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2011 Nonlinearity 24 3279

(http://iopscience.iop.org/0951-7715/24/12/001)

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Nonlinearity **24** (2011) 3279–3297 doi:10.1088/0951-7715/24/12/001

Singularity formation in chemotaxis systems with volume-filling effect

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Received 23 December 2010, in final form 5 July 2011
Published 28 October 2011
Online at stacks.iop.org/Non/24/3279

Recommended by J A Glazier

Abstract
A parabolic–elliptic model of chemotaxis which takes into account volume-filling effects is considered under the assumption that there is an *a priori* threshold for the cell density. For a wide range of nonlinear diffusion operators including singular and degenerate ones it is proved that if the taxis force is strong enough with respect to diffusion and the initial data are chosen properly then there exists a classical solution which reaches the threshold at the maximal time of its existence, no matter whether the latter is finite or infinite. Moreover, we prove that the threshold may even be reached in finite time provided the diffusion of cells is non-degenerate.

Mathematics Subject Classification: 35K55, 35K65, 34B15, 34C25, 92C17

(Some figures may appear in colour only in the online journal)

1. Introduction

This paper is concerned with processes of singularity formation in a parabolic–elliptic model of chemotaxis which takes into account volume-filling effects. In this class of models arbitrarily high densities of biological cells are precluded due to the finite size of cells and volume limitations. The density of cells is then assumed to be *a priori* bounded by the threshold corresponding to the tight packing state. This concept was first developed by Painter and Hillen [19] with a further investigation by Wang and Hillen [24] (cf also [9, 29] for a broader survey on volume-filling effects in chemotaxis models).

More precisely, we let u denote the cell density and v represent the concentration of chemoattractant. Then adopting the well-accepted assumption that chemicals diffuse much faster than cells [13], we shall consider the parabolic–elliptic problem

$$\begin{cases} u_t = \nabla \cdot \{D(u)\nabla u - uh(u)\nabla v\}, & x \in \Omega,\ t > 0, \\ 0 = \Delta v - m + u, & x \in \Omega,\ t > 0, \\ \displaystyle\int_\Omega v(x,t)\,\mathrm{d}x = 0, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $\frac{\partial}{\partial \nu}$ denotes differentiation with respect to the outward normal vector field on $\partial\Omega$, and

$$m = m(u_0) = \bar{u}_0 := \frac{1}{|\Omega|}\int_\Omega u_0(x)\,\mathrm{d}x. \tag{1.2}$$

Here the cell flux consists of diffusive flux corresponding to the collective diffusion (in the terminology of [7]) of cells, and chemotactic flux (taxis) which is due to the motion of cells towards the concentration gradient of chemoattractant.

Guided by various approaches to incorporating volume-filling effects [9, 29] it seems reasonable to assume that the parameter functions in (1.1) exhibit a power-type behaviour near the threshold density, the latter being normalized so as to be at $u = 1$. We therefore have in mind functional choices of the form

$$D(u) \simeq (1-u)^{-\alpha}, \qquad u \simeq 1 \tag{1.3}$$

and

$$h(u) \simeq (1-u)^{\beta}, \qquad u \simeq 1, \tag{1.4}$$

with certain constants α and β.

A chemotaxis model with singular diffusion (or superdiffusion) was recently derived in [17] as a macroscopic limit of microscopic cellular Potts model, where u is interpreted as a volume fraction, and this particular model corresponds to $\alpha = 2$ and $\beta = 0$. On the other hand, in the model investigated in [24] some fast diffusion appears in the sense that $\alpha = 1 - r$ and $\beta = r$ for some $r \in (0, 1)$. We moreover refer the reader to [3] where the parabolic–elliptic model of chemotaxis with linear diffusion $\alpha = 0$ and $\beta = 1$ as well as the case where $D(u)$ degenerates at $u = 0$ and $u = 1$ are studied both in the whole space $\mathbb{R}^n$ and on bounded domains.

Singularity formation. The particular focus of this work is on the question whether it may occur that solutions approach the threshold value $u = 1$ during evolution, which might be interpreted as cell aggregation in the sense of reaching the tight packing state. Here a considerable history indicates that in the analysis of chemotaxis models, detecting singularity formation seems much more delicate than proving its absence.

This observation already applies to the so-called minimal chemotaxis model which was introduced by Patlak [20] and Keller and Segel [14]. A parabolic–elliptic version thereof can formally be obtained upon setting $D(u) \equiv$ const and $h(u) \equiv \chi =$ const in (1.1), whereas apart from that in the full parabolic–parabolic minimal model the second equation becomes

$$v_t = \Delta v + u - v. \tag{1.5}$$

Again, the most appealing feature of this minimal model is its ability to describe spontaneous singularity formation, which in this case corresponds to the occurrence of solutions blowing up with respect to the norm in $L^\infty(\Omega)$. Indeed, such unbounded solutions are known to exist in both parabolic–parabolic and parabolic–elliptic cases, provided that either $n = 2$ and the initial mass $m(u_0)$ of cells or the chemosensitivity constant χ is big enough, or if $n \geqslant 3$; on the other hand, if $n \geqslant 2$ and the initial data are appropriately small then the solution remains bounded for all times. While the literature addressing the latter bounded solutions is rather comprehensive [9], there is less knowledge on possible blow-up mechanisms, and the mathematical techniques are much more sophisticated [8, 12, 27], some of these being restricted to parabolic–elliptic simplifications only [2, 13, 18, 28, 10, 11].

Accordingly, previous work on chemotaxis systems with nonlinear ingredients as specified in (1.3), (1.4) concentrates on identifying assumptions, essentially on the parameters, which rule out a singularity formation in the sense that u remains bounded away from the value 1. For instance, in the recent paper [30] the parabolic–parabolic system corresponding to (1.1) with the second equation in (1.1) replaced by (1.5) is investigated with functions D and h satisfying

$$D(u) \geqslant K_D(1-u)^{-\alpha} \qquad \text{for all } u \in (1-\delta, 1) \quad (\alpha > 0) \tag{1.6}$$

and

$$h(u) > 0 \quad \text{and} \quad h(u) \leqslant K_h(1-u)^{\beta} \qquad \text{for all } u \in (1-\delta, 1) \quad (\beta \geqslant 0) \tag{1.7}$$

with constants $\delta \in (0, 1)$, $K_D > 0$ and $K_h > 0$. Generalizing a result from [1] where existence of global solutions to the model in [17] is shown, it is proved in [30] that under the assumption

$$\alpha + \beta \geqslant 1 \tag{1.8}$$

solutions exist globally whenever the unique local-in-time classical solution satisfies

$$\sup_{t\in[0,T]\cap[0,T_{\max})} \|(1-u(\cdot,t))^{-1}\|_{L^p(\Omega)} < +\infty \tag{1.9}$$

for some $p > \alpha$ and any $T > 0$. Without imposing the latter condition on the solution itself, it is shown there that a slightly stronger condition than (1.8), namely

$$\frac{\alpha}{2} + \beta \geqslant 1, \tag{1.10}$$

implies the existence of a unique global classical solution.

All such solutions satisfy

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^\infty(\Omega)} < 1$$

for any $T > 0$, and if $\beta > 2$ then this inequality even holds with $T = +\infty$. The case of $\beta = 0$ was previously studied in [5] and the existence of classical global solutions was proved there for $\alpha \geqslant 2$, which is in agreement with (1.10).

The only available result which indicates the possibility of reaching the threshold 1 by a solution of the chemotaxis model with density threshold is contained in [16], where in the spatially one-dimensional case the existence of a stationary solution attaining the value $u = 1$ is proved, provided that the diffusion of cells is degenerate at $u = 1$, that is, when $\alpha < 0$ in (1.3).

Main results. Let us state the hypotheses on the data that we will use in the following, as well as the main results. The functions D and h are assumed to belong to $C^2([0, 1))$ and to satisfy

$$D(u) > 0 \qquad \text{for all } u \in [0, 1). \tag{1.11}$$

In our main theorem we require $\Omega := B_R(0) \subset \mathbb{R}^n$ to be the ball in $\mathbb{R}^n$, centred at the origin, with radius $R > 0$, and the initial data $0 \not\equiv u_0 \in C^0(\bar{\Omega})$ are supposed to be radially symmetric with respect to $x = 0$. Moreover, we assume that

$$0 \leqslant u_0(x) < 1 \qquad \text{for all } x \in \bar{\Omega}, \tag{1.12}$$

and that accordingly the constant m given by (1.2) satisfies $0 < m < 1$.

Our goal is to make sure that if chemotactic flux (taxis) in (1.1) is sufficiently strong as compared with diffusion, then a singularity formation will occur if the initial data are chosen properly. Here, similar to the case when both $D(u)$ and $h(u)$ are defined and positive for all $u \geqslant 0$ in [4, 22, 26], it turns out that in this respect the *ratio* of h and D is the crucial quantity, as commented in [23]. In fact, the validity of the lower estimate

$$\frac{h(u)}{D(u)} \geqslant c_{hD}(1-u)^{\lambda} \qquad \text{for all } u \in (0,1) \quad \text{with some } \lambda \in (0,1) \text{ and } c_{hD} > 0 \tag{1.13}$$

is sufficient to allow for solutions approaching the singular value 1 in finite or infinite time, as stated in the first part of the following main theorem. The second part of the theorem ensures that indeed finite-time singularity formation can be achieved under the additional hypothesis (1.15) asserting that diffusion in (1.1) is non-degenerate:

Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is a ball with radius R and (1.11) holds. Then*

(i) For any $\lambda \in (0, 1)$ there exists $c = c(R, \lambda, n, m) > 0$ with the following property. If (1.13) holds with some

$$c_{hD} \geqslant c, \tag{1.14}$$

then there exist radially symmetric initial data $u_0 \in C^\infty(\bar{\Omega})$ fulfilling (1.12) such that the problem (1.1) has a unique classical solution (u, v) in $\Omega \times (0, T)$ for some $T \in (0, \infty]$ which satisfies $0 \leqslant u < 1$ in $\bar{\Omega} \times [0, T)$ and

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \to 1 \qquad \text{as } t \nearrow T.$$

(ii) If in addition to (1.13) we assume that

$$D(u) \geqslant c_D \qquad \text{for all } u \in [0,1) \quad \text{with some } c_D > 0, \tag{1.15}$$

then the above conclusion holds with some $T < \infty$.

It is worth underlining that for our prototype problem obtained upon the choices $D(u) = (1-u)^{-\alpha}$ and $h(u) = c_{hD}(1-u)^{\beta}$, the condition (1.13) is equivalent to the contradiction of (1.8) with $\lambda = \alpha + \beta$.

Indeed, in this framework the assumption (1.13) is critical: in the forthcoming paper [25], it will be shown that if in (1.3) and (1.4) we have $\alpha + \beta > 1$, then solutions exist globally and remain bounded away from $u = 1$, uniformly for all times; in the critical case $\alpha + \beta = 1$, solutions exist globally if in addition $\alpha < 0$.

It may also be noticed that in the case of singular (fast) diffusion (i.e. $\alpha \in (0, 1)$) on one hand the existence of global weak solutions may be proved in a similar way as that in ([30]) and on the other hand theorem 1.1 may be applied to show that there are global-in-time weak solutions which do attain the value 1, provided that (1.13) and (1.15) hold. We furthermore refer to remark 3.2 for detailed formulae which exhibit how the constant c in (1.14) depends on the space dimension n.

There is a number of questions that have to be left open in this work. For instance, it would be interesting to see whether, given $\lambda \in (0, 1)$, one can find a critical value of $c_{hD} > 0$ in (1.13) such that singular solutions occur only above this threshold, and that possibly all solutions remain bounded away from $u = 1$ at least when $(1-u)^{-\lambda} \cdot \frac{h(u)}{D(u)}$ is small throughout $(0, 1)$. As mentioned before, the question of criticality of the value $\lambda = 1$ will be addressed in [25], but in the borderline case when $\alpha + \beta = 1$ it is not clear whether singularities may occur at least for large initial data.

Evidently, this work concentrates on the simplest conceivable setting in respect of the structure of the PDE system as well as the geometry of solutions. Accordingly, some natural next steps consist of possible extensions to both the nonradial framework and the case of corresponding parabolic–parabolic Keller–Segel systems.

2. Local existence and uniqueness

Lemma 2.1. *Suppose that $u_0 \in W^{1,\infty}(\Omega)$ satisfies (1.12), and that m is given by (1.2). Then there exists $T_{\max} \in (0, \infty]$ and a uniquely determined pair of functions (u, v), each belonging to $C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max}))$ such that (u, v) solves (1.1) in the classical sense in $\Omega \times (0, T_{\max})$ with $0 \leqslant u < 1$. Moreover, we have the following dichotomy:*

$$\text{Either } T_{\max} = \infty, \qquad \text{or} \quad \limsup_{t \searrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = 1. \tag{2.1}$$

Proof. The proof is based on rather routine arguments and therefore we may confine ourselves with presenting a sketch only. Let $M = \|u_0\|_{L^\infty(\Omega)}$ and $\eta \in (0\,, 1 - M)$. We define a set

$$X_T := \left\{ w \in C^0(\bar{\Omega} \times [0, T]) \;\middle|\; 0 \leqslant w \leqslant M + \eta < 1 \text{ and } \frac{1}{|\Omega|} \int_\Omega w(x, t)\, \mathrm{d}x = m \text{ for all } t \in (0, T) \right\}$$

and a mapping $\Phi : X_T \mapsto X_T$ such that given $\tilde{u} \in X_T$, $\Phi(\tilde{u}) = u$ where u is a L^2-weak solution to

$$\begin{cases} u_t = \nabla \cdot (D(\tilde{u}) \nabla u - u h(\tilde{u}) \nabla v), & x \in \Omega,\ t \in (0, T), \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.2}$$

with v defined to be the solution of

$$\begin{cases} -\Delta v = -m + \tilde{u}, & x \in \Omega,\ t \in (0, T), \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0, T), \end{cases} \tag{2.3}$$

along with the condition

$$\int_\Omega v(x\,, t)\, \mathrm{d}x = 0 \qquad \text{for any } t \in [0\,, T]. \tag{2.4}$$

Next using the Schauder fixed point theorem one can show that for $T = T_0$ small enough Φ has a fixed point u and then the classical regularity theories of elliptic [6, theorem 8.34] and parabolic equations [15, theorem V1.1] ensure that a pair $(u\,, v)$ which solves (1.1) in a weak sense is more regular. Indeed by elliptic regularity theory, for any $t \in (0\,, T_0]$

$$v(\cdot\,, t) \in C^{2+\gamma}(\bar{\Omega})$$

for some $\gamma \in (0\,, 1)$ and it is easy to check that in fact

$$v \in C^{2+\gamma, \frac{\gamma}{2}}(\bar{\Omega} \times [\tau, T_0]) \qquad \text{for all } \tau \in (0, T_0).$$

Parabolic regularity theory [15, theorem V.6.1] thus entails

$$u \in C^{2+\gamma, 1+\frac{\gamma}{2}}(\bar{\Omega} \times [\tau, T_0]) \qquad \text{for all } \tau \in (0, T_0).$$

The solution may be prolonged in the interval $[0\,, T_{\max})$, and either $T_{\max} = \infty$ or $T_{\max} < \infty$, where in the latter case necessarily

$$\|u(\cdot\,, t)\|_\infty \to 1 \qquad \text{when } t \to T_{\max}.$$

The uniqueness of solutions to problem (1.1) follows easily from the fact that the solutions are L^∞-bounded functions and D and h are locally Lipschitz. Finally the non-negativity of u follows from the classical maximum principle if we rewrite the first equation in (1.1) in the non-divergence form. □

Remark 2.1. The above uniqueness statement entails that the assumed radial symmetry of u_0 is inherited by both solution components u and v. Accordingly, without any danger of confusion we may write $u_0 = u_0(r)$ and $u = u(r,t)$ whenever this appears to be convenient in the following.

3. Radial monotonicity

Let us first make sure by means of a comparison argument that (downward) radial monotonicity of the initial data is inherited by the solution of (1.1).

Lemma 3.1. *Suppose that $\Omega = B_R(0)$ for some $R > 0$, that $u_0 = u_0(r)$ belongs to $C^2(\bar{\Omega})$ and satisfies (1.12), and that m is given by (1.2). Then if*

$$u_{0r}(r) \leqslant 0 \qquad \text{for all } r \in (0,R) \quad \text{and} \quad u_{0r}(R) = 0, \tag{3.1}$$

the solution $u = u(r,t)$ of (1.1) satisfies

$$u_r(r,t) \leqslant 0 \qquad \text{for all } r \in (0,R) \quad \text{and} \quad t \in (0, T_{\max}). \tag{3.2}$$

Proof. Let $T \in (0, T_{\max})$. Then since $0 \leqslant u < 1$ in $\bar{\Omega} \times [0,T]$, from (1.11) and our regularity assumptions on D and h we see that there exist positive constants c_1, c_2 and c_3 such that

$$D(u) \geqslant c_1 \qquad \text{in } \Omega \times (0,T) \tag{3.3}$$

and

$$|D'(u)| \leqslant c_2 \qquad \text{and} \qquad |D''(u)| \leqslant c_2 \quad \text{in } \Omega \times (0,T) \tag{3.4}$$

as well as

$$|S(u)| \leqslant c_3, \quad |S'(u)| \leqslant c_3 \qquad \text{and} \qquad |S''(u)| \leqslant c_3 \quad \text{in } \Omega \times (0,T), \tag{3.5}$$

where

$$S(\xi) := \xi h(\xi), \qquad \xi \in [0,1).$$

We now fix $\kappa > 0$ large fulfilling

$$\left(3 + \frac{(n+1)m}{n} + \frac{n-1}{n}\right) \cdot c_3 < \frac{\kappa}{2} \tag{3.6}$$

and define

$$\zeta(r,t) := \mathrm{e}^{-\kappa t} \cdot u_r(r,t), \qquad r \in [0,R], \ t \in [0,T].$$

Then in view of standard parabolic regularity theory ([15]), (3.1) ensures that ζ belongs to $C^0([0,R] \times [0,T]) \cap C^{2,1}((0,R) \times (0,T))$ and satisfies

$$\zeta(r,0) \leqslant 0 \quad \text{for all } r \in [0,R] \quad \text{and} \quad \zeta(0,t) = \zeta(R,t) = 0 \quad \text{for all } t \in [0,T]. \tag{3.7}$$

To complete our choice of parameters, we finally pick $\delta > 0$ small enough such that

$$c_2 \mathrm{e}^{\kappa T} R\delta < c_1 \tag{3.8}$$

and

$$c_2 \mathrm{e}^{2\kappa T} \delta^2 + \frac{m+1}{n} c_3 \mathrm{e}^{\kappa T} R\delta < \frac{\kappa}{2}. \tag{3.9}$$

Let us now assume that $\zeta_+ \not\equiv 0$ in $(0,R) \times (0,T)$. Then since ζ is continuous, we can find $t_0 \in [0,T)$ and $r_0 \in [0,R]$ such that

$$0 < \max_{(r,t) \in [0,R] \times [0,t_0]} \zeta(r,t) = \zeta(r_0,t_0) < \delta, \tag{3.10}$$

where (3.7) implies that actually $t_0 > 0$ and $0 < r < R$. Hence, at (r_0, t_0) we have

$$\zeta_r = 0, \quad \zeta_{rr} \leqslant 0 \qquad \text{and} \qquad \zeta_t \geqslant 0. \tag{3.11}$$

On the other hand, from (1.1) we easily derive, dropping the argument u in $D(u)$, $S(u)$ etc, that

$$u_t = Du_{rr} + D'u_r^2 + \frac{n-1}{r}Du_r - mS + uS - \frac{m}{n}rS'u_r + \frac{1}{r^{n-1}}S'Uu_r \qquad \text{in } (0,R)\times(0,T)$$

holds with

$$U(r,t) := \int_0^r \rho^{n-1}u(\rho,t)\,\mathrm{d}\rho, \qquad r\in[0,R],\ t\in[0,T].$$

A straightforward differentiation yields

$$\zeta_t = D\zeta_{rr} + A(r,t)\zeta_r + D''\mathrm{e}^{2\kappa t}\zeta^3 + \left\{\frac{n-1}{r}D' - \frac{m}{n}rS'' + \frac{1}{r^{n-1}}S''U\right\}\cdot \mathrm{e}^{\kappa t}\zeta^2$$
$$+ \left\{-\kappa - \frac{n-1}{r^2}D + S - \frac{(n+1)m}{n}S' + 2uS' - \frac{n-1}{r^n}S'U\right\}\cdot\zeta$$

in $(0,R)\times(0,T)$, where

$$A(r,t) := 2D'u_r + \frac{n-1}{r}D - \frac{m}{n}rS' + \frac{1}{r^{n-1}}S'U, \qquad r\in(0,R),\ t\in(0,T).$$

Therefore, (3.11) says that at $(r,t) = (r_0,t_0)$ we have

$$0 \leqslant D''\mathrm{e}^{2\kappa t}\zeta^3 + \left\{\frac{n-1}{r}D' - \frac{m}{n}rS'' + \frac{1}{r^{n-1}}S''U\right\}\cdot \mathrm{e}^{\kappa t}\zeta^2$$
$$+ \left\{-\kappa - \frac{n-1}{r^2}D + S - \frac{(n+1)m}{n}S' + 2uS' - \frac{n-1}{r^n}S'U\right\}\cdot\zeta,$$

that is,

$$\kappa + \frac{n-1}{r^2}D \leqslant D''\mathrm{e}^{2\kappa t}\zeta^2 + \left\{\frac{n-1}{r}D' - \frac{m}{n}rS'' + \frac{1}{r^{n-1}}S''U\right\}\cdot \mathrm{e}^{\kappa t}\zeta$$
$$+ \left\{S - \frac{(n+1)m}{n}S' + 2uS' - \frac{n-1}{r^n}S'U\right\}.$$

Since $0 \leqslant u(r,t) < 1$ and hence $0 \leqslant U(r,t) < \frac{r^n}{n}$ for all $(r,t)\in(0,R)\times(0,T)$, in view of (3.3)–(3.5) and (3.10) this entails that at $(r,t) = (r_0,t_0)$,

$$\kappa + \frac{n-1}{r^2}c_1 \leqslant c_2\mathrm{e}^{2\kappa T}\delta^2 + \left\{\frac{n-1}{r}c_2 + \frac{m}{n}Rc_3 + c_3\cdot\frac{R}{n}\right\}\cdot \mathrm{e}^{\kappa T}\delta$$
$$+ \left\{c_3 + \frac{(n+1)m}{n}c_3 + 2c_3 + \frac{n-1}{n}c_3\right\}. \tag{3.12}$$

Now (3.8) ensures that

$$\frac{n-1}{r}\cdot c_2\mathrm{e}^{\kappa T}\delta \leqslant \frac{n-1}{r^2}\cdot c_1 \qquad \text{for all } r\in(0,R),$$

whence (3.12) implies

$$\kappa \leqslant c_2\mathrm{e}^{2\kappa T}\delta^2 + \frac{m+1}{n}\cdot c_3\mathrm{e}^{\kappa T}\cdot R\delta + \left\{3 + \frac{(n+1)m}{n} + \frac{n-1}{n}\right\}\cdot c_3,$$

which clearly contradicts (3.9) and (3.6). Therefore, we must have $\zeta_+ \equiv 0$ in $(0,R)\times(0,T)$ and thus conclude that $u_r \leqslant 0$ in $(0,R)\times(0,T_{\max})$, because $T\in(0,T_{\max})$ was arbitrary. □

3.1. Transformation to a scalar parabolic equation

Following [13], we let

$$z(s,t) := n \cdot \int_0^{s^{\frac{1}{n}}} \rho^{n-1}(1-u(\rho,t))\,\mathrm{d}\rho, \qquad s \in [0,R^n],\ t \in [0,T_{\max}). \tag{3.13}$$

Lemma 3.1 implies

$$0 < z(s,t) \leqslant (1-m)s \qquad \text{for all } s \in (0,R^n] \quad \text{and} \quad t \in [0,T_{\max}). \tag{3.14}$$

$$z_s(s,t) = 1 - u(s^{\frac{1}{n}},t) \qquad \text{and} \qquad z_{ss}(s,t) = -\frac{1}{n}s^{\frac{1}{n}-1}u_r(s^{\frac{1}{n}},t). \tag{3.15}$$

Since $0 \leqslant u < 1$, we infer that

$$0 < z_s \leqslant 1 \qquad \text{in } (0,R^n) \times (0,T_{\max}). \tag{3.16}$$

Next we compute

$$z_t = -ns^{1-\frac{1}{n}}D(u)u_r + ns^{1-\frac{1}{n}}uh(u)v_r.$$

It is easy to check that

$$r^{1-n}(r^{n-1}v_r)_r = m - u$$

implies

$$ns^{1-\frac{1}{n}}v_r(s^{\frac{1}{n}},t) = (m-1)s + z(s,t),$$

and therefore it follows that

$$z_t = n^2 s^{2-\frac{2}{n}}D(1-z_s)z_{ss} - \big((1-m)s - z\big)\cdot(1-z_s)\cdot h(1-z_s) \quad \text{in } (0,R^n)\times(0,T_{\max}).$$

For convenience in notation, in the following we shall write

$$D_1(\xi) := D(1-\xi) \qquad \text{and} \qquad h_1(\xi) := h(1-\xi) \quad \text{for } \xi \in (0,1],$$

and thus see that the function z defined by (3.13) satisfies the scalar degenerate parabolic problem

$$\begin{cases} z_t = n^2 s^{2-\frac{2}{n}}D_1(z_s)z_{ss} - \big((1-m)s - z\big), & s \in (0,R^n),\ t \in (0,T_{\max})\cdot(1-z_s)\cdot h_1(z_s), \\ z(0,t) = 0, & z(R^n,t) = (1-m)R^n, \quad t \in (0,T_{\max}), \\ z(s,0) = z_0(s), & s \in (0,R^n), \end{cases} \tag{3.17}$$

where

$$z_0(s) := n \cdot \int_0^{s^{\frac{1}{n}}} \rho^{n-1}(1-u_0(\rho))\,\mathrm{d}\rho, \qquad s \in [0,R^n]. \tag{3.18}$$

Clearly, both u and u_0 can be reconstructed from z and z_0 via (3.15) and (3.18) according to

$$u(r,t) = 1 - z_s(r^n,t), \qquad r \in [0,R],\ t \in [0,T_{\max}), \qquad \text{and}$$
$$u_0(r) = 1 - z_{0s}(r^n), \qquad r \in [0,R]. \tag{3.19}$$

As an immediate consequence of (3.15) and lemma 3.1, we state the following assertion on conservation of convexity.

Corollary 3.2. *If $R > 0$ and $u_0 = u_0(r)$ and m satisfy (1.12) and (1.2), and if*

$$z_{0ss}(s) \geqslant 0 \qquad \text{for all } s \in (0,R^n) \qquad \text{as well as} \quad z_{0ss}(R^n) = 0,$$

then the solution z of (3.17) determined by (3.13) has the property

$$z_{ss}(s,t) \geqslant 0 \qquad \text{for all } s \in (0,R^n) \quad \text{and} \quad t \in (0,T_{\max}).$$

3.2. Time monotone solutions

We now again use the maximum principle to construct initial data for which the function z defined by (3.13) decreases with time and lies below the steady state $s \mapsto Z_c(s) := (1-m)s$ of (3.17) corresponding to the constant equilibrium of (1.1). As we shall see below, our construction is possible only when c_{hD} is suitably large. But we do not know if a similar statement on instability of Z_c from below is valid also for small c_{hD}.

Lemma 3.3. *Assume that (1.11) and (1.13) hold, and that $m \in (0,1)$. Suppose that*

$$c_{hD} > \frac{2^{\lambda+1} n^2 \pi^2}{R^2 m (1-m)^\lambda}. \tag{3.20}$$

Then there exists $\varepsilon_0 > 0$ with the following property: whenever $\varepsilon \in (0, \varepsilon_0]$, the relations

$$z_0(s) := (1-m)s - \varepsilon R^n \cdot \sin \frac{\pi s}{R^n}, \qquad s \in [0, R^n], \tag{3.21}$$

and (3.19) define a radial function $u_0 \in C^\infty(\bar{B}_R(0))$ such that (1.12) holds and the number $m(u_0)$ in (1.2) satisfies $m(u_0) = m$, and such that for the corresponding solution z of (3.17) in $(0, R^n) \times (0, T_{\max})$ given by (3.13) we have

$$z_t \leqslant 0 \qquad \text{in } (0, R^n) \times (0, T_{\max}). \tag{3.22}$$

Proof. With c_{hD} and $\lambda \in (0,1)$ taken from (1.13), we assume (3.20) and define

$$\varepsilon_0 := \min\left\{\frac{1-m}{2\pi}, \frac{m}{2\pi n}\right\}. \tag{3.23}$$

Then for $\varepsilon \in (0, \varepsilon_0]$, writing $\varphi(s) := \sin \frac{\pi s}{R^n}$, $s \in [0, R^n]$, we have

$$|\varepsilon R^n \varphi_s(s)| \leqslant \pi \varepsilon \leqslant \min\left\{\frac{1-m}{2}, \frac{m}{2}\right\} \qquad \text{for all } s \in (0, R^n) \tag{3.24}$$

and

$$\varphi_{ss}(s) = -\frac{\pi^2}{R^{2n}} \varphi(s) \qquad \text{for all } s \in (0, R^n). \tag{3.25}$$

Hence, by (3.24) the function z_0 defined by (3.21) satisfies $z_{0s}(s) \geqslant \frac{1-m}{2}$ and $\frac{1-m}{2} s \leqslant z_0(s) \leqslant (1-m)s$ for all $s \in (0, R^n)$, and moreover (3.24) guarantees that

$$1 - z_{0s}(s) = 1 - (1-m) + \varepsilon R^n \varphi_s(s) \geqslant \frac{m}{2} \qquad \text{for all } s \in (0, R^n). \tag{3.26}$$

Consequently, for

$$\mathcal{A} z_0 := n^2 s^{2-\frac{2}{n}} z_{0ss} - \big((1-m)s - z_0\big) \cdot (1 - z_{0s}) \cdot \frac{h_1(z_{0s})}{D_1(z_{0s})},$$

in view of (1.13) we have

$$\begin{aligned}
\mathcal{A} z_0 &= -n^2 \cdot \varepsilon R^n \cdot s^{2-\frac{2}{n}} \varphi_{ss} - \varepsilon R^n \varphi \cdot (1 - z_{0s}) \cdot \frac{h_1(z_{0s})}{D_1(z_{0s})} \\
&\leqslant -n^2 \cdot \varepsilon R^n \cdot s^{2-\frac{2}{n}} \varphi_{ss} - \varepsilon R^n \varphi \cdot \frac{m}{2} \cdot \frac{h_1(z_{0s})}{D_1(z_{0s})} \\
&\leqslant -n^2 \cdot \varepsilon R^n \cdot s^{2-\frac{2}{n}} \varphi_{ss} - \varepsilon R^n \varphi \cdot \frac{m}{2} \cdot c_{hD} \cdot z_{0s}^\lambda \\
&= -n^2 \cdot \varepsilon R^n \cdot s^{2-\frac{2}{n}} \varphi_{ss} - \frac{m c_{hD}}{2} \cdot \varepsilon R^n \cdot \big(1 - m - \varepsilon R^n \varphi_s\big)^\lambda \\
&= n^2 \varepsilon R^n s^{2-\frac{2}{n}} \cdot \left\{-\frac{\varphi_{ss}}{\varphi} - \frac{m c_{hD}}{2n^2} \cdot s^{-2+\frac{2}{n}} \cdot \big(1 - m - \varepsilon R^n \varphi_s\big)^\lambda\right\}
\end{aligned} \tag{3.27}$$

for all $s \in (0, R^n)$. Now estimating $s^{-2+\frac{2}{n}} \geqslant R^{-2n+2}$ and using (3.25) and (3.24), we obtain

$$-\frac{\varphi_{ss}}{\varphi} - \frac{mc_{hD}}{2n^2} \cdot s^{-2+\frac{2}{n}} \cdot \left(1 - m - \varepsilon R^n \varphi_s\right)^{\lambda} \leqslant \frac{\pi^2}{R^{2n}} - \frac{mc_{hD}}{2n^2} \cdot R^{-2n+2} \cdot \left(\frac{1-m}{2}\right)^{\lambda}$$
$$= \frac{\pi^2}{R^{2n}} \cdot \left\{1 - \frac{mc_{hD}}{2n^2\pi^2} \cdot \left(\frac{1-m}{2}\right)^{\lambda} \cdot R^2\right\}$$
$$< 0 \qquad \text{for all } s \in (0, R^n)$$

according to (3.20). Thereupon, (3.27) entails that

$$\mathcal{A}z_0 < 0 \qquad \text{in } (0, R^n),$$

so that $z_t \leqslant 0$ holds initially, that is, in $(0, R^n) \times \{0\}$. Hence, a well-known comparison argument (cf [21, chapter 52] for details) implies that (3.22) holds. □

3.3. *Nonexistence of small regular steady states for c_{hD} large enough*

We proceed to exclude the existence of steady states of (3.17) which lie below the initial data considered in lemma 3.3, again provided that c_{hD} is large.

Lemma 3.4. *Suppose that (1.11) and (1.13) are valid, and let $\varepsilon > 0$. Assume that for some $\delta \in (0, 1)$ we have*

$$c_{hD} > \frac{n^2(1-\delta m)^{1-\lambda}}{\varepsilon c_2 \delta (1-\lambda) R^2 m}, \tag{3.28}$$

where

$$c_2 := \int_0^{\mu} \sigma^{-2+\frac{2}{n}} \sin \pi\sigma \, d\sigma \tag{3.29}$$

with $\mu := \frac{(1-\delta)m}{1-\delta m}$. Then the stationary problem

$$\begin{cases} 0 = n^2 s^{2-\frac{2}{n}} D_1(Z_s) Z_{ss} - ((1-m)s - Z) \cdot (1 - Z_s) \cdot h_1(Z_s), & s \in (0, R^n), \\ Z(0) = 0, \quad Z(R^n) = (1-m)R^n, \end{cases} \tag{3.30}$$

does not possess any positive nondecreasing classical solution $Z \in C^0([0, R^n]) \cap C^2((0, R^n))$ which satisfies

$$Z(s) \leqslant (1-m)s - \varepsilon R^n \cdot \sin \frac{\pi s}{R^n} \qquad \text{for all } s \in [0, R^n]. \tag{3.31}$$

Proof. Given $\varepsilon > 0$, we fix any $\delta \in (0, 1)$ and assume that (3.28) holds. We note that $\mu < m$ because $m \in (0, 1)$ and $\delta \in (0, 1)$, so that c_2 is positive. Note that (3.28) may be rewritten in the form

$$\frac{1}{\delta m} \cdot \frac{1}{1-\lambda} \cdot (1-\delta m)^{1-\lambda} < c_1 c_2 R^2, \tag{3.32}$$

where $c_1 := \frac{c_{hD}\varepsilon}{n^2}$. Now suppose that Z is a classical solution of (3.30) satisfying (3.31). Then thanks to the nonnegativity of Z, from the mean-value theorem we infer that there exists $s_\star \in [\mu R^n, R^n]$ such that

$$Z_s(s_\star) = \frac{Z(R^n) - Z(\mu R^n)}{R^n - \mu R^n} \leqslant \frac{Z(R^n)}{(1-\mu)R^n} = \frac{1-m}{1-\mu} = 1 - \delta m. \tag{3.33}$$

In particular, considering the initial-value problem for the ODE in (3.30) with given initial data for Z_s at $s_\star$ with $0 < Z_s(s_\star) < 1$, we gain from (3.33) upon a uniqueness argument that $Z_s(s) \neq 1$ for all $s \in (0, R^n)$ and hence

$$Z_s(s) < 1 \qquad \text{for all } s \in (0, R^n). \tag{3.34}$$

Accordingly, using (3.31), (3.30) and (1.13) we see that Z satisfies

$$\begin{aligned} Z_{ss} &= \frac{1}{n^2} s^{-2+\frac{2}{n}} \big((1-m)s - Z\big) \cdot (1 - Z_s) \cdot \frac{h_1(Z_s)}{D_1(Z_s)} \\ &\geqslant \frac{1}{n^2} \cdot \varepsilon R^n \cdot s^{-2+\frac{2}{n}} \cdot \sin\frac{\pi s}{R^n} \cdot (1 - Z_s) \cdot \frac{h_1(Z_s)}{D_1(Z_s)} \\ &\geqslant c_1 R^n \cdot s^{-2+\frac{2}{n}} \cdot \sin\frac{\pi s}{R^n} \cdot (1 - Z_s) Z_s^\lambda \qquad \text{for all } s \in (0, R^n). \end{aligned} \tag{3.35}$$

Now since Z is positive in $(0, R^n)$ and $Z(0) = 0$, there must exist a sequence of numbers $s_k \searrow 0$ such that $Z_s(s_k) > 0$, which in conjunction with the convexity of Z, as asserted by (3.35) and (3.34), implies that actually $Z_s > 0$ in $(0, R^n)$. Therefore, the inequality (3.35) can be integrated so as to yield

$$\begin{aligned} \Phi(Z_s(s)) - \Phi(Z_s(s_k)) &\geqslant c_1 R^n \cdot \int_{s_k}^{s} \tau^{-2+\frac{2}{n}} \sin\frac{\pi\tau}{R^n} \mathrm{d}\tau \\ &= c_1 R^2 \cdot \int_{s_k R^{-n}}^{s R^{-n}} \sigma^{-2+\frac{2}{n}} \sin\pi\sigma \mathrm{d}\sigma \qquad \text{for all } s \in (s_k, R^n) \end{aligned} \tag{3.36}$$

for each $k \in \mathbb{N}$, where

$$\Phi(\xi) := \int_0^\xi \frac{\mathrm{d}\eta}{\eta^\lambda (1-\eta)}, \qquad \xi \in (0, 1), \tag{3.37}$$

defines a nonnegative increasing function $\Phi \in C^1([0, 1))$ due to the fact that $\lambda < 1$.

We next evaluate (3.36) at $s = s_\star$ and let $k \to \infty$ to conclude that

$$\begin{aligned} c_1 R^2 \cdot \int_0^\mu \sigma^{-2+\frac{2}{n}} \sin\pi\sigma \,\mathrm{d}\sigma \leqslant c_1 R^2 \cdot \int_0^{s_\star R^{-n}} \sigma^{-2+\frac{2}{n}} \sin\pi\sigma \,\mathrm{d}\sigma \\ = c_1 R^2 \cdot \limsup_{k\to\infty} \int_{s_k R^{-n}}^{s_\star R^{-n}} \sigma^{-2+\frac{2}{n}} \sin\pi\sigma \,\mathrm{d}\sigma \\ \leqslant \Phi(Z_s(s_\star)) \\ \leqslant \Phi(1 - \delta m) \end{aligned} \tag{3.38}$$

by (3.33) and the monotonicity and nonnegativity of Φ. However, since $1 - \eta \geqslant \delta m$ for all $\eta \leqslant 1 - \delta m$, (3.37) shows that

$$\Phi(1 - \delta m) \leqslant \frac{1}{\delta m} \cdot \int_0^{1-\delta m} \frac{\mathrm{d}\eta}{\eta^\lambda} = \frac{1}{\delta m} \cdot \frac{1}{1-\lambda} \cdot (1 - \delta m)^{1-\lambda}.$$

Hence, in light of the definition (3.29) and (3.32), (3.38) turns into the inequality

$$c_1 c_2 R^2 \leqslant \Phi(1 - \delta m) < c_1 c_2 R^2,$$

which is false. This contradiction rules out the existence of such a solution Z. □

Remark 3.1. It is worth pointing out that lemmas 3.3 and 3.4 essentially depend on the sign in front of the cross-diffusion term. The results seem to be invalid for the case of chemorepulsive interaction which corresponds to the case where the sign in front of the cross-diffusion term is negative.

The next statement essentially says that all possible limits of $z(\cdot, t)$ as $t \to \infty$ must either be positive steady states of (3.17), or vanish in some subinterval of $(0, R^n)$, provided that c_{hD} is large enough and the initial data are chosen as in (3.21).

Lemma 3.5. *Assume that (1.11) and (1.13) hold, that $m \in (0, 1)$ and that (3.20) is valid. With ε_0 as provided by lemma 3.3, let $\varepsilon \in (0, \varepsilon_0]$ and suppose that z_0 and u_0 are defined through (3.21) and (3.19), respectively. Let z_0 and u_0 be defined through (3.21) and (3.19), respectively. Moreover, assume that the corresponding solution (u, v) be global in time. Then the solution z of (3.17) defined by (3.13) satisfies*

$$z(\cdot, t) \to Z \quad \text{in } C^0([0, R^n]) \qquad \text{as } t \to \infty,$$

where $Z \in C^0([0, R^n])$ is some nondecreasing nonnegative function. If moreover $Z > 0$ in $(0, R^n)$, then $Z \in C^2((0, R^n))$ is a classical solution of (3.30) and strictly increases on $(0, R^n)$.

Proof. From lemma 3.3 we know that

$$Z(s, t) := \lim_{t \to \infty} z(s, t), \qquad s \in [0, R^n]$$

defines a nonnegative function Z which clearly satisfies $Z(0) = 0$ and $Z(R^n) = (1 - m)R^n$. Since $0 < z_s \leqslant 1$ in $(0, R^n) \times (0, \infty)$, we furthermore have

$$z(\cdot, t) \to Z \qquad \text{in } C^0([0, R^n]) \tag{3.39}$$

as $t \to \infty$. We thus know that Z is continuous, nondecreasing and nonnegative, so that it remains to verify the claimed properties of Z in the case when $Z > 0$ in $(0, R^n)$. In order to achieve this, we first observe that

$$\int_0^T \int_0^{R^n} |z_t| = \int_0^{R^n} z_0 - \int_0^{R^n} z(\cdot, T) \leqslant \int_0^{R^n} z_0 \qquad \text{for all } T > 0,$$

which implies that $\int_0^\infty \int_0^R |z_t| < \infty$, and hence we can pick a sequence of times $t_k \to \infty$ along which

$$z_t(\cdot, t_k) \to 0 \qquad \text{in } L^1((0, R^n)). \tag{3.40}$$

Next, using (3.16) we infer from (3.17) that

$$z_{ss} = \frac{1}{n^2 s^{2-\frac{2}{n}} D_1(z_s)} \cdot \Big\{ z_t + \big((1 - m)s - z\big) \cdot (1 - z_s) \cdot h_1(z_s) \Big\} \quad \text{in } (0, R^n) \times (0, \infty), \tag{3.41}$$

so that using $z_t \leqslant 0$ and corollary 3.2 we easily arrive at the two-sided inequality

$$\begin{aligned} 0 \leqslant z_{ss} &\leqslant \frac{1}{n^2} \cdot s^{-2+\frac{2}{n}} \cdot \big((1 - m)s - z\big) \cdot (1 - z_s) \cdot \frac{h_1(z_s)}{D_1(z_s)} \\ &\leqslant \frac{1}{n^2} \cdot s^{-2+\frac{2}{n}} \cdot (1 - m) \cdot \frac{h_1(z_s)}{D_1(z_s)} \qquad \text{in } (0, R^n) \times (0, \infty), \end{aligned} \tag{3.42}$$

because by (3.14) we have $0 \leqslant z(s, t) \leqslant (1 - m)s$ in $(0, R^n) \times (0, \infty)$. In order to turn this into an estimate for z in $C^2_{\mathrm{loc}}((0, R^n])$, we fix any $s_0 \in (0, R^n)$ and then infer from the mean-value theorem that for each $t > 0$ there exists $s(t) \in (0, s_0)$ such that

$$z_s(s(t), t) = \frac{z(s_0, t) - z(0, t)}{s_0} = \frac{z(s_0, t)}{s_0}.$$

Since $z_{ss} \geqslant 0$ and $z(\cdot, t) \geqslant Z$, this shows that

$$z_s(s, t) \geqslant \frac{Z(s)}{s} \qquad \text{for all } s \in (0, R^n) \quad \text{and} \quad t > 0, \tag{3.43}$$

and thus the positivity of Z in $(0, R^n]$ in conjunction with (3.42) and the boundedness of $\frac{h_1}{D_1}$ on $(0, 1]$ asserts that for all $s_0 \in (0, R^n)$ there exists $c_1(s_0)$ such that

$$|z_{ss}(s,t)| \leqslant c_1(s_0) \qquad \text{for all } s \in (s_0, R^n) \text{ and } t > 0. \tag{3.44}$$

By the Arzelá-Ascoli theorem, (3.39) can therefore be improved so as to read

$$z(\cdot,t) \to Z \quad \text{in } C^0([0,R^n]) \cap C^1_{\mathrm{loc}}((0,R^n]) \qquad \text{as } t \to \infty. \tag{3.45}$$

Now multiplying (3.41) by an arbitrary $\psi \in C_0^\infty((0,R^n))$ and integrating by parts with respect to $s \in (0, R^n)$ yields

$$-\int_0^{R^n} z_s\psi_s = \frac{1}{n^2}\cdot\int_0^{R^n} \frac{1}{s^{2-\frac{2}{n}}D_1(z_s)}\cdot z_t\cdot\psi$$
$$+\frac{1}{n^2}\cdot\int_0^{R^n} s^{-s+\frac{2}{n}}\cdot\big((1-m)s - z\big)\cdot(1-z_s)\cdot\frac{h_1(z_s)}{D_1(z_s)}\cdot\psi$$

for all $t > 0$. According to (3.45), we may evaluate this at $t = t_k$ and let $k \to \infty$ to infer that

$$-\int_0^{R^n} Z_s\psi = \int_0^{R^n} A(s)\cdot\psi \qquad \text{for all } \psi \in C_0^\infty((0,R^n)),$$

where

$$A(s) := \frac{1}{n^2}\cdot s^{-2+\frac{2}{n}}\cdot\big((1-m)s - Z(s)\big)\cdot(1 - Z_s(s))\cdot\frac{h_1(Z_s(s))}{D_1(Z_s(s))}, \qquad s \in (0, R^n).$$

Since A is locally Lipschitz continuous in $(0, R^n]$ due to (3.45), (3.43) and (3.44), standard elliptic regularity theory ([6]) ensures that Z belongs to $C^2((0,R^n))$ and satisfies $Z_{ss} = A$ classically in $(0, R^n)$. □

3.4. Proof of theorem 1.1 (i)

We are now in the position to prove that (1.13) and the mere parabolicity assumption (1.11) are sufficient to guarantee that a singularity formation occurs at least in infinite time for some initial data, provided that c_{hD} is large enough.

Proof of theorem 1.1 (i). Let ε_0 be as provided by lemma 3.3 and fix some $\varepsilon \in (0, \varepsilon_0]$ and $\delta \in (0, 1)$ in lemma 3.4. We then assume that c_{hD} satisfies the hypothesis (3.20) from lemma 3.3 as well as assumption (3.28) of lemma 3.4. That is

$$c_{hD} \geqslant c = \max\left\{\frac{2^{\lambda+1}n^2\pi^2}{R^2m(1-m)^\lambda}, \frac{n^2(1-\delta m)^{1-\lambda}}{R^2 m\varepsilon\delta(1-\lambda)}\right\}. \tag{3.46}$$

We assume also that z_0 and u_0 are defined via (3.21) and (3.19), respectively. It can then easily be checked that u_0 is smooth on $\bar\Omega = \bar B_R(0)$ and satisfies (1.12), and lemma 3.3 says that the corresponding solution z of (3.17) defined by (3.13) satisfies $z_t \leqslant 0$ in $\Omega \times (0, T_{\max})$. Now if $T_{\max} < \infty$, the claim immediately results from (2.1), whereas in the case $T_{\max} = \infty$ we may apply lemma 3.5 to infer that $z(\cdot,t) \to Z$ in $C^0([0,R^n])$ as $t \to \infty$ for some nonnegative Z which is either positive on $(0, R^n]$ or vanishes in $[0, s_0]$ for some $s_0 > 0$. However, lemma 3.4 says that the former alternative is impossible. This means that $z(\cdot,t) \to 0$ uniformly in $(0, s_0)$ and thus $u(\cdot,t) \to 1$ in $L^1(B_{s_0^{1/n}}(0))$ as $t \to \infty$. □

Remark 3.2.

(1) By a simple comparison argument, the same conclusion holds if z_0 is replaced by any smooth positive function $\tilde z_0$ which only satisfies the weaker conditions $\tilde z_0 \leqslant z_0$ and $0 < \tilde z_{0s} \leqslant 1$ in $[0, R^n]$ as well as $\tilde z_0(R^n) = (1-m)R^n$, where z_0 is the function given by (3.21).

(2) When $n = 2$, it is easy to calculate that $R^2 m = \frac{1}{\pi}\int_\Omega u_0(x)\,\mathrm{d}x$. If $n = 3$, then $R^2 m = \frac{3}{4R\pi}\int_\Omega u_0(x)\,\mathrm{d}x$. Accordingly, our sufficient conditions for singularity formation read

$$c_{hD} \geqslant c = \max\left\{\frac{2^{\lambda+1}n^2\pi^3}{(1-m)^\lambda\int_\Omega u_0(x)\,\mathrm{d}x}, \frac{\pi n^2(1-\delta m)^{1-\lambda}}{\varepsilon\delta(1-\lambda)\int_\Omega u_0(x)\,\mathrm{d}x}\right\} \text{ if } n = 2,$$

$$c_{hD} \geqslant c = \max\left\{\frac{2^{\lambda+3}n^2\pi^3 R}{3(1-m)^\lambda\int_\Omega u_0(x)\,\mathrm{d}x}, \frac{4\pi Rn^2(1-\delta m)^{1-\lambda}}{3\varepsilon\delta(1-\lambda)\int_\Omega u_0(x)\,\mathrm{d}x}\right\} \text{ if } n = 3.$$

Hence the critical value of chemosensitivity c_{hD} is inverse to the initial cell mass, which means that chemosensitivity needs to be large if the cell initial mass is small so that the solution reaches the singular value. In the three-dimensional space, this indicates that the solution is more likely to reach the singular value when the domain is small.

4. Finite-time blow-up for non-degenerate diffusion

Our next goal is to show that under the non-degeneracy condition (1.15) the singularity formation asserted by theorem 1.1(i) in fact already occurs within finite time. To achieve this, we first prove the following statement on locally uniform positivity of global solutions of (3.17) in that case.

Lemma 4.1. *Assume that D and h meet the requirements (1.15) and (1.13). Suppose that $m \in (0,1)$, $R > 0$ and that z is a global classical solution of (3.17) satisfying $0 < z_s \leqslant 1$ and $z_t \leqslant 0$, for which in addition there exists $c > 0$ such that*

$$z(s,t) \leqslant (1-m)s - c\,\sin\frac{\pi s}{R^n} \qquad \text{for all } s \in (0, R^n) \quad \text{and} \quad t > 0. \tag{4.1}$$

Then

$$\lim_{t\to\infty} z(s,t) > 0 \qquad \text{for all } s \in (0, R^n]. \tag{4.2}$$

Proof. If (4.2) is false then since $z_t \leqslant 0$, we can find $s_0 \in (0, R^n)$ such that

$$z(s_0, t) \to 0 \qquad \text{as } t \to \infty. \tag{4.3}$$

We claim that then there exists $T > 0$ such that

$$z(s,T) = 0 \qquad \text{for all } s \in \left[0, \frac{s_0}{2}\right] \tag{4.4}$$

which will be incompatible with the global existence assumption and thereby prove the lemma. To see this, let us first note that in view of (4.1) there exists $c_1 \in (0, 1-m)$ such that

$$z(s,t) \leqslant (1-m-c_1)s \qquad \text{for all } s \in (0, s_0) \quad \text{and} \quad t > 0. \tag{4.5}$$

With $c_{hD} > 0$ and $\lambda \in (0,1)$ as in (1.13) and $c_2 > 0$ small enough fulfilling

$$(A+B)^\lambda \geqslant c_2(A^\lambda + B^\lambda) \qquad \text{for all } A \geqslant 0 \quad \text{and} \quad B \geqslant 0, \tag{4.6}$$

we can then choose $k > 0$ small enough such that

$$\left(\frac{3(1-\lambda)s_0^2 k}{8}\right)^{\frac{1}{1-\lambda}} \leqslant c_1 \tag{4.7}$$

and

$$k \leqslant \frac{c_1 c_2 c_{hD} m}{n^2 s_0^{2-\frac{2}{n}}}. \tag{4.8}$$

Next we proceed to construct a supersolution $\bar{z}$ which satisfies (4.4). To this end we define a function $\psi : [0, s_0] \to \mathbb{R}$ by setting

$$\psi(s) := \begin{cases} 0 & \text{if } s \in \left[0, \frac{s_0}{2}\right], \\ \left(\frac{(1-\lambda)k}{2}\right)^{\frac{1}{1-\lambda}} \cdot \int_{\frac{s_0}{2}}^{s} \left(\sigma^2 - \frac{s_0^2}{4}\right)^{\frac{1}{1-\lambda}} & \text{if } s \in \left(\frac{s_0}{2}, s_0\right]. \end{cases} \tag{4.9}$$

Observe that for $s \in (\frac{s_0}{2}, s_0]$ we have

$$\psi_s(s) = \left(\frac{(1-\lambda)k}{2}\right)^{\frac{1}{1-\lambda}} \cdot \left(s^2 - \frac{s_0^2}{4}\right)^{\frac{1}{1-\lambda}} \tag{4.10}$$

and

$$\frac{\psi_{ss}(s)}{\psi_s^\lambda(s)} = \frac{\left(\frac{(1-\lambda)k}{2}\right)^{\frac{1}{1-\lambda}} \cdot \frac{2}{1-\lambda} \cdot \left(s^2 - \frac{s_0^2}{4}\right)^{\frac{1}{1-\lambda}-1} \cdot s}{\left(\frac{(1-\lambda)k}{2}\right)^{\frac{\lambda}{1-\lambda}} \cdot \left(s^2 - \frac{s_0^2}{4}\right)^{\frac{\lambda}{1-\lambda-1}}} = ks,$$

whence it follows that $\psi \in C^2([0, s_0])$ and

$$\psi_{ss} = ks\psi_s^\lambda \qquad \text{in } [0, s_0]. \tag{4.11}$$

We next let $y_0 := 1 - m - c_1 > 0$ and $c_3 := \psi_s(s_0)$ and pick $\gamma > 0$ small such that

$$\gamma < c_1 c_2 c_{hD} m c_D, \tag{4.12}$$

with c_D taken from (1.15). Finally, since $z(s_0, t) \to 0$ as $t \to \infty$ by our assumption (4.3), we can find $t_0 > 0$ such that

$$z(s_0, t) \leqslant \psi(s_0) \qquad \text{for all } t \geqslant t_0. \tag{4.13}$$

With these parameters fixed henceforth, we let $y = y(t)$ denote the solution of the initial-value problem

$$\begin{cases} y' = -\gamma y^\lambda, & t \in (t_0, T), \\ y(t_0) = y_0, \end{cases} \tag{4.14}$$

that is, we define

$$y(t) := \left\{ y_0^{1-\lambda} - \gamma(1-\lambda)(t - t_0) \right\}^{\frac{1}{1-\lambda}}, \qquad t \in [t_0, T],$$

where

$$T := t_0 + \frac{y_0^{1-\lambda}}{\gamma(1-\lambda)}$$

denotes the extinction time of y. Then

$$\bar{z}(s, t) := \psi(s) + y(t) \cdot s, \qquad s \in [0, s_0],\ t \in [t_0, T]$$

satisfies

$$\bar{z}(s, t_0) = \psi(s) + y_0 \cdot s \geqslant y_0 \cdot s = (1 - m - c_1) \cdot s \geqslant z(s, t_0) \qquad \text{for all } s \in (0, s_0) \tag{4.15}$$

by (4.5) and, clearly,

$$\bar{z}(0, t) = 0 \geqslant z(0, t) \qquad \text{for all } t \in (t_0, T) \tag{4.16}$$

as well as

$$\bar{z}(s_0, t) = \psi(s_0) + y(t) \cdot s_0 \geqslant \psi(s_0) \geqslant z(s_0, t) \qquad \text{for all } t \in (t_0, T) \tag{4.17}$$

according to (4.13). In order to derive an appropriate parabolic inequality for $\overline{z}$, let us first note that since y decreases, we have

$$\overline{z}_s(s,t) = \psi_s(s) + y(t) \leqslant \psi_s(s) + y_0,$$

and therefore (4.10) and (4.7) ensure that

$$\overline{z}(s,t) \leqslant \psi_s(s_0) + y_0 \leqslant c_1 + y_0 = 1 - m \qquad \text{for all } s \in (0, s_0) \text{ and } t \in (t_0, T).$$

Consequently, in

$$\begin{aligned} \mathcal{P}\overline{z} &:= \overline{z}_t - n^2 s^{2-\frac{2}{n}} D_1(\overline{z}_s)\overline{z}_{ss} + \big((1-m)s - z\big) \cdot (1 - \overline{z}_s) \cdot h_1(\overline{z}_s) \\ &=: I_1 + I_2 + I_3, \end{aligned}$$

using (4.5) we can estimate

$$I_3 \geqslant c_1 m s h_1(\overline{z}_s) \qquad \text{in } (0, s_0) \times (t_0, T).$$

Since evidently $\overline{z}_{ss} \geqslant 0$, we moreover have

$$I_2 \geqslant -n^2 s_0^{2-\frac{2}{n}} D_1(\overline{z}_s)\overline{z}_{ss} \qquad \text{in } (0, s_0) \times (t_0, T),$$

so that (4.11) entails that

$$\begin{aligned} \mathcal{P}\overline{z} &\geqslant \overline{z}_t - n^2 s_0^{2-\frac{2}{n}} D_1(\overline{z}_s)\overline{z}_{ss} + c_1 m s h_1(\overline{z}_s) \\ &= y' \cdot s - n^2 s_0^{2-\frac{2}{n}} \cdot k \cdot D_1(\overline{z}_s) \cdot s\psi_s^\lambda + c_1 m s h_1(\overline{z}_s) \\ &= sD_1(\overline{z}_s) \cdot \left\{ \frac{1}{D_1(\overline{z}_s)} \cdot y' - n^2 s_0^{2-\frac{2}{n}} k\psi_s^\lambda + c_1 m \cdot \frac{h_1(\overline{z}_s)}{D_1(\overline{z}_s)} \right\} \\ &= sD_1(\overline{z}_s) \cdot J(s,t) \qquad \text{in } (0, s_0) \times (t_0, T) \end{aligned} \tag{4.18}$$

with

$$J(s,t) := \frac{1}{D_1(\overline{z}_s)} \cdot y' - n^2 s_0^{2-\frac{2}{n}} k\psi_s^\lambda + c_1 m \cdot \frac{h_1(\overline{z}_s)}{D_1(\overline{z}_s)}, \qquad s \in (0, s_0), \quad t \in (t_0, T).$$

Here, using (1.15) and (1.13) and again the fact that y decreases, we see that

$$\begin{aligned} J &\geqslant \frac{1}{c_D} \cdot y' - n^2 s_0^{2-\frac{2}{n}} k\psi_s^\lambda + c_1 c_{hD} m(\psi_s + y)^\lambda \\ &\geqslant \frac{1}{c_D} \cdot y' - n^2 s_0^{2-\frac{2}{n}} k\psi_s^\lambda + c_1 c_2 c_{hD} m(\psi_s^\lambda + y^\lambda) \qquad \text{in } (0, s_0) \times (t_0, T) \end{aligned}$$

because of (4.6). Thus, recalling (4.14) we obtain

$$J \geqslant -\frac{\gamma}{c_D} \cdot y^\lambda - n^2 s_0^{2-\frac{2}{n}} k\psi_s^\lambda + c_1 c_2 c_{hD} m\psi_s^\lambda + c_1 c_2 c_{hD} m \cdot y^\lambda \qquad \text{in } (0, s_0) \times (t_0, T),$$

so that (4.8) and (4.12) ensure that J is positive in $(0, s_0) \times (t_0, T)$, whence (4.18) entails that $\mathcal{P}\overline{z} > 0$ in $(0, s_0) \times (t_0, T)$. Since obviously $\mathcal{P}z \equiv 0$ in $(0, s_0) \times (t_0, T)$, from this and (4.15)–(4.17) we conclude upon a comparison argument based on [28, theorem 4.1 on p 1055] that $z \leqslant \overline{z}$ in $[0, s_0] \times [t_0, T]$. This in particular implies (4.4) and thereby completes the proof. $\square$

4.1. Proof of theorem 1.1 (ii)

Now our final statement is an almost trivial consequence.

Proof of theorem 1.1 (ii). Choosing R, c_{hD}, z_0 and u_0 in the same way as in the proof of theorem 1.1, we claim that for the corresponding solutions u and z of (1.1) and (3.17), respectively, we have $T_{\max} = \infty$. Then in view of lemma 4.1, lemma 3.5 now says that $z(\cdot, t)$ decreases to a positive solution Z of (3.30) satisfying (3.31). However, lemma 3.4 asserts nonexistence of such an equilibrium. Therefore $T_{\max} < \infty$, and hence (2.1) completes the proof. □

5. Numerical simulation

This section is devoted to numerically illustrating that the solution u of model (1.1) may reach the singular value 1 in either finite time or infinite time for suitable initial data and parameter values. It is helpful to remark that for the solution component u to reach the value 1 it is necessary and sufficient that the transformed variable z reaches zero. Recall that for given parameters satisfying the inequalities (3.20) and (3.28), and for initial data z_0 fulfilling (3.21), we know that the solution z reaches zero in finite time if the diffusion $D(u)$ satisfies (1.13), (1.15), and that it reaches zero in either finite time or infinite time for $D(u)$ satisfying (1.13) only. Let us first specify some appropriate initial data u_0 which result in a singular solution. For simplicity, we only explore the numerical solutions in space dimension $n = 1$ and assume $\Omega = (-R, R)$, $R > 0$. Then from (3.21) and (3.19), we derive that the initial condition for $u(x, t)$ is

$$u_0(x) = m + \varepsilon \cos \frac{\pi x}{R}, \qquad x \in [-R, R], \tag{5.1}$$

which satisfies the Neumann boundary condition and mass conservation (1.2). For the purpose of numerical computation, using (1.1) we also compute

$$v_0(x) = -\frac{R^2}{\pi^2} \varepsilon \cos \frac{\pi x}{R}, \qquad x \in [-R, R]. \tag{5.2}$$

With these initial conditions, we implement the finite element based computing package COMSOL multiphysics to perform the numerical computations, choosing the domain size $R = 20$.

Figure 1 illustrates the process of solution u of model (1.1) approaching the singular value one, where $D(u)$ satisfies conditions (1.13) and (1.15), and the parameter values are chosen to satisfy the inequalities (3.20) and (3.28). Figure 1(a) shows the dynamics of the solution u approaching one when time increases. Figure 1(b) plots the time evolution of the maximum of solution u for different values of chemosensitivity c_{hD}, and shows that the solution u with larger chemosensitivity approaches $u = 1$ faster.

Figure 2 shows the numerical simulation of solution u to (1.1) with singular diffusion, where $D(u)$ and $h(u)$ fulfil the condition (1.13). For illustration, we choose $D(u) = (1-u)^{-\alpha}$ and $h(u) = c_{hD}(1-u)^{\beta}$ with $0 < \alpha, \beta < 1$ and $\alpha + \beta = \lambda < 1$. In this case, the diffusion $D(u) > 1$ increases with respect to the cell density u with a growth rate parameter α. By theorem 1.1 (ii), the solution u will reach the singular value one in finite time. The numerical simulation in figure 2(b) shows that the maximum of solution u grows monotonically and indeed approaches $u = 1$.

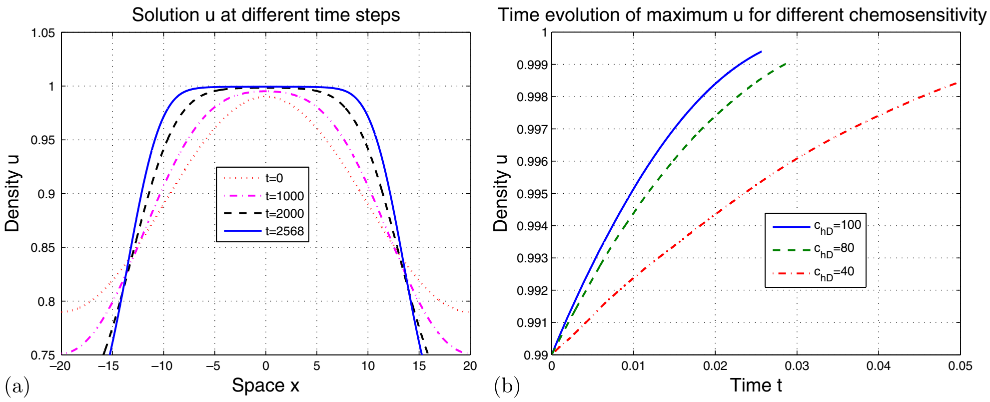

Figure 1. (a) Numerical simulation of the solution u approaching singular value one in finite time for the model (1.1) with $D(u) = 1$, $h(u) = c_{hD}(1-u)^{\beta}$. Here t represents the time step and parameter values are $m = 0.89$, $\varepsilon = 0.1$, $c_{hD} = 100$, $\beta = 0.6$, $R = 20$. (b) The plot of time evolution of maximum value of solution u for different values of chemosensitivity c_{hD}. Other parameter are the same as those in (a).

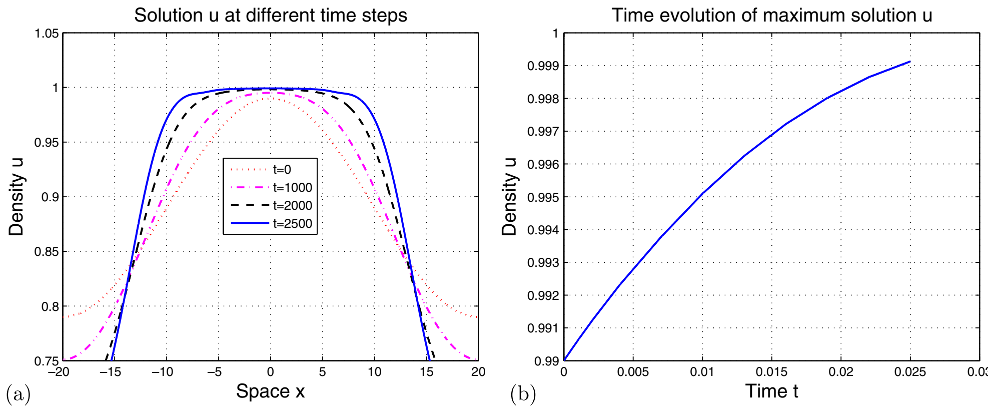

Figure 2. (a) Numerical illustration of the evolution of solution u to model (1.1) with singular diffusion, where $D(u) = (1-u)^{-\alpha}$ and $h(u) = c_{hD}(1-u)^{\beta}$ with $\alpha, \beta > 0$ and $\alpha + \beta = \lambda < 1$. The parameter values are $m = 0.89$, $\varepsilon = 0.1$, $c_{hD} = 100$, $\alpha = 0.2$, $\beta = 0.6$, $R = 20$ and t denotes the time step. (b) The plot of the time evolution of maximum value of solution, where the parameter values are the same as those in (a).

Acknowledgments

The first author is supported by a start-up fund from the Hong Kong Polytechnic University. The third author would like to thank the Department of Applied Mathematics of Hong Kong Polytechnic University for the financial support and the hospitality during his stay in 7–11 November, 2010. His work was also supported by the Polish Ministry of Science and Higher Education under the grant N201547638.

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