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Global solvability of a class of reaction–diffusion systems with cross-diffusion

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This paper is concerned with a class of reaction–diffusion systems with cross-diffusion which have various applications such as autocatalytic chemical reaction, predator–prey interactions, combustion and so on. By imposing some suitable structure assumptions, we establish the global existence and asymptotic behavior of solutions of the system in a two-dimensional bounded domain with Neumann boundary conditions.

1. Introduction

This paper is concerned with the following reaction–diffusion system with cross-diffusion

$$\begin{cases} u_t = \Delta(d(v)u) + \theta u^\alpha F(v), & x \in \Omega, \quad t > 0, \\ v_t = D\Delta v - u^\alpha F(v), & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ denote the concentration of two substances, $\theta \geq 0, D > 0$ and $0 < \alpha \leq 1$ are constants, $d(v)$ is a density-dependent diffusion coefficient satisfying

$$d \in C^3([0,+\infty)),\ d > 0,\ d'(v) \neq 0 \text{ on } [0,+\infty). \tag{1.2}$$

The function $F(v)$ satisfies

$$F(v) \in C^1([0,\infty)),\ F(0) \geq 0,\ F(v) > 0 \text{ in } (0,\infty) \text{ and } F'(v) > 0 \text{ on } [0,\infty). \tag{1.3}$$

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When $d(v)$ is constant, the system (1.1) has various applications. For instance, when $\alpha = \theta = 1, F(v) = v^\beta(\beta > 1)$, the model (1.1) was the well-known isothermal autocatalytic system describing the autocatalytic chemical reaction of order $m+1$ (cf. [1–3]). When $\alpha = 1$ and $\theta > 0$, equations in (1.1) can be regarded as a simplest predator–prey system with the trophic function $F(v)$ (cf. [4]). When $\theta > 0, F(v) = e^v$, the model can describe the exothermic combustion (cf. [5]). The above-mentioned are only partial applications of (1.1) with constant $d(v)$ and more applications can be found in a survey paper [6]. There are many interesting analytical results obtained on the global solvability of (1.1) with constant $d(v)$ (see [6]) but many interesting questions still remain open. When $d(v)$ is non-constant, (1.1) becomes a cross-diffusion system and its global well-posedness problem was one of the open questions proposed in [6]. As we know, there are no results in the literature addressing the global solvability of (1.1) with non-constant $d(v)$. The purpose of this paper is to establish the global existence and asymptotic behavior of classical solutions to (1.1) with $d(v)$ satisfying (1.2) and $F(v)$ fulfilling (1.3). We remark that the Chapman's cross-diffusion $\Delta(d(v)u)$ has various applications in biological modelings which have attracted attentions in mathematical analysis recently, for example the preytaxis [7,8], density-suppressed motility (cf. [9–11]), chemotaxis [12,13] and competition system [14,15].

The main results of this paper are stated in the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume $(u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$ with $u_0, v_0 \gneqq 0$, $d(v)$ and $F(v)$ satisfy* (1.2) and (1.3)*, respectively. Then the problem* (1.1) *has a unique global classical solution $(u, v) \in [C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))]^2$ satisfying $u, v > 0$ for all $t > 0$ and*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \le C,$$

where $C > 0$ is a constant independent of t. Moreover, if $F(0) = 0$, then the solution (u, v) satisfies: $u(x, t) \to u_*$ and $v(\cdot, t) \to 0$ in $L^\infty(\Omega)$, with $u_* = \frac{1}{|\Omega|}\int_\Omega u_0 dx + \frac{\theta}{|\Omega|}\int_\Omega v_0 dx$.

2. Proof of Theorem 1.1

In the sequel, we shall use c, C or C_i $(i = 1, 2, \ldots)$ to denote a generic positive constant which may vary in the context. For simplicity, the integration variables x and t will be omitted, for instance $\int_0^T \int_\Omega f(x, t)dxdt$ will be abbreviated as $\int_0^T \int_\Omega f(x, t)$. We will write $\|f\|_{L^p(\Omega)}$ as $\|f\|_{L^p}$. We first present the existence of local solutions of (1.1), which can be readily proved by the Amann's theorem [16,17].

Lemma 2.1 (*Local Existence*)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume $(u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$ with $u_0, v_0 \gneqq 0$ for all $x \in \Omega$, then there exist a constant $T_{max} \in (0, \infty]$ and a pair (u, v) of non-negative functions*

$$(u, v) \in [C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))]^2,$$

which solves (1.1) *in the classical sense in $\Omega \times (0, T_{max})$. Moreover*

$$\text{if } T_{max} < \infty, \text{ then } \|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}} \to \infty \text{ as } t \nearrow T_{max}. \tag{2.1}$$

Proof. Denote $z = (u, v)$. Then the system (1.1) can be written as

$$\begin{cases} z_t = \nabla \cdot (P(z)\nabla z) + Q(z), & x \in \Omega, \quad t > 0, \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ z(\cdot, 0) = (u_0, v_0), & x \in \Omega, \end{cases} \tag{2.2}$$

where $P(z) = \begin{pmatrix} d(v) & d'(v)u \\ 0 & D \end{pmatrix}$, $\quad Q(z) = \begin{pmatrix} \theta u^\alpha F(v) \\ -u^\alpha F(v) \end{pmatrix}$. The matrix $P(z)$ is positive definite for the given initial data, which indicates that system (2.2) is normally parabolic. Then the results of [16, Theorem 7.3] give a $T_{max} > 0$ such that system (2.2) possesses a unique solution $(u, v) \in [C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))]^2$.

Next, we will prove that $u, v \geq 0$ by the maximum principle. To this end, we rewrite the first equation of (1.1) as $u_t = d(v)\Delta u + d'(v)\nabla v \cdot \nabla u + \theta u^\alpha F(v)$. Then by the strong maximum principle, we derive that $u > 0$ for all $(x, t) \in \Omega \times (0, T_{max})$ due to $u_0 \not\equiv 0$. Similarly, we can show $v > 0$ for any $(x, t) \in \Omega \times (0, T_{max})$ by applying the strong maximum principle to the second equation of system (1.1). Since $P(z)$ is an upper triangular matrix, we obtain (2.1) by [17, Theorem 5.2] directly. Therefore the proof of Lemma 2.1 is completed. □

Lemma 2.2. *The solution (u, v) of (1.1) satisfies $\|v(\cdot, t)\|_{L^\infty} \leq \|v_0\|_{L^\infty}$ and*

$$\|u(\cdot, t)\|_{L^1} + \theta\|v(\cdot, t)\|_{L^1} = \|u_0\|_{L^1} + \theta\|v_0\|_{L^1} \text{ for all } t \in (0, T_{max}). \tag{2.3}$$

Proof. By the maximum principle applied to the second equation of (1.1), we get $\|v(\cdot, t)\|_{L^\infty} \leq \|v_0\|_{L^\infty}$ immediately. Multiplying the second equation (1.1) by θ and adding the resulting equation to the first equation of (1.1), we obtain (2.3) directly upon an integration. □

Lemma 2.3. *Let (u, v) be a solution of (1.1). Then there exist constants $c, C > 0$ such that*

$$c \leq \int_\Omega u \leq C \text{ for all } t \in (0, T_{max}). \tag{2.4}$$

Proof. Integrating the first equation in (1.1) over Ω with the boundary conditions, we have

$$\frac{d}{dt}\int_\Omega u \geq 0 \text{ and } \int_\Omega u \geq \int_\Omega u_0.$$

We multiply the second equation of (1.1) by θ and add the resulting equation to the first equation of (1.1). Then integrating the result over Ω by parts along with the boundary conditions, we get $\frac{d}{dt}\left(\int_\Omega u + \theta \int_\Omega v\right) = 0$, which indicates $\int_\Omega u + \theta \int_\Omega v = \int_\Omega u_0 + \theta \int_\Omega v_0$. Therefore we obtain (2.4) from the non-negativity of u and v. □

Second, we aim to derive the bound of u in space–time L^2 norm by the classical duality-based arguments (see [18,19]). For convenience, we first introduce some notation. Let A_0 denote the self-adjoint operator of $-\Delta$ defined in the Hilbert space

$$D(A_0) = \left\{\phi \in W^{2,2}(\Omega) \Big| \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \partial\Omega \text{ and } \int_\Omega \phi = 0\right\}.$$

By the classical duality-based arguments, we derive the following boundedness of u in space–time L^2.

Lemma 2.4. *Let (u, v) be a solution of (1.1). Then there exists a constant $C > 0$ such that*

$$\int_t^{t+\tau}\int_\Omega u^2 \leq C \text{ for all } t \in [0, \tilde{T}_{max}),$$

where

$$\tau := \min\left\{1, \frac{1}{2}T_{max}\right\} \text{ and } \tilde{T}_{max} := \begin{cases} T_{max} - \tau, & \text{if } T_{max} < \infty, \\ \infty, & \text{if } T_{max} = \infty. \end{cases} \tag{2.5}$$

Proof. By (1.3) and maximum principle, we derive from the second equation in (1.1) that $v \le K_0 = \|v_0\|_{L^\infty}$. It follows from (1.2) that there exist positive constants C_1 and C_2 such that $0 < C_1 \le d(v) \le C_2$. Multiplying the second equation of (1.1) by θ and adding the result equation to the first equation of (1.1), one has

$$(u+\theta v-\bar{u}-\theta\bar{v})_t = -A_0(d(v)u+\theta Dv-\overline{d(v)u}-\theta D\bar{v}), \tag{2.6}$$

where $\bar{u}=\frac{1}{|\Omega|}\int_\Omega u$. In view of the fact $\int_\Omega(u+\theta v-\bar{u}-\theta\bar{v})=0$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |A_0^{-\frac{1}{2}}(u+\theta v-\bar{u}-\theta\bar{v})|^2 = \int_\Omega A_0^{-\frac{1}{2}}(u+\theta v-\bar{u}-\theta\bar{v})[A_0^{-\frac{1}{2}}(u+\theta v-\bar{u}-\theta\bar{v})]_t\\
&=\int_\Omega A_0^{-1}(u+\theta v-\bar{u}-\theta\bar{v})(u+\theta v-\bar{u}-\theta\bar{v})_t\\
&=-\int_\Omega A_0^{-1}(u+\theta v-\bar{u}-\theta\bar{v})A_0(d(v)u+\theta Dv-\overline{d(v)u}-\theta D\bar{v})\\
&=-\int_\Omega (u+\theta v-\bar{u}-\theta\bar{v})(d(v)u+\theta Dv-\overline{d(v)u}-\theta D\bar{v})\\
&=-\int_\Omega d(v)(u-\bar{u})^2-\bar{u}\int_\Omega d(v)(u-\bar{u})-\theta\int_\Omega(D+d(v))(u-\bar{u})(v-\bar{v})\\
&\quad -\theta\bar{u}\int_\Omega d(v)(v-\bar{v})-\theta^2 D\int_\Omega(v-\bar{v})^2\\
&\le -C_1\int_\Omega(u-\bar{u})^2+C_2|\Omega|\bar{u}^2+\theta\int_\Omega(D+d(v))(u\bar{v}+\bar{u}v)+\theta\int_\Omega d(v)\bar{u}\bar{v}-\theta^2 D\int_\Omega(v-\bar{v})^2\\
&\le -C_1\int_\Omega(u-\bar{u})^2+C_2|\Omega|\bar{u}^2+2\theta(D+C_2)|\Omega|\bar{u}\bar{v}+C_2\theta|\Omega|\bar{u}\bar{v}-\theta^2 D\int_\Omega(v-\bar{v})^2.\\
&\le -C_1\int_\Omega(u-\bar{u})^2-\theta^2 D\int_\Omega(v-\bar{v})^2+C_3,
\end{aligned}$$

where we have used the boundedness of $\bar{u},\bar{v}$, (2.4) and the fact $0<C_1\le d(v)\le C_2$. Therefore,

$$\frac{d}{dt}\int_\Omega |A_0^{-\frac{1}{2}}(u+\theta v-\bar{u}-\theta\bar{v})|^2+2C_1\int_\Omega(u-\bar{u})^2+2\theta^2 D\int_\Omega(v-\bar{v})^2\le 2C_3. \tag{2.7}$$

Applying the Poincaré inequality and the fact $\int_\Omega A_0^{-\frac{1}{2}}(u+\theta v-\bar{u}-\theta\bar{v})=0$, we can find a constant $C_4>0$ such that

$$\begin{aligned}
&\int_\Omega |A_0^{-\frac{1}{2}}(u+\theta v-\bar{u}-\theta\bar{v})|^2\\
&\le C_4\int_\Omega|\nabla A_0^{-\frac{1}{2}}(u+\theta v-\bar{u}-\theta\bar{v})|^2 = C_4\int_\Omega|u+\theta v-\bar{u}-\theta\bar{v}|^2\\
&\le 2C_4\int_\Omega(u-\bar{u})^2+2C_4\theta^2\int_\Omega(v-\bar{v})^2\le 2C_4\int_\Omega(u-\bar{u})^2+2C_4\theta^2|\Omega|\|v_0\|_{L^\infty}^2,
\end{aligned}$$

which combined with (2.7) implies there exists a constant $C_5>0$ such that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |A_0^{-\frac{1}{2}}(u+\theta v-\bar{u}-\theta\bar{v})|^2 &+ \frac{C_1}{2C_4}\int_\Omega |A_0^{-\frac{1}{2}}(u+\theta v-\bar{u}-\theta\bar{v})|^2\\
&+C_1\int_\Omega(u-\bar{u})^2+2\theta^2 D\int_\Omega(v-\bar{v})^2\le C_5.
\end{aligned} \tag{2.8}$$

By Grönwall's inequality, we find a constant $C_6>0$ such that $\int_\Omega|A_0^{-\frac{1}{2}}(u+\theta v-\bar{u}-\theta\bar{v})|^2\le C_6$. Integrating (2.8) over $(t,t+\tau)$, yields a constant $C_7>0$ such that $\int_t^{t+\tau}\int_\Omega(u-\bar{u})^2\le C_7$. Then by Lemma 2.3, we obtain

$$\int_t^{t+\tau}\int_\Omega u^2\le 2\int_t^{t+\tau}\int_\Omega(u-\bar{u})^2+2\int_t^{t+\tau}\int_\Omega\bar{u}^2\le C_7+2\bar{u}^2|\Omega|\le C_8.$$

This completes the proof of Lemma 2.4. □

Lemma 2.5. *Let (u,v) be a solution of (1.1). Then there exists a constant $C>0$ such that*

$$\int_\Omega |\nabla v|^2 \le C \text{ and } \int_t^{t+\tau}\int_\Omega |\Delta v|^2 \le C \text{ for all } t\in(0,\tilde{T}_{max}), \tag{2.9}$$

where τ and $\tilde{T}_{max}$ are defined in (2.5).

Proof. By simple computations, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 = -D\int_\Omega |\Delta v|^2 + \int_\Omega u^\alpha F(v)\Delta v \le -\frac{D}{2}\int_\Omega |\Delta v|^2 + \frac{1}{2D}\int_\Omega u^{2\alpha}F^2(v).$$

Then by the boundedness of v and (1.3), we can find a constant $C_1>0$ such that

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 + D\int_\Omega |\Delta v|^2 \le \frac{1}{D}\int_\Omega u^{2\alpha}F^2(v) \le C_1\int_\Omega u^{2\alpha} \le C_1\int_\Omega u^2. \tag{2.10}$$

Using the Gagliardo–Nirenberg inequality (see [11, Lemma 2.5]) and boundedness of v, one has

$$\int_\Omega |\nabla v|^2 dx = \|\nabla v\|_{L^2}^2 \le C_2(\|\Delta v\|_{L^2}\|v\|_{L^2} + \|v\|_{L^2}^2) \le \frac{D}{2}\|\Delta v\|_{L^2}^2 + C_3. \tag{2.11}$$

Substituting (2.11) into (2.10), we get

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 + \int_\Omega |\nabla v|^2 + \frac{D}{2}\int_\Omega |\Delta v|^2 \le C_1\int_\Omega u^2 + C_3. \tag{2.12}$$

Denote $y(t) := \int_\Omega |\nabla v(\cdot,t)|^2 dx$ and $z(t) = C_1\int_\Omega u^2 + C_3$, from (2.12) we have

$$y'(t) + y(t) \le z(t) \text{ for all } t\in(0,T_{max}).$$

By Lemma 2.4 and [20, Lemma 3.4], we derive the first inequality in (2.9). Then integrating (2.12) over $(t,t+\tau)$ along with the first inequality in (2.9) and Lemma 2.4, we obtain the second inequality in (2.9). Therefore, we finish the proof of Lemma 2.5. □

Lemma 2.6. *Let (u,v) be a solution of (1.1). There exist constants $c,C>0$ such that for any $p\ge 2$, and for all $t\in(0,T_{max})$, we have*

$$\frac{d}{dt}\int_\Omega u^p + cp(p-1)\int_\Omega u^{p-2}|\nabla u|^2 \le Cp(p-1)\int_\Omega u^p|\nabla v|^2 + Cp(p-1)\int_\Omega u^p.$$

Proof. By the boundedness of v (see Lemma 2.2), the conditions (1.2) and (1.3), we obtain that there exist constants $C_1,C_2,C_3>0$ such that

$$0\le F(v)\le C_1,\ d(v)\ge C_2,\ \frac{|d'(v)|}{d(v)} \le C_3. \tag{2.13}$$

Multiplying the first equation of (1.1) by u^{p-1} with $p\ge 2$, integrating the resulting equation by parts and using Young's inequality, we derive

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p &= \int_\Omega u^{p-1}\Delta(d(v)u) + \theta\int_\Omega u^{p+\alpha-1}F(v)\\ &\le -(p-1)\int_\Omega d(v)u^{p-2}|\nabla u|^2 + (p-1)\int_\Omega d'(v)u^{p-1}\nabla u\nabla v + C_1\theta\int_\Omega u^{p+\alpha-1}\\ &\le -\frac{p-1}{2}\int_\Omega d(v)u^{p-2}|\nabla u|^2 + \frac{p-1}{2}\int_\Omega \frac{|d'(v)|^2}{d(v)}u^p|\nabla v|^2 + C_1\theta\int_\Omega u^p,\end{aligned}$$

which along with (2.13), yields

$$\frac{d}{dt}\int_\Omega u^p + \frac{p(p-1)}{2}C_2\int_\Omega u^{p-2}|\nabla u|^2 \le \frac{p(p-1)}{2}C_3\int_\Omega u^p|\nabla v|^2 + C_1\theta p\int_\Omega u^p.$$

This completes the proof of Lemma 2.6. □

Based on Lemma 2.5 and the Gagliardo–Nirenberg inequality, we will derive the a priori L^2 estimate of the solution component u.

Lemma 2.7. *Let (u, v) be a solution of (1.1). Then there exists a constant $C > 0$ such that*

$$\int_\Omega u^2 \le C \text{ for all } t \in (0, T_{max}).$$

Proof. Let $p = 2$ in Lemma 2.6, we get the following estimate

$$\frac{d}{dt}\int_\Omega u^2 + C_1\int_\Omega |\nabla u|^2 \le C_2\int_\Omega u^2|\nabla v|^2 + C_2\int_\Omega u^2 \tag{2.14}$$

for some constants $C_1, C_2 > 0$. We apply Gagliardo–Nirenberg inequality to derive that

$$\|u\|^2_{L^4} \le C_3\left(\|\nabla u\|^{\frac12}_{L^2}\|u\|^{\frac12}_{L^2} + \|u\|_{L^2}\right)^2, \tag{2.15}$$

and we use [11, Lemma 2.5] and (2.9) to derive that

$$\|\nabla v\|^2_{L^4} \le C_3(\|\Delta v\|_{L^2}\|\nabla v\|_{L^2} + \|\nabla v\|^2_{L^2}) \le C_4(\|\Delta v\|_{L^2} + 1), \tag{2.16}$$

where the spatial dimension $n = 2$ has been used.

Then we use (2.15) and (2.16), we obtain that

$$\begin{aligned}\int_\Omega u^2|\nabla v|^2 \le \|u\|^2_{L^4}\|\nabla v\|^2_{L^4} &\le C_5\left(\|\nabla u\|^{\frac12}_{L^2}\|u\|^{\frac12}_{L^2} + \|u\|_{L^2}\right)^2(\|\Delta v\|_{L^2} + 1)\\ &\le \frac{C_1}{2C_2}\|\nabla u\|^2_{L^2} + C_6(1 + \|\Delta v\| + \|\Delta v\|^2_{L^2})\|u\|^2_{L^2}\\ &\le \frac{C_1}{2C_2}\|\nabla u\|^2_{L^2} + C_6\left(1 + \frac{1 + \|\Delta v\|^2}{2} + \|\Delta v\|^2_{L^2}\right)\|u\|^2_{L^2}\\ &\le \frac{C_1}{2C_2}\|\nabla u\|^2_{L^2} + C_7(1 + \|\Delta v\|^2_{L^2})\|u\|^2_{L^2}.\end{aligned}$$

Therefore,

$$C_2\int_\Omega u^2|\nabla v|^2 \le \frac{C_1}{2}\|\nabla u\|^2_{L^2} + C_2C_7(1 + \|\Delta v\|^2_{L^2})\|u\|^2_{L^2}.$$

Substituting the above inequality into (2.14), we derive that there exists a constant $C_8 > 0$ such that

$$\frac{d}{dt}\int_\Omega u^2 + \frac{C_1}{2}\int_\Omega |\nabla u|^2 \le C_8(1 + \|\Delta v\|^2_{L^2})\int_\Omega u^2.$$

An application of Lemma 2.4, Lemma 2.5 and uniform Grönwall inequality ([21, P91, Lemma 1.1] gives the desired result. □

Lemma 2.8. *Let (u, v) be a solution of (1.1). For any $p \ge 1$, there exists a constant $C > 0$ such that*

$$\int_\Omega |\nabla v|^p \le C \text{ for all } t \in (0, T_{max}).$$

Proof. Lemma 2.8 is a direct consequence of [22, Lemma 1] and Lemma 2.7. □

Lemma 2.9. *Let (u, v) be a solution of (1.1). Then there exists a constant $C > 0$ such that*

$$\|u\|_{L^\infty} \le C \text{ for all } t \in (0, T_{max}).$$

Proof. By Lemma 2.6, we can find constants $C_1, C_2 > 0$ such that for any $p \ge 2$

$$\frac{d}{dt}\int_\Omega u^p + C_1\frac{p-1}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \le C_2 p(p-1)\int_\Omega u^p|\nabla v|^2 + C_2 p(p-1)\int_\Omega u^p. \tag{2.17}$$

As a consequence of Lemma 2.8 and Young's inequality, there exists $C_3 > 0$ such that

$$C_2 p(p-1)\int_\Omega u^p|\nabla v|^2 \le C_2 p(p-1)\left(\int_\Omega u^{2p}\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla v|^4\right)^{\frac{1}{2}} \le C_3 p(p-1)\left(\int_\Omega u^{2p}\right)^{\frac{1}{2}}$$

and

$$(C_2+1)p(p-1)\int_\Omega u^p \le C_3 p(p-1)\left(\int_\Omega u^{2p}\right)^{\frac{1}{2}},$$

which combined with (2.17) gives that

$$\frac{d}{dt}\int_\Omega u^p + C_1\frac{p-1}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + p(p-1)\int_\Omega u^p \le 2C_3 p(p-1)\left(\int_\Omega u^{2p}\right)^{\frac{1}{2}}.$$

The Gagliardo–Nirenberg inequality and Young's inequality provide constants $C_4, C_5 > 0$ such that

$$\begin{aligned} 2C_3\left(\int_\Omega u^{2p}\right)^{\frac{1}{2}} = 2C_3\|u^{\frac{p}{2}}\|_{L^4}^2 &\le C_4\left(\|\nabla u^{\frac{p}{2}}\|_{L^2}^{\frac{3}{4}}\|u^{\frac{p}{2}}\|_{L^1}^{\frac{1}{4}} + \|u^{\frac{p}{2}}\|_{L^1}\right)^2 \\ &\le 2C_4\left(\|\nabla u^{\frac{p}{2}}\|_{L^2}^{\frac{3}{2}}\|u^{\frac{p}{2}}\|_{L^1}^{\frac{1}{2}} + \|u^{\frac{p}{2}}\|_{L^1}^2\right) \\ &\le \frac{C_1}{2p^2}\|\nabla u^{\frac{p}{2}}\|_{L^2}^2 + C_5(1+p^6)\|u^{\frac{p}{2}}\|_{L^1}^2. \end{aligned} \tag{2.18}$$

Substituting (2.18) into (2.17) and noting $1 + p^6 \le (1 + p)^6$, we obtain

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p \le C_5 p(p-1)(1+p)^6\left(\int_\Omega u^{\frac{p}{2}}\right)^2.$$

Then employing a standard Moser iteration (cf. [23]) or a similar argument as in [24] to the above inequality, the desired result can be obtained. □

Proof of Theorem 1.1. Theorem 1.1 is a consequence of Lemma 2.9, [22, Lemma 1] and the extension criterion Lemma 2.1. The asymptotic stability result asserted in Theorem 1.1 can be proved by the same argument as for Theorem 1.2 in [18] and we omit the details here for brevity.

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