system (1.3). Moreover, compared to the usual decay rate derived in H^k space (e.g. see [2]), our result demonstrates that the $\dot{H}^{-s}$ norm of initial data enhances the decay rate of the solution by $\frac{s}{2}$. Precisely, we have the following decay rate estimates.

Theorem 1.2 *(Decay rates). Let the assumptions in Theorem 1.1 hold. If we further assume that* $(u_0 - \bar{u}, \mathbf{v}_0) \in \dot{H}^{-s}(\mathbb{R}^d) \times \dot{H}^{-s}(\mathbb{R}^d)$ *for some* $s \in (0, \frac{d}{2})$, *then for any* $t \ge 0$, *the solution* $(u, \mathbf{v})$ *of* (1.3) *obtained in Theorem 1.1 with suitably small* η *has the following decay rates:*

$$\begin{aligned} &\|\nabla^\ell (u - \bar{u})\|_{L^2(\mathbb{R}^d)} + \|\nabla^\ell \mathbf{v}\|_{L^2(\mathbb{R}^d)} \le C\big(1+t\big)^{-\frac{s+\ell}{2}}, \qquad (\ell = 0, 1, \cdots, k-1) \\ &\|\nabla^k (u - \bar{u})\|_{L^2(\mathbb{R}^d)} + \|\nabla^k \mathbf{v}\|_{L^2(\mathbb{R}^d)} \le C\big(1+t\big)^{-\frac{s+k-1}{2}}, \end{aligned} \tag{1.5}$$

where C *is a constant independent of* t.

Remark 1.2. We note that the decay rate of solutions obtained in Theorem 1.2 is optimal in the sense that it attains the decay rate of solutions to the linearized system (see Section 2.2). Indeed, the decay rate of solutions in Theorem 1.2 has the same decay rates as the solutions of heat equations (e.g., see [6, Theorem 1.1]).

Remark 1.3. Taking $k = 2$ in Theorem 1.2, we can use Sobolev embedding and the interpolation inequality to deduce that $\|u - \bar{u}\|_{L^\infty} \to 0$ and $\|\mathbf{v}\|_{L^\infty} \to 0$ as $t \to +\infty$, which implies that $(u, \mathbf{v})$ will converges to the constant ground state $(\bar{u}, 0)$ as $t \to +\infty$.

Next we present the convergence of solutions from (1.3) with $\varepsilon > 0$ to (2.2) with $\varepsilon = 0$ for any given positive time. We remark that in Theorem 1.1 and Theorem 1.2, the constant C can be independent of ε if we restrict $\varepsilon \in [0, K]$ for any given $K > 0$. This result allows us to prove the following convergence rate estimates with respect to ε.

Theorem 1.3 *(Diffusion convergence rate). Let $(u^\varepsilon, \mathbf{v}^\varepsilon)$ denote the solution of system (1.3) for $\varepsilon \geq 0$ given by Theorem 1.1. Then it holds that*

$$\|u^\varepsilon(t) - u^0(t)\|^2_{H^{k-2}(\mathbb{R}^d)} + \|\mathbf{v}^\varepsilon(t) - \mathbf{v}^0(t)\|^2_{H^{k-2}(\mathbb{R}^d)} \leq \varepsilon^2 e^{Ct} \qquad \text{for any } \ t \in [0, \infty),$$

where C is a positive constant independent of ε and t.

Remark 1.4. The diffusion convergence rate independent of time t as $\varepsilon \to 0$ was derived in [24] where the convergence rate is $\mathcal{O}(\sqrt{\varepsilon})$. Here we improve the convergence rate to $\mathcal{O}(\varepsilon)$ but with a price that the convergence rate depends on time t. Currently we are unable to remove the time-dependence condition for such scenario.

Finally, we transfer the results back to the original system (1.1) via the Cole–Hopf transformation (1.2) and obtain the following results for the original system (1.1).

Theorem 1.4. *Let $(u_0, \nabla \ln c_0) \in H^k(\mathbb{R}^d) \times H^k(\mathbb{R}^d)$ for some integer $k \geq 2$, where $(u_0, c_0)(x) = (u, c)(x, 0)$ satisfying the compatibility condition $\mathbf{v}_0 = -\nabla \ln c_0$. Then for any constant $M_0 > 0$ with $\|\nabla^2 u_0\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^3 \ln c_0\|^2_{L^2(\mathbb{R}^d)} \leq M_0$, there exists a positive constant η depending on M_0 such that if*

$$\|u_0 - \bar{u}\|^2_{H^1(\mathbb{R}^d)} + \|\nabla \ln c_0\|^2_{H^1(\mathbb{R}^d)} \leq \eta^2,$$

for some constant $\bar{u} > 0$, then system (1.1) admits a unique global classical solution.

Furthermore, if $(u_0, \nabla \ln c_0) \in \dot{H}^{-s}(\mathbb{R}^d) \times \dot{H}^{-s}(\mathbb{R}^d)$ for $s \in (0, \frac{d}{2})$, then there exists a positive constant C independent of t such that the solution has the following decay rates in time:

$$\|u - \bar{u}\|_{L^\infty(\mathbb{R}^d)} \leq C(1+t)^{-\frac{1+s}{2}},$$
$$\|c\|_{L^\infty(\mathbb{R}^d)} \leq Ce^{-\bar{u}t}.$$

Moreover, let $(u^\varepsilon, c^\varepsilon)$ denote the unique solution of (1.1) with $\varepsilon \geq 0$. Then for any fixed time $t > 0$, the following convergence rate with respect to ε holds:

$$\|u^\varepsilon(t) - u^0(t)\|^2_{H^{k-2}(\mathbb{R}^d)} + \|c^\varepsilon(t) - c^0(t)\|^2_{H^{k-2}(\mathbb{R}^d)} \le C(t)\varepsilon^2,$$

where $C(t)$ is an increasing function of t.

The rest of this paper is organized as follows. In Section 2, we study the global well-posedness of solutions to the nonlinear system (2.2), and give the proof of Theorem 1.1. In Section 3, we are devoted to deriving the decay estimates and proving Theorem 1.2. In Section 3, we investigate the convergence rate of solutions with respect to ε and prove Theorem 1.3. Finally, we convert the results of the transformed system back to the original system (1.1) and prove Theorem 1.4.

Notations: Throughout this paper, ∇^ℓ with an integer $\ell \ge 0$ stands for the usual spatial derivatives of order ℓ. The letters c and C denote generic positive constants which may vary in the context.

2. Global well-posedness

In this section, we investigate the global well-posedness of solutions to the nonlinear system (2.2). We will use the following two basic facts:

$$\Delta \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) \qquad \text{and} \qquad \|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^d)} \simeq \|\nabla^\ell \nabla \cdot \mathbf{v}\|_{L^2(\mathbb{R}^d)} \tag{2.1}$$

for any nonnegative integer ℓ, where "$\simeq$" means the two norms are equivalent. The former follows from the fact that $\mathbf{v}$ is a gradient field $\mathbf{v} = -\nabla \ln c$ and hence $\nabla \times \mathbf{v} = 0$, while the latter can be obtained by the L^2 boundedness of Riesz transform.

2.1. Reformulation of the problem

For simplicity, we shall take $\bar{u} = 1$ in the sequel. Then by setting $p = u - 1$ and $p_0 = u_0 - 1$, we can rewrite (1.3) as

$$\begin{cases} \partial_t p - \Delta p - \nabla \cdot \mathbf{v} = \nabla \cdot (p\mathbf{v}), & x \in \mathbb{R}^d,\ t > 0, \\ \partial_t \mathbf{v} - \varepsilon \Delta \mathbf{v} - \nabla p = -\varepsilon \nabla |\mathbf{v}|^2, & x \in \mathbb{R}^d,\ t > 0, \\ (p, \mathbf{v})(x, 0) = (p_0, \mathbf{v}_0)(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.2}$$

where $d = 2, 3$. Then we turn to consider the problem (2.2)

We first give the local well-posedness of the Cauchy problem (2.2).

Lemma 2.1 *(Local well-posedness). Assume that $(p_0, \mathbf{v}_0)$ satisfies $(p_0, \mathbf{v}_0) \in H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$ for some $s > \frac{d}{2}$. Then there exist a time $T = T\big(\|p_0\|_{H^s(\mathbb{R}^d)}, \|\mathbf{v}_0\|_{H^s(\mathbb{R}^d)}\big) > 0$ such that the system (2.2) has a unique solution $(p, \mathbf{v}) \in C\big([0, T), H^s(\mathbb{R}^d)\big)$.*

Proof. By a standard energy argument, it is easy to prove the conclusion for $s > \frac{d}{2} + 1$ (see [13]). If $s > \frac{d}{2}$, the results can be proved similarly with the help of the higher order commutator estimates (see [3]). We omit the details here for brevity. □

2.2. Linear energy dissipation

The idea of showing the global well-posedness of solutions to the nonlinear system (2.2) is partially motivated by Guo and Wang [6,30] on the compressible Navier–Stokes equations, and Ren, Wu, Zhang and the second author [25,26] on the incompressible MHD equations without magnetic diffusion. To illustrate the main idea of our proof, we visit the linear part of nonlinear system (2.2):

$$\begin{cases} \partial_t p - \Delta p - \nabla\cdot\mathbf{v} = 0, \\ \partial_t \mathbf{v} - \varepsilon\Delta\mathbf{v} - \nabla p = 0. \end{cases} \tag{2.3}$$

For any nonnegative integer ℓ, the standard ℓ-th level energy identity of (2.3) reads as

$$\frac{1}{2}\frac{d}{dt}\Big(\|\nabla^\ell p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^\ell\mathbf{v}\|^2_{L^2(\mathbb{R}^d)}\Big) + \|\nabla^{\ell+1}p\|^2_{L^2(\mathbb{R}^d)} + \varepsilon\|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} = 0. \tag{2.4}$$

In (2.4), there is a dissipation term $\varepsilon\|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)}$ for $\mathbf{v}$. However this dissipation term cannot be employed since we anticipate our results hold for $\varepsilon = 0$. Therefore we have to pursue other ways to find the dissipation for $\mathbf{v}$. The idea here is to construct an interactive energy functional between p and $\mathbf{v}$ by using the dissipative structure of $\mathbf{v}$ in the first equation of (2.3). To this end, we apply ∇^ℓ to equations $(2.3)_1$ and $(2.3)_2$, take the inner product with $-\nabla^\ell\nabla\cdot\mathbf{v}$ and $\nabla^{\ell+1}p$, respectively. Then integrating the results and adding up, we have

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^d} \nabla^{\ell+1}p\cdot\nabla^\ell\mathbf{v}dx + \|\nabla^\ell\nabla\cdot\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} \\ &\quad + (1+\varepsilon)\int_{\mathbb{R}^d}\nabla^{\ell+2}p\cdot\nabla^{\ell+1}\mathbf{v}dx - \|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^d)} = 0. \end{aligned} \tag{2.5}$$

However the integral terms in (2.5) cannot be estimated using (2.4) and (2.5). To overcome this difficulty, we investigate the $(\ell+1)$-th level dissipation for p:

$$\frac{1}{2}\frac{d}{dt}\Big(\|\nabla^{\ell+1} p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)}\Big) + \|\nabla^{\ell+2}p\|^2_{L^2(\mathbb{R}^d)} + \varepsilon\|\nabla^{\ell+2}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} = 0. \tag{2.6}$$

Now collecting (2.4), (2.5) and (2.6), we obtain for any constant $\delta > 0$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\nabla^\ell p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+1}p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^\ell\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} + 2\delta\int_{\mathbb{R}^d}\nabla^{\ell+1}p\cdot\nabla^\ell\mathbf{v}dx\Big) \\ &+ \Big(\|\nabla^{\ell+1}p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+2}p\|^2_{L^2(\mathbb{R}^d)} + \delta\|\nabla^\ell\nabla\cdot\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} + \varepsilon\|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} \\ &+ \varepsilon\|\nabla^{\ell+2}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} + (1+\varepsilon)\delta\int_{\mathbb{R}^d}\nabla^{\ell+2}p\nabla^\ell\nabla\cdot\mathbf{v}dx - \delta\|\nabla^{\ell+1}p\|^2_{L^2(\mathbb{R}^d)}\Big) = 0. \end{aligned}$$

By taking δ suitably small and using (2.1), we can control the cross term and thus establish the ε-independent dissipation for both p and $\mathbf{v}$ (e.g. see Lemma 2.3 below).

2.3. Nonlinear energy dissipation structure

Motivated by the above analysis on the linear system (2.3), we introduce the following energy for the nonlinear system (2.2) by using the ℓth and $(\ell+1)$th level dissipations:

$$
\begin{aligned}
\mathcal{E}_\ell(t) := {}& \|\nabla^\ell p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+1} p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^\ell \mathbf{v}\|^2_{L^2(\mathbb{R}^d)} \\
& + \|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} + 2\delta \int_{\mathbb{R}^d} \nabla^{\ell+1} p \cdot \nabla^\ell \mathbf{v} dx, \\
\mathcal{F}_\ell(t) := {}& \|\nabla^{\ell+1} p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+2} p\|^2_{L^2(\mathbb{R}^d)} + \delta\|\nabla^\ell \nabla\cdot \mathbf{v}\|^2_{L^2(\mathbb{R}^d)} + \varepsilon\|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} \\
& + \varepsilon\|\nabla^{\ell+2}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} + \delta(1+\varepsilon)\int_{\mathbb{R}^d} \nabla^{\ell+2} p \nabla^\ell \nabla\cdot\mathbf{v} dx - \delta\|\nabla^{\ell+1} p\|^2_{L^2(\mathbb{R}^d)}
\end{aligned}
\tag{2.7}
$$

for any nonnegative integer ℓ, where $0 < \delta < 1$ is a constant. For the nonlinear energy (2.7), we have the following basic dissipation structure.

Lemma 2.2. *Let $\mathcal{E}_\ell(t)$ and $\mathcal{F}_\ell(t)$ be defined by (2.7). Then for any $\ell \geq 0$, it holds that*

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\mathcal{E}_\ell(t) + \mathcal{F}_\ell(t) \\
& = \int_{\mathbb{R}^d} \nabla^\ell p \nabla^\ell \nabla\cdot(p\mathbf{v}) dx - \int_{\mathbb{R}^d} \left(\nabla^{\ell+2} p + \delta\nabla^\ell\nabla\cdot\mathbf{v}\right)\nabla^\ell\nabla\cdot(p\mathbf{v}) dx \\
& \quad - \varepsilon\int_{\mathbb{R}^d} \nabla^\ell \mathbf{v}\cdot\nabla\nabla^\ell|\mathbf{v}|^2 dx + \varepsilon\int_{\mathbb{R}^d}\left(\nabla^{\ell+1}\nabla\cdot\mathbf{v} - \delta\nabla^{\ell+1}p\right)\nabla^{\ell+1}|\mathbf{v}|^2 dx.
\end{aligned}
\tag{2.8}
$$

Proof. Applying ∇^ℓ to equations $(2.2)_1$ and $(2.2)_2$, and taking the inner product with $\nabla^\ell p$ and $\nabla^\ell \mathbf{v}$, respectively, we obtain

$$
\frac{1}{2}\frac{d}{dt}\|\nabla^\ell p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+1} p\|^2_{L^2(\mathbb{R}^d)} - \int_{\mathbb{R}^d} \nabla^\ell p \nabla^\ell\nabla\cdot\mathbf{v} dx = \int_{\mathbb{R}^d} \nabla^\ell p\nabla^\ell\nabla\cdot(p\mathbf{v}) dx
$$

and

$$
\frac{1}{2}\frac{d}{dt}\|\nabla^\ell \mathbf{v}\|^2_{L^2(\mathbb{R}^d)} + \varepsilon\|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} - \int_{\mathbb{R}^d} \nabla^{\ell+1} p\cdot\nabla^\ell\mathbf{v} dx = -\varepsilon\int_{\mathbb{R}^d}\nabla^\ell\mathbf{v}\cdot\nabla\nabla^\ell|\mathbf{v}|^2 dx,
$$

which implies that the ℓ-th level dissipation holds:

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left(\|\nabla^\ell p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^\ell\mathbf{v}\|^2_{L^2(\mathbb{R}^d)}\right) + \left(\|\nabla^{\ell+1} p\|^2_{L^2(\mathbb{R}^d)} + \varepsilon\|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)}\right) \\
& = \int_{\mathbb{R}^d} \nabla^\ell p\nabla^\ell\nabla\cdot(p\mathbf{v}) dx - \varepsilon\int_{\mathbb{R}^d}\nabla^\ell\mathbf{v}\cdot\nabla\nabla^\ell|\mathbf{v}|^2 dx.
\end{aligned}
\tag{2.9}
$$

Similarly, we can apply $\nabla^{\ell+1}$ to equations (2.2)$_1$ and (2.2)$_2$, and then take the inner product with $\nabla^{\ell+1}p$ and $\nabla^{\ell+1}\mathbf{v}$, respectively, to obtain the $\ell+1$ level dissipation:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^d)}^2+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^d)}^2\Big)+\Big(\|\nabla^{\ell+2}p\|_{L^2(\mathbb{R}^d)}^2+\varepsilon\|\nabla^{\ell+2}\mathbf{v}\|_{L^2(\mathbb{R}^d)}^2\Big)\\&\quad=\int\limits_{\mathbb{R}^d}\nabla^{\ell+1}p\nabla^{\ell+1}\nabla\cdot(p\mathbf{v})dx-\varepsilon\int\limits_{\mathbb{R}^d}\nabla^{\ell+1}\mathbf{v}\cdot\nabla\nabla^{\ell+1}|\mathbf{v}|^2dx.\end{aligned}\tag{2.10}$$

On the other hand, to establish a dissipation independent of ε, we apply ∇^ℓ to equations (2.2)$_1$ and (2.2)$_2$, take the inner product with $-\nabla^\ell\nabla\cdot\mathbf{v}$ and $\nabla^{\ell+1}p$, respectively, and then have

$$-\int\limits_{\mathbb{R}^d}(\nabla^\ell p)_t\nabla^\ell\nabla\cdot\mathbf{v}dx+\int\limits_{\mathbb{R}^d}\nabla^{\ell+2}p\nabla^\ell\nabla\cdot\mathbf{v}dx+\|\nabla^\ell\nabla\cdot\mathbf{v}\|_{L^2(\mathbb{R}^d)}^2=-\int\limits_{\mathbb{R}^d}\nabla^\ell\nabla\cdot(p\mathbf{v})\nabla^\ell\nabla\cdot\mathbf{v}dx$$

and

$$\int\limits_{\mathbb{R}^d}(\nabla^\ell\mathbf{v})_t\cdot\nabla^{\ell+1}pdx-\varepsilon\int\limits_{\mathbb{R}^d}\nabla^{\ell+2}\mathbf{v}\cdot\nabla^{\ell+1}pdx-\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^d)}^2=-\varepsilon\int\limits_{\mathbb{R}^d}\nabla^{\ell+1}p\cdot\nabla^{\ell+1}|\mathbf{v}|^2dx.$$

This yields that

$$\begin{aligned}&\frac{d}{dt}\int\limits_{\mathbb{R}^d}\nabla^\ell\mathbf{v}\cdot\nabla^{\ell+1}pdx+\|\nabla^\ell\nabla\cdot\mathbf{v}\|_{L^2(\mathbb{R}^d)}^2+(1+\varepsilon)\int\limits_{\mathbb{R}^d}\nabla^{\ell+2}p\nabla^\ell\nabla\cdot\mathbf{v}dx-\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^d)}^2\\&\quad=-\int\limits_{\mathbb{R}^d}\nabla^\ell\nabla\cdot(p\mathbf{v})\nabla^\ell\nabla\cdot\mathbf{v}dx-\varepsilon\int\limits_{\mathbb{R}^d}\nabla^{\ell+1}p\cdot\nabla^{\ell+1}|\mathbf{v}|^2dx.\end{aligned}\tag{2.11}$$

Thus, combining (2.9), (2.10) and (2.11), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla^\ell p\|_{L^2(\mathbb{R}^d)}^2+\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^d)}^2+\|\nabla^\ell\mathbf{v}\|_{L^2(\mathbb{R}^d)}^2+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^d)}^2+2\delta\int\limits_{\mathbb{R}^d}\nabla^{\ell+1}p\cdot\nabla^\ell\mathbf{v}dx\Big)\\&\quad+\Big(\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^d)}^2+\|\nabla^{\ell+2}p\|_{L^2(\mathbb{R}^d)}^2+\delta\|\nabla^\ell\nabla\cdot\mathbf{v}\|_{L^2(\mathbb{R}^d)}^2+\varepsilon\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^d)}^2\\&\quad+\varepsilon\|\nabla^{\ell+2}\mathbf{v}\|_{L^2(\mathbb{R}^d)}^2+\delta(1+\varepsilon)\int\limits_{\mathbb{R}^d}\nabla^{\ell+2}p\nabla^\ell\nabla\cdot\mathbf{v}dx-\delta\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^d)}^2\Big)\\&=\int\limits_{\mathbb{R}^d}\nabla^\ell p\nabla^\ell\nabla\cdot(p\mathbf{v})dx+\int\limits_{\mathbb{R}^d}\nabla^{\ell+1}p\nabla^{\ell+1}\nabla\cdot(p\mathbf{v})dx-\delta\int\limits_{\mathbb{R}^d}\nabla^\ell\nabla\cdot\mathbf{v}\nabla^\ell\nabla\cdot(p\mathbf{v})dx\\&\quad-\varepsilon\int\limits_{\mathbb{R}^d}\nabla^\ell\mathbf{v}\cdot\nabla\nabla^\ell|\mathbf{v}|^2dx-\varepsilon\int\limits_{\mathbb{R}^d}\nabla^{\ell+1}\mathbf{v}\cdot\nabla\nabla^{\ell+1}|\mathbf{v}|^2dx-\delta\varepsilon\int\limits_{\mathbb{R}^d}\nabla^{\ell+1}p\cdot\nabla^{\ell+1}|\mathbf{v}|^2dx.\end{aligned}$$

Then (2.8) follows from the integration by parts to the above identity. This completes the proof of Lemma 2.2. □

Moreover, for the nonlinear energy (2.7), we have the following property.

Lemma 2.3. *For any $\delta \in (0,1)$, there exist two constants c_0 and $\hat{c}_0$ such that*

$$\mathcal{E}_\ell(t) \simeq \|\nabla^\ell p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+1} p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^\ell \mathbf{v}\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)}$$
$$\mathcal{F}_\ell(t) \simeq \|\nabla^{\ell+1} p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+2} p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+1} \mathbf{v}\|^2_{L^2(\mathbb{R}^d)} + \varepsilon\|\nabla^{\ell+2}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)}$$

for any integer $\ell \geq 0$, where $A \simeq B \Longleftrightarrow c_0 B \leq A \leq \hat{c}_0 B$.

Proof. By Young's inequality and (2.1), we have

$$\left|\int_{\mathbb{R}^d} \nabla^{\ell+1} p \cdot \nabla^\ell \mathbf{v} dx\right| \leq \|\nabla^{\ell+1} p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^\ell \mathbf{v}\|^2_{L^2(\mathbb{R}^d)}$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \nabla^{\ell+2} p \cdot \nabla^{\ell+1} \mathbf{v} dx\right| &\leq \|\nabla^{\ell+2} p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+1} \mathbf{v}\|^2_{L^2(\mathbb{R}^d)} \\ &\leq \|\nabla^{\ell+2} p\|^2_{L^2(\mathbb{R}^d)} + C\|\nabla^\ell \nabla\cdot \mathbf{v}\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Thus the constants c_0 and c_1 can be readily found such that the desired conclusion holds. □

2.4. Nonlinear energy estimates

In this subsection, we derive the *a priori* estimates for the solutions of system (1.3). For simplicity, we define the following quantities

$$\begin{cases} \mathcal{K}_1 := \int_{\mathbb{R}^d} \nabla^\ell p \nabla^\ell \nabla\cdot(p\mathbf{v}) dx \\ \mathcal{K}_2 := -\int_{\mathbb{R}^d} \left(\nabla^{\ell+2} p + \delta \nabla^\ell \nabla\cdot\mathbf{v}\right) \nabla^\ell\nabla\cdot(p\mathbf{v}) dx \\ \mathcal{K}_3 := -\varepsilon\int_{\mathbb{R}^d} \nabla^\ell \mathbf{v}\cdot\nabla\nabla^\ell|\mathbf{v}|^2 dx \\ \mathcal{K}_4 := \varepsilon\int_{\mathbb{R}^d} \left(\nabla^{\ell+1}\nabla\cdot\mathbf{v} - \delta\nabla^{\ell+1} p\right)\nabla^{\ell+1}|\mathbf{v}|^2 dx. \end{cases}$$

Then (2.8) can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}_\ell(t) + \mathcal{F}_\ell(t) = \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 + \mathcal{K}_4. \tag{2.12}$$

Now it is the key to estimating the terms $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$. In the paper, we shall employ the technique of *a priori* assumption. That is, we first assume that the solution $(p, \mathbf{v})$ of equations (2.2) satisfies for any $t \in [0, T]$

$$\|\nabla^2 p(t)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^2 \mathbf{v}(t)\|^2_{L^2(\mathbb{R}^d)} \le \frac{2\hat{c}_0}{c_0} M_0^2 \tag{2.13}$$

and

$$\|p(t)\|^2_{H^1(\mathbb{R}^d)} + \|\mathbf{v}(t)\|^2_{H^1(\mathbb{R}^d)} \le \kappa_0^2 \tag{2.14}$$

where c_0 and $\hat{c}_0$ are from Lemma 2.3, and then derive the *a priori* estimates to obtain global solutions. Finally, we show the obtained global solutions satisfy the above *a priori* assumptions and close our argument.

We divide our analysis into two cases: $d = 3$ and $d = 2$. We first give the derivation for $d = 3$.

Lemma 2.4. *Let the solution $(p, \mathbf{v})$ of equations (2.2) satisfy (2.13) and (2.14) with $d = 3$. For any given $M_0 > 0$, if κ_0 is suitably small, then there is a constant $c_1 > 0$ such that for all $t \in [0, T]$ we have*

$$\frac{d}{dt}\mathcal{E}_\ell(t) + c_1 \mathcal{F}_\ell(t) \le 0 \tag{2.15}$$

for $\ell = 0, 1, \cdots, k-1$.

Proof. The proof is split into three steps.

Step 1 ($\ell = 0$)**:** In the following, we shall frequently use the following inequalities:

$$\|f\|_{L^3(\mathbb{R}^3)} \le C\|f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)} \|\nabla f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}, \ \|f\|_{L^6(\mathbb{R}^3)} \le C\|\nabla f\|_{L^2(\mathbb{R}^3)} \tag{2.16}$$

where the former is obtained by the Gagliardo–Nirenberg inequality and the latter follows from the Sobolev inequality. The assumptions (2.13)–(2.14) will be used often in the sequel without mention of them. Then for the term $\mathcal{K}_1$, we use the integration by parts, (2.16) and the Hölder's inequality to obtain

$$\begin{aligned}
|\mathcal{K}_1| &= \left| -\int_{\mathbb{R}^3} \nabla p \cdot (p\mathbf{v}) dx \right| \\
&\le \|\nabla p\|_{L^2(\mathbb{R}^3)} \|p\mathbf{v}\|_{L^2(\mathbb{R}^3)} \\
&\le \|\nabla p\|_{L^2(\mathbb{R}^3)} \|p\|_{L^3(\mathbb{R}^3)} \|\mathbf{v}\|_{L^6(\mathbb{R}^3)} \\
&\le C\|p\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)} \|\nabla p\|^{\frac{3}{2}}_{L^2(\mathbb{R}^3)} \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)} \\
&\le C\kappa_0 \Big(\|\nabla p\|^2_{L^2(\mathbb{R}^3)} + \|\nabla \mathbf{v}\|^2_{L^2(\mathbb{R}^3)} \Big).
\end{aligned} \tag{2.17}$$

For $\mathcal{K}_2$, by (2.16) and the Hölder's inequality, we have

$$\begin{aligned}
|\mathcal{K}_2| &= \left| \int_{\mathbb{R}^3} \Big(\nabla^2 p + \delta \nabla \cdot \mathbf{v}\Big) \Big(\nabla p \cdot \mathbf{v} + p \nabla \cdot \mathbf{v}\Big) dx \right| \\
&\le C\Big(\|\nabla^2 p\|_{L^2(\mathbb{R}^3)} + \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)}\Big)\Big(\|\nabla p\|_{L^3(\mathbb{R}^3)} \|\mathbf{v}\|_{L^6(\mathbb{R}^3)} + \|p\|_{L^\infty(\mathbb{R}^3)} \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le C\Big(\|\nabla^2 p\|_{L^2(\mathbb{R}^3)} + \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)}\Big)\|\nabla p\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\nabla^2 p\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)} \\
&\le C\|\nabla p\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\Big(\|\nabla^2 p\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)}^2\Big) \\
&\le C\kappa_0\Big(\|\nabla^2 p\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)}^2\Big) \qquad (2.18)
\end{aligned}
$$

where the following inequality has been used:

$$
\|f\|_{L^\infty(\mathbb{R}^3)} \le C\|f\|_{L^6(\mathbb{R}^3)}^{\frac{1}{2}}\|\nabla^2 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \le C\|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\nabla^2 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}. \qquad (2.19)
$$

For the terms $\mathcal{K}_3$ and $\mathcal{K}_4$, we use (2.16) and Hölder's inequality to drive that

$$
\begin{aligned}
|\mathcal{K}_3| &= \varepsilon\left|\int_{\mathbb{R}^3} \mathbf{v}\cdot\nabla|\mathbf{v}|^2 dx\right| = \varepsilon\left|-\int_{\mathbb{R}^3} |\mathbf{v}|^2\nabla\cdot\mathbf{v}dx\right| \\
&\le \varepsilon\|\mathbf{v}\|_{L^3(\mathbb{R}^3)}\|\mathbf{v}\|_{L^6(\mathbb{R}^3)}\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)} \\
&\le C\|\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{5}{2}} \\
&\le C\kappa_0\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)}^2 \qquad (2.20)
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathcal{K}_4| &= \varepsilon\left|\int_{\mathbb{R}^3}\Big(\nabla(\nabla\cdot\mathbf{v}) - \delta\nabla p\Big)\cdot\nabla|\mathbf{v}|^2 dx\right| \\
&\le C\varepsilon\Big(\|\nabla^2\mathbf{v}\|_{L^2(\mathbb{R}^3)} + \|\nabla p\|_{L^2(\mathbb{R}^3)}\Big)\|\nabla \mathbf{v}\|_{L^6(\mathbb{R}^3)}\|\mathbf{v}\|_{L^3(\mathbb{R}^3)} \\
&\le C\varepsilon\Big(\|\nabla^2\mathbf{v}\|_{L^2(\mathbb{R}^3)} + \|\nabla p\|_{L^2(\mathbb{R}^3)}\Big)\|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^3)}\|\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \\
&\le C\|\mathbf{v}\|_{H^1(\mathbb{R}^3)}\Big(\|\nabla p\|_{L^2(\mathbb{R}^3)}^2 + \varepsilon\|\nabla^2\mathbf{v}\|_{L^2(\mathbb{R}^3)}^2\Big) \\
&\le C\kappa_0\Big(\|\nabla p\|_{L^2(\mathbb{R}^3)}^2 + \varepsilon\|\nabla^2\mathbf{v}\|_{L^2(\mathbb{R}^3)}^2\Big). \qquad (2.21)
\end{aligned}
$$

Substituting (2.17)–(2.21) into (2.12) and using Lemma 2.3, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\mathcal{E}_0(t) + \mathcal{F}_0(t) &\le C_0\kappa_0\Big(\|\nabla p\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla^2 p\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)}^2 + \varepsilon\|\nabla^2\mathbf{v}\|_{L^2(\mathbb{R}^3)}^2\Big) \\
&\le \frac{C_0}{c_0}\kappa_0\mathcal{F}_0(t).
\end{aligned}
$$

If we let κ_0 be suitably small such that $\frac{C_0}{c_0}\kappa_0 \le \frac{1}{2}$, we can find a positive constant c_1 such that

$$
\frac{1}{2}\frac{d}{dt}\mathcal{E}_0(t) + c_1\mathcal{F}_0(t) \le 0.
$$

Step 2 ($\ell = 1$)**:** In this case, with Leibniz's formula, Hölder's inequality and (2.16), we have

$$
\begin{aligned}
|\mathcal{K}_1| &= \left| \int_{\mathbb{R}^3} \nabla p \cdot \nabla\nabla \cdot (p\mathbf{v}) dx \right| = \left| -\int_{\mathbb{R}^3} \nabla^2 p \left(\nabla p \cdot \mathbf{v} + p \nabla \cdot \mathbf{v} \right) dx \right| \\
&\le \|\nabla^2 p\|_{L^2(\mathbb{R}^3)} \Big(\|\nabla p\|_{L^6(\mathbb{R}^3)} \|\mathbf{v}\|_{L^3(\mathbb{R}^3)} + \|p\|_{L^3(\mathbb{R}^3)} \|\nabla \mathbf{v}\|_{L^6(\mathbb{R}^3)} \Big) \\
&\le C \|\nabla^2 p\|_{L^2(\mathbb{R}^3)} \Big(\|\nabla^2 p\|_{L^2(\mathbb{R}^3)} \|\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} + \|p\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla p\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^3)} \Big) \\
&\le C \Big(\|p\|_{H^1(\mathbb{R}^3)} + \|\mathbf{v}\|_{H^1(\mathbb{R}^3)} \Big) \Big(\|\nabla^2 p\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^3)} \Big) \\
&\le C\kappa_0 \Big(\|\nabla^2 p\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^3)} \Big)
\end{aligned} \tag{2.22}
$$

and

$$
\begin{aligned}
|\mathcal{K}_2| &= \left| \int_{\mathbb{R}^3} \left(\nabla^3 p + \delta \nabla\nabla \cdot \mathbf{v} \right) \cdot \left(\nabla\nabla p \cdot \mathbf{v} + \nabla p \cdot \nabla \mathbf{v} + \nabla p \nabla \cdot \mathbf{v} + p \nabla\nabla \cdot \mathbf{v} \right) dx \right| \\
&\le C \Big(\|\nabla^3 p\|_{L^2(\mathbb{R}^3)} + \|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^3)} \Big) \\
&\quad \cdot \Big(\|\nabla^2 p\|_{L^6(\mathbb{R}^3)} \|\mathbf{v}\|_{L^3(\mathbb{R}^3)} + \|\nabla p\|_{L^3(\mathbb{R}^3)} \|\nabla \mathbf{v}\|_{L^6(\mathbb{R}^3)} + \|p\|_{L^\infty(\mathbb{R}^3)} \|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^3)} \Big) \\
&\le C \Big(\|\nabla^3 p\|_{L^2(\mathbb{R}^3)} + \|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^3)} \Big) \\
&\quad \cdot \Big(\|\nabla^3 p\|_{L^2(\mathbb{R}^3)} \|\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} + \|\nabla p\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla^2 p\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^3)} \Big) \\
&\le C \Big(\|\nabla p\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla^2 p\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} + \|\mathbf{v}\|_{H^1(\mathbb{R}^3)} \Big) \Big(\|\nabla^3 p\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^3)}^2 \Big) \\
&\le C \Big(\kappa_0^{\frac{1}{2}} M_0^{\frac{1}{2}} + \kappa_0 \Big) \Big(\|\nabla^3 p\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^3)}^2 \Big).
\end{aligned} \tag{2.23}
$$

Similarly, for $\mathcal{K}_3$ and $\mathcal{K}_4$, we use $\varepsilon \in [0, 1]$ to deduce that

$$
\begin{aligned}
|\mathcal{K}_3| &= \varepsilon \left| \int_{\mathbb{R}^3} \nabla^2 \mathbf{v} \cdot \nabla |\mathbf{v}|^2 dx \right| \\
&\le C \|\mathbf{v}\|_{L^3(\mathbb{R}^3)} \|\nabla \mathbf{v}\|_{L^6(\mathbb{R}^3)} \|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^3)} \\
&\le C \|\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^3)}^2 \\
&\le C\kappa_0 \|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^3)}^2
\end{aligned} \tag{2.24}
$$

and

$$
\begin{aligned}
|\mathcal{K}_4| &= 2\varepsilon\left|\int\limits_{\mathbb{R}^3}\Big(\nabla^2(\nabla\cdot\mathbf{v})-\delta\nabla^2 p\Big)\Big(\nabla^2\mathbf{v}\cdot\mathbf{v}+\nabla\mathbf{v}\cdot\nabla\mathbf{v}\Big)dx\right| \\
&\le C\varepsilon\Big(\|\nabla^3\mathbf{v}\|_{L^2}+\|\nabla^2 p\|_{L^2}\Big)\Big(\|\nabla^2\mathbf{v}\|_{L^2}\|\mathbf{v}\|_{L^\infty}+\|\nabla\mathbf{v}\|_{L^6}\|\nabla\mathbf{v}\|_{L^3}\Big) \\
&\le C\varepsilon\Big(\|\nabla^3\mathbf{v}\|_{L^2}+\|\nabla^2 p\|_{L^2}\Big)\|\nabla^2\mathbf{v}\|_{L^2}\|\nabla\mathbf{v}\|^{\frac12}_{L^2(\mathbb{R}^3)}\|\nabla^2\mathbf{v}\|^{\frac12}_{L^2(\mathbb{R}^3)} \\
&\le C\|\nabla\mathbf{v}\|^{\frac12}_{L^2(\mathbb{R}^3)}\|\nabla^2\mathbf{v}\|^{\frac12}_{L^2(\mathbb{R}^3)}\Big(\|\nabla^2 p\|^2_{L^2}+\|\nabla^2\mathbf{v}\|^2_{L^2}+\varepsilon\|\nabla^3\mathbf{v}\|^2_{L^2}\Big) \\
&\le C\kappa_0^{\frac12}M_0^{\frac12}\Big(\|\nabla^2 p\|^2_{L^2}+\|\nabla^2\mathbf{v}\|^2_{L^2}+\varepsilon\|\nabla^3\mathbf{v}\|^2_{L^2}\Big).
\end{aligned}
\tag{2.25}
$$

Substituting (2.22)–(2.25) into (2.12) and using Lemma 2.3 yields

$$
\begin{aligned}
&\frac12\frac{d}{dt}\mathcal{E}_1(t)+\mathcal{F}_1(t) \\
&\le C_1\Big(\kappa_0+\kappa_0^{\frac12}M_0^{\frac12}\Big)\Big(\|\nabla^2 p\|^2_{L^2(\mathbb{R}^3)}+\|\nabla^3 p\|^2_{L^2(\mathbb{R}^3)}+\|\nabla^2\mathbf{v}\|^2_{L^2(\mathbb{R}^3)}+\varepsilon\|\nabla^3\mathbf{v}\|^2_{L^2(\mathbb{R}^3)}\Big) \\
&\le \frac{2C_1}{c_0}\kappa_0^{\frac12}M_0^{\frac12}\mathcal{F}_1(t).
\end{aligned}
$$

If we allow κ_0 to be suitably small such that $\frac{2C_1}{c_0}\kappa_0^{\frac12}M_0^{\frac12}\le\frac12$, we can find a positive constant c_1 such that

$$
\frac12\frac{d}{dt}\mathcal{E}_1(t)+c_1\mathcal{F}_1(t)\le 0.
$$

Step 3 ($\ell\ge 2$)**:** In this case, the estimates are more delicate. First of all, for the term $\mathcal{K}_1$, we use the integration by parts and Leibniz's formula to obtain

$$
\mathcal{K}_1=-\int\limits_{\mathbb{R}^3}\nabla^{\ell+1}p\cdot\nabla^\ell(p\mathbf{v})dx=-\sum_{j=0}^{\ell}C_\ell^j\int\limits_{\mathbb{R}^3}\nabla^{\ell+1}p\cdot\nabla^j p\nabla^{\ell-j}\mathbf{v}dx,
$$

where $C_\ell^j:=\begin{pmatrix} l\\ j\end{pmatrix}$ denotes the binomial coefficient. This, together with Hölder's inequality, gives that

$$
|\mathcal{K}_1|\le C\sum_{j=0}^{\ell}\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^3)}\|\nabla^j p\nabla^{\ell-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)}.
$$

We need to estimate the second factor on the right hand side. To this end, we notice that for any $\ell\ge 2$ we have either $j\le\frac{3\ell-1}{4}$ or $j\ge\frac{\ell}{2}+1$ for any nonnegative integer j. If $j\le\frac{3\ell-1}{4}$, we first use the interpolation inequality (2.19) and the Gagliardo–Nirenberg inequality to derive that

$$
\|\nabla^j p\|_{L^\infty(\mathbb{R}^3)}\le C\|\nabla^{\alpha_1}p\|^{1-\frac{j+1}{\ell+1}}_{L^2(\mathbb{R}^3)}\|\nabla^{\ell+1}p\|^{\frac{j+1}{\ell+1}}_{L^2(\mathbb{R}^3)}\le C\|p\|^{\frac{3\ell-4j-1}{4(\ell+1)}}_{L^2(\mathbb{R}^3)}\|\nabla^2 p\|^{\frac14}_{L^2(\mathbb{R}^3)}\|\nabla^{\ell+1}p\|^{\frac{j+1}{\ell+1}}_{L^2(\mathbb{R}^3)},
$$

where

$$\alpha_1 := \frac{\ell+1}{2(\ell-j)} \in [0,2].$$

Furthermore the Gagliardo–Nirenberg inequality yields that

$$\|\nabla^{\ell-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)} \le C\|\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{j+1}{\ell+1}}\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{1-\frac{j+1}{\ell+1}}.$$

Thus we have

$$\begin{aligned}
&\|\nabla^j p\nabla^{\ell-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\\
&\quad\le \|\nabla^j p\|_{L^\infty(\mathbb{R}^3)}\|\nabla^{\ell-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\\
&\quad\le C\|p\|_{L^2(\mathbb{R}^3)}^{\frac{3\ell-4j-1}{4(\ell+1)}}\|\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{j+1}{\ell+1}}\|\nabla^2 p\|_{L^2(\mathbb{R}^3)}^{\frac14}\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^3)}^{\frac{j+1}{\ell+1}}\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{1-\frac{j+1}{\ell+1}}\\
&\quad\le C\kappa_0^{\frac34}M_0^{\frac14}\Big(\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^3)}+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\Big).
\end{aligned} \tag{2.26}$$

On the other hand, if $j \ge \frac{\ell}{2}+1$, we can use the interpolation to deduce that

$$\|\nabla^j p\|_{L^2(\mathbb{R}^3)} \le C\|\nabla p\|_{L^2(\mathbb{R}^3)}^{1-\frac{j-1}{\ell}}\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^3)}^{\frac{j-1}{\ell}}$$

and

$$\|\nabla^{\ell-j}\mathbf{v}\|_{L^\infty(\mathbb{R}^3)} \le C\|\nabla^{\alpha_2}\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{j-1}{\ell}}\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{1-\frac{j-1}{\ell}} \le C\|\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac12}\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{2(j-1)-\ell}{2\ell}}\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{1-\frac{j-1}{\ell}},$$

where

$$\alpha_2 := \frac{2(j-1)-\ell}{2(j-1)} \in [0,1].$$

Thus we obtain

$$\begin{aligned}
&\|\nabla^j p\nabla^{\ell-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\\
&\quad\le \|\nabla^j p\|_{L^2(\mathbb{R}^3)}\|\nabla^{\ell-j}\mathbf{v}\|_{L^\infty(\mathbb{R}^3)}\\
&\quad\le C\|\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac12}\|\nabla p\|_{L^2(\mathbb{R}^3)}^{1-\frac{j-1}{\ell}}\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{2(j-1)-\ell}{2\ell}}\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^3)}^{\frac{j-1}{\ell}}\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{1-\frac{j-1}{\ell}}\\
&\quad\le C\kappa_0\Big(\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^3)}+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\Big).
\end{aligned} \tag{2.27}$$

Hence combining (2.26) and (2.27), for any integer $j \ge 0$, we have

$$\|\nabla^j p\nabla^{\ell-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)} \le C\Big(\kappa_0^{\frac34}M_0^{\frac14}+\kappa_0\Big)\Big(\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^3)}+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\Big) \tag{2.28}$$

which gives rise to

$$
\begin{aligned}
|\mathcal{K}_1| &\le C\Big(\kappa_0^{\frac{3}{4}} M_0^{\frac{1}{4}} + \kappa_0\Big)\|\nabla^{\ell+1} p\|_{L^2(\mathbb{R}^3)}\Big(\|\nabla^{\ell+1} p\|_{L^2(\mathbb{R}^3)} + \|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\Big) \\
&\le C\Big(\kappa_0^{\frac{3}{4}} M_0^{\frac{1}{4}} + \kappa_0\Big)\Big(\|\nabla^{\ell+1} p\|^2_{L^2(\mathbb{R}^3)} + \|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^3)}\Big). \qquad (2.29)
\end{aligned}
$$

Next, we turn to the estimate of term $\mathcal{K}_2$. Considering that

$$
\mathcal{K}_2 = -\sum_{j=0}^{\ell+1} C_{\ell+1}^{j} \int_{\mathbb{R}^3} \Big(\nabla^{\ell+2} p + \delta \nabla^{\ell}\nabla\cdot\mathbf{v}\Big)\nabla^j p \nabla^{\ell+1-j}\mathbf{v}dx
$$

by Leibniz's formula, we first use Hölder's inequality to obtain

$$
|\mathcal{K}_2| \le C\sum_{j=0}^{\ell+1}\Big(\|\nabla^{\ell+2} p\|_{L^2(\mathbb{R}^3)} + \|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\Big)\|\nabla^j p\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)}.
$$

If $j \le \frac{3\ell}{4}$, Hölder's inequality and the interpolation give that

$$
\begin{aligned}
\|\nabla^j p\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)} &\le \|\nabla^j p\|_{L^\infty(\mathbb{R}^3)}\|\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)} \\
&\le C\|\nabla^{\alpha_3} p\|^{1-\frac{j}{\ell}}_{L^2(\mathbb{R}^3)}\|\nabla^{\ell+2} p\|^{\frac{j}{\ell}}_{L^2(\mathbb{R}^3)}\|\nabla\mathbf{v}\|^{\frac{j}{\ell}}_{L^2(\mathbb{R}^3)}\|\nabla^{\ell+1}\mathbf{v}\|^{1-\frac{j}{\ell}}_{L^2(\mathbb{R}^3)} \\
&\le C\|p\|^{\frac{1}{4}}_{L^2(\mathbb{R}^3)}\|\nabla^2 p\|^{\frac{3\ell-4j}{4\ell}}_{L^2(\mathbb{R}^3)}\|\nabla^{\ell+2} p\|^{\frac{j}{\ell}}_{L^2(\mathbb{R}^3)}\|\nabla\mathbf{v}\|^{\frac{j}{\ell}}_{L^2(\mathbb{R}^3)}\|\nabla^{\ell+1}\mathbf{v}\|^{1-\frac{j}{\ell}}_{L^2(\mathbb{R}^3)},
\end{aligned}
$$

where

$$
\alpha_3 := \frac{3\ell-4j}{2(\ell-j)} \in [0,2].
$$

It then follows from Young's inequality that

$$
\begin{aligned}
\|\nabla^j p\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)} &\le C\kappa_0^{\frac{1}{4}} M_0^{\frac{3}{4}}\|\nabla^{\ell+2} p\|^{\frac{j}{\ell}}_{L^2(\mathbb{R}^3)}\|\nabla^{\ell+1}\mathbf{v}\|^{1-\frac{j}{\ell}}_{L^2(\mathbb{R}^3)} \\
&\le C\kappa_0^{\frac{1}{4}} M_0^{\frac{3}{4}}\Big(\|\nabla^{\ell+2} p\|_{L^2(\mathbb{R}^3)} + \|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\Big). \qquad (2.30)
\end{aligned}
$$

On the other hand, if $j \ge \frac{\ell+3}{2}$, we set

$$
\alpha_4 := \frac{\ell+1}{2(j-1)} \in [0,1]
$$

and perform a similar procedure as above to obtain

$$\begin{aligned}\|\nabla^j p\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)} &\le \|\nabla^j p\|_{L^2(\mathbb{R}^3)}\|\nabla^{\ell+1-j}\mathbf{v}\|_{L^\infty(\mathbb{R}^3)}\\ &\le C\|\nabla p\|_{L^2(\mathbb{R}^3)}^{1-\frac{j-1}{\ell+1}}\|\nabla^{\ell+2}p\|_{L^2(\mathbb{R}^3)}^{\frac{j-1}{\ell+1}}\|\nabla^{\alpha_4}\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{j-1}{\ell+1}}\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{1-\frac{j-1}{\ell+1}}\\ &\le C\kappa_0\Big(\|\nabla^{\ell+2}p\|_{L^2(\mathbb{R}^3)}+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\Big). \end{aligned} \tag{2.31}$$

Notice that for any $\ell \ge 3$ or $\ell = 2$ and $j \ne 2$, we have either $j \le \frac{3\ell}{4}$ or $j \ge \frac{\ell+3}{2}$ for any nonnegative j. Thus we need to consider the case $\ell = 2$ and $j = 2$. Indeed, in this case, we have

$$\begin{aligned}\|\nabla^j p\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)} &\le \|\nabla^2 p\|_{L^\infty(\mathbb{R}^3)}\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^3)}\\ &\le C\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^3)}\|\nabla^3 p\|_{L^2(\mathbb{R}^3)}^{\frac12}\|\nabla^4 p\|_{L^2(\mathbb{R}^3)}^{\frac12}\\ &\le C\kappa_0\Big(\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^3)}+\|\nabla^{\ell+2}p\|_{L^2(\mathbb{R}^3)}\Big). \end{aligned} \tag{2.32}$$

By (2.30)–(2.32), we find that for any $j \ge 0$, it holds that

$$\|\nabla^j p\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)} \le C\Big(\kappa_0^{\frac14}M_0^{\frac34}+\kappa_0\Big)\Big(\|\nabla^{\ell+2}p\|_{L^2(\mathbb{R}^3)}+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\Big) \tag{2.33}$$

which upon the substitution gives

$$|\mathcal{K}_2| \le C\Big(\kappa_0^{\frac14}M_0^{\frac34}+\kappa_0\Big)\Big(\|\nabla^{\ell+2}p\|_{L^2(\mathbb{R}^3)}^2+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}^2\Big). \tag{2.34}$$

For the term $\mathcal{K}_3$, we first use the integration by parts and Leibniz's formula to obtain

$$\mathcal{K}_3 = \varepsilon\int_{\mathbb{R}^3}\nabla^\ell(\nabla\cdot\mathbf{v})\cdot\nabla^\ell|\mathbf{v}|^2dx = \varepsilon\sum_{j=0}^{\ell}C_\ell^j\int_{\mathbb{R}^3}\nabla^\ell(\nabla\cdot\mathbf{v})\cdot\nabla^j\mathbf{v}\nabla^{\ell-j}\mathbf{v}dx,$$

which, along with the Hölder's inequality, gives that

$$|\mathcal{K}_3| \le C\sum_{j=0}^{\ell}\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\|\nabla^j\mathbf{v}\nabla^{\ell-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)}.$$

Then taking a similar procedure to that of estimating $\mathcal{K}_1$, namely, replacing p with $\mathbf{v}$ in (2.28), we can deduce that

$$|\nabla^j\mathbf{v}\nabla^{\ell-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)} \le C\Big(\kappa_0^{\frac34}M_0^{\frac14}+\kappa_0\Big)\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)},$$

which implies that

$$|\mathcal{K}_3| \le C\Big(\kappa_0^{\frac34}M_0^{\frac14}+\kappa_0\Big)\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}^2. \tag{2.35}$$

Finally, we estimate the term $\mathcal{K}_4$. By Leibniz's formula, we have

$$\mathcal{K}_4 = \varepsilon \sum_{j=0}^{\ell+1} C_{\ell+1}^j \int_{\mathbb{R}^3} \left(\nabla^{\ell+1}\nabla\cdot\mathbf{v} - \delta\nabla^{\ell+1}p\right)\nabla^j\mathbf{v}\nabla^{\ell+1-j}\mathbf{v}dx$$

which, along with the Hölder's inequality, yields that

$$|\mathcal{K}_4| \le C\varepsilon \sum_{j=0}^{\ell+1}\left(\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^3)} + \|\nabla^{\ell+2}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\right)\|\nabla^j\mathbf{v}\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)}. \tag{2.36}$$

Same as estimating $\mathcal{K}_2$, namely replacing p by $\mathbf{v}$ in (2.33), we end up with for any integer $j \ge 0$:

$$\|\nabla^j\mathbf{v}\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)} \le C\varepsilon\left(\kappa_0^{\frac{1}{4}}M_0^{\frac{3}{4}} + \kappa_0\right)\left(\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)} + \|\nabla^{\ell+2}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\right) \tag{2.37}$$

Substituting (2.37) into (2.36) gives

$$|\mathcal{K}_4| \le C\left(\kappa_0^{\frac{1}{4}}M_0^{\frac{3}{4}} + \kappa_0\right)\left(\|\nabla^{\ell+1}p\|^2_{L^2(\mathbb{R}^3)} + \|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^3)} + \varepsilon\|\nabla^{\ell+2}\mathbf{v}\|^2_{L^2(\mathbb{R}^3)}\right). \tag{2.38}$$

Substituting the estimates (2.29), (2.34), (2.35) and (2.38) into (2.12), we conclude that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\mathcal{E}_\ell(t) + \mathcal{F}_\ell(t)\\
&\le C_2\kappa_0^{\frac{1}{4}}M_0^{\frac{3}{4}}\left(\|\nabla^{\ell+1}p\|^2_{L^2(\mathbb{R}^3)} + \|\nabla^{\ell+2}p\|^2_{L^2(\mathbb{R}^3)} + \|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^3)} + \varepsilon\|\nabla^{\ell+2}\mathbf{v}\|^2_{L^2(\mathbb{R}^3)}\right)\\
&\le \frac{C_2}{c_0}\kappa_0^{\frac{1}{4}}M_0^{\frac{3}{4}}\mathcal{F}_\ell(t)
\end{aligned} \tag{2.39}$$

by Lemma 2.3. If we let κ_0 small such that $\frac{C_2}{c_0}\kappa_0^{\frac{1}{4}}M_0^{\frac{3}{4}} \le \frac{1}{2}$, we can find a positive constant c_1 such that

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}_\ell(t) + c_1\mathcal{F}_\ell(t) \le 0 \qquad \text{for any } \ \ell \ge 2.$$

Thus we complete the proof Lemma 2.4. □

Lemma 2.5. *Let the solution $(p, \mathbf{v})$ of equations (2.2) satisfy (2.13) and (2.14) with $d = 2$. For any given $M_0 > 0$, if κ_0 is suitably small, then the same energy estimates (2.15) holds.*

Proof. In this case, we shall frequently use the following inequalities:

$$\|f\|_{L^4(\mathbb{R}^2)} \le C\|f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)} \quad \text{and} \quad \|f\|_{L^\infty(\mathbb{R}^2)} \le C\|f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla^2 f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}, \tag{2.40}$$

which are obtained by the Gagliardo–Nirenberg inequality. We divide the proof into two steps.

Step 1 ($\ell = 0$)**:** We shall estimate $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$ one by one. First, for $\mathcal{K}_1$ and $\mathcal{K}_2$, we use the integration by parts, Hölder's inequality and (2.40) to derive that

$$
\begin{aligned}
|\mathcal{K}_1| &= \left| \int_{\mathbb{R}^2} p \nabla \cdot (p\mathbf{v}) dx \right| = \left| \int_{\mathbb{R}^2} p \nabla p \cdot \mathbf{v} dx \right| \\
&\le \|p\|_{L^4(\mathbb{R}^2)} \|\nabla p\|_{L^2(\mathbb{R}^2)} \|\mathbf{v}\|_{L^4(\mathbb{R}^2)} \\
&\le C \|p\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla p\|_{L^2(\mathbb{R}^2)}^{\frac{3}{2}} \|\mathbf{v}\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \\
&\le C\kappa_0 \Big(\|\nabla p\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)}^2 \Big)
\end{aligned} \tag{2.41}
$$

and

$$
\begin{aligned}
|\mathcal{K}_2| &= \left| \int_{\mathbb{R}^2} \Big(\nabla^2 p + \delta \nabla \cdot \mathbf{v} \Big) \Big(\nabla p \cdot \mathbf{v} + p \nabla \cdot \mathbf{v} \Big) dx \right| \\
&\le C \Big(\|\nabla^2 p\|_{L^2(\mathbb{R}^2)} + \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)} \Big) \Big(\|\nabla p\|_{L^4(\mathbb{R}^2)} \|\mathbf{v}\|_{L^4(\mathbb{R}^2)} + \|p\|_{L^\infty} \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)} \Big) \\
&\le C \Big(\|\nabla^2 p\|_{L^2(\mathbb{R}^2)} + \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)} \Big) \\
&\quad \cdot \Big(\|\nabla p\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla^2 p\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\mathbf{v}\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} + \|p\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla^2 p\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)} \Big) \\
&\le C \Big(\|p\|_{H^1(\mathbb{R}^2)} + \|\mathbf{v}\|_{H^1(\mathbb{R}^2)} \Big) \Big(\|\nabla^2 p\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)}^2 \Big) \\
&\le C\kappa_0 \Big(\|\nabla^2 p\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)}^2 \Big).
\end{aligned} \tag{2.42}
$$

Similarly, we estimate $\mathcal{K}_3$ and $\mathcal{K}_4$ as follows:

$$
\begin{aligned}
|\mathcal{K}_3| &= \varepsilon \left| \int_{\mathbb{R}^2} |\mathbf{v}|^2 \nabla \cdot \mathbf{v} dx \right| \\
&\le \|\mathbf{v}\|_{L^4(\mathbb{R}^2)}^2 \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)} \\
&\le C \|\mathbf{v}\|_{L^2(\mathbb{R}^2)} \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)}^2 \\
&\le C\kappa_0 \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)}^2
\end{aligned} \tag{2.43}
$$

and

$$
\begin{aligned}
|\mathcal{K}_4| &= \varepsilon \left| \int_{\mathbb{R}^2} \Big(\nabla(\nabla \cdot \mathbf{v}) - \delta \nabla p \Big) \cdot \nabla |\mathbf{v}|^2 dx \right| \\
&\le C\varepsilon \Big(\|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^2)} + \|\nabla p\|_{L^2(\mathbb{R}^2)} \Big) \|\nabla \mathbf{v}\|_{L^4(\mathbb{R}^2)} \|\mathbf{v}\|_{L^4(\mathbb{R}^2)} \\
&\le C\varepsilon \Big(\|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^2)} + \|\nabla p\|_{L^2(\mathbb{R}^2)} \Big) \|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\mathbf{v}\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)} \\
&\le C\varepsilon \|\mathbf{v}\|_{H^1(\mathbb{R}^2)} \Big(\|\nabla p\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^2)}^2 \Big) \\
&\le C\kappa_0 \Big(\|\nabla p\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)}^2 + \varepsilon \|\nabla^2 \mathbf{v}\|_{L^2(\mathbb{R}^2)}^2 \Big).
\end{aligned} \tag{2.44}
$$

Substituting (2.41)–(2.44) into (2.12) and using Lemma 2.3, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathcal{E}_0(t)+\mathcal{F}_0(t) &\le C_3\kappa_0\Big(\|\nabla p\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^2 p\|^2_{L^2(\mathbb{R}^2)}+\|\nabla \mathbf{v}\|^2_{L^2(\mathbb{R}^2)}+\varepsilon\|\nabla^2 \mathbf{v}\|^2_{L^2(\mathbb{R}^2)}\Big)\\ &\le \frac{C_3}{c_0}\kappa_0\mathcal{F}_0(t).\end{aligned}$$

If we take κ_0 suitably small such that $\frac{C_3}{c_0}\kappa_0 \le \frac{1}{2}$, we can find a positive constant c_1 such that

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}_0(t)+c_1\mathcal{F}_0(t)\le 0.$$

Step 2 ($\ell \ge 1$)**:** In this case, the estimates for $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$ are more subtle. First of all, for $\mathcal{K}_1$, we use the integration by parts and Leibniz's formula to obtain

$$\mathcal{K}_1=-\int_{\mathbb{R}^2}\nabla^{\ell+1}p\cdot\nabla^{\ell}(p\mathbf{v})dx=-\sum_{j=0}^{\ell}C_\ell^j\int_{\mathbb{R}^2}\nabla^{\ell+1}p\cdot\nabla^j p\nabla^{\ell-j}\mathbf{v}dx.$$

It then follows from Hölder's inequality that

$$|\mathcal{K}_1|\le C\sum_{j=0}^{\ell}\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^2)}\|\nabla^j p\nabla^{\ell-j}\mathbf{v}\|_{L^2(\mathbb{R}^2)}. \tag{2.45}$$

We need estimate the second factor on the right hand side of inequality (2.45). For $0\le j\le \ell-1$, we use Hölder's inequality, Gagliardo–Nirenberg interpolation inequality and Young's inequality to obtain

$$\begin{aligned}\|\nabla^j p\nabla^{\ell-j}\mathbf{v}\|_{L^2(\mathbb{R}^2)} &\le \|\nabla^j p\|_{L^\infty(\mathbb{R}^2)}\|\nabla^{\ell-j}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\\ &\le C\|p\|^{1-\frac{j+1}{\ell+1}}_{L^2(\mathbb{R}^2)}\|\nabla^{\ell+1}p\|^{\frac{j+1}{\ell+1}}_{L^2(\mathbb{R}^2)}\|\mathbf{v}\|^{\frac{j+1}{\ell+1}}_{L^2(\mathbb{R}^2)}\|\nabla^{\ell+1}\mathbf{v}\|^{1-\frac{j+1}{\ell+1}}_{L^2(\mathbb{R}^2)}\\ &\le C\Big(\|p\|_{L^2(\mathbb{R}^2)}+\|\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big)\Big(\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big).\end{aligned}$$

On the other hand, for $j=\ell$, we perform a similar procedure and obtain that

$$\begin{aligned}\|\nabla^j p\nabla^{\ell-j}\mathbf{v}\|_{L^2(\mathbb{R}^2)} &\le \|\nabla^\ell p\|_{L^2(\mathbb{R}^2)}\|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)}\\ &\le C\|p\|^{\frac{1}{\ell+1}}_{L^2(\mathbb{R}^2)}\|\nabla^{\ell+1}p\|^{\frac{\ell}{\ell+1}}_{L^2(\mathbb{R}^2)}\|\mathbf{v}\|^{\frac{\ell}{\ell+1}}_{L^2(\mathbb{R}^2)}\|\nabla^{\ell+1}\mathbf{v}\|^{\frac{1}{\ell+1}}_{L^2(\mathbb{R}^2)}\\ &\le C\Big(\|p\|_{L^2(\mathbb{R}^2)}+\|\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big)\Big(\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big).\end{aligned}$$

In summary, for any $0\le j\le l$, it has that

$$
\begin{aligned}
&\|\nabla^j p\nabla^{\ell-j}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\\
&\quad\le C\Big(\|p\|_{L^2(\mathbb{R}^2)}+\|\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big)\Big(\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big).
\end{aligned}
\tag{2.46}
$$

Substituting (2.46) into (2.45), we have

$$
\begin{aligned}
|\mathcal{K}_1| &\le C\Big(\|p\|_{L^2(\mathbb{R}^2)}+\|\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big)\Big(\|\nabla^{\ell+1}p\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^2)}\Big)\\
&\le C\kappa_0\Big(\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big).
\end{aligned}
\tag{2.47}
$$

We now estimate the term $\mathcal{K}_2$. Since

$$
\mathcal{K}_2=-\sum_{j=0}^{\ell+1}C_{\ell+1}^j\int_{\mathbb{R}^2}\Big(\nabla^{\ell+2}p+\delta\nabla^{\ell}\nabla\cdot\mathbf{v}\Big)\nabla^j p\nabla^{\ell+1-j}\mathbf{v}dx
$$

by Leibniz's formula, we use Hölder's inequality to obtain

$$
|\mathcal{K}_2|\le C\sum_{j=0}^{\ell+1}\Big(\|\nabla^{\ell+2}p\|_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big)\|\nabla^j p\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^2)}.
\tag{2.48}
$$

To estimate the second factor on the right hand side of (2.48), we employ the Hölder's inequality and (2.40) to deduce that if $j=0$, then

$$
\begin{aligned}
\|\nabla^j p\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^2)} &= \|p\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\\
&\le \|p\|_{L^\infty(\mathbb{R}^2)}\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\\
&\le \|p\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\nabla^2 p\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

If $1\le j\le\ell$, we have

$$
\begin{aligned}
\|\nabla^j p\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^2)} &\le \|\nabla^j p\|_{L^\infty(\mathbb{R}^2)}\|\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\\
&\le C\|\nabla p\|^{1-\frac{j}{\ell+1}}_{L^2(\mathbb{R}^2)}\|\nabla^{\ell+2}p\|^{\frac{j}{\ell+1}}_{L^2(\mathbb{R}^2)}\|\mathbf{v}\|^{\frac{j}{\ell+1}}_{L^2(\mathbb{R}^2)}\|\nabla^{\ell+1}\mathbf{v}\|^{1-\frac{j}{\ell+1}}_{L^2(\mathbb{R}^2)},
\end{aligned}
$$

while if $j=\ell+1$, then

$$
\begin{aligned}
\|\nabla^{\ell+1}p\mathbf{v}\|_{L^2(\mathbb{R}^2)} &\le \|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^2)}\|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)}\\
&\le C\|p\|^{\frac{1}{\ell+2}}_{L^2(\mathbb{R}^2)}\|\nabla^{\ell+2}p\|^{\frac{\ell+1}{\ell+2}}_{L^2(\mathbb{R}^2)}\|\nabla^{\alpha_5}\mathbf{v}\|^{\frac{\ell+1}{\ell+2}}_{L^2(\mathbb{R}^2)}\|\nabla^{\ell+1}\mathbf{v}\|^{\frac{1}{\ell+2}}_{L^2(\mathbb{R}^2)},
\end{aligned}
$$

where $\alpha_5:=\frac{1}{\ell+1}\in(0,1)$. Therefore for $0\le j\le\ell+1$, we obtain

$$\|\nabla^j p\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^2)}$$
$$\le C\Big(\|p\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\nabla^2 p\|^{\frac12}_{L^2(\mathbb{R}^2)}+\|p\|_{H^1(\mathbb{R}^2)}+\|\mathbf{v}\|_{H^1(\mathbb{R}^2)}\Big)\Big(\|\nabla^{\ell+2}p\|_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big). \tag{2.49}$$

Substituting (2.49) into (2.48) and using (2.13)–(2.14), one has

$$|\mathcal{K}_2|$$
$$\le C\Big(\|p\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\nabla^2 p\|^{\frac12}_{L^2(\mathbb{R}^2)}+\|p\|_{H^1(\mathbb{R}^2)}+\|\mathbf{v}\|_{H^1(\mathbb{R}^2)}\Big)\Big(\|\nabla^{\ell+2}p\|_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big)^2$$
$$\le C\Big(\kappa_0^{\frac12}M_0^{\frac12}+\kappa_0\Big)\Big(\|\nabla^{\ell+2}p\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^2)}\Big). \tag{2.50}$$

The terms $\mathcal{K}_3$ and $\mathcal{K}_4$ can be similarly. Indeed, for $\mathcal{K}_3$, we have

$$\begin{aligned}\mathcal{K}_3&=\varepsilon\sum_{j=0}^{\ell}C_\ell^j\int_{\mathbb{R}^2}\nabla^\ell(\nabla\cdot\mathbf{v})\cdot\nabla^j\mathbf{v}\nabla^{\ell-j}\mathbf{v}dx\\&\le C\sum_{j=0}^{\ell}\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\|\nabla^j\mathbf{v}\nabla^{\ell-j}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\\&\le C\|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^2)}\|\mathbf{v}\|_{L^2(\mathbb{R}^2)}\\&\le C\kappa_0\|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^2)}\end{aligned} \tag{2.51}$$

where we have used the estimate $\|\nabla^j\mathbf{v}\nabla^{\ell-j}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\le C\|\mathbf{v}\|_{L^2(\mathbb{R}^2)}\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^2)}$ for all $0\le j\le l$ (see the estimates of (2.46)). For $\mathcal{K}_4$, we have

$$\begin{aligned}\mathcal{K}_4&=-\varepsilon\sum_{j=0}^{\ell+1}C_{\ell+1}^j\int_{\mathbb{R}^2}\Big(\nabla^{\ell+1}\nabla\cdot\mathbf{v}-\delta\nabla^{\ell+1}p\Big)\nabla^j\mathbf{v}\nabla^{\ell+1-j}\mathbf{v}dx\\&\le C\varepsilon\sum_{j=0}^{\ell+1}\Big(\|\nabla^{\ell+2}\mathbf{v}\|_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^2)}\Big)\|\nabla^j\mathbf{v}\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

By the same derivation of (2.49), one has

$$\begin{aligned}&\|\nabla^j\mathbf{v}\nabla^{\ell+1-j}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\\&\le C\Big(\|\mathbf{v}\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\nabla^2\mathbf{v}\|^{\frac12}_{L^2(\mathbb{R}^2)}+\|\mathbf{v}\|_{H^1(\mathbb{R}^2)}\Big)\Big(\|\nabla^{\ell+2}\mathbf{v}\|_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big)\\&\le C\Big(\kappa_0^{\frac12}M_0^{\frac12}+\kappa_0\Big)\Big(\|\nabla^{\ell+2}\mathbf{v}\|_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big)\end{aligned}$$

for $0\le j\le\ell+1$, which implies that

$$
\begin{aligned}
|\mathcal{K}4| &\le C\varepsilon\Big(\kappa_0^{\frac{1}{2}}M_0^{\frac{1}{2}}+\kappa_0\Big)\Big(\|\nabla^{\ell+2}\mathbf{v}\|_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}p\|_{L^2(\mathbb{R}^2)}\Big)\Big(\|\nabla^{\ell+2}\mathbf{v}\|_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big)\\
&\le C\Big(\kappa_0^{\frac{1}{2}}M_0^{\frac{1}{2}}+\kappa_0\Big)\Big(\|\nabla^{\ell+1}p\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^2)}+\varepsilon\|\nabla^{\ell+2}\mathbf{v}\|^2_{L^2(\mathbb{R}^2)}\Big).
\end{aligned}
\tag{2.52}
$$

Substituting the estimates (2.47), (2.50), (2.51) and (2.52) into (2.12), we conclude that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\mathcal{E}_\ell(t)+\mathcal{F}_\ell(t)\\
&\quad\le C_4\Big(\kappa_0^{\frac{1}{2}}M_0^{\frac{1}{2}}+\kappa_0\Big)\Big(\|\nabla^{\ell+1}p\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^2)}+\varepsilon\|\nabla^{\ell+2}\mathbf{v}\|^2_{L^2(\mathbb{R}^2)}\Big)\\
&\quad\le \frac{2C_4}{c_0}\kappa_0^{\frac{1}{2}}M_0^{\frac{1}{2}}\mathcal{F}_\ell(t)
\end{aligned}
\tag{2.53}
$$

by Lemma 2.3. If we take κ_0 suitably small such that $\frac{2C_4}{c_0}\kappa_0^{\frac{1}{2}}M_0^{\frac{1}{2}}\le\frac{1}{2}$, we can find a positive constant c_1 such that

$$
\frac{1}{2}\frac{d}{dt}\mathcal{E}_\ell(t)+c_1\mathcal{F}_\ell(t)\le 0 \qquad \text{for any } \ \ell\ge 1.
$$

The proof of Lemma 2.5 is completed. □

Now we are in a position to give the *a priori* estimates for solutions of (2.2).

Lemma 2.6 *(A priori estimates). Suppose that the solution* $(p,\mathbf{v})$ *of system* (2.2) *satisfies the assumption of* Lemma 2.4. *Then it holds that*

$$
\begin{aligned}
&\|\nabla^\ell p(t)\|^2_{L^2(\mathbb{R}^d)}+\|\nabla^{\ell+1}p(t)\|^2_{L^2(\mathbb{R}^d)}+\|\nabla^\ell\mathbf{v}(t)\|^2_{L^2(\mathbb{R}^d)}+\|\nabla^{\ell+1}\mathbf{v}(t)\|^2_{L^2(\mathbb{R}^d)}\\
&\qquad+c_1\int_0^t\Big(\|\nabla^{\ell+1}p(\tau)\|^2_{L^2(\mathbb{R}^d)}+\|\nabla^{\ell+2}p(\tau)\|^2_{L^2(\mathbb{R}^d)}\\
&\qquad+\|\nabla^{\ell+1}\mathbf{v}(\tau)\|^2_{L^2(\mathbb{R}^d)}+\varepsilon\|\nabla^{\ell+2}\mathbf{v}(\tau)\|^2_{L^2(\mathbb{R}^d)}\Big)d\tau\\
&\quad\le\frac{\hat{c}_0}{c_0}\Big(\|\nabla^\ell p_0\|^2_{L^2(\mathbb{R}^d)}+\|\nabla^{\ell+1}p_0\|^2_{L^2(\mathbb{R}^d)}+\|\nabla^\ell\mathbf{v}_0\|^2_{L^2(\mathbb{R}^d)}+\|\nabla^{\ell+1}\mathbf{v}_0\|^2_{L^2(\mathbb{R}^d)}\Big)
\end{aligned}
\tag{2.54}
$$

for any $t\in[0,T]$ *and* $\ell=0,1,\cdots,k-1$.

Proof. The conclusion follows from Lemmas 2.3, 2.4 and 2.5 directly. □

2.5. Proof of Theorem 1.1

For any $t\in[0,T]$, by taking $\ell=0$ in (2.54), we have

$$
\|p(t)\|^2_{H^1(\mathbb{R}^d)}+\|\mathbf{v}(t)\|^2_{H^1(\mathbb{R}^d)}\le\frac{\hat{c}_0}{c_0}\Big(\|p_0\|^2_{H^1(\mathbb{R}^d)}+\|\mathbf{v}_0\|^2_{H^1(\mathbb{R}^d)}\Big)\le\frac{\hat{c}_0}{c_0}\eta^2.
$$

If η is suitably small such that $\frac{\hat{c}_0}{c_0}\eta^2 \le \kappa_0^2$, we can deduce that

$$\|p(t)\|^2_{H^1(\mathbb{R}^d)} + \|\mathbf{v}(t)\|^2_{H^1(\mathbb{R}^d)} \le \kappa_0^2, \qquad t \in [0, T],$$

which closes the *a priori* assumption (2.14).

By (2.54) with $\ell = 1$, we also have

$$\begin{aligned}
&\|\nabla^2 p(t)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^2 \mathbf{v}(t)\|^2_{L^2(\mathbb{R}^d)} \\
&\le \frac{\hat{c}_0}{c_0}\Big(\|\nabla p_0\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^2 p_0\|^2_{L^2(\mathbb{R}^d)} + \|\nabla \mathbf{v}_0\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^2 \mathbf{v}_0\|^2_{L^2(\mathbb{R}^d)}\Big) \\
&\le \frac{\hat{c}_0}{c_0}(\eta^2 + M_0^2) \le \frac{2\hat{c}_0}{c_0} M_0^2
\end{aligned}$$

for any $t \in [0, T]$, which closes the *a priori* assumption (2.13).

On the other hand, by summing up (2.54) from $\ell = 0$ to $k - 1$, we can deduce that

$$\begin{aligned}
&\|p(t)\|^2_{H^k(\mathbb{R}^d)} + \|\mathbf{v}(t)\|^2_{H^k(\mathbb{R}^d)} \\
&\quad + \int_0^t \Big(\|\nabla p(\tau)\|^2_{H^k(\mathbb{R}^d)} + \|\nabla \mathbf{v}(\tau)\|^2_{H^{k-1}(\mathbb{R}^d)} + \varepsilon\|\nabla^{k+1}\mathbf{v}(\tau)\|^2_{L^2(\mathbb{R}^d)}\Big) d\tau \\
&\le C_5\Big(\|p_0\|^2_{H^k(\mathbb{R}^d)} + \|\mathbf{v}_0\|^2_{H^k(\mathbb{R}^d)}\Big)
\end{aligned}$$

for any $t \in [0, T]$, which gives (1.4).

Finally the standard continuity argument concludes the global existence of solution $(p, \mathbf{v})$ from the local existence in Lemma 2.1 and the *a priori* estimates given in Lemma 2.6. This completes the proof of Theorem 1.1 by noticing that $p = u - \bar{u}$. □

3. Decay estimates

In this section, we prove Theorem 1.2 by using the energy methods. Without loss of generality, we may assume that there exists a positive constant $M_1 > 1$ such that

$$\|p_0\|^2_{\dot{H}^{-s}(\mathbb{R}^d)} + \|\mathbf{v}_0\|^2_{\dot{H}^{-s}(\mathbb{R}^d)} \le M_1^2, \tag{3.1}$$

since $(u_0 - \bar{u}, \mathbf{v}_0) \in \dot{H}^{-s}(\mathbb{R}^d) \times \dot{H}^{-s}(\mathbb{R}^d)$. For simplicity, we set $\Lambda := \sqrt{-\Delta}$, which is defined by the Fourier transform $\widehat{\Lambda f}(\xi) = |\xi|\hat{f}(\xi)$.

Lemma 3.1. *Suppose that the solution $(p, \mathbf{v})$ of system (2.2) satisfies the assumption of Lemma 2.4. Assume that*

$$\|p(t)\|^2_{\dot{H}^{-s}(\mathbb{R}^d)} + \|\mathbf{v}(t)\|^2_{\dot{H}^{-s}(\mathbb{R}^d)} \le 2M_1^2, \qquad t \in [0, T], \tag{3.2}$$

where $0 < s < \frac{d}{2}$. Then

- *for any $t \in [0, T]$ and all $\ell = 0, 1, \cdots, k-1$, we have*

$$\|\nabla^\ell p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+1} p\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^\ell \mathbf{v}\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+1} \mathbf{v}\|^2_{L^2(\mathbb{R}^d)} \le C M_1^2 (1+t)^{-(s+\ell)}; \tag{3.3}$$

- *for some positive constant $\kappa > 1$ and any $t \in [0, T]$, we have*

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\Lambda^{-s} p\|^2_{L^2(\mathbb{R}^d)} + \|\Lambda^{-s} \mathbf{v}\|^2_{L^2(\mathbb{R}^d)}\Big) + \|\Lambda^{-s}\nabla p\|^2_{L^2(\mathbb{R}^d)} + \varepsilon\|\Lambda^{-s}\nabla \mathbf{v}\|^2_{L^2(\mathbb{R}^d)} \\ &\quad \le C M_1^2 \eta^{\frac{s}{4}} \|\Lambda^{-s} p\|_{L^2(\mathbb{R}^d)} (1+t)^{-\kappa}, \end{aligned} \tag{3.4}$$

where C is a positive constant independent of t.

Proof. To derive (3.3), we first use the interpolation to obtain that

$$\|\nabla^\ell p\|_{L^2(\mathbb{R}^d)} \le C\|p\|^{\frac{1}{s+\ell+1}}_{\dot H^{-s}(\mathbb{R}^d)} \|\nabla^{\ell+1} p\|^{\frac{s+\ell}{s+\ell+1}}_{L^2(\mathbb{R}^d)} \le 2C M_1^{\frac{1}{s+\ell+1}} \|\nabla^{\ell+1} p\|^{\frac{s+\ell}{s+\ell+1}}_{L^2(\mathbb{R}^d)}$$

and

$$\|\nabla^\ell \mathbf{v}\|_{L^2(\mathbb{R}^d)} \le C\|\mathbf{v}\|^{\frac{1}{s+\ell+1}}_{\dot H^{-s}(\mathbb{R}^d)} \|\nabla^{\ell+1} \mathbf{v}\|^{\frac{s+\ell}{s+\ell+1}}_{L^2(\mathbb{R}^d)} \le 2C M_1^{\frac{1}{s+\ell+1}} \|\nabla^{\ell+1} \mathbf{v}\|^{\frac{s+\ell}{s+\ell+1}}_{L^2(\mathbb{R}^d)}.$$

Similarly, we can deduce that

$$\|\nabla^{\ell+1} p\|_{L^2(\mathbb{R}^d)} \le C\|\nabla p\|^{\frac{1}{s+\ell+1}}_{\dot H^{-s}(\mathbb{R}^d)} \|\nabla^{\ell+2} p\|^{\frac{s+\ell}{s+\ell+1}}_{L^2(\mathbb{R}^d)} \le 2C M_1^{\frac{1}{s+\ell+1}} \|\nabla^{\ell+2} p\|^{\frac{s+\ell}{s+\ell+1}}_{L^2(\mathbb{R}^d)}$$

where we have used the interpolation inequality

$$\|\nabla p\|_{\dot H^{-s}(\mathbb{R}^d)} = \|p\|_{\dot H^{1-s}(\mathbb{R}^d)} \le C\|p\|^{\frac{s}{s+1}}_{\dot H^{-s}(\mathbb{R}^d)} \|p\|^{\frac{1}{s+1}}_{\dot H^{1}(\mathbb{R}^d)} \le C M_1.$$

Moreover, the *a priori* estimate (2.15) in Lemma 2.4 implies that

$$\begin{aligned} &\|\nabla^{\ell+1}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} \\ &\quad \le C_5\Big(\|\nabla^\ell p_0\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+1} p_0\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^\ell \mathbf{v}_0\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+1} \mathbf{v}_0\|^2_{L^2(\mathbb{R}^d)}\Big) \le C \end{aligned}$$

and thus

$$\|\nabla^{\ell+1}\mathbf{v}\|_{L^2(\mathbb{R}^d)} \le C\|\nabla^{\ell+1}\mathbf{v}\|^{\frac{s+\ell}{s+\ell+1}}_{L^2(\mathbb{R}^d)}.$$

Then by collecting the above estimates and using Lemma 2.3, we obtain

$$\mathcal{E}_\ell(t) \le C M_1^{\frac{2}{s+\ell+1}} \mathcal{F}_\ell^{\frac{s+\ell}{s+\ell+1}}(t),$$

which, together with (2.15) in Lemma 2.4 again, yields that

$$\frac{d}{dt}\mathcal{E}_\ell(t) + cM_1^{-\frac{2}{s+\ell}}\mathcal{E}_\ell^{\frac{s+\ell+1}{s+\ell}}(t) \le 0.$$

By a direct calculation, we can deduce that

$$\mathcal{E}_\ell(t) \le CM_1^2\big(\mathcal{E}_\ell(0)^{-\frac{1}{s+\ell}} + t\big)^{-(s+\ell)} \le CM_1^2\big(1+t\big)^{-(s+\ell)},$$

which, together with Lemma 2.3, gives (3.3).

We now turn to prove (3.4). For this purpose, we apply Λ^{-s} to equations $(2.2)_1$ and $(2.2)_2$, and take the inner product with $\Lambda^{-s}p$ and $\Lambda^{-s}\mathbf{v}$, respectively, to obtain

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|\Lambda^{-s}p\|^2_{L^2(\mathbb{R}^d)} + \|\Lambda^{-s}\nabla p\|^2_{L^2(\mathbb{R}^d)} - \int_{\mathbb{R}^d}\Lambda^{-s}p\Lambda^{-s}\nabla\cdot\mathbf{v}dx = \int_{\mathbb{R}^d}\Lambda^{-s}p\Lambda^{-s}\nabla\cdot(p\mathbf{v})dx, \\ \frac{1}{2}\frac{d}{dt}\|\Lambda^{-s}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} + \varepsilon\|\Lambda^{-s}\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} - \int_{\mathbb{R}^d}\Lambda^{-s}\nabla p\cdot\Lambda^{-s}\mathbf{v}dx \\ \quad = -\varepsilon\int_{\mathbb{R}^d}\Lambda^{-s}\mathbf{v}\cdot\Lambda^{-s}\nabla|\mathbf{v}|^2dx, \end{cases}$$

which, along with integration by parts, implies that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\Lambda^{-s}p\|^2_{L^2(\mathbb{R}^d)} + \|\Lambda^{-s}\mathbf{v}\|^2_{L^2(\mathbb{R}^d)}\Big) + \|\Lambda^{-s}\nabla p\|^2_{L^2(\mathbb{R}^d)} + \varepsilon\|\Lambda^{-s}\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^d)} \\ &= \int_{\mathbb{R}^d}\Lambda^{-s}p\Lambda^{-s}\nabla\cdot(p\mathbf{v})dx - \varepsilon\int_{\mathbb{R}^d}\Lambda^{-s}\mathbf{v}\cdot\Lambda^{-s}\nabla|\mathbf{v}|^2dx \\ &:= \mathcal{S}_1 + \mathcal{S}_2. \end{aligned} \tag{3.5}$$

To estimate $\mathcal{S}_1$ and $\mathcal{S}_2$, we will use the following L^p–L^q estimate

$$\|\Lambda^{-\alpha}f\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)} \quad \text{with} \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}, \quad 0<\alpha<d \quad \text{and} \quad 1\le p<q<+\infty \tag{3.6}$$

(Stein [28], page 119, Theorem 1). We divide the proof into two cases: $d=3$ and $d=2$.

Case 1 ($d=3$)**.** In this case, for $\mathcal{S}_1$, it follows from Hölder's inequality and the L^p–L^q estimate (3.6) that

$$\begin{aligned} \mathcal{S}_1 &\le C\|\Lambda^{-s}p\|_{L^2(\mathbb{R}^3)}\big\|\Lambda^{-s}(\nabla p\cdot\mathbf{v} + p\nabla\cdot\mathbf{v})\big\|_{L^2(\mathbb{R}^3)} \\ &\le C\|\Lambda^{-s}p\|_{L^2(\mathbb{R}^3)}\Big(\|\nabla p\cdot\mathbf{v}\|_{L^{\frac{6}{2s+3}}(\mathbb{R}^3)} + \|p\nabla\cdot\mathbf{v}\|_{L^{\frac{6}{2s+3}}(\mathbb{R}^3)}\Big) \\ &\le C\|\Lambda^{-s}p\|_{L^2(\mathbb{R}^3)}\Big(\|\nabla p\|_{L^{\frac{3}{s+1}}(\mathbb{R}^3)}\|\mathbf{v}\|_{L^6(\mathbb{R}^3)} + \|p\|_{L^6(\mathbb{R}^3)}\|\nabla\cdot\mathbf{v}\|_{L^{\frac{3}{s+1}}(\mathbb{R}^3)}\Big). \end{aligned}$$

Then by Sobolev embedding and the interpolation, we see

$$\begin{aligned}\mathcal{S}_1 &\le C\|\Lambda^{-s}p\|_{L^2(\mathbb{R}^3)}\Big(\|p\|_{L^2(\mathbb{R}^3)}^{\frac{2s+1}{4}}\|\nabla^2 p\|_{L^2(\mathbb{R}^3)}^{\frac{3-2s}{4}}\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)} \\ &\quad + \|\nabla p\|_{L^2(\mathbb{R}^3)}\|\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{2s+1}{4}}\|\nabla^2\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{3-2s}{4}}\Big) \\ &\le C\|\Lambda^{-s}p\|_{L^2(\mathbb{R}^3)}\big(\|p\|_{L^2(\mathbb{R}^3)}+\|\mathbf{v}\|_{L^2(\mathbb{R}^3)}\big)^{\frac{2s+1}{4}}\big(\|\nabla p\|_{L^2(\mathbb{R}^3)}+\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^3)}\big) \\ &\quad \cdot \big(\|\nabla^2 p\|_{L^2(\mathbb{R}^3)}+\|\nabla^2\mathbf{v}\|_{L^2(\mathbb{R}^3)}\big)^{\frac{3-2s}{4}}. \end{aligned} \tag{3.7}$$

The term $\mathcal{S}_2$ can be estimated similarly as follows:

$$\mathcal{S}_2 \le C\|\Lambda^{-s}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\|\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{2s+1}{4}}\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^3)}\|\nabla^2\mathbf{v}\|_{L^2(\mathbb{R}^3)}^{\frac{3-2s}{4}}, \tag{3.8}$$

which is obtained simply by replacing p with $\mathbf{v}$ in the estimate of $\mathcal{S}_1$.

Substituting (3.7)–(3.8) into (3.5), and using the decay estimate (3.3) with $\ell = 1$ and $\ell = 2$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\Lambda^{-s}p\|_{L^2(\mathbb{R}^3)}^2+\|\Lambda^{-s}\mathbf{v}\|_{L^2(\mathbb{R}^3)}^2\Big)+\|\Lambda^{-s}\nabla p\|_{L^2(\mathbb{R}^3)}^2+\varepsilon\|\Lambda^{-s}\nabla\mathbf{v}\|_{L^2(\mathbb{R}^3)}^2 \\ &\le CM_1^2\eta^{\frac{2s+1}{8}}\Big(\|\Lambda^{-s}p\|_{L^2(\mathbb{R}^3)}+\|\Lambda^{-s}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\Big)(1+t)^{-\frac{s+1}{2}}(1+t)^{-\frac{s+2}{2}\cdot\frac{3-2s}{4}} \\ &\le CM_1^2\eta^{\frac{s}{4}}\Big(\|\Lambda^{-s}p\|_{L^2(\mathbb{R}^3)}+\|\Lambda^{-s}\mathbf{v}\|_{L^2(\mathbb{R}^3)}\Big)(1+t)^{-\kappa}, \end{aligned} \tag{3.9}$$

where

$$\kappa := \frac{s+1}{2}+\frac{s+2}{2}\cdot\frac{3-2s}{4} > 1$$

by $s \in (0, \frac{3}{2})$.

Case 2 ($d = 2$)**.** The procedure of estimating $\mathcal{S}_1$ and $\mathcal{S}_2$ is analogous with the case of $d = 3$. Indeed, by Hölder's inequality, the L^p–L^q estimate (3.6) and the interpolation, we have that

$$\begin{aligned}\mathcal{S}_1 &\le \|\Lambda^{-s}p\|_{L^2(\mathbb{R}^2)}\big\|\Lambda^{-s}(\nabla p\cdot\mathbf{v}+p\nabla\cdot\mathbf{v})\big\|_{L^2(\mathbb{R}^2)} \\ &\le C\|\Lambda^{-s}p\|_{L^2(\mathbb{R}^2)}\Big(\|\nabla p\cdot\mathbf{v}\|_{L^{\frac{2}{s+1}}(\mathbb{R}^2)}+\|p\nabla\cdot\mathbf{v}\|_{L^{\frac{2}{s+1}}(\mathbb{R}^2)}\Big) \\ &\le C\|\Lambda^{-s}p\|_{L^2(\mathbb{R}^2)}\Big(\|\nabla p\|_{L^2(\mathbb{R}^2)}\|\mathbf{v}\|_{L^{\frac{2}{s}}(\mathbb{R}^2)}+\|p\|_{L^{\frac{2}{s}}(\mathbb{R}^2}\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big) \\ &\le C\|\Lambda^{-s}p\|_{L^2(\mathbb{R}^2)}\Big(\|\nabla p\|_{L^2(\mathbb{R}^2)}\|\mathbf{v}\|_{L^2(\mathbb{R}^2)}^{s}\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)}^{1-s}+\|p\|_{L^2(\mathbb{R}^2)}^{s}\|\nabla p\|_{L^2(\mathbb{R}^2)}^{1-s}\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big)\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}_2 &\le \varepsilon\|\Lambda^{-s}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\big\|\Lambda^{-s}(\mathbf{v}\nabla\mathbf{v})\big\|_{L^2(\mathbb{R}^2)} \\ &\le C\|\Lambda^{-s}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\|\mathbf{v}\nabla\mathbf{v}\|_{L^{\frac{2}{s+1}}(\mathbb{R}^2)} \\ &\le C\|\Lambda^{-s}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\|\mathbf{v}\|_{L^{\frac{2}{s}}(\mathbb{R}^2}\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)} \\ &\le C\|\Lambda^{-s}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\|\mathbf{v}\|_{L^2(\mathbb{R}^2)}^{s}\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)}^{2-s}.\end{aligned}$$

Then employing the decay estimates (3.3) with $\ell = 1$, we have

$$\frac{1}{2}\frac{d}{dt}\Big(\|\Lambda^{-s}p\|^2_{L^2(\mathbb{R}^2)} + \|\Lambda^{-s}\mathbf{v}\|^2_{L^2(\mathbb{R}^2)}\Big) + \|\Lambda^{-s}\nabla p\|^2_{L^2(\mathbb{R}^2)} + \varepsilon\|\Lambda^{-s}\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^2)}$$
$$\le CM_1^2\eta^s\Big(\|\Lambda^{-s}p\|_{L^2(\mathbb{R}^2)} + \|\Lambda^{-s}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big)(1+t)^{-\frac{(s+1)(2-s)}{2}}.$$

Taking $\kappa := \frac{(s+1)(2-s)}{2}$, we have $\kappa > 1$ by $s \in (0, 1)$. Thus,

$$\frac{1}{2}\frac{d}{dt}\Big(\|\Lambda^{-s}p\|^2_{L^2(\mathbb{R}^2)} + \|\Lambda^{-s}\mathbf{v}\|^2_{L^2(\mathbb{R}^2)}\Big) + \|\Lambda^{-s}\nabla p\|^2_{L^2(\mathbb{R}^2)}$$
$$\le CM_1^2\eta^{\frac{s}{4}}\Big(\|\Lambda^{-s}p\|_{L^2(\mathbb{R}^2)} + \|\Lambda^{-s}\mathbf{v}\|_{L^2(\mathbb{R}^2)}\Big)(1+t)^{-\kappa}.$$

This completes the proof of Lemma 3.1. $\square$

We now turn to the proof of Theorem 1.2 on the decay estimates of solutions.

Proof of Theorem 1.2. By Lemma 3.1, the decay estimates (1.5) can be obtained from (3.3) provided that we can close the *a priori* assumption (3.2) for some constant $M_1 > 0$. Now we show (3.2) holds in fact. For this purpose, we first use (3.4) to obtain

$$\begin{aligned}
&\|\Lambda^{-s}p(t)\|^2_{L^2(\mathbb{R}^d)(\mathbb{R}^d)} + \|\Lambda^{-s}\mathbf{v}(t)\|^2_{L^2(\mathbb{R}^d)} \\
&\le \Big(\|\Lambda^{-s}p_0\|^2_{L^2(\mathbb{R}^d)} + \|\Lambda^{-s}\mathbf{v}_0\|^2_{L^2(\mathbb{R}^d)}\Big) \\
&\quad + CM_1^2\eta^{\frac{s}{4}}\int_0^t \Big(\|\Lambda^{-s}p(\tau)\|_{L^2(\mathbb{R}^d)} + \|\Lambda^{-s}\mathbf{v}(\tau)\|_{L^2(\mathbb{R}^d)}\Big)(1+\tau)^{-\kappa}d\tau \\
&\le \Big(\|\Lambda^{-s}p_0\|^2_{L^2(\mathbb{R}^d)} + \|\Lambda^{-s}\mathbf{v}_0\|^2_{L^2(\mathbb{R}^d)}\Big) \\
&\quad + CM_1^2\eta^{\frac{s}{4}}\sup_{\tau\in[0,t]}\Big(\|\Lambda^{-s}p(\tau)\|^2_{L^2(\mathbb{R}^d)} + \|\Lambda^{-s}\mathbf{v}(\tau)\|^2_{L^2(\mathbb{R}^d)}\Big)^{\frac{1}{2}}\int_0^t (1+\tau)^{-\kappa}d\tau \\
&\le \Big(\|\Lambda^{-s}p_0\|^2_{L^2(\mathbb{R}^d)} + \|\Lambda^{-s}\mathbf{v}_0\|^2_{L^2(\mathbb{R}^d)}\Big) \\
&\quad + CM_1^2\eta^{\frac{s}{4}}\sup_{\tau\in[0,t]}\Big(\|\Lambda^{-s}p(\tau)\|^2_{L^2(\mathbb{R}^d)} + \|\Lambda^{-s}\mathbf{v}(\tau)\|^2_{L^2(\mathbb{R}^d)}\Big)^{\frac{1}{2}}
\end{aligned}$$

by $\kappa > 1$. For simplicity, we first set

$$\mathcal{M}(t) := \sup_{\tau\in[0,t]}\Big(\|\Lambda^{-s}p(\tau)\|^2_{L^2(\mathbb{R}^d)} + \|\Lambda^{-s}\mathbf{v}(\tau)\|^2_{L^2(\mathbb{R}^d)}\Big)^{\frac{1}{2}}$$

and then use Young's inequality to obtain

$$\mathcal{M}^2(t) \le M_1^2 + CM_1^2\eta^{\frac{s}{4}}\mathcal{M}(t) \le \frac{1}{4}\mathcal{M}^2(t) + M_1^2 + C_6M_1^4\eta^{\frac{s}{2}}$$

for some positive constant C_6 independent of η and M_1. Then, by taking η suitably small such that $C_6\eta^{\frac{s}{2}}M_1^2 \le \frac{1}{2}$, we can deduce

$$\|\Lambda^{-s}p(t)\|^2_{L^2(\mathbb{R}^d)(\mathbb{R}^d} + \|\Lambda^{-s}\mathbf{v}(t)\|^2_{L^2(\mathbb{R}^d)} \le \mathcal{M}^2(t) \le 2M_1^2,$$

which closes the *a priori* assumption (3.2).

Thus the standard continuity argument gives the desired estimate (1.5). This completes the proof of Theorem 1.2. □

4. Convergence rate of diffusion

In this section, we will use the energy methods to derive that the solutions of (2.2) with $\varepsilon > 0$ converge to that of (2.2) with $\varepsilon = 0$ as $\varepsilon \to 0$. Without loss of generality, we may assume $\varepsilon \in [0, 1]$ throughout this section.

Proof of Theorem 1.3. Let $(p^\varepsilon, \mathbf{v}^\varepsilon)$ be the solution of system (2.2) with $\varepsilon \ge 0$ obtained in Theorem 1.1. We define

$$P = p^\varepsilon - p^0, \qquad V = \mathbf{v}^\varepsilon - \mathbf{v}^0.$$

Substituting them into (1.3), we end up with

$$\begin{cases} P_t - \Delta P - \nabla\cdot V = \nabla\cdot(P\mathbf{v}^\varepsilon + p^0V), & x \in \mathbb{R}^d,\ t > 0,\\ V_t - \epsilon\Delta\mathbf{v}^\varepsilon - \nabla P = -\epsilon\nabla|\mathbf{v}^\varepsilon|^2, & x \in \mathbb{R}^d,\ t > 0,\\ P(x,0) = 0, \quad V(x,0) = 0, & x \in \mathbb{R}^d. \end{cases} \tag{4.1}$$

For any $0 \le \ell \le k-2$, applying ∇^ℓ to equations $(4.1)_1$ and $(4.1)_2$, and taking the inner product with $\nabla^\ell P$ and $\nabla^\ell V$, respectively, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla^\ell P\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^{\ell+1}P\|^2_{L^2(\mathbb{R}^d)} - \int_{\mathbb{R}^d} \nabla^\ell P\nabla^\ell\nabla\cdot V\,dx = -\int_{\mathbb{R}^d} \nabla^{\ell+1}P\cdot\nabla^\ell\big(P\mathbf{v}^\varepsilon + p^0V\big)dx$$

and

$$\frac{1}{2}\frac{d}{dt}\|\nabla^\ell V\|^2_{L^2(\mathbb{R}^d)} - \varepsilon\int_{\mathbb{R}^d} \nabla^\ell V\cdot\nabla^{\ell+2}\mathbf{v}^\varepsilon dx - \int_{\mathbb{R}^d} \nabla^{\ell+1}P\cdot\nabla^\ell V\,dx = -\varepsilon\int_{\mathbb{R}^d} \nabla^\ell V\cdot\nabla^\ell\nabla|\mathbf{v}^\varepsilon|^2dx.$$

Then the sum of above two identities gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\nabla^\ell P\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^\ell V\|^2_{L^2(\mathbb{R}^d)}\Big) + \|\nabla^{\ell+1}P\|^2_{L^2(\mathbb{R}^d)}\\ &= \varepsilon\int_{\mathbb{R}^d} \nabla^\ell V\cdot\nabla^{\ell+2}\mathbf{v}^\varepsilon dx - \int_{\mathbb{R}^d} \nabla^{\ell+1}P\cdot\nabla^\ell\Big(P\mathbf{v}^\varepsilon + p^0V\Big)dx - \varepsilon\int_{\mathbb{R}^d} \nabla^\ell V\cdot\nabla^\ell\nabla|\mathbf{v}^\varepsilon|^2dx. \end{aligned} \tag{4.2}$$

Summing up (4.2) from $\ell = 0$ to $\ell = k-2$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|P\|^2_{H^{k-2}(\mathbb{R}^d)} + \|V\|^2_{H^{k-2}(\mathbb{R}^d)}\Big) + \|\nabla P\|^2_{H^{k-2}(\mathbb{R}^d)} \\
&\quad = \varepsilon \sum_{\ell=0}^{k-2} \int_{\mathbb{R}^d} \nabla^\ell V \cdot \nabla^{\ell+2} \mathbf{v}^\varepsilon dx - \sum_{\ell=0}^{k-2} \int_{\mathbb{R}^d} \nabla^{\ell+1} P \cdot \nabla^\ell \Big(P\mathbf{v}^\varepsilon + p^0 V\Big) dx \\
&\qquad - \varepsilon \sum_{\ell=0}^{k-2} \int_{\mathbb{R}^d} \nabla^\ell V \cdot \nabla^\ell \nabla |\mathbf{v}^\varepsilon|^2 dx \\
&\quad := \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned} \tag{4.3}$$

First $\mathcal{I}_1$ can be estimated by the Hölder's inequality and (1.4) as follows:

$$\mathcal{I}_1 \le \varepsilon \sum_{\ell=0}^{k-2} \|\nabla^\ell V\|_{L^2(\mathbb{R}^d)} \|\nabla^{\ell+2}\mathbf{v}^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C\varepsilon \|V\|_{H^{k-2}(\mathbb{R}^d)} \|\mathbf{v}^\varepsilon\|_{H^k(\mathbb{R}^d)} \le C\varepsilon \|V\|_{H^{k-2}(\mathbb{R}^d)}. \tag{4.4}$$

For $\mathcal{I}_2$, we use Hölder's inequality, the product estimate and the interpolation to infer that

$$\begin{aligned}
\mathcal{I}_2 &\le \sum_{\ell=0}^{k-2} \|\nabla^{\ell+1} P\|_{L^2(\mathbb{R}^d)} \big\|\nabla^\ell \big(P\mathbf{v}^\varepsilon + p^0 V\big)\big\|_{L^2(\mathbb{R}^d)} \\
&\le C\|\nabla P\|_{H^{k-2}(\mathbb{R}^d)} \Big(\|P\mathbf{v}^\varepsilon\|_{H^{k-2}(\mathbb{R}^d)} + \|p^0 V\|_{H^{k-2}(\mathbb{R}^d)}\Big) \\
&\le C\|\nabla P\|_{H^{k-2}(\mathbb{R}^d)} \Big(\|P\|_{H^{k-2}(\mathbb{R}^d)} \|\mathbf{v}^\varepsilon\|_{H^k(\mathbb{R}^d)} + \|p^0\|_{H^k(\mathbb{R}^d)} \|V\|_{H^{k-2}(\mathbb{R}^d)}\Big) \\
&\le C\|\nabla P\|_{H^{k-2}(\mathbb{R}^d)} \Big(\|P\|_{H^{k-2}(\mathbb{R}^d)} + \|V\|_{H^{k-2}(\mathbb{R}^d)}\Big).
\end{aligned} \tag{4.5}$$

Similarly, for $\mathcal{I}_3$, we have

$$\mathcal{I}_3 \le \varepsilon \sum_{\ell=0}^{k-2} \|\nabla^\ell V\|_{L^2(\mathbb{R}^d)} \|\nabla^\ell |\mathbf{v}^\varepsilon|^2\|_{L^2(\mathbb{R}^d)} \le C\varepsilon \|V\|_{H^{k-2}(\mathbb{R}^d)} \|\mathbf{v}^\varepsilon\|^2_{H^k(\mathbb{R}^d)} \le C\varepsilon \|V\|_{H^{k-2}(\mathbb{R}^d)}. \tag{4.6}$$

Substituting (4.4), (4.5) and (4.6) into (4.3), and using Young's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|P\|^2_{H^{k-2}(\mathbb{R}^d)} + \|V\|^2_{H^{k-2}(\mathbb{R}^d)}\Big) + \|\nabla P\|^2_{H^{k-2}(\mathbb{R}^d)} \\
&\quad \le C\varepsilon \|V\|_{H^{k-2}(\mathbb{R}^d)} + C\|\nabla P\|_{H^{k-2}(\mathbb{R}^d)} \Big(\|P\|_{H^{k-2}(\mathbb{R}^d)} + \|V\|_{H^{k-2}(\mathbb{R}^d)}\Big) \\
&\quad \le \frac{1}{2}\|\nabla P\|^2_{H^{k-2}(\mathbb{R}^d)} + C\Big(\|P\|^2_{H^{k-2}(\mathbb{R}^d)} + \|V\|^2_{H^{k-2}(\mathbb{R}^d)}\Big) + C\varepsilon^2,
\end{aligned}$$

which yields that

$$\frac{d}{dt}\Big(\|P\|^2_{H^{k-2}(\mathbb{R}^d)}+\|V\|^2_{H^{k-2}(\mathbb{R}^d)}\Big)\le C_7\Big(\|P\|^2_{H^{k-2}(\mathbb{R}^d)}+\|V\|^2_{H^{k-2}(\mathbb{R}^d)}\Big)+C_7\varepsilon^2,$$

where C_7 is a positive constants independent of ε. Hence, by Gronwall's inequality, we get

$$\|P\|^2_{H^{k-2}(\mathbb{R}^d)}+\|V\|^2_{H^{k-2}(\mathbb{R}^d)}\le\varepsilon^2 e^{C_7 t}\quad\text{for any}\ \ t\in[0,\infty).$$

This completes the proof of Theorem 1.3. □

5. Proof of Theorem 1.4

We are in a position to prove Theorem 1.4. Since u remains the same in the original model (1.1) and the transformed system (1.3), the result for u is straightforward. With the Cole–Hopf transformation (1.2), the existence of global classical solutions of (1.1) results from Theorem 1.1 directly with parabolic regularity theory. Next we derive the time convergence rate for u which follows from (1.5) and the Gagliardo–Nirenberg inequality for $d=2,3$ that

$$\begin{aligned}\|u-\bar u\|_{L^\infty(\mathbb{R}^d)}&=\|p\|_{L^\infty(\mathbb{R}^d)}\\&\le C\|p\|^{\frac12}_{L^2(\mathbb{R}^d)}\|\nabla^d p\|^{\frac12}_{L^2(\mathbb{R}^d)}\\&\le C\|p\|^{\frac12}_{L^2(\mathbb{R}^d)}\|\nabla^2 p\|^{\frac12}_{L^2(\mathbb{R}^d)}\\&\le C(1+t)^{-\frac{1+s}{2}}\end{aligned}$$

where we have used the fact that higher derivative has steeper decay in time (see Theorem 1.2). We proceed to examine the decay rate for the chemical concentration c. From the second equation of (1.1) and the Cole–Hopf transformation (1.2), we can derive that

$$(\ln c)_t=-\varepsilon\nabla\mathbf{v}+\varepsilon\mathbf{v}^2-u$$

which, upon the integration, yields

$$c(x,t)=c_0(x)\exp\bigg(-\bar u t+\int_0^t(\bar u-u+\varepsilon(\mathbf{v}^2-\nabla\mathbf{v}))d\tau\bigg).\tag{5.1}$$

From (1.5) and Gagliardo–Nirenberg inequality, we have

$$\begin{aligned}\int_0^t\|u-\bar u\|_{L^\infty(\mathbb{R}^d)}d\tau&\le C\int_0^t\|p\|^{\frac12}_{L^2(\mathbb{R}^d)}\|\nabla^d p\|^{\frac12}_{L^2(\mathbb{R}^d)}d\tau\\&\le C\int_0^t\|p\|^{\frac12}_{L^2(\mathbb{R}^d)}\|\nabla^2 p\|^{\frac12}_{L^2(\mathbb{R}^d)}d\tau\end{aligned}$$

$$\begin{aligned}&\le C\int_0^t (1+\tau)^{-\frac{1+s}{2}}d\tau\\&\le C(1+t)^{\frac{1-s}{2}}.\end{aligned}\tag{5.2}$$

In a similar way, we may readily derive that

$$\int_0^t \|\mathbf{v}^2\|_{L^\infty(\mathbb{R}^d)}d\tau \le C(1+t)^{-s}\tag{5.3}$$

and

$$\int_0^t \|\nabla\mathbf{v}\|_{L^\infty(\mathbb{R}^d)}d\tau \le C(1+t)^{\frac{1-s}{2}}.\tag{5.4}$$

Substituting (5.2)–(5.4) into (5.1), we can derive that

$$\|c\|_{L^\infty(\mathbb{R}^d)} \le Ce^{-\bar{u}(1+t)[1-C(1+t)^{-\frac{1+s}{2}}-C(1+t)^{-1-s}]} \le Ce^{-\bar{u}t}.$$

Now we let $(u^\varepsilon, c^\varepsilon)$ denote the solution of (1.1) with $\varepsilon \ge 0$, and derive the estimate $v^\varepsilon - c^0$ with respect to ε. To this end, we let $q = v^\varepsilon - c^0$ and obtain the equation for q from the second equation of (1.1) as follows:

$$\begin{cases} q_t = \varepsilon\Delta q - u^0 q - (u^\varepsilon - u^0)c^\varepsilon, \; x\in\mathbb{R}^d, \; t>0\\ q(x,0)=0.\end{cases}$$

Then applying ∇^ℓ to the equation, taking inner product with $\nabla^\ell q$ and adding up the results from $l=0$ to $l=k-2$, we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}\|q\|^2_{H^{k-2}(\mathbb{R}^d)}+\varepsilon\|q\|^2_{H^{k-1}(\mathbb{R}^d)}\\&=-\sum_{l=0}^{k-2}\int_{\mathbb{R}^d}\nabla^\ell q\nabla^\ell(qu^0)dx-\sum_{l=0}^{k-2}\int_{\mathbb{R}^d}\nabla^\ell q\nabla^\ell((u^\varepsilon-u^0)c^\varepsilon)dx := \mathcal{J}_1+\mathcal{J}_2.\end{aligned}$$

For $\mathcal{J}_1$, we have the following estimates

$$\begin{aligned}\mathcal{J}_1 &\le \sum_{l=0}^{k-2}\|\nabla^\ell q\|_{L^2(\mathbb{R}^d)}\|\nabla^\ell(qu^0)\|_{L^2(\mathbb{R}^d)}\\&\le C\|q\|_{H^{k-2}(\mathbb{R}^d)}\|qu^0\|_{H^{k-2}(\mathbb{R}^d)}\\&\le \|q\|^2_{H^{k-2}(\mathbb{R}^d)}\|u^0\|_{L^\infty(\mathbb{R}^d)}\\&\le \|q\|^2_{H^{k-2}(\mathbb{R}^d)}.\end{aligned}$$

Similarly, $\mathcal{J}_2$ is estimated as:

$$
\begin{aligned}
\mathcal{J}_2 &\le \sum_{l=0}^{k-2} \|\nabla^\ell q\|_{L^2(\mathbb{R}^d)} \|\nabla^\ell (u^\varepsilon - u^0) c^\varepsilon\|_{L^2(\mathbb{R}^d)} \\
&\le C\|q\|_{H^{k-2}(\mathbb{R}^d)} \|u^\varepsilon - u^0\|_{H^{k-2}(\mathbb{R}^d)} \|c^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \\
&\le C\|q\|^2_{H^{k-2}(\mathbb{R}^d)} + \|u^\varepsilon - u^0\|^2_{H^{k-2}(\mathbb{R}^d)} \|c^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)} \\
&\le C(\|q\|^2_{H^{k-2}(\mathbb{R}^d)} + \varepsilon^2 e^{Ct}).
\end{aligned}
$$

Thus it follows that

$$
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} \|q\|^2_{H^{k-2}(\mathbb{R}^d)} + \varepsilon\|q\|^2_{H^{k-1}(\mathbb{R}^d)} \le C(\|q\|^2_{H^{k-2}(\mathbb{R}^d)} + \varepsilon^2 e^{Ct})
$$

which, along with the Gronwall's inequality, gives the desired results and the proof is completed.

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