SIAM J. APPL. MATH.
Vol. 70, No. 5, pp. 1522–1541
© 2009 Society for Industrial and Applied Mathematics

NONLINEAR STABILITY OF TRAVELING WAVES TO A HYPERBOLIC-PARABOLIC SYSTEM MODELING CHEMOTAXIS*

TONG LI† AND ZHI-AN WANG‡

Abstract. We prove nonlinear stability of traveling waves of arbitrary amplitudes to a hyperbolic-parabolic system modeling repulsive chemotaxis. In contrast to the previous related results, where various smallness conditions on wave strengths were imposed, we are able to prove the nonlinear stability of the traveling waves with arbitrary amplitudes under small perturbations in spite of partial diffusion in the model. Moreover, we perform numerical experiments to verify our theoretical results. Finally, the biological implications are discussed. Our results indicate that when the dissipative effect is not negligible, the cell density distribution approaches a smooth viscous shock profile asymptotically if the chemotaxis is repulsive.

Key words. nonlinear stability, hyperbolic-parabolic system, chemotaxis, viscous shock waves, large amplitudes, energy estimates

AMS subject classifications. 35G25, 35M10, 35L65, 92C17

DOI. 10.1137/09075161X

1. Introduction.

In this paper, we study the nonlinear stability of traveling wave solutions to the following hyperbolic-parabolic system describing the repulsive chemotaxis:

$$\begin{cases} p_t - (pq)_x = Dp_{xx}, \\ q_t - p_x = 0 \end{cases} \tag{1.1}$$

for $x \in R$ and $t > 0$, subject to the initial data

$$(p, q)(x, 0) = (p_0, q_0)(x) \to (p^\pm, q^\pm) \quad \text{as} \quad x \to \pm\infty, \tag{1.2}$$

where $p_0 > 0$ and $p_\pm > 0$.

The model (1.1) is derived from the following chemotaxis model:

$$\begin{cases} p_t = D\dfrac{\partial}{\partial x}\left(p\dfrac{\partial}{\partial x}\left(\ln\left(\dfrac{p}{\Phi(w)}\right)\right)\right), \quad x \in \Omega,\ t > 0, \\ w_t = \varphi(p, w), \end{cases} \tag{1.3}$$

where $p(x, t)$ denotes the particle density and $w(x, t)$ is the concentration of chemicals. $D > 0$ is the diffusion rate of particles. The function Φ is commonly referred to as the chemotactic potential and φ denotes the chemical kinetics. They are given by

$$\Phi(w) = w^{-\alpha}, \quad \varphi(p, w) = \lambda pw - \mu w, \tag{1.4}$$

*Received by the editors March 4, 2009; accepted for publication (in revised form) October 5, 2009; published electronically December 11, 2009.
http://www.siam.org/journals/siap/70-5/75161.html

†Department of Mathematics, University of Iowa, Iowa City, IA 52242 (tli@math.uiowa.edu). The work of this author was partially supported by a Career Development Award at the University of Iowa.

‡Corresponding author. Institute for Mathematics and Its Applications, University of Minnesota, Minneapolis, MN 55455 (zhiwang@ima.umn.edu). Current address: Department of Mathematics, Vanderbilt University, Nashville, TN 37240 (zhian.wang@vanderbilt.edu). The work of this author was supported by the IMA postdoctoral fellowship.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

where $\alpha \in R$, $\lambda > 0$, and $\mu \geq 0$.

The model (1.3) was proposed by Othmer and Stevens [11] to describe the chemotactic movement of particles, where the chemicals are nondiffusible and can modify the local environment for succeeding passages. For example, myxobacteria produce slime over which their cohorts can move more readily, and ants can follow trails left by predecessors [11]. Another direct application of (1.3) is to model haptotaxis, where cells move towards an increasing concentration of immobilized signals such as surface or matrix-bound adhesive molecules. The model (1.3), with (1.4), was given in [6] and a comprehensive qualitative and numerical analysis was provided there. Next we briefly present the derivation of (1.1) from (1.3), (1.4).

In fact, the substitution of (1.4) into (1.3) yields the following system:

$$\begin{cases} p_t = D\left(p_{xx} + \alpha\left(p\dfrac{w_x}{w}\right)_x\right), \\ w_t = \lambda p w - \mu w, \end{cases} \tag{1.5}$$

which is a special case of the Keller–Segel model (see Horstmann [3]) that describes the chemotactic motion in the macroscopic level. We say that the chemotaxis is attractive if $\alpha < 0$ and repulsive for $\alpha > 0$.

Furthermore, making the transformation

$$\bar{p} = \lambda p, \quad q = \frac{w_x}{w} = \frac{\partial}{\partial x} \ln w, \tag{1.6}$$

substituting it into (1.5), and omitting the bar over p for convenience, we obtain

$$\begin{cases} p_t = Dp_{xx} + D\alpha(pq)_x, \\ q_t = p_x. \end{cases} \tag{1.7}$$

Let us consider the repulsive case $\alpha > 0$. Nondimensionalizing system (1.7) by setting

$$\xi = D\alpha,\ \tilde{t} = \xi t,\ \tilde{x} = \xi x,\ \tilde{p} = \xi p,\ \tilde{q} = \xi q,\ \tilde{D} = D^2\alpha \tag{1.8}$$

and dropping the tilde for convenience, we see that model (1.7) returns to model (1.1).

In the case of attractive chemotaxis $\alpha < 0$, the global existence and blowup of solutions for (1.3), (1.4) subject to zero flux boundary condition were investigated in [14, 15]. The existence and stability of spike solutions of (1.3), (1.4) were established for bounded domain in [12]. For the case of repulsive chemotaxis $\alpha > 0$, the global existence of solutions for (1.5) was given in [16] for bounded domain with zero flux boundary condition.

Motivated by the previous qualitative and numerical results [6], Wang and Hillen [13] established the existence of the viscous traveling waves of (1.3), (1.4) and their convergence to the shock waves of (1.3), (1.4) when the viscosity vanishes ($D \to 0$) for both attractive and repulsive cases. The aim of this paper is to investigate the asymptotic behavior of solutions of the Cauchy problem to the repulsive case of chemotaxis model (1.3), (1.4), which is equivalent to (1.1). Specifically, we show that the solutions of (1.1), (1.2) converge to the translated traveling waves as time tends to infinity provided that the initial data are small perturbations of the traveling waves. With the partial diffusion in the hyperbolic-parabolic system (1.1), we are able to prove the nonlinear stability of the traveling wave solutions of arbitrary amplitudes in contrast to the previous related results, e.g., [7], [8], and [16], where various smallness

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

conditions on wave strengths were imposed. As far as we know, nonlinear stability of traveling waves to chemotaxis models has not been studied before.

Notation. In what follows, C denotes a generic constant that can change from one line to another. An integral lacking limits of integration means an integral over the whole real line. For a weight function $w \geq 0$, L^2_w denotes the space of measurable functions f so that $\sqrt{w}f \in L^2$ with norm

$$\|f\|_{L^2_w} = \left(\int w(x)|f(x)|^2 dx \right)^{1/2}.$$

The rest of the paper is organized as follows. In section 2, we state our main results. In section 3, we reformulate our problem in terms of perturbation quantities. Then we prove our main results in section 4. In section 5, we perform numerical simulations to verify our theoretical results. Finally, a brief summary and open questions are presented in section 6.

2. Main results. System (1.1) is the so-called hyperbolic-parabolic system in which the viscosity matrix is non–negative-definite. Other hyperbolic-parabolic systems include the isentropic compressible Navier–Stokes equations in one spatial dimension [8]. System (1.1) is hyperbolic when $D = 0$. Indeed, when $D = 0$ the eigenvalues of the Jacobian of (1.1) are given by

$$\lambda_1(p,q) = -\frac{q}{2} - \frac{\sqrt{q^2+4p}}{2} \leq -\frac{q}{2} + \frac{\sqrt{q^2+4p}}{2} = \lambda_2(p,q). \tag{2.1}$$

It is strictly hyperbolic if $p > 0$. Assuming $p_0 > 0$, the positivity of p will be established later. Further calculation shows that both characteristic families are genuinely nonlinear,

$$\nabla\lambda_1(p,q) \cdot \vec{r_1}(p,q) = -1 - \frac{q}{\sqrt{q^2+4p}} < 0, \quad \nabla\lambda_2(p,q) \cdot \vec{r_2}(p,q) = -1 + \frac{q}{\sqrt{q^2+4p}} < 0,$$

provided that $p > 0$, where $\vec{r_1}(p,q)$ and $\vec{r_2}(p,q)$ are the right eigenvectors corresponding to eigenvalues $\lambda_1(p,q)$ and $\lambda_2(p,q)$, respectively.

Now we look for the traveling wave solutions to system (1.1). Substituting the traveling wave ansatz

$$(p,q)(x,t) = (P,Q)(z), \qquad z = x - st,$$

into system (1.1), where s denotes the traveling wave speed and z is the traveling wave variable, one has the following system of differential equations:

$$\begin{cases} -sP_z - (PQ)_z = DP_{zz}, \\ -sQ_z - P_z = 0. \end{cases} \tag{2.2}$$

The boundary conditions of (2.2) are imposed as

$$(P,Q)(z) \to (p^\pm, q^\pm) \text{ as } z \to \pm\infty, \tag{2.3}$$

where $p_\pm > 0$.

Integrating system (2.2) with respect to z over $(\pm\infty, z)$ and using the fact that $P_z \to 0$ as $z \to \pm\infty$, we have

$$\begin{cases} s(p^+ - p^-) = -p^+q^+ + p^-q^-, \\ s(q^+ - q^-) = -p^+ + p^-, \end{cases} \tag{2.4}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

which corresponds to the Rankine–Hugoniot jump conditions of shock waves for hyperbolic system (1.1) with $D = 0$. In this paper, we restrict our attention to the case of $s > 0$, and the analysis for $s < 0$ is similar. It should be noted that in our study, $s > 0$ corresponds to the second characteristic field of shocks, whereas $s < 0$ corresponds to the first characteristic field.

The existence of the traveling waves of (1.1) when $D > 0$, and the convergence of the traveling waves to shock waves of (1.1) as $D \to 0$, have been established in paper [13] under the Lax entropy conditions [5]

$$\lambda_2(p^+, q^+) < s < \lambda_2(p^-, q^-), \tag{2.5}$$

where the second characteristic family of shocks is considered.

Solving for speed s in (2.4) and taking the positive one, we have

$$s = -\frac{q^+}{2} + \frac{\sqrt{{q^+}^2 + 4p^-}}{2} > 0. \tag{2.6}$$

Moreover, from the second equation of (2.2), one derives that

$$sQ + P = A,$$

where A is a constant defined as

$$A = sq^- + p^- = sq^+ + p^+. \tag{2.7}$$

For the sake of later use, we cite the main result of [13] in the following lemma.

Lemma 2.1. *Let* (2.4) *and* (2.5) *hold. Then* (2.2), (2.3) *admits a monotone shock profile* $(P, Q)(x - st)$*, which is unique up to a translation, satisfying* $P_z < 0$, $Q_z > 0$, *and*

$$\begin{cases} sDP_z = P^2 - (A + s^2)P - (p^+)^2 + (A + s^2)p^+, \\ sQ = A - P. \end{cases} \tag{2.8}$$

Remark 1. Due to $P_z < 0$ and $Q_z > 0$, it follows that $0 < p^+ \le P \le p^-$ and $q^- \le Q \le q^+$. Thus (2.8) implies that $|P_z|$ and $|Q_z|$ are bounded for given diffusion rate $D > 0$. Moreover, $\frac{1}{P}$ and its derivatives are bounded.

Remark 2. A direct consequence of Lemma 2.1 is the convergence of the traveling wave solution (P, Q), as $D \to 0$, to the following entropic shock solution of (1.1) with $D = 0$:

$$(p, q) = \begin{cases} (p^-, q^-), & x - st < 0, \\ (p^+, q^+), & x - st > 0. \end{cases}$$

See also [13].

Remark 3. The traveling wave solution (P, Q) can be obtained explicitly from (2.8); see [13]. Then the traveling wave solutions of (1.5) can be recovered via the transformation (1.6) and (1.8).

Now we turn to show that the traveling wave solutions of system (1.1) are asymptotically stable under small perturbations. We assume the integrability of the initial perturbation and write the integral in the form [4]

$$\int_{-\infty}^{+\infty} \begin{pmatrix} p_0(x) - P(x) \\ q_0(x) - Q(x) \end{pmatrix} dx = x_0 \begin{pmatrix} p^+ - p^- \\ q^+ - q^- \end{pmatrix} + \beta \vec{r_1}(p^-, q^-),$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

where x_0 and β are uniquely determined by the initial data $(p_0(x), q_0(x))$ through the above equation.

In this paper, we shall study the case, as in [4], where the initial perturbation is prescribed such that $\beta = 0$, and the case of $\beta \neq 0$ remains open. Then by the conservation laws in (1.1) and (2.2), one readily derives that

$$\int_{-\infty}^{+\infty} \begin{pmatrix} p(x,t) - P(x+x_0-st) \\ q(x,t) - Q(x+x_0-st) \end{pmatrix} dx = \int_{-\infty}^{+\infty} \begin{pmatrix} p_0(x) - P(x+x_0) \\ q_0(x) - Q(x+x_0) \end{pmatrix} dx$$
$$= \int_{-\infty}^{+\infty} \begin{pmatrix} p_0(x) - P(x) \\ q_0(x) - Q(x) \end{pmatrix} dx + \int_{-\infty}^{+\infty} \begin{pmatrix} P(x) - P(x+x_0) \\ Q(x) - Q(x+x_0) \end{pmatrix} dx$$
$$= \int_{-\infty}^{+\infty} \begin{pmatrix} p_0(x) - P(x) \\ q_0(x) - Q(x) \end{pmatrix} dx - x_0 \begin{pmatrix} p^+ - p^- \\ q^+ - q^- \end{pmatrix} = \vec{0}.$$

We decompose the solution (p, q) of (1.1), (1.2) into the form

$$(p,q)(x,t) = (P,Q)(x-st+x_0) + (\phi_x, \psi_x)(x,t), \tag{2.9}$$

where

$$\phi(x,t) = \int_{-\infty}^{x} (p(y,t) - P(y+x_0-st))dy,$$
$$\psi(x,t) = \int_{-\infty}^{x} (q(y,t) - Q(y+x_0-st))dy$$

for all $x \in R$ and $t \geq 0$.

It is clear that

$$\phi(\pm\infty, t) = 0,\ \psi(\pm\infty, t) = 0 \text{ for all } t \geq 0.$$

We assume that the translation $x_0 = 0$ without loss of generality, namely,

$$\int_{-\infty}^{+\infty} \begin{pmatrix} p_0(x) - P(x) \\ q_0(x) - Q(x) \end{pmatrix} dx = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.10}$$

Let us denote the initial value of the perturbation (ϕ, ψ) as

$$(\phi,\psi)(x,0) = (\phi_0, \psi_0)(x) = \int_{-\infty}^{x} (p_0 - P, q_0 - Q)(y)dy, \quad x \in R. \tag{2.11}$$

The main theorem of this paper is stated as follows.

THEOREM 2.2. *Let* $(P,Q)(x-st)$ *from Lemma* 2.1 *be a traveling wave solution. There exists a constant* $\varepsilon_0 > 0$ *such that if* $\|p_0 - P\|_1 + \|q_0 - Q\|_1 + \|(\phi_0, \psi_0)\| \leq \varepsilon_0$, *then the Cauchy problem* (1.1), (1.2) *has a unique global solution* $(p,q)(x,t)$ *satisfying* $p(x,t) \geq \delta_0 > 0$ *for all* $x \in R$, $t \geq 0$ *for some* $\delta_0 > 0$ *and*

$$(p-P, q-Q) \in C([0,\infty); H^1) \cap L^2([0,\infty); H^1).$$

Furthermore, the solution has the following asymptotic stability:

$$\sup_{x \in R} |(p,q)(x,t) - (P,Q)(x-st)| \to 0 \quad as \quad t \to +\infty. \tag{2.12}$$

Remark 4. The above nonlinear stability results hold true regardless of the strengths of the waves; i.e., the wave amplitude $|p^+ - p^-| + |q^+ - q^-|$ can be arbitrarily large.

Remark 5. The stability of the traveling wave solutions for system (1.1) implies the stability of the traveling wave solutions for system (1.5) through relations (1.6) and (1.8).

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

3. Reformulation of the problem. The proof of Theorem 2.2 is based on iterative L^2 energy estimates due to partial diffusion in hyperbolic-parabolic system (1.1). In view of (2.9), we shall seek a solution of the form

$$(p,q)(x,t) = (P,Q)(x-st) + (\phi_x, \psi_x)(x,t) = (P,Q)(z) + (\bar{\phi}_z, \bar{\psi}_z)(z,t) \tag{3.1}$$

with $(\bar{\phi}, \bar{\psi})$ in a certain space of integrable functions which will be defined later. For simplicity of notation, we will omit the bars in $(\bar{\phi}, \bar{\psi})$ in the rest of the paper.

Substituting (3.1) into (1.1), using (2.2), and integrating the resulting equations with respect to z, we derive that the perturbation (ϕ, ψ) satisfies

$$\begin{cases} \phi_t = D\phi_{zz} + s\phi_z + P\psi_z + Q\phi_z + \phi_z\psi_z, \\ \psi_t = s\psi_z + \phi_z \end{cases} \tag{3.2}$$

with initial data (2.11).

The asymptotic stability of the traveling wave profile (P,Q) means that the perturbation $(\phi_z, \psi_z)(z,t) \to 0$ as $t \to +\infty$.

We seek the solutions of reformulated problem (3.2), (2.11) in the solution space

$$\begin{aligned} X(0,T) = \{(\phi(z,t), \psi(z,t)) : \phi \in C([0,T);H^2),\ \psi \in C([0,T);H^2) \cap C^1((0,T);H^1), \\ \phi_z \in L^2((0,T);H^2),\ \psi_z \in L^2((0,T);H^1)\} \end{aligned}$$

with $0 < T < +\infty$.

By the Sobolev embedding theorem, if we let

$$N(t) = \sup_{0\le\tau\le t} \{\|\phi(\cdot,\tau)\|_2 + \|\psi(\cdot,\tau)\|_2\}, \tag{3.3}$$

then it follows that

$$\sup_{z\in R} \{|\phi|, |\phi_z|, |\psi|, |\psi_z|\} \le N(t). \tag{3.4}$$

Thus Theorem 2.2 is a consequence of the following theorem.

THEOREM 3.1. *Assume that the conditions of Theorem 2.2 hold. Then there exists a constant $\delta_1 > 0$ such that if*

$$N(0) \le \delta_1, \tag{3.5}$$

then the Cauchy problem (3.2), (2.11) *has a unique global solution $(\phi, \psi) \in X(0,+\infty)$ such that*

$$\begin{aligned} &\|\phi(\cdot,t)\|_2^2 + \|\psi(\cdot,t)\|_2^2 + \int_0^t \|(\phi(\cdot,\tau), \psi(\cdot,\tau))\|_{L^2_w}^2 d\tau + \int_0^t \|\phi_z(\cdot,\tau)\|_2^2 d\tau \\ &\quad + \int_0^t \|\psi_z(\cdot,\tau)\|_1^2 d\tau \\ &\le CN^2(0) \end{aligned} \tag{3.6}$$

for all $t \in [0,+\infty)$, where $w = |P_z|$. Moreover, it follows that

$$\sup_{z\in R} |(\phi_z, \psi_z)(z,t)| \to 0 \quad as \quad t \to +\infty. \tag{3.7}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

The global existence of (ϕ, ψ) announced in Theorem 3.1 follows from the local existence theorem and the a priori estimates given below.

Proposition 3.2 (local existence). *For any $\delta_2 > 0$, there exists a positive constant T depending on δ_2 such that if $(\phi_0, \psi_0) \in H^2$ with $N(0) \le \delta_2/2$, then the problem* (3.2), (2.11) *has a unique solution $(\phi, \psi) \in X(0,T)$ satisfying*

$$N(t) < 2N(0) \tag{3.8}$$

for any $0 \le t \le T$.

Proposition 3.3 (a priori estimates). *Assume that $(\phi, \psi) \in X(0,T)$ is a solution obtained in Proposition* 3.2 *for a positive constant T. Then there is a positive constant $\delta_3 > 0$, independent of T, such that if*

$$N(t) < \delta_3$$

for any $0 \le t \le T$, then the solution (ϕ, ψ) of (3.2), (2.11) *satisfies* (3.6) *for any $0 \le t \le T$.*

With the solution (ϕ, ψ) obtained in Theorem 3.1 and (P, Q) in Lemma 2.1, we have the desired solution of problem (1.1), (1.2) through relation (3.1). The local existence can be shown in a standard way (cf. [10]) and we omit the proof for brevity. Theorem 3.1 is a consequence of Propositions 3.2 and 3.3 by the continuation arguments. So it remains to prove Proposition 3.3. The following section is devoted to the proof of Proposition 3.3 based on iterative L^2 energy estimates.

4. Energy estimates.

In this section, we establish the basic energy estimates for solution (ϕ, ψ) of (3.2), (2.11) and prove Proposition 3.3.

Lemma 4.1. *Let the assumptions in Theorem* 2.2 *hold, and let $w = |P_z|$. Then there exist constants $\nu_0 > 0$ and $C > 0$ such that*

$$\begin{aligned} &\|\phi(\cdot,t)\|^2 + \|\psi(\cdot,t)\|^2 + \nu_0 \int_0^t \|(\phi(\cdot,\tau), \psi(\cdot,\tau))\|_{L_w^2}^2 d\tau + D\int_0^t \|\phi_z(\cdot,\tau)\|^2 d\tau \\ &\qquad \le C\left(\|\phi_0\|^2 + \|\psi_0\|^2 + \int_0^t \int |\phi\phi_z\psi_z| dz d\tau \right). \end{aligned} \tag{4.1}$$

Proof. Multiplying the first equation of (3.2) by ϕ/P, the second by ψ, and integrating the resulting equations with respect to z and adding them, we have

$$\frac{1}{2}\frac{d}{dt}\int \left(\frac{\phi^2}{P} + \psi^2 \right) dz = D\int \frac{\phi\phi_{zz}}{P} dz + s\int \frac{\phi\phi_z}{P} dz + \int \frac{Q}{P}\phi\phi_z dz + \int \frac{\phi\phi_z\psi_z}{P} dz. \tag{4.2}$$

Noting that

$$\frac{\phi\phi_{zz}}{P} = \left(\frac{\phi\phi_z}{P} \right)_z - \frac{\phi_z^2}{P} + \frac{P_z\phi\phi_z}{P^2}$$

and integrating (4.2) by parts, we derive that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \left(\frac{\phi^2}{P} + \psi^2 \right) dz + D\int \frac{\phi_z^2}{P} dz \\ &= -\frac{1}{2}\int \left[-\left(\frac{D}{P} \right)_{zz} + \left(\frac{s+Q}{P} \right)_z \right] \phi^2 dz + \int \frac{\phi\phi_z\psi_z}{P} dz. \end{aligned} \tag{4.3}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

By using (2.2) and (2.7), one calculates that

$$-\left(\frac{D}{P}\right)_{zz} + \left(\frac{s+Q}{P}\right)_z = \nu(z)(-P_z), \tag{4.4}$$

where

$$\nu(z) = \frac{2}{P^3}\left[P\left(s + q^- + \frac{p^-}{s}\right) - \frac{1}{s}P^2 + DP_z\right].$$

Substituting the first equation of (2.8) into the above equation yields

$$\nu(z) = \frac{2p^+}{P^3}(s + q^+).$$

Since $0 < p^+ \le P \le p^-$, it follows from (2.6) that

$$\nu(z) \ge \frac{2p^+}{(p^-)^3}(s + q^+) = \frac{p^+}{(p^-)^3}(q^+ + \sqrt{(q^+)^2 + 4p^-}) = \nu_0 > 0. \tag{4.5}$$

Inserting (4.4) and (4.5) into (4.3), noting $0 < p^+ \le P \le p^-$, and integrating the result with respect to t, we get

$$\begin{aligned} &\|\phi(\cdot,t)\|^2 + \|\psi(\cdot,t)\|^2 + \nu_0 \int_0^t \|\phi(\cdot,\tau)\|^2_{L^2_w} d\tau + D\int_0^t \|\phi_z(\cdot,\tau)\|^2 d\tau \\ &\qquad \le C\left(\|\phi_0\|^2 + \|\psi_0\|^2 + \int_0^t\int |\phi\phi_z\psi_z| dz d\tau\right), \end{aligned} \tag{4.6}$$

where $w(z) = |P_z(z)|$ for all $z \in R$ and $C > 0$ is a constant.

To finish the proof of (4.1), it remains to estimate $\int_0^t \|\psi(\cdot,\tau)\|^2_{L^2_w} d\tau$. To this end, we multiply the first equation of (3.2) by ϕ, the second by $P\psi$, and integrate the results in z to obtain that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int(\phi^2 + P\psi^2)dz + D\int \phi_z^2 dz + \frac{1}{2}\int Q_z\phi^2 dz + \frac{s}{2}\int P_z\psi^2 dz \\ &= \int P(\psi_z\phi + \psi\phi_z)dz + \int \phi\phi_z\psi_z dz. \end{aligned} \tag{4.7}$$

Integrating $2 \times (4.7)$ with respect to t and using (4.6) and $\int P(\psi_z\phi + \psi\phi_z)dz = -\int P_z\phi\psi dz$, we have that

$$\begin{aligned} s\int_0^t\int |P_z|\psi^2 dz d\tau &= \int(\phi^2 + P\psi^2)dx - \int(\phi_0^2 + P\psi_0^2)dz + 2D\int_0^t\int \phi_z^2 dz d\tau \\ &\quad + \int_0^t\int Q_z\phi^2 dz d\tau + 2\int_0^t\int P_z\phi\psi dz d\tau - 2\int_0^t\int \phi\phi_z\psi_z dz d\tau \\ &\le C(\|\phi_0\|^2 + \|\psi_0\|^2) + C\int_0^t\int |\phi\phi_z\psi_z| dz d\tau + 2\int_0^t\int P_z\phi\psi dz d\tau, \end{aligned} \tag{4.8}$$

where $Q_z = -P_z/s$ is a scalar multiple of P_z, which is bounded by a multiple of the weight $w = |P_z|$.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Using Young's inequality, we estimate the last term on the right-hand side of (4.8) as

$$2\int_0^t\int P_z\phi\psi dzd\tau \le \frac{s}{2}\int_0^t\int |P_z|\psi^2 dzd\tau + \frac{2}{s}\int_0^t\int |P_z|\phi^2 dzd\tau.$$

Substituting the above inequality into (4.8) and combining it with (4.6), we finish the proof of (4.1). ∎

The next lemma gives the estimate of the first order derivatives of (ϕ,ψ).

LEMMA 4.2. *Let the assumptions in Theorem 2.2 hold. Then there exists a constant C such that*

$$\begin{aligned}&\|\phi(\cdot,t)\|_1^2+\|\psi(\cdot,t)\|_1^2+\nu_0\int_0^t\|(\phi(\cdot,\tau),\psi(\cdot,\tau))\|_{L_w^2}^2d\tau+D\int_0^t\|\phi_z(\cdot,\tau)\|_1^2d\tau\\&\quad+\int_0^t\|\psi_z(\cdot,\tau)\|^2d\tau\\&\le C(\|\phi_0\|_1^2+\|\psi_0\|_1^2)+C\int_0^t\int(|\phi|+|\phi_z|+|\psi_z|+|\psi_{zz}|)|\phi_z\psi_z|dzd\tau,\end{aligned}\tag{4.9}$$

where ν_0 is defined as in (4.5).

Proof. Multiply the first equation of (3.2) by $-\phi_{zz}/P$, the second by $-\psi_{zz}$, integrate them with respect to z, and finally, add them to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left(\frac{\phi_z^2}{P}+\psi_z^2\right)dz+D\int\frac{\phi_{zz}^2}{P}dz\\&=-\int\left(\frac{1}{P}\right)_z\phi_t\phi_zdz-\frac{1}{2}\left(s+\frac{A}{s}\right)\int\frac{P_z}{P^2}\phi_z^2dz-\int\frac{1}{P}\phi_z\psi_z\phi_{zz}dz;\end{aligned}\tag{4.10}$$

here the second equation of (2.8) has been used.

Now we estimate the first term on the right-hand side of (4.10):

$$I=-\int\left(\frac{1}{P}\right)_z\phi_t\phi_zdz.$$

Indeed, substituting the first equation of (3.2) into I, we deduce, after integrating by parts and using the fact that Q, $\left(\frac{1}{P}\right)_z$, $\left(\frac{1}{P}\right)_{zz}$ are bounded (Lemma 2.1), that

$$\begin{aligned}I&=\int\left[\frac{D}{2}\left(\frac{1}{P}\right)_{zz}-(s+Q)\left(\frac{1}{P}\right)_z\right]\phi_z^2dz-\int P\left(\frac{1}{P}\right)_z\phi_z\psi_zdz-\int\left(\frac{1}{P}\right)_z\phi_z^2\psi_zdz\\&\le C\int\phi_z^2dz+C\int\phi_z^2|\psi_z|dz-\int P\left(\frac{1}{P}\right)_z\phi_z\psi_zdz.\end{aligned}\tag{4.11}$$

Next, we estimate the last term in (4.11). In fact, using (2.2) and the Cauchy–Schwarz inequality, as well as the fact that P is bounded away from zero, we derive that

$$\begin{aligned}-P\left(\frac{1}{P}\right)_z\phi_z\psi_z&=-\left(P\left(\frac{1}{P}\right)_z\phi_z\psi\right)_z+P\left(\frac{1}{P}\right)_z\phi_{zz}\psi+\left[P_z\left(\frac{1}{P}\right)_z\phi_z+P\left(\frac{1}{P}\right)_{zz}\phi_z\right]\psi\\&=-\left(P\left(\frac{1}{P}\right)_z\phi_z\psi\right)_z-\frac{P_z}{P}\phi_{zz}\psi+\left[\frac{1}{D}\left(s+\frac{A-2P}{s}\right)+\frac{P_z}{P}\right]\frac{P_z}{P}\phi_z\psi\\&\le-\left(P\left(\frac{1}{P}\right)_z\phi_z\psi\right)_z+\frac{D}{2}\frac{\phi_{zz}^2}{P}+C(|P_z|\psi^2+\phi_z^2)\end{aligned}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

for some $C > 0$.

Now substituting the above inequality into (4.11), one has that

$$I \leq \frac{D}{2} \int \frac{\phi_{zz}^2}{P} dz + C \int (|P_z|\psi^2 + \phi_z^2) dz + C \int \phi_z^2 |\psi_z| dz. \tag{4.12}$$

The combination of (4.10)–(4.12) gives

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int \left(\frac{\phi_z^2}{P} + \psi_z^2 \right) dz + \frac{D}{2} \int \frac{\phi_{zz}^2}{P} dz \\ &\leq C \int (|P_z|\psi^2 + \phi_z^2) dz + C \int |\phi_z \psi_z \phi_{zz}| dz + C \int \phi_z^2 |\psi_z| dz. \end{aligned}$$

Integrating the above inequality with respect to t and using (4.1), as well as the fact that $0 < p^+ \leq P \leq p^-$, we have

$$\begin{aligned} &\int (\phi_z^2 + \psi_z^2) dz + D \int_0^t \int \phi_{zz}^2 dz d\tau \\ &\leq C(\|\phi_0\|_1^2 + \|\psi_0\|_1^2) + C \int_0^t \int (|\phi| + |\phi_z| + |\phi_{zz}|) |\phi_z \psi_z| dz d\tau. \end{aligned} \tag{4.13}$$

Now we are left to estimate the term $\int_0^t \|\psi_z(\cdot, \tau)\|^2 d\tau$ in (4.9). To this end, we multiply the first equation of (3.2) by ψ_z to obtain

$$\phi_t \psi_z = D\phi_{zz}\psi_z + s\phi_z\psi_z + P\psi_z^2 + Q\phi_z\psi_z + \phi_z\psi_z^2. \tag{4.14}$$

On the other hand, we have from the second equation of (3.2) that

$$\phi_t \psi_z = (\phi\psi_z)_t - s(\phi\psi_z)_z - (\phi\phi_z)_z + s\phi_z\psi_z + \phi_z^2. \tag{4.15}$$

Equating (4.15) with (4.14) and integrating the result in z yield

$$\begin{aligned} \int P\psi_z^2 dz &= \frac{d}{dt} \int \phi\psi_z dz + \int \phi_z^2 dz - D \int \phi_{zz}\psi_z dz - \int Q\phi_z\psi_z dz - \int \phi_z \psi_z^2 dz \\ &\leq \frac{d}{dt} \int \phi\psi_z dz + \int \phi_z^2 dz + D \int |\phi_{zz}\psi_z| dz + (|q^-| + |q^+|) \int |\phi_z\psi_z| dz \\ &\quad + \int |\phi_z| \psi_z^2 dz. \end{aligned}$$

Noting $p^+ \int \psi_z^2 dz \leq \int P\psi_z^2 dz$, using Young's inequality to estimate the terms $D \int |\phi_{zz}\psi_z| dz \leq \frac{p^+}{4} \int \psi_z^2 dz + \frac{4D^2}{p^+} \int \phi_{zz}^2 dz$ and $(|q^-| + |q^+|) \int |\phi_z\psi_z| dz \leq \frac{p^+}{4} \int \psi_z^2 dz + \frac{4(|q^+|+|q^-|)^2}{p^+} \int \phi_z^2 dz$, and integrating the above inequality in t, one deduces that

$$\begin{aligned} &\int_0^t \int \psi_z^2 dz d\tau \\ &\leq C \left(\int \phi\psi_z dz - \int \phi_0 \psi_{0,z} dz \right) + C \int_0^t \int (\phi_z^2 + \phi_{zz}^2) dz d\tau + C \int_0^t \int |\phi_z| \psi_z^2 dz d\tau \\ &\leq C(\|\phi(\cdot, t)\|^2 + \|\psi_z(\cdot, t)\|^2 + \|\phi_0\|^2 + \|\psi_{0,z}\|^2) \\ &\qquad + C \int_0^t \int (\phi_z^2 + \phi_{zz}^2) dz d\tau + C \int_0^t \int |\phi_z| \psi_z^2 dz d\tau \end{aligned}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

for some $C > 0$, where $\psi_{0,z}$ means the derivative of ψ_0 with respect to z.

Applying (4.1) and (4.13) into the above inequality gives

$$\int_0^t \int \psi_z^2 dz d\tau \le C(\|\phi_0\|_1^2 + \|\psi_0\|_1^2) + C\int_0^t \int (|\phi| + |\phi_z| + |\psi_z| + |\phi_{zz}|)|\phi_z \psi_z| dz d\tau. \tag{4.16}$$

The combination of (4.13) and (4.16) gives (4.9), and the proof of Lemma 4.2 is completed. □

Next, we give the estimates of second order derivatives of (ϕ, ψ).

LEMMA 4.3. *Let the assumptions in Theorem 2.2 hold. Then there exists a constant C such that*

$$\begin{aligned} &\|\phi(\cdot,t)\|_2^2 + \|\psi(\cdot,t)\|_2^2 + \nu_0 \int_0^t \|(\phi(\cdot,\tau),\psi(\cdot,\tau))\|_{L_w^2}^2 d\tau + D\int_0^t \|\phi_z(\cdot,\tau)\|_2^2 d\tau \\ &\quad + \int_0^t \|\psi_z(\cdot,\tau)\|_1^2 d\tau \\ &\le C(\|\phi_0\|_2^2 + \|\psi_0\|_2^2) + C\int_0^t \int (|\phi| + |\phi_z| + |\psi_z| + |\phi_{zz}| + |\psi_{zz}|)|\phi_z\psi_z| dz d\tau \\ &\quad + C\int_0^t \int (|\phi_{zz}| + |\psi_{zz}| + |\phi_{zzz}|)|(\phi_z\psi_z)_z| dz d\tau, \end{aligned} \tag{4.17}$$

where ν_0 is given by (4.5).

Proof. We multiply the first equation of (3.2) by $1/P$, differentiate the resulting equation with respect to z twice, and then differentiate the second equation of (3.2) with respect to z twice, to get

$$\begin{cases} \left(\frac{1}{P}\phi_t\right)_{zz} = \left(\frac{D}{P}\phi_{zz}\right)_{zz} + \left(\frac{s}{P}\phi_z\right)_{zz} + \psi_{zzz} + \left(\frac{Q}{P}\phi_z\right)_{zz} + \left(\frac{1}{P}\phi_z\psi_z\right)_{zz}, \\ \psi_{tzz} = s\psi_{zzz} + \phi_{zzz}. \end{cases} \tag{4.18}$$

Now, multiplying the first equation of (4.18) by ϕ_{zz}, the second by ψ_{zz}, and integrating the results, we end up with

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \left(\frac{\phi_{zz}^2}{P} + \psi_{zz}^2\right) dz + \int \left[\left(\frac{1}{P}\right)_{zz}\phi_t + 2\left(\frac{1}{P}\right)_z \phi_{tz}\right]\phi_{zz} dz \\ &= D\int \left(\frac{\phi_{zz}}{P}\right)_{zz}\phi_{zz} dz + s\int \left(\frac{\phi_z}{P}\right)_{zz}\phi_{zz} dz \\ &\quad + \int \left(\frac{Q}{P}\phi_z\right)_{zz}\phi_{zz} dz + \int \left(\frac{1}{P}\phi_z\psi_z\right)_{zz}\phi_{zz} dz \\ &=: I_1 + I_2 + I_3 + \int \left(\frac{1}{P}\phi_z\psi_z\right)_{zz}\phi_{zz} dz. \end{aligned} \tag{4.19}$$

Next, we are going to estimate I_i $(i = 1, 2, 3)$, respectively. Integrating by parts, we can estimate I_1 as

$$\begin{aligned} I_1 &= D\int \left(\frac{1}{P}\right)_{zz}\phi_{zz}^2 dz + 2D\int \left(\frac{1}{P}\right)_z \phi_{zz}\phi_{zzz} dz + D\int \frac{1}{P}\phi_{zz}\phi_{zzzz} \\ &= \frac{D}{2}\int \left(\frac{1}{P}\right)_{zz}\phi_{zz}^2 dz - D\int \frac{1}{P}\phi_{zzz}^2 dz. \end{aligned} \tag{4.20}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

I_2 is estimated as follows:

$$\begin{aligned} I_2 &= s\int \Big(\frac{1}{P}\Big)_{zz}\phi_z\phi_{zz}dz + 2s\int\Big(\frac{1}{P}\Big)_z\phi_{zz}^2dz + s\int\frac{1}{P}\phi_{zz}\phi_{zzz}dz \\ &= -\frac{s}{2}\int\Big(\frac{1}{P}\Big)_{zzz}\phi_z^2dz + \frac{3s}{2}\int\Big(\frac{1}{P}\Big)_z\phi_{zz}^2dz. \end{aligned} \tag{4.21}$$

Furthermore, integration by parts gives the estimate of I_3 as

$$\begin{aligned} I_3 &= \int \Big(\frac{Q}{P}\Big)_{zz}\phi_z\phi_{zz}dz + 2\int\Big(\frac{Q}{P}\Big)_z\phi_{zz}^2dz + \int\frac{Q}{P}\phi_{zz}\phi_{zzz}dz \\ &= -\frac{1}{2}\int\Big(\frac{Q}{P}\Big)_{zzz}\phi_z^2dz + \frac{3}{2}\int\Big(\frac{Q}{P}\Big)_z\phi_{zz}^2dz. \end{aligned} \tag{4.22}$$

Substituting (4.20)–(4.22) into (4.19) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\Big(\frac{\phi_{zz}^2}{P}+\psi_{zz}^2\Big)dz + \int\Big[\Big(\frac{1}{P}\Big)_{zz}\phi_t + 2\Big(\frac{1}{P}\Big)_z\phi_{tz}\Big]\phi_{zz}dz + D\int\frac{1}{P}\phi_{zzz}^2dz \\ &= -\frac{1}{2}\int\eta_1(z)\phi_z^2dz + \frac{1}{2}\int\eta_2(z)\phi_{zz}^2dz + \int\Big(\frac{1}{P}\phi_z\psi_z\Big)_{zz}\phi_{zz}dz, \end{aligned} \tag{4.23}$$

where

$$\eta_1(z) = \Big(\frac{s+Q}{P}\Big)_{zzz}, \quad \eta_2(z) = 3\Big(\frac{s+Q}{P}\Big)_z + \Big(\frac{D}{P}\Big)_{zz}.$$

Now we are in position to estimate the term

$$\eta = \int\Big[\Big(\frac{1}{P}\Big)_{zz}\phi_t + 2\Big(\frac{1}{P}\Big)_z\phi_{tz}\Big]\phi_{zz}dz$$

on the left-hand side of (4.23).

Indeed, substituting the first equation of (3.2) into η and integrating by parts, we have that

$$\begin{aligned} \eta &= \int\eta_3(z)\phi_{zz}^2dz + \int\eta_4(z)\phi_z\phi_{zz}dz + \int P\Big(\frac{1}{P}\Big)_{zz}\psi_z\phi_{zz}dz \\ &\quad + \int\Big(\frac{1}{P}\Big)_{zz}\phi_z\psi_z\phi_{zz}dz - 2\int P\Big(\frac{1}{P}\Big)_z\psi_z\phi_{zzz}dz + 2\int\Big(\frac{1}{P}\Big)_z(\phi_z\psi_z)_z\phi_{zz}dz, \end{aligned} \tag{4.24}$$

where

$$\eta_3(z) = 2(s+Q)\Big(\frac{1}{P}\Big)_z, \quad \eta_4(z) = (s+Q)\Big(\frac{1}{P}\Big)_{zz} + 2Q_z\Big(\frac{1}{P}\Big)_z.$$

Feeding (4.24) into (4.23), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\Big(\frac{\phi_{zz}^2}{P}+\psi_{zz}^2\Big)dz + D\int\frac{1}{P}\phi_{zzz}^2dz \\ &= -\frac{1}{2}\int\eta_1(z)\phi_z^2dz + \frac{1}{2}\int\eta_5(z)\phi_{zz}^2dz - \int\eta_4(z)\phi_z\phi_{zz}dz \\ &\quad - \int P\Big(\frac{1}{P}\Big)_{zz}\psi_z\phi_{zz}dz - \int\Big(\frac{1}{P}\Big)_{zz}\phi_z\psi_z\phi_{zz}dz \\ &\quad - 2\int P\Big(\frac{1}{P}\Big)_z\psi_z\phi_{zzz}dz - 2\int\Big(\frac{1}{P}\Big)_z(\phi_z\psi_z)_z\phi_{zz}dz \\ &\quad + \int\Big(\frac{1}{P}\phi_z\psi_z\Big)_{zz}\phi_{zz}dz \end{aligned} \tag{4.25}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

with $\eta_5(z) = \eta_2(z) - 2\eta_3(z)$.

Due to the boundedness of Q, P, $\frac{1}{P}$, and their derivatives, it can be easily verified that $\eta_4(z)$, $\eta_5(z)$ are bounded for any $z \in R$. It follows from Young's inequality that $\left|-2\int P\left(\frac{1}{P}\right)_z \psi_z\phi_{zzz}dz\right| \leq \frac{D}{2}\int \frac{1}{P}\phi_{zzz}^2 dz + C\int \psi_z^2 dz$ and $\left|-\int P\left(\frac{1}{P}\right)_{zz}\psi_z\phi_{zz}dz\right| \leq C\int(\psi_z^2 + \phi_{zz}^2)dz$ for some constant $C > 0$. Substituting these into (4.25) and integrating the result with respect to t, we get

$$\begin{aligned}
&\int \left(\frac{\phi_{zz}^2}{P} + \psi_{zz}^2\right)dz + D\int_0^t \int \frac{1}{P}\phi_{zzz}^2 dz d\tau \\
&\leq C\int_0^t \int (\phi_z^2 + \phi_{zz}^2 + \psi_z^2)dz d\tau - 2\int \left(\frac{1}{P}\right)_{zz}\phi_z\psi_z\phi_{zz}dz \\
&\quad - 4\int \left(\frac{1}{P}\right)_z (\phi_z\psi_z)_z\phi_{zz}dz + 2\int \left(\frac{1}{P}\phi_z\psi_z\right)_{zz}\phi_{zz}dz.
\end{aligned} \tag{4.26}$$

Now we estimate the last term in (4.26) as follows:

$$\begin{aligned}
\int \left(\frac{1}{P}\phi_z\psi_z\right)_{zz}\phi_{zz}dz &= -\int \left(\frac{1}{P}\phi_z\psi_z\right)_z \phi_{zzz}dz \\
&= \int \left(\frac{1}{P}\right)_{zz}\phi_z\psi_z\phi_{zz}dz + \int \left(\frac{1}{P}\right)_z (\phi_z\psi_z)_z\phi_{zz}dz \\
&\quad - \int \frac{1}{P}(\phi_z\psi_z)_z\phi_{zzz}dz.
\end{aligned} \tag{4.27}$$

Then using the boundedness of $\left(\frac{1}{P}\right)_z$ as well as $\left(\frac{1}{P}\right)_{zz}$, and combining (4.26), (4.27) with (4.9), we have

$$\begin{aligned}
&\|\phi(\cdot,t)\|_2^2 + \|\psi(\cdot,t)\|_2^2 + \nu_0\int_0^t \|(\phi(\cdot,\tau),\psi(\cdot,\tau))\|_{L_w^2}^2 d\tau + D\int_0^t \|\phi_z(\cdot,\tau)\|_2^2 d\tau \\
&\quad + \int_0^t \|\psi_z(\cdot,\tau)\|^2 d\tau \\
&\leq C(\|\phi_0\|_2^2 + \|\psi_0\|_2^2) + C\int_0^t \int (|\phi| + |\phi_z| + |\psi_z| + |\phi_{zz}| + |\psi_{zz}|)|\phi_z\psi_z|dz d\tau \\
&\quad + C\int_0^t \int |(\phi_z\psi_z)_z\phi_{zz}|dz d\tau + C\int_0^t \int |(\phi_z\psi_z)_z\phi_{zzz}|dz d\tau.
\end{aligned} \tag{4.28}$$

To finish the proof of (4.17), it remains only to estimate $\int_0^t \|\psi_{zz}(\cdot,\tau)\|^2 d\tau$. For this purpose, we multiply the first equation of (3.2) by ψ_{zzz} and integrate the result in z to get

$$\begin{aligned}
\int \phi_t\psi_{zzz}dz &= -D\int \phi_{zzz}\psi_{zz}dz - \int (Q_z\phi_z + (s+Q)\phi_{zz})\psi_{zz}dz \\
&\quad - \int P\psi_{zz}^2 dz - \int P_z\psi_z\psi_{zz}dz + \int \phi_z\psi_z\psi_{zzz}dz.
\end{aligned} \tag{4.29}$$

From the second equation of (3.2), it follows that

$$\phi_t\psi_{zzz} = (\phi\psi_{zzz})_t - \phi\psi_{tzzz} = (\phi\psi_{zzz})_t - \phi(s\psi_{zzzz} + \phi_{zzzz}). \tag{4.30}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Plugging (4.30) into (4.29) and integrating the result with respect to t, one has

$$
\begin{aligned}
\int_0^t\int P\psi_{zz}^2 dzd\tau &= -\int \phi\psi_{zzz}dz + \int \phi_0\psi_{0,zzz}dz + \int_0^t\int \phi(s\psi_{zzzz}+\phi_{zzzz})dzd\tau \\
&\quad - D\int_0^t\int \phi_{zzz}\psi_{zz}dzd\tau - \int_0^t\int (Q_z\phi_z+(s+Q)\phi_{zz})\psi_{zz}dzd\tau \\
&\quad - \int_0^t\int P_z\psi_z\psi_{zz}dzd\tau + \int_0^t\int \phi_z\psi_z\psi_{zzz}dzd\tau \\
&= \int \phi_z\psi_{zz}dz + \int \phi_{0,zz}\psi_{0,z}dz + s\int_0^t\int \phi_{zz}\psi_{zz}dzd\tau \\
&\quad + \int_0^t\int \phi_{zz}^2 dz - D\int_0^t\int \phi_{zzz}\psi_{zz}dzd\tau \\
&\quad - \int_0^t\int (Q_z\phi_z+(s+Q)\phi_{zz})\psi_{zz}dzd\tau \\
&\quad - \int_0^t\int P_z\psi_z\psi_{zz}dzd\tau - \int_0^t\int (\phi_z\psi_z)_z\psi_{zz}dzd\tau.
\end{aligned} \tag{4.31}
$$

Taking into account that $P \geq p^+ > 0$ and the boundedness of Q, P_z, Q_z, we estimate the terms on the right-hand side of (4.31) by the use of Young's inequality. For example, the fifth and sixth terms can be estimated as $-D\int_0^t\int \phi_{zzz}\psi_{zz}dzd\tau \leq \frac{p^+}{8}\int_0^t\int \psi_{zz}^2 dzd\tau + \frac{8D^2}{p^+}\int_0^t\int \phi_{zzz}^2 dzd\tau$ and $-\int_0^t\int (Q_z\phi_z + (s+Q)\phi_{zz})\psi_{zz}dzd\tau \leq \frac{p^+}{8}\int_0^t\int \psi_{zz}^2 dzd\tau + C\int_0^t\int (\phi_z^2+\phi_{zz}^2)dzd\tau$, respectively, for some constant $C > 0$. Other terms can be estimated analogously. Finally, by employing (4.28), we end up with

$$
\begin{aligned}
\int_0^t\int \psi_{zz}^2 dzd\tau &\leq C(\|\phi_{0,z}\|^2 + \|\psi_{0,zz}\|^2 + \|\phi_z(\cdot,t)\|^2 + \|\psi_{zz}(\cdot,t)\|^2) \\
&\quad + C\int_0^t\int (\phi_z^2+\psi_z^2+\phi_{zz}^2+\phi_{zzz}^2)dzd\tau + C\int_0^t\int |(\phi_z\psi_z)_z\psi_{zz}|dzd\tau \\
&\leq C(\|\phi_0\|_2^2 + \|\psi_0\|_2^2) + C\int_0^t\int (|\phi|+|\phi_z|+|\psi_z|+|\phi_{zz}|+|\psi_{zz}|)|\phi_z\psi_z|dzd\tau \\
&\quad + C\int_0^t\int |(\phi_z\psi_z)_z\phi_{zzz}|dzd\tau + C\int_0^t\int |(\phi_z\psi_z)_z\phi_{zz}|dzd\tau + C\int_0^t\int |(\phi_z\psi_z)_z\psi_{zz}|dzd\tau.
\end{aligned} \tag{4.32}
$$

Finally, the combination of (4.32) and (4.28) completes the proof of (4.17). ∎

Now we are in position to prove Proposition 3.3.

In fact, we need only show that the a priori estimate (3.6) holds. Hence it remains to estimate the cubic terms in (4.17). Indeed, by applying the Sobolev embedding theorems, all these cubic terms can be bounded by $CN(t)\int_0^t \|(\phi_z(\cdot,\tau),\psi_z(\cdot,\tau))\|_2^2 d\tau$ for some constant $C > 0$. Then from Lemma 4.3, we have

$$
\begin{aligned}
N^2(t) + \int_0^t \|(\phi(\cdot,\tau),\psi(\cdot,\tau))\|_{L_w^2}^2 d\tau + \int_0^t \|\phi_z(\cdot,\tau)\|_2^2 d\tau + \int_0^t \|\psi_z(\cdot,\tau)\|_1^2 d\tau \\
\leq CN^2(0) + CN(t)\int_0^t \|(\phi_z(\cdot,\tau),\psi_z(\cdot,\tau))\|_2^2 d\tau
\end{aligned}
$$

for $t \in [0,T]$ and for some constant $C > 0$.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Therefore, letting $N(t) \le \frac{1}{2C}$, we obtain the following estimate for any $t \in [0,T]$:

$$N^2(t) + \int_0^t \|(\phi(\cdot,\tau),\psi(\cdot,\tau))\|^2_{L^2_w}\, d\tau + \int_0^t \|\phi_z(\cdot,\tau)\|_2^2 d\tau + \int_0^t \|\psi_z(\cdot,\tau)\|_1^2 d\tau \le CN^2(0),$$

which gives the desired estimate (3.6).

From global estimate (3.6), we derive

$$\|(\phi_z(\cdot,t),\psi_z(\cdot,t))\|_1 \to 0 \quad \text{as} \quad t \to +\infty. \tag{4.33}$$

Consequently, for all $z \in R$,

$$\begin{aligned} \phi_z^2(z,t) &= 2\int_{-\infty}^{z} \phi_z\phi_{zz}(y,t)dy \\ &\le 2\Big(\int_{-\infty}^{+\infty} \phi_z^2 dy\Big)^{1/2}\Big(\int_{-\infty}^{+\infty} \phi_{zz}^2 dy\Big)^{1/2} \to 0 \quad \text{as} \quad t \to +\infty. \end{aligned} \tag{4.34}$$

Applying the same argument to ψ_z leads, for all $z \in R$, to

$$\psi_z(z,t) \to 0 \quad \text{as} \quad t \to +\infty.$$

Hence (3.7) is proved.

Thus the proof of Proposition 3.3 is completed. □

Finally, we prove the positivity of p claimed in Theorem 2.2.

In fact, if the initial perturbation (2.11) satisfies (3.5), then by (3.6) there is a constant $C > 0$ such that

$$|\phi_z(z,t)| \le \sqrt{2}N(t) \le CN(0) \le C\delta_1.$$

Thus for all $z \in R$ and $t \ge 0$, it follows from (3.1) that

$$\begin{aligned} p(x,t) &= (p(x,t) - P(z)) + P(z) = \phi_z(z,t) + P(z) \ge -CN(0) + p^+ \ge -C\delta_1 + p^+ \\ &= \delta_0 > 0, \end{aligned}$$

where $\delta_0 = -C\delta_1 + p^+ > 0$ provided that δ_1 is suitably small and p^+ is positive. □

5. Numerical simulations. In this section, we show some numerical simulations for the model (1.1), (1.2) to verify our theoretical results. Since the numerical scheme is restricted to a finite domain, we consider the stability of traveling wave solutions to perturbations which vanish outside the finite domain. The boundary conditions are set as Dirichlet conditions that are compatible with the initial conditions. In the simulations, we choose different perturbations to verify our theoretical results. We use finite element package COMSOL Multiphysics to perform our numerical computations, and we plot only the quantity of cell density p that we are interested in.

Figure 1 shows the development of traveling waves as the diffusion rate D decreases. The initial data (1.2) are given as

$$\begin{aligned} p_0(x) &= \bar{P}(x) = 0.5 + \gamma/(1 + \exp(2(x-\xi))), \\ q_0(x) &= \bar{Q}(x) = 0.5 + \nu/(1 + \exp(-2(x-\xi))), \end{aligned} \tag{5.1}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

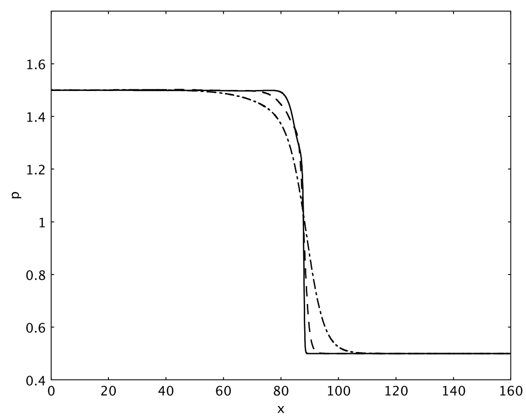


FIG. 1. *Development of traveling wave solutions of* (1.1), (5.1) *as the diffusion rate D decreases. The dash-dotted, dashed, and solid curves correspond to $D = 4, 1, 0.1$, respectively. The domain is* $[0, 160]$ *and $\gamma = \nu = 1$, $\xi = 30$.*

where parameters γ, ν, and ξ will be chosen appropriately under different circumstances. Then the left state of p is given by $0.5 + \gamma$ and the right state by 0.5. The left and right states of q are given by 0.5 and $0.5 + \nu$, respectively. That is

$$\begin{aligned} p^- &= 0.5 + \gamma, \quad p^+ = 0.5, \\ q^- &= 0.5, \quad q^+ = 0.5 + \nu. \end{aligned} \tag{5.2}$$

In these simulations, we choose $\gamma = \nu = 1$, $D = 4$, $\xi = 30$, and hence the left and right states of p are 1.5 and 0.5, respectively, as shown. As expected from Lemma 2.1, the traveling waves become steeper as the diffusion rate D decreases and finally develop into a shock wave as $D \to 0$.

Figure 2 shows the stability of the traveling waves under small perturbations. The initial perturbations are set as

$$(p_0 - \bar{P})(x) = (q_0 - \bar{Q})(x) = 0.5 \sin x/(((x - 80)/10)^2 + 1), \tag{5.3}$$

where $(\bar{P}, \bar{Q})$ and boundary conditions are the same as in (5.1) and (5.2), respectively, with $\gamma = \nu = 1$, $D = 4$, $\xi = 80$. The plot of perturbation (5.3) is given in Figure 2(a). Note that the perturbation is in $H^1(R) \times H^1(R)$ as required in Theorem 2.2. From Figure 2(b), we see that the solution of (1.1) under such a perturbation converges, as time increases, to a traveling wave with the boundary condition (5.2).

Figure 3 verifies the fact that the traveling waves are stable under small perturbations with no smallness assumptions on the amplitudes of the waves (see Remark 4). In the simulations shown in Figure 3, we choose $\gamma = \nu = 4$, $D = 4$, $\xi = 50$, and hence increase the amplitudes of the waves from 1 to 4 in Figure 2. The initial perturbations are prescribed as

$$(p_0 - \bar{P})(x) = (q_0 - \bar{Q})(x) = \sin x \exp(-0.0008(x - 50)^2), \tag{5.4}$$

as plotted in Figure 3(a). Again the perturbation is in $H^1(R) \times H^1(R)$. Through numerical simulations as shown in Figure 3(b), we find that the traveling waves with such a large amplitude are stable under the perturbation (5.4).

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

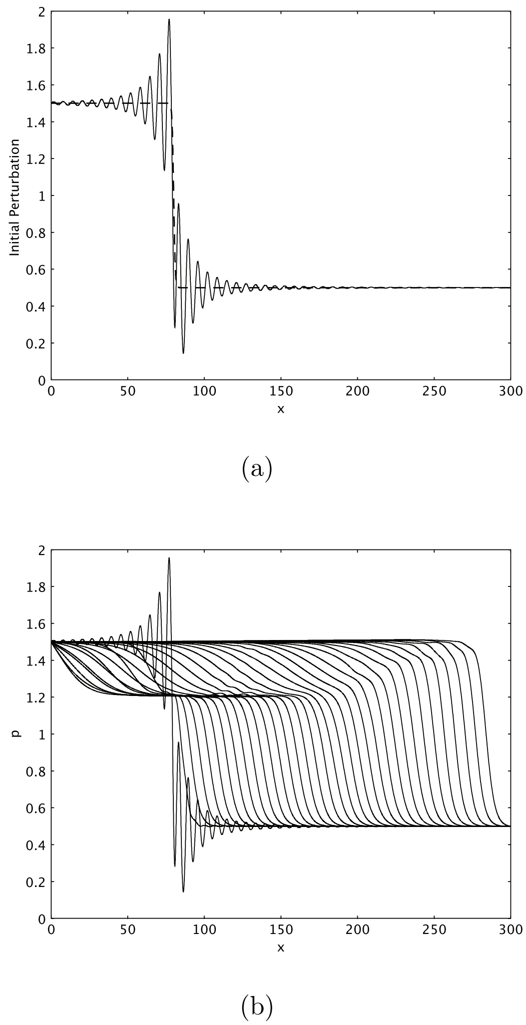


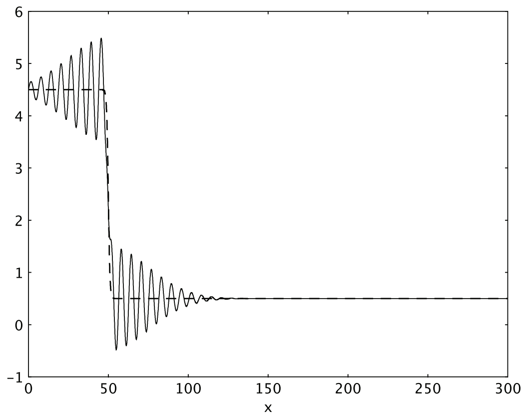
FIG. 2. *Numerical simulations of stability of traveling waves of* (1.1) *with small perturbations* (5.3). *The domain is* $[0, 300]$. (a) *The plot of initial perturbation (solid curve) around the initial traveling wave (dashed curve) of cell density* p*, where* $\gamma = \nu = 1$*,* $D = 4$*,* $\xi = 80$. (b) *The evolution of the solution subject to the perturbation. The wave propagates from left to right.*

In Figure 4, we present a numerical observation that the traveling waves are stable under large perturbations, which is not, however, proved analytically in the paper. Particularly, the traveling waves are stable when the perturbations are larger than the amplitudes of the waves. In our simulation, the amplitude of the wave, as plotted in Figure 4(a), is chosen as 1, and the initial perturbations are given by

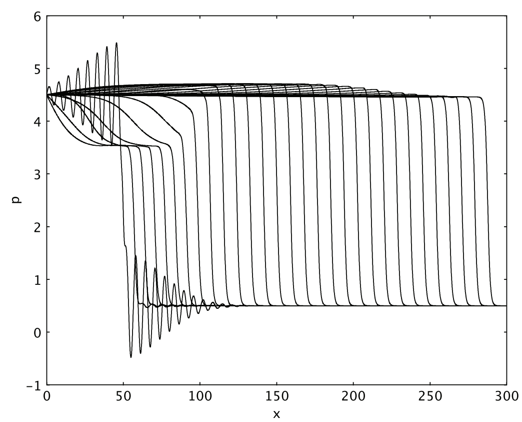
$$(5.5)\quad (p_0 - \bar{P})(x) = (q_0 - \bar{Q})(x) = 1.2 \sin x \exp(-0.0001(x-80)^2)/(((x-80)/10)^2+1)$$

so that the perturbation is in $H^1(R) \times H^1(R)$ and the amplitude of the perturbation is 2.4, which is larger than the amplitude of the wave, where $\gamma = \nu = 1$, $D = 4$, $\xi = 80$. With the simulations shown in Figure 4(b), we observe that the solution subject to a large perturbation such as (5.5) converges to a traveling wave as time evolves.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.



(a)

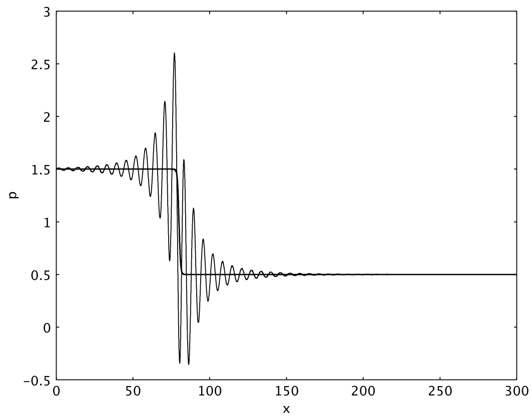


(b)

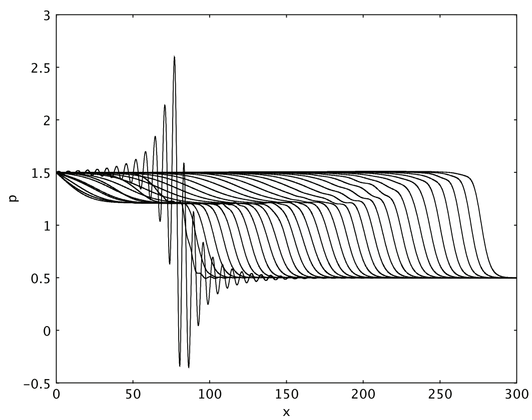
FIG. 3. *An illustration of the stability of traveling waves of* (1.1) *with large amplitudes under small perturbations* (5.4)*, where* $\gamma = \nu = 4$*,* $D = 4$*, and* $\xi = 50$*.* (a) *Initial perturbation (solid curve) about the traveling waves (dashed curve) in cell density* p*, as given in* (5.4)*.* (b) *The evolution of the solution under the perturbation. The wave propagates from left to right.*

6. Conclusion. We established the nonlinear stability of traveling waves of arbitrary amplitudes to a hyperbolic-parabolic system modeling repulsive chemotaxis. For the system (1.1) with physical viscosity, based on the L^2 energy methods, we proved that the traveling waves with arbitrary amplitudes are stable subject to small perturbation. Our result is distinctively different from the previous related results [7, 8], where strengths of the waves were assumed to be small. Particularly, we improved the results in [16], where the existence of global solutions of (1.1) was established under the assumption that the initial data are small. Moreover, we performed the numerical experiments to complement our theoretical results. We found numerically that the traveling waves are also stable under large perturbations, which is not, however, shown in the present paper analytically and remains open.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.



(a)



(b)

FIG. 4. *Numerical simulations of the stability of traveling waves of* (1.1) *under large perturbations, where* $\gamma = \nu = 1$*,* $D = 4$*, and* $\xi = 80$*.* (a) *The plot of the initial perturbation (solid curve) of cell density as given in* (5.5)*. The amplitude of the perturbation is larger than the amplitude of the traveling wave (dashed curve).* (b) *The evolution of the solution under this large perturbation. The wave propagates from left to right.*

The combination of numerical results in [6] and analytical analysis in [13] showed that when the chemotactic effect dominates the dissipative effect, i.e., when D is small, the sharp transition (shocks) of cell density will result. The cell density then undergoes drastic changes. The results in this paper further indicated that, when the dissipative effect is not negligible, the cell density will approach a smooth steady state profile asymptotically if the chemotaxis is repulsive.

In this paper we did not address the stability of traveling waves in the case of attractive chemotaxis, which was not studied in [16]. It appears that the L^2 energy method used in this paper does not directly apply to the case of attractive chemotaxis. Biologically, attractive and repulsive chemotaxes use very distinctive mechanisms (see [1]) for migrations and this may result in different phenomena [2, 9].

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

We will explore the stability and instability of the traveling waves for the attractive case in a forthcoming paper.

Acknowledgments. The authors would like to express their thanks for the hospitality and support of the Institute for Mathematics and Its Applications (IMA), University of Minnesota, where this work was initiated, and would like to thank the referees for their valuable comments towards the improvement of the paper.

REFERENCES

[1] M. Eisenbach, *Chemotaxis*, Imperial College Press, London, 2004.
[2] S. Gueron and N. Liron, *A model of herd grazing as a traveling wave: Chemotaxis and stability*, J. Math. Biol., 27 (1989), pp. 595–608.
[3] D. Horstmann, *From* 1970 *until present: The Keller-Segel model in chemotaxis and its consequences:* I, Jahresber. Deutsch. Math.-Verein., 105 (2003), pp. 103–165.
[4] S. Kawashima and A. Matsumura, *Stability of shock profiles in viscoelasticity with non-convex constitutive relations*, Comm. Pure Appl. Math., 47 (1994), pp. 1547–1569.
[5] P. Lax, *Hyperbolic systems of conservation laws.* II, Comm. Pure Appl. Math., 10 (1957), pp. 537–566.
[6] H. A. Levine and B. D. Sleeman, *A system of reaction diffusion equations arising in the theory of reinforced random walks*, SIAM J. Appl. Math., 57 (1997), pp. 683–730.
[7] T. Li, *Stability of traveling waves in quasi-linear hyperbolic systems with relaxation and diffusion*, SIAM J. Math. Anal., 40 (2008), pp. 1058–1075.
[8] A. Matsumura and K. Nishihara, *On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math., 2 (1985), pp. 17–25.
[9] T. Nagai and T. Ikeda, *Traveling waves in a chemotaxis model*, J. Math. Biol., 30 (1991), pp. 169–184.
[10] T. Nishida, *Nonlinear Hyperbolic Equations and Related Topics in Fluid Dynamics*, Publications Mathématiques d'Orsay 78-02, Département de Mathématique, Université de Paris-Sud, Orsay, France, 1978.
[11] H. G. Othmer and A. Stevens, *Aggregation, blowup, and collapse: The ABC's of taxis in reinforced random walks*, SIAM J. Appl. Math., 57 (1997), pp. 1044–1081.
[12] B. D. Sleeman, M. J. Ward, and J. C. Wei, *The existence and stability of spike patterns in a chemotaxis model*, SIAM J. Appl. Math., 65 (2005), pp. 790–817.
[13] Z. A. Wang and T. Hillen, *Shock formation in a chemotaxis model*, Math. Methods Appl. Sci., 31 (2008), pp. 45–70.
[14] Y. Yang, H. Chen, and W. Liu, *On existence of global solutions and blow-up to a system of the reaction-diffusion equations modelling chemotaxis*, SIAM J. Math. Anal., 33 (2001), pp. 763–785.
[15] Y. Yang, H. Chen, W. Liu, and B. D. Sleeman, *The solvability of some chemotaxis systems*, J. Differential Equations, 212 (2005), pp. 432–451.
[16] M. Zhang and C. J. Zhu, *Global existence of solutions to a hyperbolic-parabolic system*, Proc. Amer. Math. Soc., 135 (2007), pp. 1017–1027.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.