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COMPETING EFFECTS OF ATTRACTION VS. REPULSION IN CHEMOTAXIS

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We consider the attraction–repulsion chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w), & x \in \Omega, \quad t > 0, \\ \tau v_t = \Delta v + \alpha u - \beta v, & x \in \Omega, \quad t > 0, \\ \tau w_t = \Delta w + \gamma u - \delta w, & x \in \Omega, \quad t > 0, \end{cases}$$

under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where $\chi \geq 0, \xi \geq 0, \alpha > 0, \beta > 0, \gamma > 0, \delta > 0$ and $\tau = 0, 1$. We study the global solvability, boundedness, blow-up, existence of non-trivial stationary solutions and asymptotic behavior of the system for various ranges of parameter values. Particularly, we prove that the system with $\tau = 0$ is globally well-posed in high dimensions if repulsion prevails over attraction in the sense that $\xi\gamma - \chi\alpha > 0$, and that the system with $\tau = 1$ is globally well-posed in two dimensions if repulsion dominates over attraction in the sense that $\xi\gamma - \chi\alpha > 0$ and $\beta = \delta$. Hence our results confirm that the attraction–repulsion is a plausible mechanism to regularize the classical Keller–Segel model whose solution may blow up in higher dimensions.

Keywords: Chemotaxis; attraction–repulsion; boundedness; stationary solutions; convergence; entropy inequality.

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1. Introduction

Chemotaxis describes oriented movement of cells along the concentration gradient of a chemical signal produced by the cells. A well-known chemotaxis model was initially proposed by Keller and Segel[27] and has been extensively studied in the past four decades from various perspectives.[19,22,44,18,25,26,51,53] Among the theoretical results, the blow-up of solutions in finite time is a striking indication of the spontaneous formation of cell aggregates. However, in many biological processes, cells often interact with a combination of repulsive and attractive signaling chemicals to produce various interesting biological patterns.[8,12,16,39] This work aims to understand the competition between attractive and repulsive signals, and we consider the following dimensionless attraction–repulsion chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u \nabla v) + \nabla\cdot(\xi u\nabla w), & x\in\Omega, \quad t>0, \\ \tau v_t = \Delta v + \alpha u - \beta v, & x\in\Omega, \quad t>0, \\ \tau w_t = \Delta w + \gamma u - \delta w, & x\in\Omega, \quad t>0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega, \quad t>0, \\ u(x,0) = u_0(x), \quad \tau v(x,0) = \tau v_0(x), & \\ \tau w(x,0) = \tau w_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ denotes the cell density, $v = v(x,t)$ represents the concentration of an attractive cue, and $w = w(x,t)$ is the concentration of a repulsive signal; $\chi \geq 0$, $\xi \geq 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$ and $\tau = 0, 1$ are parameters. The first cross-diffusive (i.e. chemotactic) term in the first equation implies that the cell movement is directed toward the *increasing* chemoattractant concentration, whereas the second cross-diffusive term indicates that cells move away from the *increasing* chemorepellent concentration. The parameters χ and ξ measure the strength of the attraction and repulsion, respectively. The second and third equations in (1.1) state that chemoattractant and chemorepellent are released by cells and undergo decay.

The attraction–repulsion chemotaxis model (1.1) with $\tau = 1$ was proposed in Ref. 33 to describe the aggregation of microglia observed in Alzheimer's disease and in Ref. 39 to address the quorum effect in the chemotactic process. In their approaches, it is assumed that there exists a secondary chemical, denoted by w, which behaves as a chemorepellent to mediate the cell's chemotactic response to the chemoattractant v accordingly. In general chemicals diffuse much faster than cells since the chemical molecules are much smaller than cells in size. Hence the attraction–repulsion chemotaxis model (1.1) can be approximated by setting $\tau = 0$. Such quasi-steady-state approximation was extensively employed in the past to study the classical chemotaxis model without repulsive signal (cf. Refs. 26, 40 and 46).

In the absence of chemorepulsive chemical (i.e. chemorepellent), namely $\xi = 0$, w is decoupled from the system (1.1) and first two equations of (1.1) comprises a classical Keller–Segel model

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u\nabla v), & x\in\Omega, \quad t>0,\\ \tau v_t = \Delta v + \alpha u - \beta v, & x\in\Omega, \quad t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega, \quad t>0,\\ u(x,0)=u_0(x), \quad \tau v(x,0)=\tau v_0(x), & x\in\Omega, \end{cases} \tag{1.2}$$

whose solution behavior has been extensively studied by various researchers (see Ref. 22 for detailed results). In summary, the solution of system (1.2) never blows up when $n = 1$ (see Ref. 38) whereas there is finite-time or infinite-time blow-up when $n \geq 3$ (see Ref. 53). The case $n = 2$ is a borderline. Nagai[35] showed that when Ω is a ball in $\mathbb{R}^2$, the radial solution of (1.2) with $\tau = 0$ blows up in finite time at the origin if the initial mass $\int_\Omega u_0(x) > 8\pi/(\chi\alpha)$ and $\int_\Omega u_0(x)|x|^2$ is sufficiently small. For a general domain Ω, Nagai[36] further showed that finite-time blow-up of non-radial solutions occurs under the condition $\int_\Omega u_0(x)|x-y|^2dx$ is sufficiently small provided that $\int_\Omega u_0(x) > 8\pi/(\chi\alpha)$ if y is an interior point of Ω or $\int_\Omega u_0(x) > 4\pi/(\chi\alpha)$ if y is on $\partial\Omega$. If $n = 2$ and the initial mass $\int_\Omega u_0(x)$ is large in some sense, then the solution of (1.2) with $\tau = 1$ blows up either in finite or in infinite time provided Ω is simply connected;[24] in the particular framework of radially symmetric solution in a planar disk, solutions may even blow up in finite time.[18]

Since the blow-up is an extreme case, a large amount of efforts were devoted to modifying the classical Keller–Segel model (1.2) such that the modified models allow global bounded solutions and hence generate pattern formation which are applicable in reality. These modified models are referred to as *regularized* models. The existing *regularized* models were extensively reviewed by Hillen and Painter[19] in which the attraction–repulsion chemotaxis model was, however, not included. The attraction–repulsions mechanism was proposed previously in Ref. 39, but has not been mathematically confirmed. Recently, Liu and Wang[32] studied the global existence of solutions and non-trivial steady states of the attraction–repulsion model (1.1) with $\tau = 1$ in one dimension. However, the result of Liu and Wang[32] does not exclude the possibility of blow-up at infinity time. More importantly the classical Keller–Segel intrinsically does not blow up in one dimension either. Hence to confirm whether or not the attraction–repulsion mechanism may regularize the classical Keller–Segel model, it is crucial to prove whether the blow-up occurs in two or higher dimensions, which is the purpose of present paper. We shall show that attraction–repulsion chemotaxis model (1.1) with $\tau = 0$ has a unique uniformly bounded global solution in high dimensions if the repulsion prevails over attraction in the sense that $\xi\gamma - \chi\alpha > 0$, and that the attraction–repulsion chemotaxis model (1.1) with $\tau = 1$ in two dimensions has a unique uniformly bounded global classical solution if the repulsion prevails over attraction in the sense $\xi\gamma - \chi\alpha > 0$ and $\beta = \delta$. Hence our results confirm that the attraction–repulsion chemotaxis model can prevent blow-up if the repulsion is strong enough. Therefore the attraction–repulsion mechanism may regularize the classical Keller–Segel model. We should mention that the global existence of the attraction–repulsion chemotaxis model (1.1) has been previously attempted in Ref. 20 without giving definite answer. In this paper, we give the affirmative results not only on the global existence, but also on the blow-up, boundedness, stationary solutions and large-time behavior of solutions. A number of mathematical techniques, such as Moser iteration, Lyapunov functional, energy estimates, maximum principle and variable transformations, will be employed to derive our results. We should also note that the flux-limited chemotaxis model recently proposed by Bellomo *et al.*[4] might be regarded as a regularized model of the classical Keller–Segel model (1.2) with $\tau = 1$, since the diffusion and chemotaxis fluxes of the population are both uniformly bounded. However, the mathematical analysis of the model in Ref. 4 is very challenging due to degeneracy and strong nonlinearity.

We should mention a previous mathematical work[10] on a parabolic–parabolic repulsion chemotaxis model. For the space dimension $n = 2$, the authors in Ref. 10 asserted the global existence of smooth solutions and the convergence to steady states based on a Lyapunov functional approach; for $n = 3, 4$, they proved global existence of weak solutions. Before concluding this section, we want to mention some mathematical works related to chemotaxis models with multi-species and/or multi-stimuli (i.e. chemical signals). The works can be classified into three categories: (i) multi-species and one stimulus, e.g. see Refs. 30, 31 and 13; (ii) one species and multi-chemicals, e.g. see Refs. 6, 42 and 41; (iii) multi-species and multi-stimuli, e.g. see Refs. 14 and 56. More references may be found in Ref. 23 where general multi-species chemotaxis models have been proposed, and Lyapunov functions and steady states of proposed models were discussed. In the case (ii), most of models considered one species responding to multiple chemoattractive signals, such as Refs. 6 and 42. The model (1.1) considered in the present paper is mostly closed to a recent model studied in Ref. 41, where one species reacting two opposite signals (i.e. one chemoattractant and one chemorepellent). But there are two significant differences. First, Ref. 41 employed the volume-filling mechanism to the chemoattractive flux, which is however not included in the present paper. Second, Ref. 41 studied the traveling wave solutions, but we consider the global and blow-up solutions here. As far as we know, the results in Ref. 41 and in the present paper as well as some results in Ref. 23 are only mathematical works considering the time-dependent solutions of chemotaxis models with multi-species and multi-stimuli, in contrast to only stationary problems considered in previous works (see Ref. 23 and references therein). It is also worth pointing out that some numerics of chemotaxis models with multi-stimuli have been explored in Refs. 42 and 41 to generate traveling wave solutions.

2. Main Results

Before presenting our main results, we shall introduce some notations. For simplicity, the variable of integration in an integral will be omitted without ambiguity, e.g.

the integral $\int_\Omega f(x)dx$ is written as $\int_\Omega f(x)$. Hereafter, c_i denotes a generic constant which may change from one section to another, where $i = 1, 2, 3, \ldots$.

Our first result is

Theorem 2.1. *Suppose that $u_0(x) \in W^{1,\infty}(\Omega)$ is a non-negative function. Assume that*

$$\xi\gamma - \chi\alpha > 0. \tag{2.1}$$

Then, for any $n \geq 2$, there exists a unique triple (u, v, w) of non-negative bounded functions belonging to $C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$, which solves (1.1) *with $\tau = 0$ classically.*

In Theorem 2.1, we need a restriction that $\xi\gamma - \chi\alpha > 0$ for the boundedness of solutions of (1.1) with $\tau = 0$. This restriction is necessary in the sense that if $\xi\gamma - \chi\alpha < 0$, then the solution of (1.1) with $\tau = 0$ might blow up in finite time. Actually, as a consequence of Nagai's results in Ref. 36, we will have the following result.

Proposition 2.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, and let $x_0 \in \Omega$ be a fixed point. Assume that*

$$\xi\gamma - \chi\alpha < 0 \quad \text{and} \quad \beta = \delta \tag{2.2}$$

and that $\int_\Omega u_0(x) > 8\pi/(\chi\alpha - \xi\gamma)$. If $\int_\Omega u_0(x)|x - x_0|^2 dx$ is sufficiently small, then the solution component u of (1.1) *with $\tau = 0$ blows up in finite time.*

The existence or non-existence of non-trivial stationary solutions to (1.1) is mathematically and biologically interesting. In respect of this, we have the following result.

Proposition 2.3. (1) *If $\xi\gamma - \chi\alpha < 0$ and $\beta = \delta$, then for any $n \geq 1$, there exist non-trivial stationary solutions (u, v, w) to* (1.1) *with $\gamma v = \alpha w$.*

(2) *If $\xi\gamma - \chi\alpha \geq 0$ and $\beta = \delta$, then, for any $n \geq 1$, problem* (1.1) *has only one trivial stationary solution $(\overline{u}_0, \frac{\alpha}{\beta}\overline{u}_0, \frac{\gamma}{\beta}\overline{u}_0)$, where*

$$\overline{u}_0 := \frac{1}{|\Omega|} \int_\Omega u_0(x). \tag{2.3}$$

When $\xi\gamma - \chi\alpha > 0$, Theorem 2.1 warrantees the global existence and boundedness of classical solutions to (1.1) with $\tau = 0$, and Proposition 2.3 asserts that (1.1) has only one trivial positive stationary solution under an additional assumption that $\beta = \delta$. The above two results raise the following interesting question: Can we establish some connection between the global solution and the unique positive trivial stationary solution? The following proposition will address this question.

Proposition 2.4. *Let Ω be a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Assume that*

$$u_0(x) \in W^{1,\infty}(\Omega), \quad u_0 > 0 \quad in\ \bar{\Omega} \tag{2.4}$$

and that

$$\xi\gamma - \chi\alpha > 0 \quad and \quad \beta = \delta. \tag{2.5}$$

Then the global solution (u, v, w) obtained in Theorem 2.1 converges to $(\overline{u}_0, \frac{\alpha}{\beta}\overline{u}_0, \frac{\gamma}{\beta}\overline{u}_0)$ exponentially as $t \to +\infty$.

Remark 2.5. Propositions 2.3 and 2.4 imply that when repulsion dominates over attraction (in the sense that $\xi\gamma - \chi\alpha > 0$), there is no pattern formation since the solution converges to a constant. This is consistent with the biological intuitions since the repulsion acts as a diffusion which helps to stabilize the system. However if attraction dominates over repulsion, non-trivial stationary solutions arise and pattern formation may be expected.

The above results can be generalized to the full attraction–repulsion chemotaxis model (1.1) in two dimensions. Precisely, we have the following result.

Proposition 2.6. *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. In addition to* (2.4) *and* (2.5), *we also assume that*

$$v_0(x) \in W^{1,\infty}(\Omega), \quad w_0(x) \in W^{1,\infty}(\Omega), \quad v_0 \geq 0 \quad and \quad w_0 \geq 0 \quad in\ \bar{\Omega}. \tag{2.6}$$

Then there exists a unique triple (u, v, w) of non-negative bounded functions belong to $C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))$ which solves (1.1) *with $\tau = 1$ classically. Moreover, this global solution converges to $(\overline{u}_0, \frac{\alpha}{\beta}\overline{u}_0, \frac{\gamma}{\beta}\overline{u}_0)$ exponentially as $t \to +\infty$.*

Proposition 2.6 says that if repulsion dominates over attraction in the sense that $\xi\gamma - \chi\alpha > 0$ and $\beta = \delta$, then (1.1) with $\tau = 1$ admits a unique uniformly bounded global solution for arbitrarily *large* initial data u_0. However, if $\beta \neq \delta$, we will need an additional *smallness* assumption on the initial data u_0 for the global solvability of (1.1) with $\tau = 1$. In this regard, we have the following result.

Theorem 2.7. *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. In addition to* (2.1), (2.4) *and* (2.6), *we assume that*

$$\frac{\chi^2\alpha^2(\beta-\delta)^2}{2\beta^2(\xi\gamma - \chi\alpha)} \cdot \int_\Omega u_0(x) \leq C(\Omega), \tag{2.7}$$

where $C(\Omega)$ is some positive constant depending only on Ω. Then there exists a unique triple (u, v, w) of non-negative functions belong to $C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))$ which solve (1.1) *with $\tau = 1$ classically.*

Remark 2.8. It should be stressed that the results of Theorem 2.1 and Propositions 2.4 and 2.6 can be readily extended to the borderline case:

$$\xi\gamma - \chi\alpha = 0 \quad \text{and} \quad \beta = \delta. \tag{2.8}$$

Indeed, if we set $z := \xi w - \chi v$, then we obtain from (2.8) and (1.1) that a pair (u, z) satisfies

$$\begin{cases} u_t = \Delta u + \nabla \cdot (u \nabla z), & x \in \Omega, \quad t > 0, \\ \tau z_t = \Delta z - \beta z, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad \tau z(x,0) = \tau(\xi w_0(x) - \chi v_0(x)), & x \in \Omega. \end{cases} \tag{2.9}$$

From the maximum principle and $\beta > 0$, we infer that $z \equiv 0$ for $\tau = 0$. This in conjunction with (2.9) yields that u actually satisfies a standard heat equation with the Neumann boundary condition. Hence, Theorem 2.1 and Proposition 2.4 can include the borderline case (2.8). Since the z-equation in (2.9) is decoupled with the variable u, it is easy to obtain that

$$\int_0^t \int_\Omega |\nabla z|^4 \leq C \quad \text{for all } t > 0,$$

for $\tau = 1$ (see Sec. 6 below) and therefore Proposition 2.6 can also include the borderline case (2.8). For simplicity, we do not particularly address this trivial case in Theorem 2.1 and Propositions 2.4 and 2.6 and in their proofs.

3. Local Existence and Preliminaries

The local solvability of (1.1) for $u_0(x) \in C^0(\bar{\Omega})$, $\tau v_0(x) \in W^{1,p}(\Omega)$ and $\tau w_0(x) \in W^{1,p}(\Omega)$ $(p > n)$ can be proved by adapting approaches that are well-established in the context of classical chemotaxis models (cf. Refs. 9, 11, 25 and 57). However, in order to unify and simplify the proofs of our local existence for both the case $\tau = 1$ and the case $\tau = 0$, we shall assume that $(u_0(x), \tau v_0(x), \tau w_0(x)) \in (W^{1,\infty}(\Omega))^3$. The present proof of local existence is inspired by an approach developed in Ref. 47.

Lemma 3.1. *Assume that* $u_0 \in W^{1,\infty}(\Omega)$, $\tau v_0(x) \in W^{1,\infty}(\Omega)$ *and* $\tau w_0(x) \in W^{1,\infty}(\Omega)$ *are non-negative. Then there exist* $T_{\max} \in (0, \infty]$ *and a unique triple* (u, v, w) *of non-negative functions from* $C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max}))$ *solving* (1.1) *classically in* $\Omega \times (0, T_{\max})$. *Moreover*

$$\text{if } T_{\max} < \infty, \text{ then } \|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \nearrow T_{\max}. \tag{3.1}$$

Proof. We give only the proof for the case $\tau = 1$, because the proof for the case $\tau = 0$ can be proceeded similarly.

(i) *Existence and uniqueness.* Define

$$R := \|u_0\|_{L^\infty(\Omega)} + 1.$$

With this R and $T \in (0, 1)$ to be specified below, in the Banach space

$$X := C^0(\bar{\Omega} \times [0, T]),$$

we consider the closed convex set

$$S_T := \{u \in X \mid \|u(\cdot,t)\|_{L^\infty(\Omega)} \le R \text{ for all } t \in [0,T]\}$$

and introduce a mapping $\Phi : S_T \mapsto S_T$ such that given $\tilde{u} \in S_T, \Phi(\tilde{u}) = u$ where u is the solution to

$$\begin{cases} u_t = \Delta u + \nabla \cdot [(-\chi \nabla v + \xi \nabla w)u], & x \in \Omega, \quad t \in (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{3.2}$$

with v being the solution of

$$\begin{cases} v_t - \Delta v + \beta v = \alpha \tilde{u}, & x \in \Omega, \quad t \in (0,T), \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0,T), \\ v(x,0) = v_0(x), & x \in \Omega \end{cases} \tag{3.3}$$

and w defined the solution of

$$\begin{cases} w_t - \Delta w + \delta w = \gamma \tilde{u}, & x \in \Omega, \quad t \in (0,T), \\ \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0,T), \\ w(x,0) = w_0(x), & x \in \Omega. \end{cases} \tag{3.4}$$

We shall show that for T small enough Φ has a unique fixed point. From the standard L^p and Schauder theories of linear parabolic equation[29] we infer that there exists a unique solution $v(x,t) \in C^{1+\theta, \frac{1+\theta}{2}}(\Omega \times (0,T))$ to (3.3) for each $\theta \in (0,1)$. Similarly, there is a unique solution $w(x,t) \in C^{1+\theta,\frac{1+\theta}{2}}(\Omega \times (0,T))$ to (3.4). Since $(-\chi\nabla v + \xi \nabla w) \in L^\infty(\Omega \times (0,T))$, and since u_0 was assumed to be Hölder continuous in $\bar{\Omega}$ due to the Sobolev embedding: $W^{1,\infty}(\Omega) \hookrightarrow C^\theta(\bar{\Omega})$ for each $\theta \in (0,1)$ (see Ref. 17), we may apply Ref. 29 to conclude that there is a unique solution $u \in C^{\theta,\frac{\theta}{2}}(\bar{\Omega} \times [0,T])$ to (3.2) and

$$\|u\|_{C^{\theta,\frac{\theta}{2}}(\bar{\Omega}\times[0,T])} \le K, \tag{3.5}$$

for some $\theta \in (0,1)$ and $K > 0$, where K depends on $\|\nabla v\|_{L^\infty((0,T);\, C^\theta(\bar{\Omega}))}$ and $\|\nabla w\|_{L^\infty((0,T);\, C^\theta(\bar{\Omega}))}$. Since the latter two quantities can be controlled by $\|\tilde{u}\|_{L^\infty(\Omega)} \le R$, it follows that $K = K(R)$. So, we have

$$\|u(\cdot,t) - u_0\|_{L^\infty(\Omega)} \le K(R) t^{\frac{\theta}{2}}$$

and thus

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + K(R) t^{\frac{\theta}{2}}.$$

From this we deduce that if we fix $T = T_0 < (\frac{1}{K(R)})^{\frac{2}{\theta}}$, then we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le R := \|u_0\|_{L^\infty(\Omega)} + 1 \quad \text{for all } t \in [0, T_0]. \tag{3.6}$$

Hence, $u \in S_T$ and this proves that Φ maps S_T into itself. By a straightforward adaptation of the above reasoning, one can easily deduce that if T is further diminished then Φ in fact becomes a contraction on S_T. For such T we therefore conclude from the contraction mapping principle[17] that there exists a unique $u \in S_T$ such that $\Phi(u) = u$.

(ii) *Regularity and non-negativity.* By $u \in C^{\theta,\frac{\theta}{2}}(\bar{\Omega} \times [0,T])$ and the classical regularity of parabolic equations[29] we obtain $v(x,t), w(x,t) \in C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega} \times [\eta, T])$ for all $\eta \in (0, T_0]$. This in conjunction with the first equation in (1.1) further entails that

$$u(x,t) \in C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega} \times [\eta, T]) \quad \text{for all } \eta \in (0, T_0].$$

The solution may be prolonged in the interval $[0, T_{\max})$ with either $T_{\max} = \infty$ or $T_{\max} < \infty$, where in the latter case

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \nearrow T_{\max}.$$

Finally, the non-negativity of u, v and w follows from the classical maximum principle since (u, v, w) is a smooth solution of (1.1). □

The following important property on mass can be easily derived.

Lemma 3.2. *The solution* (u, v, w) *of* (1.1) *satisfies the following properties*:

$$\|u(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \qquad \textit{for all } t \in (0, T_{\max}), \tag{3.7}$$

$$\|v(\cdot,t)\|_{L^1(\Omega)} \le \tau\|v_0\|_{L^1(\Omega)} + \frac{\alpha}{\beta}\|u_0\|_{L^1(\Omega)} \quad \textit{for all } t \in (0, T_{\max}), \tag{3.8}$$

$$\|w(\cdot,t)\|_{L^1(\Omega)} \le \tau\|w_0\|_{L^1(\Omega)} + \frac{\gamma}{\delta}\|u_0\|_{L^1(\Omega)} \quad \textit{for all } t \in (0, T_{\max}), \tag{3.9}$$

where the equalities in (3.8) *and* (3.9) *hold when* $\tau = 0$. *Moreover, under the additional assumption that* $u_0 > 0$, *we have*

$$u > 0 \quad \textit{for all } x \in \Omega,\ \ t > 0. \tag{3.10}$$

Proof. Integrating each equation of (1.1) with respect to $x \in \Omega$, we get that $\frac{d}{dt}\int_\Omega u \equiv 0$, $\tau\frac{d}{dt}\int_\Omega v + \beta\int_\Omega v = \alpha\int_\Omega u$, and that $\tau\frac{d}{dt}\int_\Omega w + \delta\int_\Omega w = \gamma\int_\Omega u$ for $t \in (0, T_{\max})$. These yield (3.7)–(3.9). Inequality (3.10) follows from the maximum principle. □

The proofs of our main results (Theorems 2.1 and 2.7) will be based on some *a priori* estimates. To derive these estimates, we shall use the Gagliardo–Nirenberg interpolation inequality. For readers' convenience, let us recall it[15]: Let $\Omega \subset \mathbb{R}^n$

be a bounded domain with smooth boundary, let l, k be any integers satisfying $0 \le l < k$, and let $1 \le q, r \le \infty$, and $p \in \mathbb{R}^+$, $\frac{l}{k} \le a \le 1$ such that

$$\frac{1}{p} - \frac{l}{n} = a\left(\frac{1}{q} - \frac{k}{n}\right) + (1-a)\frac{1}{r}. \tag{3.11}$$

Then, for any $u(x) \in W^{k,q}(\Omega) \cap L^r(\Omega)$, there exist two positive constants c_1 and c_2 depending only on Ω, q, k, r and n such that the following inequality holds:

$$\|D^l u\|_{L^p(\Omega)} \le c_1 \|D^k u\|^a_{L^q(\Omega)} \|u\|^{1-a}_{L^r(\Omega)} + c_2 \|u\|_{L^r(\Omega)} \tag{3.12}$$

with the following exception: If $1 < q < \infty$ and $k - l - \frac{n}{q}$ is a non-negative integer, then (3.11) holds only for a satisfying $\frac{l}{k} \le a < 1$.

Taking $l = 0, k = 1$ and $q = 2$, we infer from (3.12) that for any $u(x) \in W^{1,2}(\Omega) \cap L^r(\Omega)$,

$$\|u\|_{L^p(\Omega)} \le c_1 \|\nabla u\|^a_{L^2(\Omega)} \|u\|^{1-a}_{L^r(\Omega)} + c_2 \|u\|_{L^r(\Omega)}, \tag{3.13}$$

with $a \in (0,1)$ satisfying

$$\frac{n}{p} = a\left(\frac{n}{2} - 1\right) + \frac{n}{r}(1-a). \tag{3.14}$$

The Gagliardo–Nirenberg inequality (3.13) will be frequently used in our analysis below.

We shall also need to use (3.12) involving the second-order derivative of u. Taking $l = 0$ and $k = 2$, we infer from (3.12) that for $u \in W^{2,q}(\Omega) \cap L^r(\Omega)$,

$$\|u\|_{L^p(\Omega)} \le c_1 \|D^2 u\|^a_{L^q(\Omega)} \|u\|^{1-a}_{L^r(\Omega)} + c_2 \|u\|_{L^r(\Omega)}, \tag{3.15}$$

with $a \in (0,1)$ satisfying

$$\frac{n}{p} = a\left(\frac{n}{q} - 2\right) + \frac{n}{r}(1-a). \tag{3.16}$$

We shall also need the following variation of the standard Gagliardo–Nirenberg inequality involving L^r space with $r < 1$ (see Ref. 37): Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, let $p > 1$ and $r \in (0, p)$. Then, for any $u(x) \in W^{1,2}(\Omega) \cap L^r(\Omega)$, there exist two positive constants c_1 and c_2 depending only on Ω, r and n such that the inequality (3.13) holds, with $a \in (0,1)$ satisfying (3.14).

Finally, we mention that, based on Hölder inequality, Winkler[52] also proved a variation of the standard Gagliardo–Nirenberg inequality involving L^r space with $r < 1$ in the whole space $\mathbb{R}^n$.

4. Boundedness for $\xi\gamma - \chi\alpha > 0$

The following lemma is the core of the argument concerning global existence and boundedness.

Lemma 4.1. *Let* (2.1) *hold. Then, for any* $p > \max(\frac{n}{2}, 1)$, *there exists a constant* $C > 0$ *such that the solution of* (1.1) *with* $\tau = 0$ *satisfies*

$$\int_\Omega u^p \le C \quad \textit{for all } t \in (0, T_{\max}). \tag{4.1}$$

Proof. Using u^{p-1} as a test function for the first equation in (1.1), integrating by parts and employing the second and the third equations in (1.1) with $\tau = 0$, we obtain

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p &= \int_\Omega u^{p-1}\Delta u - \int_\Omega u^{p-1}\nabla\cdot(\chi u\nabla v) + \int_\Omega u^{p-1}\nabla\cdot(\xi u\nabla w) \\
&= -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \chi(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v \\
&\quad - \xi(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla w \\
&= -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{\chi(p-1)}{p}\int_\Omega \nabla u^p\cdot\nabla v - \frac{\xi(p-1)}{p}\int_\Omega \nabla u^p\cdot\nabla w \\
&= -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{p-1}{p}\cdot\int_\Omega u^p(-\chi\Delta v + \xi\Delta w) \\
&= -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{p-1}{p}\cdot\int_\Omega u^p[\xi\delta w - (\xi\gamma - \chi\alpha)u - \chi\beta v]
\end{aligned}$$

for all $t \in (0, T_{\max})$. This, along with $v \ge 0$, yields

$$\frac{d}{dt}\int_\Omega u^p \le -(\xi\gamma - \chi\alpha)(p-1)\int_\Omega u^{p+1} + \xi\delta(p-1)\int_\Omega u^p w \quad \text{for all } t \in (0, T_{\max}). \tag{4.2}$$

We need to further estimate the last term in (4.2). By (2.1) and the Young inequality:

$$ab \le \varepsilon a^q + (\varepsilon q)^{-\frac{r}{q}} r^{-1} b^r \quad \text{for any } a, b \ge 0, \quad \varepsilon > 0, \quad q, r > 0, \quad \frac{1}{q} + \frac{1}{r} = 1,$$

we see that

$$\begin{aligned}
\xi\delta\int_\Omega u^p w &\le \frac{\xi\gamma - \chi\alpha}{2}\cdot\int_\Omega u^{p+1} \\
&\quad + \xi\delta\cdot\left[\frac{2\xi\delta p}{(\xi\gamma - \chi\alpha)(p+1)}\right]^p \frac{1}{p+1}\int_\Omega w^{p+1} \quad \text{for all } t \in (0, T_{\max}).
\end{aligned} \tag{4.3}$$

Collecting (4.2) and (4.3) we obtain

$$\frac{d}{dt}\int_\Omega u^p \le -\frac{(\xi\gamma - \chi\alpha)(p-1)}{2}\int_\Omega u^{p+1} + c_1\int_\Omega w^{p+1} \quad \text{for all } t \in (0, T_{\max}), \tag{4.4}$$

where $c_1 := \xi\delta\cdot[\frac{2\xi\delta p}{(\xi\gamma-\chi\alpha)(p+1)}]^p\cdot\frac{p-1}{p+1}$.

In the following we will show that $\int_\Omega w^{p+1}$ can be controlled by $\varepsilon\int_\Omega u^{p+1}+c$ for sufficiently small $\varepsilon>0$ and some constant $c>0$. Noting that w solves

$$\begin{cases} -\Delta w+\delta w=\gamma u, & x\in\Omega, \quad t\in(0,T_{\max}),\\ \dfrac{\partial w}{\partial\nu}=0, & x\in\partial\Omega, \quad t\in(0,T_{\max}), \end{cases}$$

where $\delta>0$, and applying the Agmon–Douglis–Nirenberg L^p estimates[1,2] on linear elliptic equations with the (zero) Neumann boundary condition, we find that there exists some constant $c_2>0$ such that

$$\|w(\cdot,t)\|_{W^{2,p}(\Omega)}\le c_2\|u(\cdot,t)\|_{L^p(\Omega)} \quad \text{for all } t\in(0,T_{\max}). \tag{4.5}$$

We interpolate using the Gagliardo–Nirenberg inequality[15] (3.9) and (4.5) to obtain some constants $c_3>0$ and $c_4>0$ such that

$$\begin{aligned}\int_\Omega w^{p+1} &= \|w\|_{L^{p+1}(\Omega)}^{p+1}\\ &\le c_3\|D^2w\|_{L^p(\Omega)}^{(p+1)\theta}\|w\|_{L^1(\Omega)}^{(p+1)(1-\theta)}+c_3\|w\|_{L^1(\Omega)}^{p+1}\\ &\le c_4\|u\|_{L^p(\Omega)}^{(p+1)\theta}+c_4 \quad \text{for all } t\in(0,T_{\max}),\end{aligned} \tag{4.6}$$

where

$$\theta:=\frac{1-\frac{1}{p+1}}{1+\frac{2}{n}-\frac{1}{p}}=\frac{np^2}{(p+1)[(n+2)p-n]}\in(0,1)$$

due to $p>\frac{n}{2}$. Furthermore, by $p>\frac{n}{2}$ again, it is easily checked that

$$(p+1)\theta<p. \tag{4.7}$$

Therefore, we use the Young inequality to further estimate

$$\begin{aligned}\int_\Omega w^{p+1} &\le c_4\|u\|_{L^p(\Omega)}^{(p+1)\theta}+c_4\\ &\le c_4(\|u\|_{L^p(\Omega)}^{p}+1)+c_4\\ &= c_4\int_\Omega u^p+2c_4\\ &\le c_4\left(\varepsilon\int_\Omega u^{p+1}+\left[\frac{p}{\varepsilon(p+1)}\right]^p\cdot\frac{1}{p+1}|\Omega|\right)+2c_4\\ &=: \varepsilon c_4\int_\Omega u^{p+1}+c_5(\varepsilon) \quad \text{for all } t\in(0,T_{\max}).\end{aligned} \tag{4.8}$$

Taking

$$\varepsilon=\frac{(\xi\gamma-\chi\alpha)(p-1)}{4c_1c_4}$$

and inserting (4.8) into (4.4), we obtain

$$\frac{d}{dt}\int_\Omega u^p \le -\frac{(\xi\gamma-\chi\alpha)(p-1)}{4}\int_\Omega u^{p+1} + c_6 \quad \text{for all } t\in(0,T_{\max}), \tag{4.9}$$

where $c_6 := c_1c_5$. Now, adding the term $\int_\Omega u^p$ on both sides of (4.9) yields

$$\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \le -\frac{(\xi\gamma-\chi\alpha)(p-1)}{4}\int_\Omega u^{p+1} + \int_\Omega u^p + c_6 \quad \text{for all } t\in(0,T_{\max}). \tag{4.10}$$

Again, using the Young inequality we have

$$\int_\Omega u^p \le \frac{(\xi\gamma-\chi\alpha)(p-1)}{4}\int_\Omega u^{p+1} + \left[\frac{4p}{(\xi\gamma-\chi\alpha)(p^2-1)}\right]^p\cdot\frac{1}{p+1}|\Omega| \tag{4.11}$$

for all $t\in(0,T_{\max})$. Combining (4.10) and (4.11) we obtain

$$\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \le c_7 \quad \text{for all } t\in(0,T_{\max}), \tag{4.12}$$

where $c_7 := c_6 + [\frac{4p}{(\xi\gamma-\chi\alpha)(p^2-1)}]^p\cdot\frac{1}{p+1}|\Omega|$. This, together with the Gronwall inequality, yields

$$\int_\Omega u^p \le e^{-t}\int_\Omega u_0^p + c_7(1-e^{-t}) \le \int_\Omega u_0^p + c_7 \quad \text{for all } t\in(0,T_{\max})$$

and the proof of (4.1) is complete. □

We are now in the position to prove Theorem 2.1.

Proof of Theorem 2.1. First, it follows from Lemma 4.1 and the L^p estimate (4.5) that there exists some $c_1>0$ such that

$$\|w(\cdot,t)\|_{W^{2,p}(\Omega)} \le c_1 \quad \text{for all } t\in(0,T_{\max}). \tag{4.13}$$

This, along with the Sobolev embedding[17]: $W^{2,p}(\Omega)\hookrightarrow C_B^1(\Omega) := \{u\in C^1(\Omega)\,|\, Du\in L^\infty(\Omega)\}$ if $p>n$, yields some $c_2>0$ such that

$$\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} \le c_2 \quad \text{for all } t\in(0,T_{\max}). \tag{4.14}$$

Similarly, we can obtain some $c_3>0$ such that

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le c_3 \quad \text{for all } t\in(0,T_{\max}). \tag{4.15}$$

Next, using u^{p-1} as a test function for the first equation in (1.1), integrating over Ω, noting (4.14) and (4.15) and applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p &= -\frac{4(p-1)}{p^2}\int_\Omega|\nabla u^{\frac{p}{2}}|^2 + (p-1)\int_\Omega u^{p-1}\nabla u\cdot(\chi\nabla v-\xi\nabla w)\\ &\le -\frac{4(p-1)}{p^2}\int_\Omega|\nabla u^{\frac{p}{2}}|^2 + (p-1)(\chi c_3+\xi c_2)\int_\Omega u^{p-1}|\nabla u|\end{aligned}$$

$$
\begin{aligned}
&= -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + (p-1)(\chi c_3+\xi c_2)\cdot\frac{2}{p}\int_\Omega u^{\frac{p}{2}}\cdot|\nabla u^{\frac{p}{2}}| \\
&\le -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + (\chi c_3+\xi c_2)\cdot\frac{2(p-1)}{p}\Bigg(\frac{1}{(\chi c_3+\xi c_2)p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \\
&\quad + \frac{(\chi c_3+\xi c_2)p}{4}\int_\Omega u^p\Bigg) \\
&= -\frac{2(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{(\chi c_3+\xi c_2)^2}{2}(p-1)\int_\Omega u^p \qquad (4.16)
\end{aligned}
$$

for all $t\in(0,T_{\max})$, where c_2, c_3 and in what follows c_i $(i\ge 4)$ are constants being independent of p. This further yields

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p \le -\frac{2(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_4 p(p-1)\int_\Omega u^p \quad (4.17)$$

for all $t\in(0,T_{\max})$ and for all $p\ge 2$, where $c_4 := 1+\frac{(\chi c_3+\xi c_2)^2}{2}$. Then, it follows from (4.17) and the well-known Moser–Alikakos iteration procedure (cf. Ref. 3 or Ref. 45) that there exists a constant $c>0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le c \quad \text{for all } t\in(0,T_{\max}). \qquad (4.18)$$

For completeness, we give the details of the proof of (4.18) here. In fact, we shall show that the last term on the right-hand side of (4.17) can be controlled by $\varepsilon\int_\Omega |\nabla u^{\frac{p}{2}}|^2$ and $(\int_\Omega u^{\frac{p}{2}})^2$ for some small $\varepsilon>0$. To this end, we need the following interpolation inequality[29]: For any $U\in W^{1,2}(\Omega)$,

$$\|U-\bar U\|^2_{L^2(\Omega)} \le c_5\|\nabla U\|^{2b}_{L^2(\Omega)}\|U\|^{2(1-b)}_{L^1(\Omega)},$$

where $\bar U = \frac{1}{|\Omega|}\int_\Omega U$, $b=\frac{n}{n+2}$, and c_5 is a constant depending only on n and Ω. This, along with Young's inequality ($yz\le \varepsilon y^p + c\varepsilon^{-\frac{q}{p}}z^q,\ y,z>0,\ p,q>0,\ \frac{1}{p}+\frac{1}{q}=1$), entails that

$$\|U\|^2_{L^2(\Omega)} \le \varepsilon\|\nabla U\|^2_{L^2(\Omega)} + c_6(1+\varepsilon^{-\frac{n}{2}})\|U\|^2_{L^1(\Omega)} \quad \text{for any } \varepsilon>0, \qquad (4.19)$$

where $c_6>0$ depends on n and Ω, but is independent of ε. Applying interpolation inequality (4.19) with $U=u^{\frac{p}{2}}$ and $\varepsilon=\frac{2}{p^2c_4}$, we obtain

$$c_4p(p-1)\int_\Omega u^p \le \frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_7p(p-1)(1+p^n)\left(\int_\Omega u^{\frac{p}{2}}\right)^2, \qquad (4.20)$$

where $c_7 := c_4\max\{1,(\frac{c_4}{2})^{\frac{n}{2}}\}$. Inserting (4.20) into (4.17) and noting $1+p^n\le(1+p)^n$, we obtain

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p \le c_7p(p-1)(1+p)^n\left(\int_\Omega u^{\frac{p}{2}}\right)^2. \qquad (4.21)$$

Thus,

$$\frac{d}{dt}\left[e^{p(p-1)t}\int_\Omega u^p\right] \le c_7 e^{p(p-1)t} p(p-1)(1+p)^n \left(\int_\Omega u^{\frac{p}{2}}\right)^2. \tag{4.22}$$

Integrating (4.22) over the time interval $[0,t]$ for $0 < t < T_{\max}$, we obtain

$$\int_\Omega u^p(x,t) \le \int_\Omega u_0^p(x) + c_7(1+p)^n \sup_{0\le t\le T_{\max}} \left(\int_\Omega u^{\frac{p}{2}}(x,t)\right)^2. \tag{4.23}$$

Define

$$K(p) := \max\left\{\|u_0\|_{L^\infty(\Omega)}, \sup_{0\le t\le T_{\max}} \left(\int_\Omega u^p(x,t)\right)^{\frac{1}{p}}\right\}.$$

Then, (4.23) entails

$$K(p) \le [c_8(1+p)^n]^{\frac{1}{p}} K\left(\frac{p}{2}\right) \quad \text{for all } p \ge 2, \tag{4.24}$$

where $c_8 := |\Omega| + c_7$. Taking $p = 2^j$, $j = 1, 2, \ldots$, one obtains

$$\begin{aligned} K(2^j) &\le c_8^{2^{-j}}(1+2^j)^{2^{-j}n} K(2^{j-1}) \\ &\;\;\vdots \\ &\le c_8^{2^{-j}+\cdots+2^{-1}}(1+2^j)^{2^{-j}n}\cdots(1+2)^{2^{-1}n}K(1) \\ &\le c_8[2^{j2^{-j}n}(2^{-j}+1)^{2^{-j}n}]\cdots[2^{2^{-1}n}(2^{-1}+1)^{2^{-1}n}]K(1) \\ &\le c_8 2^{[j2^{-j}+(j-1)2^{-(j-1)}+\cdots+2^{-1}]n}\cdot 2^{(2^{-j}+2^{-(j-1)}+\cdots+2^{-1})n}K(1) \\ &\le c_8 2^{3n}K(1). \end{aligned}$$

Letting $j \to \infty$ and using (3.7), we finally conclude that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le c_8 2^{3n} K(1) \le c_8 2^{3n} \max\{\|u_0\|_{L^\infty(\Omega)}, \|u_0\|_{L^1(\Omega)}\} \le c.$$

This proves (4.18). Finally, the assertion of Theorem 2.1 is an immediate consequence of (4.18), Lemma 3.1 and the extensibility criterion therein. □

Next, we prove Proposition 2.2.

Proof of Proposition 2.2. Introducing the following scalings

$$\tilde{t} = \beta t, \quad \tilde{x} = \sqrt{\beta}x,$$

setting

$$z := \chi v - \xi w$$

and using (2.2), we obtain from (1.1) with $\tau = 0$ that

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla z), & x \in \Omega, \quad t > 0, \\ 0 = \Delta z - z + \dfrac{\chi\alpha - \xi\gamma}{\beta} u, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{4.25}$$

where the tildes have been dropped without confusion. Noticing that $\chi\alpha - \xi\gamma > 0$ and the scaling $\tilde{x} = \sqrt{\beta} x$, then the assertion of Proposition 2.2 is an immediate consequence of Ref. 36 applied to system (4.25).

Since the finite-time blow-up is a very important issue in chemotaxis, let us recall the main ideas of the proof in Ref. 36, which is based on a so-called moment method.

We divide the proof into four steps.

Step 1. We choose an appropriate weight function $\Psi(x)$ for a moment of u.

Let $0 < r_1 < r_2 < \operatorname{dist}(x_0, \partial\Omega)$, where $\operatorname{dist}(x_0, \partial\Omega)$ is the distance between x_0 and $\partial\Omega$. We define the function $\psi \in C^1([0,\infty)) \cap W^{2,\infty}((0,\infty))$ by

$$\psi(r) = \begin{cases} r^2, & \text{if } 0 \le r \le r_1, \\ a_1 r^2 + a_2 r + a_3, & \text{if } r_1 \le r \le r_2, \\ r_1 r_2, & \text{if } r > r_2, \end{cases} \tag{4.26}$$

where

$$a_1 = -\frac{r_1}{r_2 - r_1}, \quad a_2 = \frac{2 r_1 r_2}{r_2 - r_1}, \quad a_3 = -\frac{r_1^2 r_2}{r_2 - r_1}.$$

Next we define $\Psi \in C^1(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$ by

$$\Psi(x) = \psi(|x|).$$

Then one can check that $\Psi(x)$ has the following properties (cf. Ref. 36):

$$|\nabla \Psi(x)| \le 2(\Psi(x))^{\frac{1}{2}}, \tag{4.27}$$

$$|\Delta \Psi(x)| \le 4, \tag{4.28}$$

$$\{\nabla \Psi(x) - \nabla \Psi(y)\} \cdot \nabla N(x-y) = -\frac{1}{\pi} \quad \text{for } (x,y) \in B_1 \times B_1, \tag{4.29}$$

$$\{\nabla \Psi(x) - \nabla \Psi(y)\} \cdot \nabla N(x-y) \le \frac{r_1}{\pi(r_2 - r_1)} \quad \text{for } (x,y) \notin B_1 \times B_1, \tag{4.30}$$

$$\frac{\partial \Psi(x)}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{4.31}$$

where $B_j := \{x \in \mathbb{R}^2 : |x| < r_j\}$ for $j = 1, 2$ and $N(x-y) := -\frac{1}{2\pi} \ln |x-y|$.

Step 2. We derive a basic integral inequality.

Multiplying the first equation in (4.25) by Ψ and integrating over Ω, we obtain

$$\frac{d}{dt}\int_\Omega u(x,t)\Psi(x)dx = \int_\Omega u(x,t)\Delta\Psi(x)dx + \int_\Omega u(x,t)\nabla\Psi(x)\cdot\nabla z(x,t)dx.$$

This, in conjunction with (4.28) and $\int_\Omega u(x,t)dx = \int_\Omega u_0 dx$, entails that

$$\frac{d}{dt}\int_\Omega u(x,t)\Psi(x)dx \le 4\int_\Omega u_0 dx + \int_\Omega u(x,t)\nabla\Psi(x)\cdot\nabla z(x,t)dx. \tag{4.32}$$

Step 3. We derive a moment differential inequality.

The moment of u is defined by

$$M_\Psi(t) := \int_\Omega u(x,t)\Psi(x)dx.$$

The crucial technical ingredient is to estimate the last integral in (4.32). To this end, we note that z solves a Poisson equation: $-\Delta z = -z + \sigma u$, where $\sigma := \frac{\chi\alpha-\xi\gamma}{\beta}$, and therefore it has an integral representation with an integrated function involving u, z and a Green function $G(x,t)$ of $-\Delta$ on a ball with zero Dirichlet boundary conditions (here we need to introduce a cutoff function on the ball; cf. Ref. 36 for details). Making full use of this fact, recalling (4.27)–(4.30) and using the symmetry properties of some related integral, one can estimate properly the integral $\int_\Omega u(x,t)\nabla\Psi(x)\cdot\nabla z(x,t)dx$ to obtain that

$$\begin{aligned}\int_\Omega u(x,t)\nabla\Psi(x)\cdot\nabla z(x,t)dx \le &-\frac{\sigma}{2\pi}\left(\int_\Omega u_0 dx\right)^2\\ &+ c_1\left(\int_\Omega u_0 dx\right)\cdot\left(\int_\Omega u(x,t)\Psi(x)dx\right)\\ &+ c_2\left(\int_\Omega u_0 dx\right)^{\frac{3}{2}}\cdot\left(\int_\Omega u(x,t)\Psi(x)dx\right)^{\frac{1}{2}},\end{aligned} \tag{4.33}$$

for some $c_1 > 0$ and $c_2 > 0$. Inserting (4.33) into (4.32) and recalling the definition of $M_\Psi(t)$ we obtain

$$\frac{d}{dt}M_\Psi(t) \le H(M_\Psi(t)), \tag{4.34}$$

where $H(s)$ is the continuous function on $[0,\infty)$ defined by

$$\begin{aligned}H(s) := &\frac{\sigma}{2\pi}\left(\int_\Omega u_0 dx\right)\left(\frac{8\pi}{\sigma} - \int_\Omega u_0 dx\right) + c_1\left(\int_\Omega u_0 dx\right)\cdot s\\ &+ c_2\left(\int_\Omega u_0 dx\right)^{\frac{3}{2}}\cdot s^{\frac{1}{2}}.\end{aligned} \tag{4.35}$$

Step 4. We prove the finite-time blow-up.

It seems that there was a gap in the original proof in Ref. 36 in this step. So we provide here more details to fill this gap.

We note that

$$H(0) = \frac{\sigma}{2\pi}\left(\int_\Omega u_0 dx\right)\left(\frac{8\pi}{\sigma} - \int_\Omega u_0 dx\right) < 0 \tag{4.36}$$

under the assumption that $\int_\Omega u_0(x)dx > \frac{8\pi}{\sigma}$. For any given $0 \le \varepsilon < \frac{1}{4}$, if we take $r_1 = \varepsilon$ and $r_2 = 2\varepsilon$, then a straightforward calculation yields

$$a_1 = -1, \quad a_2 = 4\varepsilon, \quad a_3 = -2\varepsilon^2.$$

This, along with (4.26) and the definition of $\Psi(x)$, entails

$$0 \le \Psi(x) \le 5\varepsilon^2. \tag{4.37}$$

For this $\Psi(x)$, it is easily checked that

$$0 \le M_\Psi(0) = \int_\Omega u(x,0)\Psi(x)dx \le 5\varepsilon^2 \int_\Omega u_0(x)dx. \tag{4.38}$$

From (4.38), (4.36) and the continuity of the function $H(s)$, we infer that if we take $\varepsilon > 0$ sufficiently small, then we have

$$H(M_\Psi(0)) < 0. \tag{4.39}$$

By this, (4.34) and the fact that $H(s)$ is increasing on $[0,\infty)$, we assert that the solution of (4.25) must blow up in a finite time.

In fact, suppose, on the contrary, that the solution (u,z) of (4.25) exists for all $t > 0$. Then, we easily derive from (4.39), (4.34) and the monotonicity of $H(s)$ that

$$H(M_\Psi(t)) < 0 \quad \text{for all } t > 0 \tag{4.40}$$

and thus

$$\frac{d}{dt}M_\Psi(t) < H(M_\Psi(t)) < 0 \quad \text{for all } t > 0.$$

Hence,

$$M_\Psi(t) \le M_\Psi(0) \quad \text{for all } t > 0. \tag{4.41}$$

From this, (4.34) and the monotonicity of $H(s)$ we find that

$$\begin{aligned} M_\Psi(t) &\le M_\Psi(0) + \int_0^t H(M_\Psi(s))ds \\ &\le M_\Psi(0) + \int_0^t H(M_\Psi(0))ds \\ &= M_\Psi(0) + H(M_\Psi(0)) \cdot t \to -\infty \quad \text{as } t \to \infty, \end{aligned}$$

thanks to (4.39), a contradiction to the fact that $M_\Psi(t) = \int_\Omega u(x,t)\Psi(x)dx \ge 0$. □

5. Stationary Solutions and Convergence

In this section we assume $\beta = \delta$ and begin with studying the stationary solutions of system (1.1). For readability, we introduce the following scalings

$$\tilde{v} = \frac{\beta}{\alpha}v, \quad \tilde{w} = \frac{\beta}{\gamma}w, \quad \tilde{t} = \beta t, \quad \tilde{x} = \sqrt{\beta}x, \quad \chi_v = \frac{\chi\alpha}{\beta}, \quad \chi_w = \frac{\xi\gamma}{\beta}. \tag{5.1}$$

Substitute them into system (1.1) and dropping the tildes for convenience, we obtain

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi_v u\nabla v) + \nabla\cdot(\chi_w u\nabla w), & x\in\Omega, \quad t>0,\\ \tau v_t = \Delta v + u - v, & x\in\Omega, \quad t>0,\\ \tau w_t = \Delta w + u - w, & x\in\Omega, \quad t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = \dfrac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega, \quad t>0,\\ u(x,0) = u_0(x), \quad \tau v(x,0) = \tau v_0(x), \quad \tau w(x,0) = \tau w_0(x), & x\in\Omega, \end{cases}$$

whose stationary problem is

$$\begin{cases} 0 = \Delta u - \nabla\cdot(\chi_v u\nabla v) + \nabla\cdot(\chi_w u\nabla w), & x\in\Omega,\\ 0 = \Delta v + u - v, & x\in\Omega,\\ 0 = \Delta w + u - w, & x\in\Omega,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = \dfrac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega. \end{cases} \tag{5.2}$$

We are now in the position to prove Proposition 2.3.

Proof of Proposition 2.3. Solving the first equation of (5.2) subject to the Neumann boundary conditions gives

$$u = \lambda\exp(\chi_v v - \chi_w w), \tag{5.3}$$

where $\lambda > 0$ is a constant of integration satisfying

$$\lambda = \frac{\int_\Omega u dx}{\int_\Omega e^{\chi_v v - \chi_w w}dx}. \tag{5.4}$$

Let $\phi = v - w$. It then follows from (5.2) that

$$\begin{cases} \Delta\phi - \phi = 0, & x\in\Omega,\\ \dfrac{\partial\phi}{\partial\nu} = 0, & x\in\partial\Omega. \end{cases} \tag{5.5}$$

By the maximum principle, one infers that

$$\phi \equiv 0 \quad \text{for all } x\in\Omega,$$

which indicates that

$$v(x) = w(x) \quad \text{for all } x\in\Omega. \tag{5.6}$$

Hence it follows from (5.3) that

$$u = \lambda e^{(\chi_v - \chi_w)w}. \tag{5.7}$$

Substituting (5.7) into the third equation of (5.2) yields

$$\begin{cases} \Delta w - w + \lambda e^{\eta w} = 0, & x \in \Omega, \\ \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{5.8}$$

where

$$\eta := \chi_v - \chi_w = \frac{\chi\alpha - \xi\gamma}{\beta}.$$

We have three cases to proceed as below:

Case 1: $\eta > 0$. For this case, the existence of non-trivial radially symmetric solutions for problem (5.8) in $\Omega \subset \mathbb{R}^n$ has been shown in Ref. 5. The existence of non-trivial non-radial solutions of (5.8) for $\Omega \subset \mathbb{R}^2$ were proved in Refs. 43, 21 and 50 for $\lambda\eta > 4\pi$ and for $\lambda\eta < 4\pi$ in Ref. 21.

Case 2: $\eta = 0$. In this case, it follows from the maximum principle that problem (5.8) has only trivial solution $w = \lambda$. Hence by (5.5) and (5.7), we have $u = v = w = \lambda$. From the cell mass conservation (3.7), we immediately derive that $\lambda = \bar{u}_0$ and thus $u = v = w = \bar{u}_0$. In terms of the original variables in the model (1.1), we have $u = \bar{u}_0, v = \frac{\alpha}{\beta}\bar{u}_0$ and $w = \frac{\gamma}{\beta}\bar{u}_0$ by using (5.1).

Case 3: $\eta < 0$. We claim in this case that problem (5.8) has only one trivial solution. In fact, in this case we infer from (5.2) and $v(x) = w(x)$ that

$$\begin{cases} 0 = \Delta u + \nabla \cdot (\kappa u \nabla v), & x \in \Omega, \\ 0 = \Delta v + u - v, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu}, & x \in \partial\Omega, \end{cases} \tag{5.9}$$

where $\kappa := -\eta > 0$. Therefore, we conclude from Ref. 10 that (5.9) has only one trivial solution. The proof in Ref. 10 was based on a Lyapunov functional along with the Jensen, Sobolev, Poincaré and Hölder inequalities. Here we would like to give an alternative simpler proof. Our proof easily follows from two steps as follows.

Step 1. *Existence.* Since $f(w) := w$ is continuous and increasing in $w \in [0, \infty)$ with $f(0) = 0$ and $\lim_{w\to+\infty} f(w) = +\infty$, and $g(w) := \lambda e^{\eta w}$ is continuous and decreasing in $w \in [0, \infty)$ with $g(0) = \lambda > 0$ and $\lim_{w\to+\infty} g(w) = 0$ due to $\eta < 0$, we infer that problem (5.8) has one trivial positive solution w^* which solves equation

$$w^* = \lambda e^{\eta w^*}. \tag{5.10}$$

Step 2. *Uniqueness.* Suppose that w_1 and w_2 are two solutions of (5.8). By simple calculation and the Lagrange intermediate value theorem: $\lambda e^{\eta w_1} - \lambda e^{\eta w_2} = \lambda\eta e^{\eta\tilde{w}}(w_1 - w_2)$ for some $\tilde{w}$ being between w_1 and w_2, we obtain

$$\begin{cases} \Delta(w_1 - w_2) - h(x)(w_1 - w_2) = 0, & x \in \Omega, \\ \dfrac{\partial(w_1 - w_2)}{\partial\nu} = 0, & x \in \partial\Omega, \end{cases} \tag{5.11}$$

where

$$h(x) := 1 - \lambda\eta e^{\eta\tilde{w}(x)}.$$

Since $\lambda > 0$ and $\eta < 0$, we find that

$$h(x) > 0. \tag{5.12}$$

Hence, we can apply the maximum principle to conclude from (5.11) and (5.12) that $w_1 - w_2 \equiv 0$. Hence (5.8) has only a trivial (i.e. constant) solution w^* satisfying (5.10). Now we explicitly derive this solution. Indeed by (5.5), (5.7) and (5.10), we have $u = v = w = w^*$. Due to the preservation of cell mass, see (3.7), we have $w^* = \bar{u}_0$ and hence $u = v = w = \bar{u}_0$. Returning v and w back to the original ones using (5.1), we complete the proof. □

Throughout the rest of this section we assume that Ω is a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Under the assumption that $\xi\gamma - \chi\alpha > 0$, Theorem 2.1 asserts the existence of global bounded classical solution to (1.1) with $\tau = 0$. In the remainder of this section we study the asymptotic behavior of solutions to (1.1) with $\tau = 0$ under an additional assumption that $\beta = \delta$. To this end, we set

$$s := \xi w - \chi v$$

and obtain from (1.1) with $\tau = 0$ that

$$\begin{cases} u_t = \Delta u + \nabla\cdot(u\nabla s), & x \in \Omega, \quad t > 0, \\ 0 = \Delta s + (\xi\gamma - \chi\alpha)u - \beta s, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial s}{\partial\nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{5.13}$$

We are now in the position to prove Proposition 2.4.

Proof of Proposition 2.4. Noting $\xi\gamma - \chi\alpha > 0$, we find that (5.13) is a parabolic–elliptic repulsive chemotaxis model. Our Theorem 2.1 has asserted that no blow-up can take place. Furthermore, it was proved in Ref. 34 that the global solutions of (5.13) converge to the steady state exponentially. This fact in conjunction with Proposition 2.3 yields that there exist some constants $\mu > 0$ and $c > 0$ such that

$$\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} \le ce^{-\mu t} \quad \text{for all } t > 0. \tag{5.14}$$

Next, from the second equation in (1.1) with $\tau = 0$ we infer that $\psi(x,t) := v(x,t) - \frac{\alpha}{\beta}\overline{u}_0$ satisfies

$$\begin{cases} -\Delta\psi + \beta\psi = \alpha(u - \overline{u}_0), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega, \quad t > 0. \end{cases} \tag{5.15}$$

Upon the application of the elliptic maximum principle[17] and (5.14), we obtain from (5.15) that

$$\begin{aligned} \left\| v(x,t) - \frac{\alpha}{\beta}\overline{u}_0 \right\|_{L^\infty(\Omega)} &= \|\psi\|_{L^\infty(\Omega)} \le \frac{\alpha}{\beta}\|u(\cdot,t) - \overline{u}_0\|_{L^\infty(\Omega)} \\ &\le \frac{\alpha}{\beta} c e^{-\mu t} \quad \text{for all } t > 0. \end{aligned}$$

Similarly, we can prove the convergence of w. □

6. Extension to the Two-Dimensional Full Model

In this section we consider the attraction–repulsion chemotaxis model (1.1) with $\tau = 1$ in two dimensions and aim to prove Proposition 2.6 and Theorem 2.7. For this purpose, we set

$$s := \xi w - \chi v$$

and obtain from (1.1) with $\tau = 1$ and assumption $\beta = \delta$ in (2.5) that

$$\begin{cases} u_t = \Delta u + \nabla \cdot (u \nabla s), & x \in \Omega, \quad t > 0, \\ s_t = \Delta s + (\xi\gamma - \chi\alpha)u - \beta s, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial s}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad s(x,0) = \xi w_0(x) - \chi v_0(x) := s_0(x), & x \in \Omega. \end{cases} \tag{6.1}$$

We are now in the position to prove Proposition 2.6.

Proof of Proposition 2.6. Noting $\xi\gamma - \chi\alpha > 0$, we find that (6.1) is a repulsion chemotaxis model which is closely related to a model studied in Ref. 10. However, we should emphasize that the initial data $s_0(x) := \xi w_0(x) - \chi v_0(x)$ might be *negative* for our present setting. Luckily, the analysis in Ref. 10 strongly depends on a Lyapunov function

$$F(u,s) := \int_\Omega \left(u \ln u + \frac{|\nabla s|^2}{2\beta} \right)$$

and its initial value $F(u_0, s_0)$, and the latter is independent of the sign of $s_0(x)$. Hence, we can refer to known results[10] to conclude that the solutions (u,s) of (6.1) globally exist and they converge to the steady state exponentially. This fact in conjunction with Proposition 2.3 yields that there exists some constants $\mu > 0$ and $c_1 > 0$ such that

$$\|u(\cdot, t) - \overline{u}_0\|_{L^\infty(\Omega)} \le c_1 e^{-\mu t} \quad \text{for all } t > 0. \tag{6.2}$$

Let us recall the main ideas of the proof of convergence in Ref. 10. We divide the proof into four steps.

Step 1. We introduce a Lyapunov functional and derive a functional identity.

Introducing the Lyapunov functional

$$G(u, s) := \int_\Omega u \ln \frac{u}{\bar{u}} + \frac{1}{2\beta} \int_\Omega |\nabla s|^2,$$

where $\bar{u} := \frac{1}{|\Omega|} \int_\Omega u = \frac{1}{|\Omega|} \int_\Omega u_0 = \bar{u}_0$ thanks to (3.7), we easily check that (cf. Ref. 10)

$$\frac{d}{dt} G(u(t), s(t)) = -D(u(t), s(t)), \tag{6.3}$$

where

$$D(u, s) := \int_\Omega \frac{|\nabla u|^2}{u} + \frac{1}{\beta} \int_\Omega |\Delta s|^2 + \frac{\xi\gamma - \chi\alpha}{\beta} \int_\Omega |\nabla s|^2.$$

Step 2. We chain $D(u, s)$ to $G(u, s)$.

Applying the preliminary inequality

$$r \ln r + r - 1 \le (r - 1)^2 \quad \text{for any } r \ge 0,$$

with $r = \frac{u}{\bar{u}}$ and using the Poincaré and Hölder inequalities, we can obtain some $c_0 > 0$ such that (cf. Ref. 10)

$$\int_\Omega u \ln \frac{u}{\bar{u}} \le c_0 \int_\Omega \frac{|\nabla u|^2}{u}$$

and thus

$$G(u, s) \le \frac{1}{2\mu} D(u, s) \quad \text{for some constant } \mu > 0. \tag{6.4}$$

Step 3. We prove the L^1-convergence of u.

Combining (6.3) and (6.4), we get

$$\frac{d}{dt} G(u(t), s(t)) \le -2\mu G(u(t), s(t)) \tag{6.5}$$

and thus

$$G(u(t), s(t)) \le G(u_0, s_0) e^{-2\mu t}.$$

In particular, we have

$$\int_\Omega u \ln \frac{u}{\bar{u}} \leq G(u_0, s_0) e^{-2\mu t}.$$

This, along with the Csiszár–Kullback–Pinsker inequality (see, e.g. Ref. 7)

$$\frac{1}{2\bar{u}} \|u - \bar{u}\|_1^2 \leq \int_\Omega u \ln \frac{u}{\bar{u}},$$

yields

$$\|u - \bar{u}\|_1^2 \leq 2\bar{u}_0 G(u_0, s_0) e^{-2\mu t}. \tag{6.6}$$

Step 4. We prove (6.2).

The L^∞-convergence (6.2) can be proved by (6.6) and the Moser–Alikakos iteration procedure as in the proof of Theorem 2.1.

Next, from the second equation in (1.1) with $\tau = 1$ we infer that $\psi(x,t) := v(x,t) - \frac{\alpha}{\beta}\overline{u}_0$ satisfies

$$\begin{cases} \psi_t - \Delta\psi + \beta\psi = \alpha(u - \overline{u}_0), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial \psi}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ \psi(x, 0) = v_0(x) - \dfrac{\alpha}{\beta}\overline{u}_0 := \psi_0(x), & x \in \Omega. \end{cases} \tag{6.7}$$

Let $\psi^*(t)$ be the solution of the following initial value problem:

$$\begin{cases} \psi_t^* + \beta\psi^* = c_1 e^{-\mu t}, \quad t > 0, \\ \psi^*(0) = \|\psi_0\|_{L^\infty(\Omega)}. \end{cases} \tag{6.8}$$

Upon the application of the comparison principle[15] we find that $\psi^*(t)$ is a supersolution of problem (6.7) and thus

$$\psi(x,t) \leq \psi^*(t) \quad \text{for all } x \in \Omega, \quad t > 0.$$

Similarly, we can prove that $\psi(x,t) \geq -\psi^*(t)$ for all $x \in \Omega, t > 0$. Hence, we have

$$|\psi(x,t)| \leq \psi^*(t) \quad \text{for all } x \in \Omega, \quad t > 0. \tag{6.9}$$

On the other hand, straightforward computation shows that there exist some $c_2 > 0$ and $c_3 > 0$ such that

$$0 \leq \psi^*(t) \leq c_2(1+t) e^{-\min\{\beta,\mu\} t} \leq c_3 e^{-\frac{\min\{\beta,\mu\}}{2} t} \quad \text{for all } t > 0. \tag{6.10}$$

Using this and (6.9) and recalling the definition of ψ, we prove the desired exponential convergence of v. Finally, the convergence of w can be similarly proven. □

We note that the proof of the global existence and boundedness in Proposition 2.6 can also be proved by an approach developed in the proof of Theorem 2.7 as shown below and that the convergence in Proposition 2.6 can also be proved by a new method recently developed in Ref. 49.

We next turn to consider the case $\beta \neq \delta$ and prove Theorem 2.7. To this end, we set $s := \xi w - \chi v$ as before and obtain from (1.1) with $\tau = 1$ that

$$\begin{cases} u_t = \Delta u + \nabla \cdot (u \nabla s), & x \in \Omega, \quad t > 0, \\ s_t = \Delta s - \delta s + (\xi\gamma - \chi\alpha)u + \chi(\beta - \delta)v, & x \in \Omega, \quad t > 0, \\ v_t = \Delta v + \alpha u - \beta v, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial s}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad s(x,0) = \xi w_0(x) - \chi v_0(x) := s_0(x), & \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{6.11}$$

The proof of Proposition 2.6 is based on a Lyapunov functional approach developed in Ref. 10. Unfortunately, this method cannot be applied to (1.1) with $\tau = 1$ and $\beta \neq \delta$ since (6.11) does *not* possess a Lyapunov functional for the case $\beta \neq \delta$. Therefore, we are motivated to turn to the following entropy-type inequality which is the cornerstone of our mathematical analysis of (6.11). This type of entropy inequality also plays an important role in a recent study of a chemotaxis system with consumption of chemoattractant.[49]

Lemma 6.1. *Let* (2.1), (2.4) *and* (2.6) *hold. Then the solution of* (6.11) *satisfies the entropy inequality*

$$\frac{d}{dt}\left\{\int_\Omega u \ln u + \frac{1}{2(\xi\gamma - \chi\alpha)}\int_\Omega |\nabla s|^2 + \frac{\chi^2(\beta-\delta)^2}{2\beta(\xi\gamma-\chi\alpha)}\int_\Omega v^2\right\}$$
$$+ 4\int_\Omega |\nabla u^{\frac{1}{2}}|^2 + \frac{1}{2(\xi\gamma-\chi\alpha)}\int_\Omega |\Delta s|^2 \leq \frac{\chi^2\alpha^2(\beta-\delta)^2}{2\beta^2(\xi\gamma-\chi\alpha)}\int_\Omega u^2 \tag{6.12}$$

for all $t \in (0, T_{\max})$.

Proof. First, testing the first equation in (6.11) by $\ln u$ and integrating over Ω yield

$$\frac{d}{dt}\int_\Omega u \ln u + 4\int_\Omega |\nabla u^{\frac{1}{2}}|^2 = -\int_\Omega \nabla u \cdot \nabla s \quad \text{for all } t \in (0, T_{\max}). \tag{6.13}$$

Then, testing the second equation in (6.11) by $-\frac{1}{\xi\gamma-\chi\alpha}\Delta s$, integrating over Ω and using the Cauchy's inequality entail that

$$\frac{1}{2(\xi\gamma-\chi\alpha)}\frac{d}{dt}\int_\Omega |\nabla s|^2 + \frac{1}{\xi\gamma-\chi\alpha}\int_\Omega |\Delta s|^2 + \frac{\delta}{\xi\gamma-\chi\alpha}\int_\Omega |\nabla s|^2$$
$$= \int_\Omega \nabla u \cdot \nabla s - \frac{\chi(\beta-\delta)}{\xi\gamma-\chi\alpha}\int_\Omega v\Delta s$$
$$\leq \int_\Omega \nabla u \cdot \nabla s + \frac{1}{2(\xi\gamma-\chi\alpha)}\int_\Omega |\Delta s|^2 + \frac{\chi^2(\beta-\delta)^2}{2(\xi\gamma-\chi\alpha)}\int_\Omega v^2$$

for all $t \in (0, T_{\max})$. After rearrangement, we obtain

$$\frac{1}{2(\xi\gamma - \chi\alpha)} \frac{d}{dt} \int_\Omega |\nabla s|^2 + \frac{1}{2(\xi\gamma - \chi\alpha)} \int_\Omega |\Delta s|^2 \leq \int_\Omega \nabla u \cdot \nabla s + \frac{\chi^2(\beta - \delta)^2}{2(\xi\gamma - \chi\alpha)} \int_\Omega v^2 \tag{6.14}$$

for all $t \in (0, T_{\max})$. Finally, testing the third equation in (6.11) by v and using the Cauchy's inequality yield

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_\Omega v^2 + \beta \int_\Omega v^2 + \int_\Omega |\nabla v|^2 \\ &\quad = \alpha \int_\Omega uv \leq \frac{\beta}{2} \int_\Omega v^2 + \frac{\alpha^2}{2\beta} \int_\Omega u^2 \quad \text{for all } t \in (0, T_{\max}), \end{aligned}$$

which leads to

$$\frac{\chi^2(\beta - \delta)^2}{2\beta(\xi\gamma - \chi\alpha)} \frac{d}{dt} \int_\Omega v^2 + \frac{\chi^2(\beta - \delta)^2}{2(\xi\gamma - \chi\alpha)} \int_\Omega v^2 \leq \frac{\chi^2 \alpha^2 (\beta - \delta)^2}{2\beta^2(\xi\gamma - \chi\alpha)} \int_\Omega u^2 \tag{6.15}$$

for all $t \in (0, T_{\max})$. Now, adding (6.13)–(6.15) proves (6.12). □

Lemma 6.2. *Let* (2.1), (2.4), (2.6) *and* (2.7) *hold. Then for each* $T_0 > 0$ *there exists* $C(T) > 0$, *which may depend on* $T := \min\{T_0, T_{\max}\}$, *such that the solution of* (6.11) *satisfies*

$$\int_\Omega |\nabla s(\cdot, t)|^2 \leq C(T) \quad \textit{for all } t \in (0, T) \tag{6.16}$$

and

$$\int_0^t \int_\Omega |\Delta s|^2 \leq C(T) \quad \textit{for all } t \in (0, T). \tag{6.17}$$

Proof. The proof is based on the entropy inequality (6.12). When $n = 2$, the Gagliardo–Nirenberg inequality and (3.7) assert that there is $c_1(\Omega) > 0$ such that

$$\begin{aligned} \int_\Omega u^2 &= \|u^{\frac{1}{2}}\|^4_{L^4(\Omega)} \leq c_1(\Omega) \|\nabla u^{\frac{1}{2}}\|^2_{L^2(\Omega)} \|u^{\frac{1}{2}}\|^2_{L^2(\Omega)} + c_1(\Omega) \|u^{\frac{1}{2}}\|^4_{L^2(\Omega)} \\ &= c_1(\Omega) \|u_0\|_{L^1(\Omega)} \|\nabla u^{\frac{1}{2}}\|^2_{L^2(\Omega)} + c_1(\Omega) \|u_0\|^2_{L^1(\Omega)} \end{aligned} \tag{6.18}$$

for all $t \in (0, T)$. Thus, if $\|u_0\|_{L^1(\Omega)}$ is sufficiently small such that

$$\frac{\chi^2 \alpha^2 (\beta - \delta)^2}{2\beta^2(\xi\gamma - \chi\alpha)} \cdot c_1(\Omega) \|u_0\|_{L^1(\Omega)} \leq 4, \tag{6.19}$$

then integrating both sides of (6.12) over $(0,t)$ and using (6.18)–(6.19) we obtain some $c_2 > 0$ such that

$$\begin{aligned}
&\frac{1}{2(\xi\gamma-\chi\alpha)}\int_\Omega |\nabla s(\cdot,t)|^2 + \frac{\chi^2(\beta-\delta)^2}{2\beta(\xi\gamma-\chi\alpha)}\int_\Omega v^2(\cdot,t) + \frac{1}{2(\xi\gamma-\chi\alpha)}\int_0^t\int_\Omega |\Delta s|^2 \\
&\quad \le \frac{1}{2(\xi\gamma-\chi\alpha)}\int_\Omega |\nabla s_0|^2 + \frac{\chi^2(\beta-\delta)^2}{2\beta(\xi\gamma-\chi\alpha)}\int_\Omega v_0^2 \\
&\qquad + \int_\Omega u_0\ln u_0 - \int_\Omega u\ln u + c_2\cdot T \quad \text{for all } t\in(0,T).
\end{aligned}$$

Since $\int_\Omega v^2(\cdot,t) \ge 0$ for all $t \in (0,T)$ and $-u\ln u \le \frac{1}{e}$ for all $u > 0$, this shows

$$\begin{aligned}
&\frac{1}{2(\xi\gamma-\chi\alpha)}\int_\Omega |\nabla s(\cdot,t)|^2 + \frac{1}{2(\xi\gamma-\chi\alpha)}\int_0^t\int_\Omega |\Delta s|^2 \\
&\quad \le \frac{1}{2(\xi\gamma-\chi\alpha)}\int_\Omega |\nabla s_0|^2 + \frac{\chi^2(\beta-\delta)^2}{2\beta(\xi\gamma-\chi\alpha)}\int_\Omega v_0^2 \\
&\qquad + \int_\Omega u_0\ln u_0 + \frac{|\Omega|}{e} + c_2\cdot T \quad \text{for all } t\in(0,T).
\end{aligned} \tag{6.20}$$

Therefore, (6.16)–(6.17) result from (6.20) and $\xi\gamma - \chi\alpha > 0$. □

Lemma 6.3. *Let* (2.1), (2.4), (2.6) *and* (2.7) *hold. Then for each* $T_0 > 0$ *there exists* $C(T) > 0$, *which may depend on* $T := \min\{T_0, T_{\max}\}$, *such that the solution of* (6.11) *satisfies the inequality*

$$\int_0^t\int_\Omega |\nabla s|^4 \le C(T) \quad \textit{for all } t\in(0,T). \tag{6.21}$$

Proof. When $n = 2$, the Gagliardo–Nirenberg inequality[15] leads to

$$\begin{aligned}
\int_\Omega |\nabla s|^4 = \|\nabla s\|^4_{L^4(\Omega)} &\le c_1\|\Delta s\|^2_{L^2(\Omega)}\cdot\|\nabla s\|^2_{L^2(\Omega)} + c_1\|\nabla s\|^4_{L^2(\Omega)} \\
&\le c_2(T)(\|\Delta s\|^2_{L^2(\Omega)} + 1)
\end{aligned} \tag{6.22}$$

for some $c_1 > 0, c_2(T) > 0$ and all $t \in (0,T)$, where (6.16) has been used. This yields (6.21) upon the integration over time interval $(0,t]$ in both sides of (6.22) and the application of (6.17). □

Lemma 6.4. *Let* (2.1), (2.4), (2.6) *and* (2.7) *hold. Then for any* $p > 1$ *and for each* $T_0 > 0$ *there exists* $C(T) > 0$, *which may depend on* $T := \min\{T_0, T_{\max}\}$, *such that the solution of* (6.11) *fulfills the inequality*

$$\int_\Omega u^p(x,t)dx \le C(T) \quad \textit{for all } t\in(0,T). \tag{6.23}$$

Proof. With the estimate (6.21) at hand, (6.23) can be proved with the idea of Ref. 54. We multiply the first equation in (6.11) by u^{p-1} and apply Young's inequality to find $c_1 > 0$ such that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2 \le c_1 \int_\Omega u^p|\nabla s|^2 \quad \text{for all } t \in (0,T). \tag{6.24}$$

By the Hölder inequality, it follows that

$$\int_\Omega u^p|\nabla s|^2 \le \left(\int_\Omega u^{2p}\right)^{\frac{1}{2}} \cdot \left(\int_\Omega |\nabla s|^4\right)^{\frac{1}{2}}.$$

The Gagliardo–Nirenberg inequality provides $c_2 > 0$ such that

$$\left(\int_\Omega u^{2p}\right)^{\frac{1}{2}} = \|u^{\frac{p}{2}}\|^2_{L^4(\Omega)} \le c_2(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)} \cdot \|u^{\frac{p}{2}}\|_{L^2(\Omega)} + \|u^{\frac{p}{2}}\|^2_{L^2(\Omega)}),$$

where we have used the fact that the spatial dimension $n = 2$. Then employing the idea of Ref. 52, we obtain by the Hölder inequality

$$\|f\|^2_{L^2} \le \|f\|^{4b}_{L^4} \cdot \|f\|^{\frac{2(1-b)}{p}}_{L^{\frac{2}{p}}}, \tag{6.25}$$

where

$$b := \frac{p-1}{2p-1}.$$

Then using (6.25) and the Young inequality, we further find that

$$\|f\|^2_{L^2} \le \varepsilon\|f\|^2_{L^4} + c(\varepsilon)\|f\|^2_{L^{\frac{2}{p}}}.$$

Setting $f := u^{\frac{p}{2}}$ and let ε small such that $c_2\varepsilon < 1$, we can find a constant c_3 such that

$$\|u^{\frac{p}{2}}\|^2_{L^4(\Omega)} \le c_3\left(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)} \cdot \|u^{\frac{p}{2}}\|_{L^2(\Omega)} + \|u^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)}\right).$$

Since $\|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)} = (\int_\Omega u)^{\frac{p}{2}} \equiv (\int_\Omega u_0)^{\frac{p}{2}}$, we can pick $c_4 > 0$ such that

$$c_1\int_\Omega u^p|\nabla s|^2 \le \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2 + c_4\left(\int_\Omega |\nabla s|^4\right)\cdot\left(\int_\Omega u^p + 1\right).$$

Let $y(t) := \int_\Omega u^p(x,t)dx,\ t \in [0,T)$. Then from (6.24), we see that $y(t)$ satisfies the differential inequality

$$y'(t) \le c_5\left(\int_\Omega |\nabla s|^4\right)\cdot(y(t)+1) \quad \text{for all } t \in (0,T),$$

with some $c_5 > 0$. Upon integration we infer that

$$y(t) + 1 \le (y(0)+1)\cdot e^{c_5\int_0^t\int_\Omega |\nabla s|^4} \quad \text{for all } t \in (0,T),$$

whereupon an application of (6.21) completes the proof. □

We are now in the position to prove Theorem 2.7.

Proof of Theorem 2.7. Noting that w solves

$$\begin{cases} w_t - \Delta w + \delta w = \gamma u, & x \in \Omega, \quad t \in (0,T), \\ \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0,T), \\ w(x,0) = w_0(x), & x \in \Omega, \end{cases}$$

where $\delta > 0$, using the assumption that $w_0(x) \in W^{1,\infty}(\Omega)$ and the estimate (6.23) with $p > n$ and applying the solution estimates for the heat equation with zero Neumann boundary condition (cf. Refs. 28 and 25), we obtain some $c_1(T) > 0$ such that

$$\|\nabla w\|_{L^\infty(\Omega)} \le c_1(T) \quad \text{for all } t \in (0,T). \tag{6.26}$$

Similarly, there exists some $c_2(T) > 0$ such that

$$\|\nabla v\|_{L^\infty(\Omega)} \le c_2(T) \quad \text{for all } t \in (0,T). \tag{6.27}$$

With the estimates (6.26) and (6.27) at hand, we then can carry out the Moser–Alikakos iteration procedure exactly as in the proof of Theorem 2.1 to obtain some $c_3(T) > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le c_3(T) \quad \text{for all } t \in (0,T). \tag{6.28}$$

Finally, the assertion of Theorem 2.7 is an immediate consequence of (6.28) and the extensibility criterion provided by Lemma 3.1. □

Before concluding this section we remark that in the proofs of Theorems 2.1 and 2.7, there are two crucial steps. One step is to derive an L^p estimate on u with $p > n$. In the derivation of this estimate, we essentially use the standard Gagliardo–Nirenberg inequalities (3.13) and (3.15) and a variation of (3.13) involving L^r space with $r < 1$. Therefore, for readers' convenience, we recall these inequalities in Sec. 3. The other step is to establish an L^∞ estimate on u by the Moser–Alikakos iteration procedure. Since we use three times this procedure, we present the details of it in the proof of Theorem 2.1. Finally, we mention that a generalized Moser–Alikakos iteration procedure for quasilinear non-uniformly parabolic equations was also given in Ref. 48.

7. Numerical Illustrations

In this section, we will numerically illustrate the competing effects of chemotactic attraction and repulsion for the cell movement to explain our analytical results. Without loss of generality, we consider the model (1.1) with $\tau = 1$ and perform the numerical simulations in one dimension $\Omega = (0, 20)$ with Neumann boundary conditions. The main purposes of this section include: (i) to show the solution behaviors and pattern formations of the model (1.1) with and without repulsion; (ii) to show the difference between strong and weak repulsion. The finite-difference-based Matlab Pde solvers are employed to perform the numerical computations.

We first explore the numerical solutions of model (1.1) without repulsion (i.e. $\xi = 0$) which is equivalent to the classical model (1.2). Figure 1 shows the numerical solution profiles u and v (left panel) and pattern formation of cell density u (right panel), where the initial data are chosen to be a small perturbation of the homogeneous steady state. The simulations demonstrate the chemotactic aggregation (i.e. peak solutions) and merging process of pattern formation, which are characteristic features of chemotaxis models.[19]

Then we include the repulsion into the model and consider the full model (1.1) with $\xi > 0$. The numerical results are shown in Fig. 2, where parameters fulfill condition (2.2) which asserts that the attraction prevails over repulsion. In this case

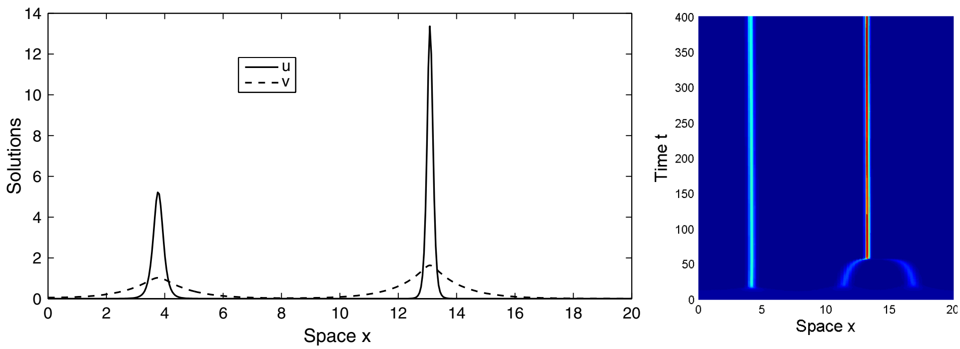

Fig. 1. Numerical simulations of model (1.1) without chemorepellent (i.e. $\xi = 0$). The left panel is a numerical plot of solution profile u and v at final time step $t = 400$ and right panel illustrates the pattern formation of cell density u. The parameter values are $\alpha = \beta = \gamma = \delta = 1, \chi = 10, \xi = 0, u_0 = 0.3, v_0 = 0.3 + r(x)$, where $r(x)$ is a 1% random spatial perturbation of the homogeneous steady state.

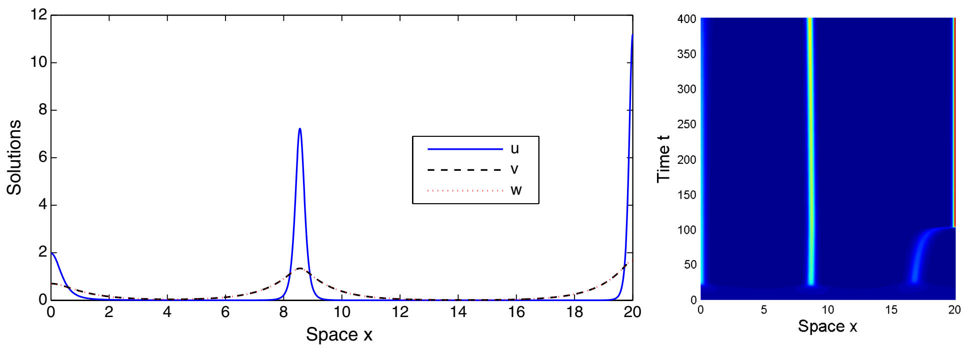

Fig. 2. Numerical simulations of model (1.1) with both chemoattractant and chemorepellent, where the attraction is stronger than the repulsion. The left panel is a numerical plot of solution profiles at time step $t = 400$ and right panel shows the pattern formation of cell density u. The parameter values are chosen as $\alpha = \beta = \gamma = \delta = 1, \chi = 10, \xi = 2, u_0 = 0.3, v_0 = w_0 = 0.3 + r(x)$, where $r(x)$ is a 1% random spatial perturbation of the homogeneous steady state. Here the stationary profile of v (dashed line) and w (dotted line) coincide due to $\beta = \delta$.

the solution globally exists in one dimension and blows up in finite time if cell mass is larger than some threshold, see Proposition 2.2. For the sake of comparison, we choose parameter values same as those in Fig. 1 except that a repulsive chemotactic coefficient ξ is incorporated. We find the qualitatively analogous aggregations and pattern formation as in Fig. 1. However a quantitative variation can also be explicitly observed, where the maximum of cell density u in Fig. 2 is smaller than that in Fig. 1. This manifests that the repulsion is a dispersal effect in chemotactic movement in contrast to the aggregation effect of attraction. In conclusion, if the attraction prevails over repulsion, the model still retains the qualitative characteristics of the classical attractive Keller–Segel model. It should be noted that in Fig. 2 solutions are stable, and hence concentration $v = w$ due to $\beta = \delta$ which is consistent with Proposition 2.3. If we continuously increase the value of ξ such that inequality in (2.2) is reversed (i.e. condition (2.5) is satisfied), namely the repulsion prevails, numerical simulations in Fig. 4 illustrate that no pattern formation develops and all solutions u, v and w stabilize to constant steady states, which is in accordance with Proposition 2.4.

In the present paper, many results, such as Propositions 2.2, 2.3(2), 2.4 and 2.6, have to assume $\beta = \delta$. The results for $\beta \neq \delta$ largely remain open and demand to be explored in the future. Figure 3 shows one example of the numerical solutions for the case $\beta \neq \delta$ where the attraction prevails. Compared with Fig. 2, similar qualitative solution profiles and pattern formations in Fig. 3 are observed except a quantitative difference where concentration of chemorepellent w is smaller than that of chemoattractant v due to more consumption of w than v (i.e. $\delta > \beta$). Figure 4 shows another example for the case $\beta \neq \delta$ where the repulsion prevails. It illustrates the same qualitative behavior as for the case $\beta = \delta$ when repulsion prevails, see Proposition 2.4, where the solutions converge to constant steady states.

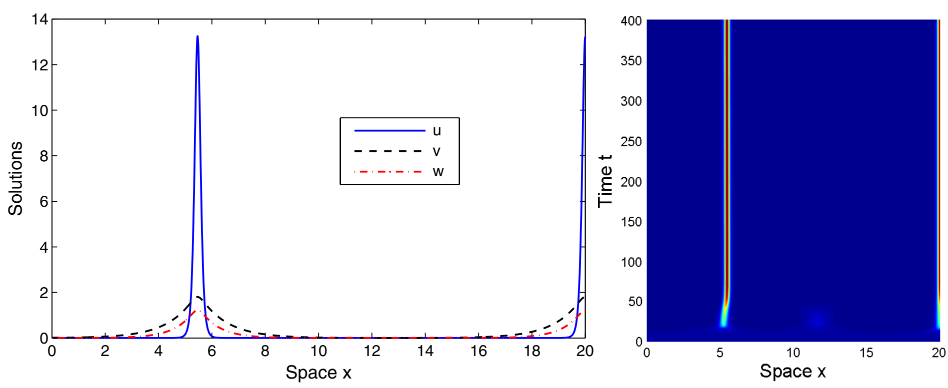

Fig. 3. Numerical simulations of model (1.1) with both chemoattractant and chemorepellent, where the attraction is stronger than the repulsion. The left panel is a numerical plot of solution profiles at time step $t = 400$ and right panel is the pattern formation of cell density u. The parameter values are $\alpha = \beta = \gamma = 1, \delta = 2, \chi = 10, \xi = 2, u_0 = 0.3, v_0 = 0.3 + r(x), w_0 = 0.15 + r(x)$, where $r(x)$ is a 1% random spatial perturbation of the homogeneous steady state.

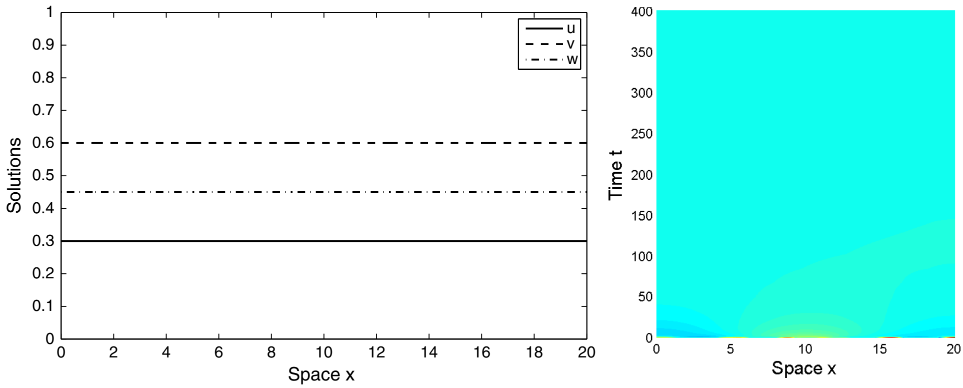

Fig. 4. Numerical simulations of model (1.1) with both chemoattractant and chemorepellent, where the attraction is weaker than the repulsion. The left panel is a plot of solution profiles at time step $t = 400$ and right panel shows no pattern formation for cell density u. The parameter values are $\alpha = 2, \beta = 1, \gamma = 3, \delta = 2, \chi = 10, \xi = 12, u_0 = 0.3, v_0 = 0.6 + r(x), w_0 = 0.45 + r(x)$, where $r(x)$ is a 1% random spatial perturbation of the homogeneous steady state.

8. Conclusions and Suggestions

The attraction–repulsion chemotaxis system includes not only a chemoattractant but also a second chemical as a chemorepellent. Hence, this system is actually a generalization of the well-known Keller–Segel model that includes only a chemoattractant, which has already been studied in many works. A striking feature of the classical chemotaxis system is the finite-time blow-up of solutions (cf. Ref. 18 for the case $n = 2$ and Ref. 55 for the case $n \geq 3$, for instance). However, our results (Theorem 2.1 and Proposition 2.4) confirm that the attraction–repulsion chemotaxis model can prevent blow-up if the repulsion is strong enough. Another striking feature of the classical chemotaxis model is the pattern formation (cf. Ref. 19). However, our numerical simulation (Fig. 4) shows that no pattern can be developed when the repulsion strongly prevails over attraction. Therefore, the attraction–repulsion mechanism may regularize the classical Keller–Segel model.

However, from the viewpoint of mathematical analysis, this work is only an early start to the study of the attraction–repulsion chemotaxis system, because there remain a few important problems unexplored. Among these open problems, in authors' opinion, the following four problems are most challenging and interesting.

Problem 1. To study the finite-time blow-up of solutions to (1.1) assuming $\xi\gamma - \chi\alpha < 0,\ \beta \neq \delta,\ \tau = 1$ and $n \geq 3$.

Problem 2. To study the global existence of solutions to (1.1) assuming $\xi\gamma - \chi\alpha > 0,\ \beta \neq \delta,\ \tau = 1$ and $n \geq 3$.

Problem 3. Can one remove the smallness assumption on the initial data $u_0(x)$ for the global existence of solutions to (1.1) assuming $\xi\gamma - \chi\alpha > 0,\ \beta \neq \delta,\ \tau = 1$ and $n = 2$?

Problem 4. To study the existence or non-existence of non-trivial stationary solutions to (1.1) assuming $\beta \neq \delta$.

As aforementioned, the finite-time blow-up is a very interesting problem for the classical chemotaxis model. However, most of existing results are only for a parabolic–elliptic simplification of the fully parabolic Keller–Segel model (cf. Refs. 36 and 35, for instance). So far the only existing result on the finite-time blow-up for the fully parabolic Keller–Segel model is that in Ref. 18, where an example of finite-time blow-up in two dimensions is shown. Here we should also mention the result in Ref. 9, where the finite-time blow-up of solutions to a one-dimensional quasilinear parabolic–parabolic Keller–Segel system with appropriately nonlinear diffusion of cells is proved. The breakthrough of the proof of the finite-time blow-up for the fully parabolic Keller–Segel model in space dimensions $n \geq 3$ has been made recently in Ref. 55 by inventing a new method, which strongly depends on the existence of a Lyapunov functional. However, there does *not* exist a Lyapunov functional for (1.1) with $\xi\gamma - \chi\alpha < 0$ and $\beta \neq \delta$. Therefore, Problem 1 becomes very challenging.

In Ref. 10 the authors proved the global existence of weak solutions to a model of chemorepulsion in space dimension $n = 3, 4$. Again, their proof strongly depends on the existence of a Lyapunov functional. The assumption that $\xi\gamma - \chi\alpha > 0$ implies that the repulsion prevails over attraction to some extent, so we might expect the global existence of solutions. However, there does *not* exist a Lyapunov functional for (1.1) with $\xi\gamma - \chi\alpha > 0$ and $\beta \neq \delta$. So, Problem 2 is interesting.

Our proof of Theorem 2.7 needs a smallness assumption (2.7) on the initial data u_0 for the global solvability of (1.1) with $\xi\gamma - \chi\alpha > 0,\ \beta \neq \delta,\ \tau = 1$ and $n = 2$. However, we do not know whether this assumption is indispensable for the global existence. Therefore, Problem 3 deserves further studying.

Proposition 2.3 addresses the existence or non-existence of nontrivial stationary solutions to (1.1) with $\beta = \delta$, and the corresponding steady-state problem is equivalent to a single semi-linear elliptic problem (5.8). However, Problem 4 will be equivalent to the study of an elliptic system which consists of two coupled semi-linear elliptic equations. To our knowledge, this problem remains open and therefore deserves exploring.

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