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Shock formation in a chemotaxis model

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SUMMARY

In this paper, we establish the existence of shock solutions for a simplified version of the Othmer–Stevens chemotaxis model (*SIAM J. Appl. Math.* 1997; **57**:1044–1081). The existence of these shock solutions was suggested by Levine and Sleeman (*SIAM J. Appl. Math.* 1997; **57**:683–730). Here, we consider the general Riemann problem and derive the shock curves in parameterized forms. By studying the travelling wave solutions, we examine the shock structure for the chemotaxis model and prove that the travelling wave speed is identical to the shock speed. Moreover, we explicitly derive an entropy–entropy flux pair to prove the uniqueness of the weak shock solutions. Some discussion is given for further study. Copyright © 2007 John Wiley & Sons, Ltd.

KEY WORDS: chemotaxis; shock solutions; travelling wave; entropy–entropy flux; shock structure; entropy condition

1. INTRODUCTION

In many biological systems, an organism navigates in response to a diffusible or otherwise transported signal. In its simplest form, this can be modelled by diffusion equations with advection terms of the form first derived by Patlak [1]. However, other systems are more accurately modelled by random walkers that deposit a non-diffusible signal that modifies the local environment for succeeding passages and there is little or no transport of the modifying substance. Examples include myxobacteria which produce slime over which their cohorts can move more readily, and ants, which follow trails left by predecessors. In either case, the question arises as to whether aggregation is possible with such strictly local modification or whether some form of longer-range communication is necessary. To answer this question, Othmer and Stevens [2] have developed a number of mathematical chemotaxis models. They illustrate that within the framework of partial

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differential equation models, stable aggregations can occur with local modulation of the transition rates, that is, without long-range signalling *via* a diffusible chemical. One of these chemotaxis models in one-space dimension reads

$$\begin{cases} \dfrac{\partial p}{\partial t} = D\dfrac{\partial}{\partial x}\left(p\dfrac{\partial}{\partial x}\left(\ln\left(\dfrac{p}{\phi(w)}\right)\right)\right) \\ \dfrac{\partial w}{\partial t} = R(p, w) \end{cases} \tag{1}$$

with no-flux boundary condition

$$\frac{\partial}{\partial x}\left(\ln\left(\frac{p}{\phi(w)}\right)\right) = 0 \quad \text{at } x = 0, l \tag{2}$$

as well as initial conditions

$$p(x, 0) = p_0(x) \geqslant 0, \quad w(x, 0) = w_0(x) > 0 \quad \text{for } 0 \leqslant x \leqslant l \tag{3}$$

Here $p(x, t)$ is the particle density of a particular species and $w(x, t)$ is the concentration of that active agent. The chemotactic potential ψ and signal reproduction and decay term R are given as

$$\phi(w) = \left(\frac{w+\beta}{w+\gamma}\right)^{\alpha}, \quad R(p, w) = \frac{\lambda p w}{k_1 + w} + \frac{\gamma_r p}{k_2 + p} - \mu w \tag{4}$$

where $\beta, \gamma, k, k_1, k_2, \lambda, \gamma_r, \mu$ and D are all non-negative constants with D and λ being strictly positive and $\alpha \neq 0$.

Othmer and Stevens [2] numerically show that a variety of dynamics of system (1)–(4) are possible, which include aggregation, blow-up or collapse depending on whether the dynamics admit stable bounded peaks, whether solutions blow up in finite time, or whether a suitable spatial norm of the density function is asymptotically less than its initial value. Levine and Sleeman [3] present the analytical results that support the numerical observations presented by Othmer and Stevens [2]. Furthermore, some additional numerical computations are made in [3]. Local and global existence of solutions of the Othmer–Stevens model (1)–(4) has been studied in [4] and in a recent paper [5]. In [5], the authors apply the existence theory of Ladyžhenskaya *et al*. [6] to obtain a very general result on local and global existence of solutions. In [7], asymptotic expansions are used to prove the existence and stability of spike solutions for the case of saturation in the signal production term.

It should be pointed out that since the first equation of (1) is parabolic in p, it is easy to observe that $p(x, t) \geqslant 0$ provided that the initial value is non-negative. To simplify model (1) and gain some insight into the Othmer–Stevens model, it is worthwhile to consider special cases which were considered in [3]. The results we obtain in this paper are for a simplified version of the Othmer–Steven model. To simplify Equations (1), we first apply the representation of $\phi(w)$ in (4) to deduce that from the first equation of (1):

$$\frac{\partial p}{\partial t} = D\left[\frac{\partial^2 p}{\partial x^2} - \frac{\partial}{\partial x}\left(\frac{\alpha(\gamma-\beta)p}{(w+\gamma)(w+\beta)}\frac{\partial w}{\partial x}\right)\right]$$

From this expression, we observe that if $\gamma \gg w \gg \beta$, the coefficient of w_x is nearly α/w, whereas if $\beta \gg w \gg \gamma$, the coefficient is $-\alpha/w$. These two extreme cases can be modelled by taking $\phi(w) = w^{-\alpha}$

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where α can be positive or negative. Throughout this paper, we consider $\gamma_r = 0$, $\phi(w) = w^{-\alpha}$ and $R(p, w) = \lambda pw - \mu w$. Substituting these choices into system (1)–(4), we end up with the following simplified system:

$$\begin{cases} p_t = D\left(p_{xx} + \alpha\left(p\dfrac{w_x}{w}\right)_x\right), & 0<x<l, \ t>0 \\ w_t = \lambda pw - \mu w \end{cases} \tag{5}$$

with boundary condition

$$\alpha\frac{w_x}{w} + \frac{p_x}{p} = 0 \quad \text{for } x = 0, l, \ t>0 \tag{6}$$

and initial data

$$p(x, 0) = p_0(x) \geqslant 0, \quad w(x, 0) = w_0(x) > 0 \quad \text{for } 0 \leqslant x \leqslant l \tag{7}$$

Here the first equation of (5) becomes a classical Patlak–Keller–Segel type. The substance w is generally refereed to as attractant for $\alpha<0$ and repellent for $\alpha>0$.

Furthermore, with these simplifications, using scaling theory by writing $t = l^2\tau/(\pi^2 D)$, $x = x'l/\pi$ and setting $\mu' = l^2\mu/(\pi^2 D)$, $\lambda' = l^2\lambda/(\pi^2 D)$, we find we may take $D = 1$ in (5). If we multiply the first equation of (5) by λ we observe that we may replace p by $p' = \lambda p$. Moreover, if we define $w' = w\exp(\mu t)$, we see that we may take $\mu = 0$ in (5) if replacing w by w'. After these rescalings, we can recast system (5)–(7) to the following initial-boundary problem by dropping the prime for convenience:

$$\begin{cases} p_t = p_{xx} + \alpha\left(p\dfrac{w_x}{w}\right)_x, & (x, t) \in (0, l) \times (0, \infty) \\ w_t = pw \end{cases} \tag{8}$$

with boundary condition

$$\alpha\frac{w_x}{w} + \frac{p_x}{p} = 0 \quad \text{for } x = 0, l, \ t>0 \tag{9}$$

and initial data

$$p(x, 0) = p_0(x) \geqslant 0, \quad w(x, 0) = w_0(x) > 0 \quad \text{for } 0 \leqslant x \leqslant l \tag{10}$$

For the hodograph analysis of system (8)–(10), we follow the argument applied in [3]. From the second equation of (8), it follows that $w(x, t)>0$ since $w_0(x)>0$ as long as the solution (p, w) exists in time. So it makes sense to let $\psi(x, t) = \ln w(x, t)$ and consequently $\psi_x = w_x/w$. Moreover, it follows from the second equation of (8) that $\psi_t = w_t/w = p$. We therefore obtain the following form from (8)–(10):

$$\begin{cases} \pounds\psi = \psi_{tt} - \alpha\psi_x\psi_{xt} - \alpha\psi_t\psi_{xx} = \psi_{xxt}, & (x, t) \in (0, l) \times (0, \infty) \\ \alpha\psi_x\psi_t + \psi_{xt} = 0 \quad \text{for } x = 0, l, \ t>0 \\ \psi(x, 0) = \psi_0(x) = \ln w_0(x) \quad \text{for } 0 \leqslant x \leqslant l \\ \psi_t(x, 0) = p_0(x) \quad \text{for } 0 \leqslant x \leqslant l \end{cases} \tag{11}$$

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The operator $\pounds$ defined by the first equation of (11) is a quasilinear second-order differential operator. The damping term ψ_{xxt} here does not really affect the overall structure of the solution. So we can specify the type of the operator $\pounds$ by determining the sign of the discriminant

$$\triangle = \alpha^2\psi_x^2(x,t) + 4\alpha\psi_t(x,t)$$

at a point (x,t). The operator $\pounds$ will be hyperbolic at the point (x,t) on a function ψ if $\triangle>0$, while elliptic if $\triangle<0$. When $\triangle=0$, we say $\pounds$ is parabolic. Since we have that $p(x,t)=\psi_t(x,t)>0$, it follows that $\triangle>0$ if $\alpha=1$ (or $\alpha>0$) and we refer to this case as hyperbolic. When $\alpha=-1$ (or $\alpha<0$), the sign of the discriminant can change and we refer to this case as mixed-type case. The hodograph plane was sketched in [8, p. 687].

When $\alpha=1$, Levine and Sleeman [3] construct solution pairs (p,w) for which $p>0$ and p collapses to a constant in finite time exponentially. When $\alpha=-1$, they show that there are solution pairs (p,w) for which $p>0$ but for which p blows up on the parabolic boundary in the 'hodograph' in finite time and the power spectrum converges to that of delta function in finite time. Furthermore, they construct an explicit family of such solutions (see Section 3 of [3]). Moreover, they conjecture that shock solutions can be obtained when $\alpha=-1$ (or $\alpha<0$). But they did not provide the rigorous justification for this contention. One of the purposes of this paper is to present the analytical justification for their assertion.

This paper is primarily concerned with system (5)–(7). The organization of the rest of this paper is as follows. In Section 2, we show that for both attractive case ($\alpha<0$) and repulsive case ($\alpha>0$), there exist shock solutions for the chemotaxis model (5)–(7). We start with the Rankine–Hugonoit condition to explicitly find the shock curves in parameterized forms that connect left and right states through a shock solution. Furthermore, we observe the difference between the attractive case and the repulsive case and plot the Hugoniot locus for both cases. In addition, we briefly discuss the general Riemann problem for system (5)–(7). The shock structures will be examined in Section 3 by studying the travelling waves to system (5)–(7) for small $D>0$. We show the existence of non-decreasing travelling waves for the attractive case and non-increasing travelling waves for the repulsive case. Essentially, we prove the travelling speed is identical to the shock speed. Numerically we confirm the existence of the travelling wave solutions. In Section 4, the entropy condition for the repulsive case ($\alpha>0$) is identified and the uniqueness of the shock solutions follows. In Section 5, we provide some discussion for further research.

2. SHOCK SOLUTIONS

2.1. The Hugoniot locus and existence of shocks

As we seek shock solutions for the Othmer and Stevens model (5)–(7), we extend the spatial domain to be $I=R$. We consider two cases where α takes different sign.

Case 1: *Attractive case* ($\alpha=-1/D<0$). Following this condition, applying the same scaling technique used in the introduction, we reformulate system (5) to the following equivalent equations:

$$\begin{cases} p_t = Dp_{xx} - \left(p\dfrac{w_x}{w}\right)_x, & (x,t)\in I\times(0,\infty) \\ w_t = pw \end{cases} \tag{12}$$

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We define $q=(\ln w)_x$ and reformulate system (12) to obtain the following form:

$$\begin{cases} p_t + pq_x + qp_x = Dp_{xx} \\ q_t = p_x \end{cases} \tag{13}$$

Let $u=(p,q)^{\mathrm{T}}$ and

$$A(u)=\begin{pmatrix} q & p \\ -1 & 0 \end{pmatrix}, \quad \bar{D}=\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

Then system (13) becomes

$$u_t + f(u)_x = u_t + A(u)u_x = \bar{D}u_{xx} \tag{14}$$

where $f(u)=f(p,q)=(pq,-p)^{\mathrm{T}}$. To study the shock formation of system (14), we let $D=0$ such that system (14) becomes the following conservation law:

$$u_t + A(u)u_x = 0, \quad (x,t)\in I\times(0,\infty) \tag{15}$$

The characteristic equation of $A(u)$ is easily computed as $\lambda^2 - q\lambda + p = 0$. Thus when $q^2-4p>0$, the matrix $A(u)$ has two real distinct eigenvalues $\lambda_1(u)<\lambda_2(u)$ given by

$$\lambda_1(u)=\frac{q}{2}-\frac{\sqrt{q^2-4p}}{2} \quad \text{and} \quad \lambda_2(u)=\frac{q}{2}+\frac{\sqrt{q^2-4p}}{2}$$

with corresponding eigenvectors which are

$$r_1(u)=(-\lambda_1(u),1)^{\mathrm{T}} \quad \text{and} \quad r_2(u)=(\lambda_2(u),-1)^{\mathrm{T}}$$

respectively. This means the conservational law (15) is strictly hyperbolic for $q^2-4p>0$. Furthermore, it is straightforward to obtain that $\nabla\lambda_1(u)\cdot r_1(u) = -q/\sqrt{q^2-4p}+1 \neq 0$ as well as $\nabla\lambda_2(u)\cdot r_2(u)=q/\sqrt{q^2-4p}+1\neq 0$ and $\lambda_1(u)<\lambda_2(u)$ due to $p>0$. Hence, the characteristic fields $(\lambda_1(u), r_1(u))$ and $(\lambda_2(u), r_2(u))$ are genuinely nonlinear which motivates us to look for shock solutions for system (15). To investigate the shock solution, we augment system (15) with Riemann initial value:

$$u(x,0)=u_0(x)=(p_0(x),q_0(x))=\begin{cases} u^-, & x<0 \\ u^+, & x>0 \end{cases} \tag{16}$$

where $u^-=(p^-,q^-)$, $u^+=(p^+,q^+)$. We suppose here that $u^+\neq u^-$. Otherwise the characteristic speeds are constant $\lambda_i(u)=\lambda_i(u^-)=\lambda_i(u^+)$ and therefore $\nabla\lambda_i(u)=0$. This is the case of linear degeneracy in which the shock wave and rarefaction wave coincide with each other and we refer to this situation as a contact discontinuity (see [9]). In this work, we restrict our attention to the case of $u^+\neq u^-$.

Recall that if a discontinuity propagating with speed s has constant u^- and u^+ on either side of the discontinuity, then the Rankine–Hugoniot jump condition must hold

$$f(u^+) - f(u^-) = s(u^+ - u^-) \tag{17}$$

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Now let us fix a state u^- and attempt to determine the set of states u^+ that can be connected to u^- by a discontinuity satisfying (17) for some s. To this end, we rewrite the Rankine–Hugoniot condition (17) as

$$\begin{cases} s(p^+ - p^-) = p^+q^+ - p^-q^- \\ s(q^+ - q^-) = -p^+ + p^- \end{cases} \tag{18}$$

Observe that system (18) gives a system of two equations in three unknowns: p^+, q^+ and s. This enables us to expect a one parameter family of solutions. Here we take q^+ as the free parameter. Then, it follows from the second equation of (18) that

$$p^+ = p^- - s(q^+ - q^-) \tag{19}$$

Substituting (19) into the first equation of (18) yields

$$-s^2(q^+ - q^-) = p^+q^+ - p^-q^- \tag{20}$$

Applying (19) into (20) gives that

$$(q^- - q^+)(s^2 - q^+s + p^-) = 0 \tag{21}$$

It is worthwhile to note here that $q^+ \neq q^-$. Otherwise, it follows from the second equation of (18) that $p^+ = p^-$ which implies that $u^+ = u^-$ and violates our assumption. Therefore, we end up with

$$s^2 - q^+s + p^- = 0$$

and get the shock speed

$$s = \frac{q^+}{2} \pm \frac{\sqrt{q^{+2} - 4p^-}}{2} \tag{22}$$

here we have assumed that $(q^+)^2 - 4p^- > 0$. As a consequence we obtain p^+ from (19)

$$p^+ = p^- - \frac{1}{2}\left(q^+ \pm \sqrt{q^{+2} - 4p^-}\right)(q^+ - q^-) \tag{23}$$

where $\pm$ signs in these equations give two solutions, one for each family. Since p^+ and s can be expressed in terms of q^+, we can parameterize these curves by taking

$$q^+ = (1+\sigma)q^- \tag{24}$$

where σ is a parameter. Therefore given $u^- \in \Omega$, we obtain the shock curves for the first characteristic fields which are parameterized by

$$S_1(\sigma, u^-) = u^- + \sigma \begin{pmatrix} -\frac{q^-}{2}\left[(1+\sigma)q^- - \sqrt{(1+\sigma)^2 q^{-2} - 4p^-}\right] \\ q^- \end{pmatrix}$$

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with shock speed

$$s_1(\sigma, u^-) = \frac{1+\sigma}{2}q^- - \frac{1}{2}\sqrt{(1+\sigma)^2 q^{-2} - 4p^-}$$

The shock curves for the second characteristic field is

$$S_2(\sigma, u^-) = u^- + \sigma \begin{pmatrix} -\dfrac{q^-}{2}\left[(1+\sigma)q^- + \sqrt{(1+\sigma)^2 q^{-2} - 4p^-}\right] \\ q^- \end{pmatrix}$$

with shock speed

$$s_2(\sigma, u^-) = \frac{1+\sigma}{2}q^- + \frac{1}{2}\sqrt{(1+\sigma)^2 q^{-2} - 4p^-}$$

Here we write $S_i(\sigma, u^-) = u_i^+(\sigma, u^-)$, $i = 1, 2$ and $u_i^+(\sigma, u^-)$ denote the solutions corresponding to the shock speed s_i. We thus obtain two shock curves through any point u^-, one for each characteristic family. By denoting the corresponding shock speed by $s_i(\sigma, u^-)$, we parameterize these curves by $S_i(\sigma, u^-)$ with $S_i(0, u^-) = u^-$. To make notations simpler, we will frequently substitute $S_i(\sigma)$ for $S_i(\sigma, u^-)$ and $s_i(\sigma)$ for $s_i(\sigma, u^-)$ when the point u^- is clearly understood. Replacing u^+, s by $S_i(\sigma), s_i(\sigma)$, respectively, in the Rankine–Hugoniot condition (17), we find that

$$f(S_i(\sigma)) - f(u^-) = s_i(\sigma)(S_i(\sigma) - u^-) \tag{25}$$

Differentiating the expression (25) with respect to σ and evaluating at $\sigma = 0$ yields

$$f'(u^-)S_i'(0) = s_i(0)S_i'(0) \tag{26}$$

so that $S_i'(0)$ must be a scalar multiple of the eigenvector $r_i(u^-)$ of $f'(u^-)$ since here the speed $s_i(0)$ coincides with the corresponding characteristic speed, i.e. $s_i(0) = \lambda_i(u^-)$. However, with the above notations, it is evident to check that

$$\frac{\partial}{\partial\sigma}S_i(0, u^-) = q^- r_i(u^-) \propto r_i(u^-), \quad s_i(0, u^-) = \lambda_i(u^-), \ i = 1, 2 \tag{27}$$

as required.

Now let us examine the conditions which the parameter σ needs to satisfy. For the shock curves $S_i(\sigma, u^-)(i = 1, 2)$ to be well defined, it is required that $(1+\sigma)^2 q^{-2} - 4p^- \geqslant 0$. But it has been required that $q^{-2} - 4p^- > 0$ to get a strictly hyperbolic conservation law which was discussed at the beginning of this section. Hence, we require that $|1+\sigma| > 1$ which implies two situations: either $\sigma > 0$ or $\sigma < -2$. For $-2 < \sigma < 0$, system (5)–(7) is not strictly hyperbolic and the Hugoniot locus has a gap (see Figure 1(a)). Therefore, when $-2 < \sigma < 0$, the shock curves are not well defined and hence we only consider the domain $\sigma > 0$.

From the preceding construction, we obtain, by standard arguments (see [9, 10]), the existence of shock solutions of the Riemann problem (15) and (16) with left and right states u^- and u^+ for the attractive case. The existence theorem is summarized in the following and the Hugonoit locus of the attractive case is plotted in Figure 1(a).

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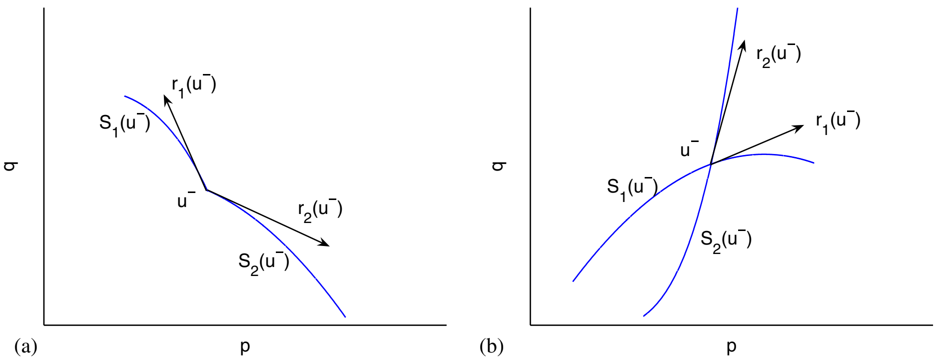

Figure 1. (a) Hugoniot locus for the left state $u^- = (2,3)$ for attractive case where $r_1(u^-) = (-1,1), r_2(u^-) = (2,-1)$ and (b) Hugoniot locus for the left state $u^- = (2,3)$ for repulsive case where $r_1(u^-) = \tilde{r}_1(u^-) = (3.56, 1), r_2(u^-) = \tilde{r}_2(u^-) = (0.56, 1), S_1(u^-) = \tilde{S}_1(u^-)$ and $S_2(u^-) = \tilde{S}_2(u^-)$.

Theorem 2.1
Let $\alpha = -1/D$ and Ω be an open set of $\mathbb{R}^2$. For each $u^- \in \Omega$ with $q^{-2} - 4p^- > 0$, there exists a parameter σ such that for each σ with $\sigma \geqslant 0$, the pair $(u^-, S_i(\sigma))$ $(i = 1, 2)$ satisfies the Rankine–Hugoniot conditions (25) and the function

$$u(x,t) = \begin{cases} u^- & \text{if } x < s_i(\sigma)t \\ S_i(\sigma) & \text{if } x > s_i(\sigma)t \end{cases} \tag{28}$$

is a weak solution of the system (15) which satisfies the Riemann condition (16) with $u^+ = S_i(\sigma)$.

Case 2: *Repulsive case* $(\alpha = 1/D > 0)$. Substituting $1/D$ for α in system (5) and rescaling the resulting system, we end up with

$$\begin{cases} p_t = Dp_{xx} + \left(p\dfrac{w_x}{w}\right)_x & (x,t) \in I \times (0,\infty) \\ w_t = pw \end{cases} \tag{29}$$

Similarly by defining $q = (\ln w)_x$, system (29) can be reduced to

$$\begin{cases} p_t - (pq)_x = Dp_{xx} \\ q_t - p_x = 0 \end{cases} \tag{30}$$

or written by

$$u_t + \tilde{A}(u)u_x = \bar{D}u_{xx} \tag{31}$$

where $u = (p,q)^{\mathrm{T}}$ and

$$\tilde{A}(u) = \begin{pmatrix} -q & -p \\ -1 & 0 \end{pmatrix}, \quad \bar{D} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

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Then system (31) for $D = 0$ becomes a conservation law

$$u_t + \tilde{f}(u)_x = u_t + \tilde{A}(u)u_x = 0, \quad (x, t) \in I \times (0, \infty) \tag{32}$$

where $\tilde{f}(u) = \tilde{f}(p, q) = (-pq, -p)^{\mathrm{T}}$ and $\tilde{f}'(u) = \tilde{A}(u)$. The characteristic equation of $\tilde{A}(u)$ is $\lambda^2 + q\lambda - p = 0$. Noticing $p>0$, it is clear that the discriminant $q^2 + 4p$ of the characteristic equation is always positive. Therefore the matrix $A(u)$ has two real distinct eigenvalues $\tilde{\lambda}_1(u)$ and $\tilde{\lambda}_2(u)$ which are given by

$$\tilde{\lambda}_1(u) = -\frac{q}{2} - \frac{\sqrt{q^2 + 4p}}{2} \quad \text{and} \quad \tilde{\lambda}_2(u) = -\frac{q}{2} + \frac{\sqrt{q^2 + 4p}}{2}$$

The corresponding eigenvectors are determined by

$$\tilde{r}_1(u) = (-\tilde{\lambda}_1(u), 1)^{\mathrm{T}} \quad \text{and} \quad \tilde{r}_2(u) = (\tilde{\lambda}_2(u), -1)^{\mathrm{T}}$$

respectively. It is obvious that $\tilde{\lambda}_1(u)<0<\tilde{\lambda}_2(u)$ which implies that the conservational law is strictly hyperbolic. Furthermore, we easily verify that $\nabla\tilde{\lambda}_1(u)\cdot\tilde{r}_1(u) = -q/\sqrt{q^2+4p} - 1<0$ and $\nabla\tilde{\lambda}_2(u)\cdot\tilde{r}_2(u) = q/\sqrt{q^2+4p} - 1<0$ because of $p>0$. Hence, the characteristic fields $(\tilde{\lambda}_i(u), \tilde{r}_1(u))$ and $(\tilde{\lambda}_2(u), \tilde{r}_2(u))$ are genuinely nonlinear. Then the Rankine–Hugoniot jump condition

$$\tilde{f}(u^+) - \tilde{f}(u^-) = \tilde{s}(u^+ - u^-)$$

takes the form

$$\begin{cases} \tilde{s}(p^+ - p^-) = -p^+q^+ + p^-q^- \\ \tilde{s}(q^+ - q^-) = -p^+ + p^- \end{cases} \tag{33}$$

System (33) consists of two equations in three unknowns: p^+, q^+ and $\tilde{s}$. We thus can regard one unknown, say q^+, as a parameter to get from the second equation of (33) that

$$p^+ = p^- - \tilde{s}(q^+ - q^-) \tag{34}$$

We substitute (34) into the first equation of (33) and obtain the following equation:

$$-\tilde{s}^2(q^+ - q^-) = -p^+q^+ + p^-q^- \tag{35}$$

Applying (34) into (35), we have

$$(q^+ - q^-)(\tilde{s}^2 + q^+\tilde{s} - p^-) = 0 \tag{36}$$

Note that $q^+ \neq q^-$. We obtain an equivalent equation to (36)

$$\tilde{s}^2 + q^+\tilde{s} - p^- = 0$$

and therefore the shock speed can be found

$$\tilde{s} = -\frac{q^+}{2} \pm \frac{\sqrt{q^{+2} + 4p^-}}{2} \tag{37}$$

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Then we substitute (37) into (34) and get

$$p^+ = p^- - \frac{1}{2}\left(-q^+ \pm \sqrt{q^{+2} + 4p^-}\right)(q^+ - q^-) \tag{38}$$

where the $\pm$ signs in these equations give two solutions, one for each family. Since p^+ and s can be expressed in terms of q^+, we can assume σ to be parameter and let

$$q^+ = (1+\sigma)q^- \tag{39}$$

Then we substitute (39) into (37) and (38) to get the shock curves. For the first characteristic field, we find the shock curves

$$\widetilde{S}_1(\sigma, u^-) = u^- + \sigma \left(\begin{array}{c} \dfrac{q^-}{2}\left[(1+\sigma)q^- + \sqrt{(1+\sigma)^2 q^{-2} + 4p^-}\right] \\ q^- \end{array} \right)$$

with shock speed

$$\tilde{s}_1(\sigma, u^-) = -\frac{1+\sigma}{2}q^- - \frac{1}{2}\sqrt{(1+\sigma)^2 q^{-2} + 4p^-}$$

For the second characteristic field, we find the shock curve

$$\widetilde{S}_2(\sigma, u^-) = u^- + \sigma \left(\begin{array}{c} \dfrac{q^-}{2}\left[(1+\sigma)q^- - \sqrt{(1+\sigma)^2 q^{-2} + 4p^-}\right] \\ q^- \end{array} \right)$$

with shock speed

$$\tilde{s}_2(\sigma, u^-) = -\frac{1+\sigma}{2}q^- + \frac{1}{2}\sqrt{(1+\sigma)^2 q^{-2} + 4p^-}$$

where we denote $\widetilde{S}_i(\sigma, u^-) = u_i^+(\sigma, u^-)$, $i = 1, 2$.

Performing the same analysis as we did for Case 1, we obtain the following theorem similar to Theorem 2.1.

Theorem 2.2
Assume $\alpha = 1/D$. For each $u^- \in \Omega$, there exists a parameter σ and $\sigma_0 > 0$ such that for each $\sigma \in [-\sigma_0, \sigma_0]$, the pair $(u^-, \widetilde{S}_i(\sigma))$ $(i = 1, 2)$ satisfies the Rankine–Hugoniot jump conditions and the function

$$u(t, x) = \begin{cases} u^- & \text{if } x < \tilde{s}_i(\sigma)t \\ \widetilde{S}_i(\sigma) & \text{if } x > \tilde{s}_i(\sigma)t \end{cases} \tag{40}$$

is a weak solution of the system (32) satisfying the Riemann condition (16) with $u^+ = S_i(\sigma)$.

Remark 2.3
We observe from the above analysis that there are shock solutions in both attractive and repulsive cases. In the first case, we need the additional assumptions $q^2 - 4p > 0$ and $q^{+2} - 4p^- > 0$ and hence $\sigma \geqslant 0$. However, in the second case, there was no such restriction on p, q and hence on σ

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to ensure the formation of the shock. Indeed, from the definition of S_i and $\widetilde{S}_i$ given above, it is evident that the real-valued solution $S_i(\sigma)$ $(i=1,2)$ exists only for $|1+\sigma|\geqslant 2\sqrt{p^-/q^-}$ while $\widetilde{S}_i(\sigma)$ takes real values for any σ.

Remark 2.4
From (27), it is clear that the Hugoniot locus $S_i(\sigma)$ is tangent to the eigenvector $r_i(u^-)$ at the point u^-. In a similar manner, the Hugoniot locus $\widetilde{S}_i(\sigma)$ is tangent to the eigenvector $\tilde{r}_i(u^-)$ at the point u^-. Then the Hugoniot locus of the state u^- for both attractive and repulsive cases can be sketched in Figure 1.

2.2. *General Riemann problem*

Next, we attempt to solve the Riemann problem graphically by drawing the Hugoniot locus for each states u^- and u^+ and looking for intersections. As illustrated in [11], we can accomplish this by finding an intermediate state u_m such that u^- and u_m are connected by a discontinuity satisfying the Rankine–Hugoniot condition and so for u_m and u^+.

Let us first examine the attractive case, i.e. the Riemann problem (15) and (16). Note that $\lambda_1(u)<\lambda_2(u)$ which requires the jump from u^- to u_m to travel more slowly than the jump from u_m to u^+. Precisely speaking, the u_m must be connected to u^- by a 1-shock S_1 while u^+ connected to u_m by a 2-shock S_2. We replace u^+ by u_m in (17) and go through the same calculation as we did in Case 1 to derive that the 1-shock connected to u_m has speed

$$s_1(\sigma,u_m)=\frac{1+\sigma}{2}q_m-\frac{1}{2}\sqrt{(1+\sigma)^2q_m^2-4p_m}<\frac{1+\sigma}{2}q_m$$

while 2-shock has speed

$$s_2(\sigma,u_m)=\frac{1+\sigma}{2}q_m+\frac{1}{2}\sqrt{(1+\sigma)^2q_m^2-4p_m}>\frac{1+\sigma}{2}q_m$$

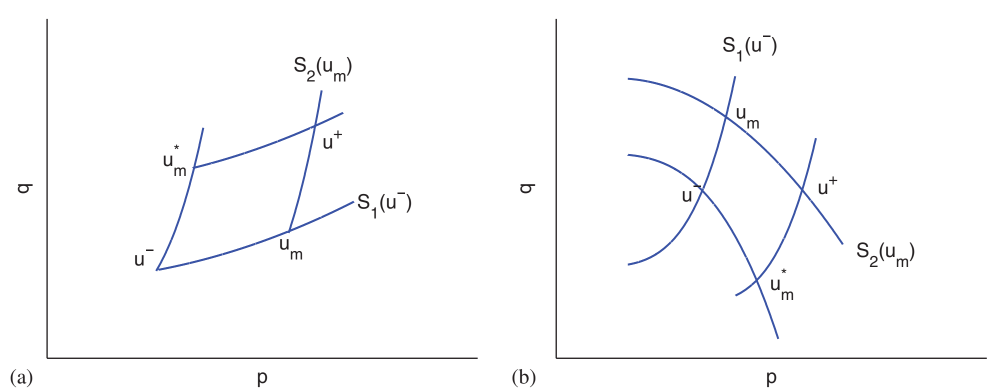

Figure 2. Construction of a shock wave for the general Riemann problem with left state u^- and right state u^+: (a) is for attractive case $\alpha=-1/D$ and (b) is for the repulsive case $\alpha=1/D$. Both (a) and (b) give two points of intersection, labelled u_m and u_m^*, but only u_m gives a single-valued solution to the Riemann problem since the requirement that the jump from u^- to u_m moves more slowly than the jump from u_m to u^+ due to $\lambda_1(u)<\lambda_2(u)$ and $\tilde{\lambda}_1(u)<\tilde{\lambda}_2(u)$.

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and consequently $s_1(\sigma, u_m) < s_2(\sigma, u_m)$ for all σ. In a similar fashion, it is straightforward to deduce that $\tilde{s}_1(\sigma, u_m) < \tilde{s}_2(\sigma, u_m)$ for the repulsive case. Figure 2 gives two points of intersection for each case, labelled u_m and u_m^*, but only u_m gives a single-valued solution to the general Riemann problem since we require the jump from u^- to u_m to travel more slowly than the jump from u_m to u^+ due to the convention $\lambda_1(u) < \lambda_2(u)$ and $\tilde{\lambda}_1(u) < \tilde{\lambda}_2(u)$.

3. TRAVELLING WAVE WITH SHOCK PROFILE

In this section, we will investigate the structure of the shock solution by considering the travelling wave for the problem

$$u_t + B(u)u_x = \bar{D}u_{xx} \tag{41}$$

where $B(u) = A(u)$ in the attractive case and $B(u) = \tilde{A}(u)$ in the repulsive case discussed in Section 2.

We define the travelling wave ansatz $u(x - ct) := u(z)$ with travelling speed c. In this paper, we restrict ourself to $c \geqslant 0$ since the shock speed s is non-negative and we shall prove that c is identical to s later to show the existence of a travelling wave with shock profile. But it turns out from our analysis that c can be negative and hence a standing wave ($c = 0$) is admitted if we ignore the biological relevance. Substituting the ansatz into Equation (41), one has that

$$(B(u) - cI_2)u' = \bar{D}u'' \tag{42}$$

where the prime means the differentiation with respect to variable z and the I_2 is the 2×2 identity matrix. Assume now that the left state u^- and the right state u^+ are given and satisfy

$$\lim_{z \to -\infty} u = u^-, \quad \lim_{z \to +\infty} u = u^+, \quad \lim_{z \to \pm\infty} u' = 0$$

Later we shall prove the travelling speed c coincides with the shock speed s, i.e. the travelling wave carries the shock profile $u(z) = u(x - st)$. Hence, if we define

$$\lim_{\bar{D} \to 0} u^{\bar{D}}(t, x) = \begin{cases} u^- & \text{if } x < st \\ u^+ & \text{if } x > st \end{cases}$$

the limit as $\bar{D} \to 0$ of solution to (42) then gives us a shock wave connecting the left state u^- and the right state u^+. The purpose of this section is to carefully study the form of s and travelling wave $u(z)$ to gain more detailed insight into the structure of the shock for positive but small D. Again, we consider two cases corresponding to the sign of α.

3.1. *Travelling wave for $\alpha < 0$*

In this subsection, we will study the travelling solution of system (41) for $\alpha < 0$. As we point out before, when $\alpha < 0$, $B(u) = A(u)$, where $A(u)$ is as defined in Case 1 in Section 2. We take up $A(u)$ and expand (42) to get

$$\begin{cases} qp' + pq' - cp' = Dp'' \\ -p' - cq' = 0 \end{cases} \tag{43}$$

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Introducing $v = p'$ and deducing from the second equation of (43) that $q' = -v/c$, we obtain equation $v' = (v/D)(q - p/c - c)$ from the first equation of (43). Coupling these equations gives rise to the following system:

$$\begin{cases} p' = v \\ q' = -\dfrac{v}{c} \\ v' = \dfrac{v}{D}\left(q - \dfrac{p}{c} - c\right) \end{cases} \tag{44}$$

Observe that p and q have an invariant of motion: $p' + cq' = 0$. Then $p + cq = \varrho_1$, where ϱ_1 is a constant determined by the left state $u^- = (p^-, q^-)$ and the right state $u^+ = (p^+, q^+)$, i.e.

$$\varrho_1 = p^- + cq^- = p^+ + cq^+ \tag{45}$$

which is an agreement with the identity (19) if $c = s$. Using the invariant of motion, system (44) is reduced to

$$\begin{cases} p' = v \\ v' = -\sigma v(p - \beta) \end{cases} \tag{46}$$

where $\sigma = 2/Dc$, $\beta = \varrho_1/2 - c^2/2$.

It is clear that system (46) has a continuum of steady states $(\theta, 0)$, where $\theta > 0$ due to the particle density $p > 0$. The corresponding community matrix about the steady state $(\theta, 0)$ is

$$J = \begin{bmatrix} 0 & 1 \\ 0 & \sigma(\beta - \theta) \end{bmatrix}$$

and hence the eigenvalues of J are

$$\lambda_1 = 0, \quad \lambda_2 = \sigma(\beta - \theta)$$

with corresponding eigenfunctions, respectively,

$$r_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\sigma(\theta - \beta) \end{bmatrix}$$

In the following, we shall study the existence of a travelling solution to nonlinear system (46) for fixed travelling speed. We give a class of equilibria in which two equilibria can be appropriately chosen to generate a non-negative heteroclinic orbit connecting the two equilibria. To this end, we first investigate the stability of the linearized system of (46).

Note that the eigenvector r_1 corresponding to zero eigenvalue λ_1 is in the direction of p axis $v = 0$ and every point $(\theta, 0)$ on the p axis is a steady state. To determine the stability of the linearized system, we only need to determine the sign of the second eigenvalue. Since $\sigma > 0$, we have the following relation:

$$\begin{aligned} \theta < \beta &\Rightarrow \lambda_2 > 0 \\ \theta = \beta &\Rightarrow \lambda_2 = 0 \\ \theta > \beta &\Rightarrow \lambda_2 < 0 \end{aligned} \tag{47}$$

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Therefore, $\theta=\beta$ is a critical point which separates the steady states into stable parts and unstable parts. So a heteroclinic connection is possible for the linearized system. Note that it has been mentioned in the Introduction that p, as the particle density, preserves the positivity. So $p>0$ and hence p^-, $p^+>0$. To have the biological relevance, we require that $\beta>0$ to obtain a real unstable manifold corresponding to λ_2. This requires that

$$c^2<\varrho_1=p^-+cq^- \tag{48}$$

In (48), we tacitly admit that $p^-+cq^->0$. Indeed from the definition of q, we know q can be negative and hence q^- and q^+ can be negative as the limits of q. Therefore, it gives an additional requirement

$$p^-+cq^->0 \tag{49}$$

Observe that inequality (49) holds true for all $q^-\geqslant 0$. We only worry about the case of $q^-<0$ which yields that from (49)

$$c<-\frac{p^-}{q^-} \tag{50}$$

Then using (50) and solving (48) give a maximum shock speed c^* such that

$$0\leqslant c<c^* \tag{51}$$

where

$$c^*=\begin{cases}\dfrac{q^-+\sqrt{(q^-)^2+4p^-}}{2} & \text{for } q^->0\\[2ex] \max\left\{-\dfrac{p^-}{q^-},\dfrac{q^-+\sqrt{(q^-)^2+4p^-}}{2}\right\} & \text{for } q^-<0\end{cases} \tag{52}$$

Then we can obtain a local stability theorem of linearization of system (46).

Lemma 3.1
Let the travelling wave speed c satisfies (51) and (52). Then $\beta>0$ and the steady state $(\theta,0)$ of the linearized system of (46) is stable for $\theta>\beta$ whereas unstable for $\theta<\beta$.

So far, we have obtained the stability of the lineariazation of the nonlinear system (46). But it is still not clear about the stability even local stability of the original nonlinear system (46) since there is a zero eigenvalue. To find an orbit connecting a stable manifold and an unstable manifold, we need to proceed to study the stability of system (46). We shall apply LaSalle's invariance principle [12, 13] through a Lyapunov function to prove the existence of a heteroclinic connection.

Since the p axis $v=0$ $(p>0)$ is a continuum of steady states, it splits the $p-v$ plane into two parts: $v>0$ and $v<0$. When $v>0$, $p'>0$ and hence p grows which requires that $p^-<p^+$. Analogously, $p^->p^+$ when $v<0$. We shall show that the monotonically decreasing travelling front wave does not exist. Whereas an increasing travelling front wave exists for $v>0$ and $p^-<p^+$ using a constructive approach. We first give the following result.

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Lemma 3.2
Let (51) and (52) be satisfied. Assume that $v \leqslant 0$ and $p^- > p^+$. Then system (46) is globally asymptotically stable and all flows of system (46), as $z \to +\infty$, converge to the following set:

$$\mathscr{L} = \{(p, v) |\ v = 0,\ p > \beta\}$$

and converge to the following set as $z \to -\infty$:

$$\mathscr{M} = \{(p, v) |\ v = 0, 0 < p < \beta\}$$

Proof
Define a function $V(p, v)$ by $V(p, v) = p$. Since $p(z) > 0$ for all z, then $V(p(z), v(z)) > 0$ and $dV/dz = p' = v \leqslant 0$ thanks to the first equation of (46). Therefore, the function $V(p, v)$ is a Lyapunov function of system (46) and hence system (46) is globally asymptotically stable. Furthermore, by LaSalle's invariance principle [12, 13], all solutions of system (45) will converge to the largest invariant set $\mathscr{M}$ which is contained in the set

$$\mathscr{L}_1 = \left\{ (p, v) \left| \frac{\mathrm{d}V}{\mathrm{d}z} = 0,\ \ p > 0, v \leqslant 0 \right. \right\} \tag{53}$$

From the first equation of (46), we have that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 0 \iff v = 0 \tag{54}$$

In addition, we know that $\lambda_2 > 0$ for all $0 < p < \beta$. Hence the manifold of system (46) corresponding to eigenvalue $\lambda_2 > 0$ is unstable and all orbits will leave the neighbourhood of the region $\mathscr{L}_2$ defined by

$$\mathscr{L}_2 = \{(p, v) |\ v \leqslant 0, 0 < p < \beta\}$$

On the other hand, from the above analysis, all trajectories will eventually enter into the set $\mathscr{L}_1$ defined in (53). Therefore, by (54), all orbits of system (46) converge to the set

$$\mathscr{L} = \mathscr{L}_1 - \mathscr{L}_2 = \{(p, v) |\ v = 0,\ p > \beta\}$$

Similarly, if we study the problem backward on variable z, we easily get the convergence to set $\mathscr{M}$ which completes the proof. □

Now we are in a position to state the non-existence theorem of decreasing travelling solutions for system (46).

Theorem 3.3
Let (51) and (52) be satisfied. Assume that $v \leqslant 0$. Then there is no travelling wave solution for system (46).

Proof (By contraction).
Assume that there exists a travelling wave solution (p, v) for system (46). Since $p' = v \leqslant 0$, the travelling wave p is non-increasing. So $p(-\infty) = p^- \geqslant p^+ = p(+\infty)$. By Lemma 3.2, it follows that $(p^+, 0) \in \mathscr{L}$ and $(p^-, 0) \in \mathscr{M}$. Thus $p^+ > \beta$ and $0 < p^- < \beta$ and hence $p^- < p^+$. This is contradictive. So system (46) has no travelling solutions. □

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Below we shall investigate the existence of travelling wave solutions of system (46) for $v>0$. We provide a constructive proof to show the existence of a travelling solution and sketch the phase portrait and numerically plot the travelling solution of system (46). To this end, we first write (46) as

$$\frac{\mathrm{d}v}{\mathrm{d}p} = -\sigma(p-\beta)$$

Integrating the equation gives rise to

$$v(p) = -\frac{\sigma}{2}p^2 + \sigma\beta p + \varrho_2 = -\frac{1}{Dc}p^2 + \frac{1}{Dc}(\varrho_1 - c^2)p + \varrho_2 \tag{55}$$

where ϱ_2 is constant of integration to be determined.

Noting that $p' = v$. Then

$$p' = -\frac{\sigma}{2}p^2 + \sigma\beta p + \varrho_2 \tag{56}$$

Solving Equation (56), one obtains the solution

$$p = a_2 + \frac{a_2 - a_1}{C_0 \exp\left(\frac{\sigma}{2}(a_2 - a_1)z\right) - 1} \tag{57}$$

where $a_1 = \beta - \sqrt{\beta^2 + 2\varrho_2/\sigma}$, $a_2 = \beta + \sqrt{\beta^2 + 2\varrho_2/\sigma}$.

Note that $a_2 - a_1 > 0$. Then, the limits of (57) are

$$p(-\infty) = a_1, \quad p(+\infty) = a_2 \tag{58}$$

By the boundary condition $p(-\infty) = p^-$, $p(+\infty) = p^+$, it follows that

$$a_1 = p^-, \quad a_2 = p^+ \tag{59}$$

Then, $p^- < p^+$ which is consistent with the fact $p' = v > 0$. Moreover, from the first equation of system (46), it follows that

$$v(\pm\infty) = \lim_{z\to\pm\infty} p' = 0$$

which implies that system (46) has a pulse wave in v. Therefore, applying (55), we have that

$$v(p^-) = v(p^+) = 0 \tag{60}$$

Recovering σ and β and applying (60) into (55) gives that

$$\varrho_2 = \frac{1}{Dc}(p^-)^2 - \frac{1}{Dc}(\varrho_1 - c^2)p^- \tag{61}$$

as well as

$$-\frac{1}{Dc}(p^-)^2 + \frac{1}{Dc}(\varrho_1 - c^2)p^- = -\frac{1}{Dc}(p^+)^2 + \frac{1}{Dc}(\varrho_1 - c^2)p^+ \tag{62}$$

Since we have $p^- < p < p^+$, then v as a quadratic of p (see (55)), is uniformly bounded. From (58), (59) and (60), we know there exists a non-negative heteroclinic orbit to system (46) connecting

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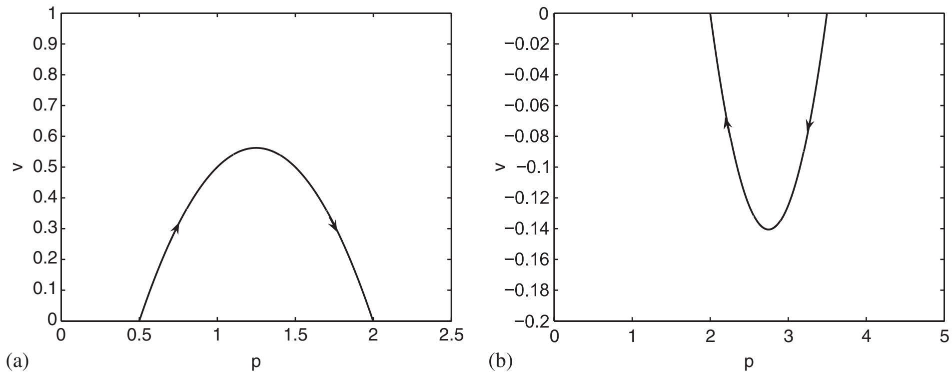

Figure 3. (a) A plot of phase portrait for system (46), where $c=1$, $D=1$ and hence $\sigma=2$. The value of $\beta=\varrho_1/2-c^2/2$ depends on the choice of ϱ_1, here we choose $\rho_1=3.5$ and then $\beta=1.25$. (b) A plot of phase portrait for system (68) with $\tilde{c}=4$, $D=1$ and $\tilde{\sigma}=0.5$. The value of $\tilde{\beta}=\tilde{\varrho}_1/2+\tilde{c}^2/2$ depends on the choice of $\tilde{\varrho}_1$, here we choose $\tilde{\varrho}_1=-10.5$ and hence $\tilde{\beta}=2.75$.

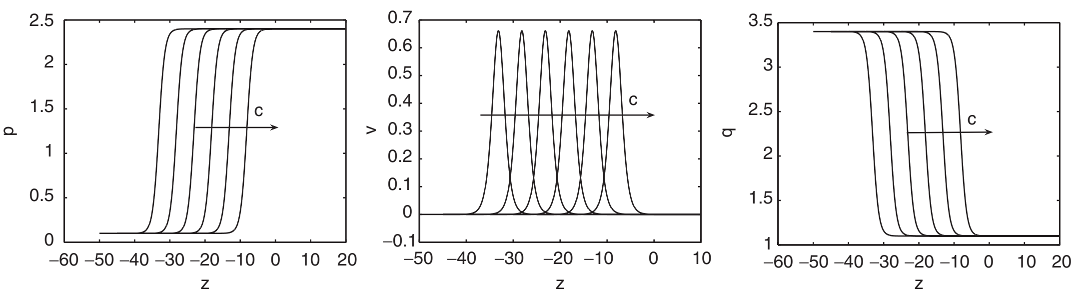

Figure 4. The travelling wave (p,q) determined by (63) for the case $\alpha=-1/D<0$, where we choose $s=1$, $D=2$, $p^-=0.5$, $q^-=3$ and time $t=0, 5, 10, 15, 20, 25$. The wave moves from left to right.

the left state $(p^-,0)$ and right state $(p^+,0)$. Given any one of end states, the other one can be determined by identity (62). The phase portrait of system (46) can be sketched by using (55) which gives rise to a parabola (see Figure 3(a)). The travelling solution of system (46) is numerically given in Figure 4, where we employed the ODE solver of Matlab to solve the equations.

By the above analysis, we obtain a travelling wave (p, v) for system (46). Utilizing the relation between v and q, we can connect the results to system (41) and obtain the following existence theorem for shock wave solutions to system (41).

Theorem 3.4
Let $\alpha=-1/D<0$, then there exists a non-decreasing travelling wave solution $u(z)=u(x-ct)$ for system (41), where s is the shock speed and $c=s$. The travelling wave $u(x-st)$ connects left state u^- and right state u^+ if and only if $u^+\in S_i(u^-)$, where $S_i(u^-)$ denotes the Hugoniot locus for left

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state u^-. Furthermore, the shock structure near $z=0$, i.e. $x=st$, is given by

$$\begin{cases} sq = \varrho_1 - p \\ p' = -\dfrac{1}{Ds}p^2 + \dfrac{1}{Ds}(\varrho_1 - s^2)p + \dfrac{1}{Ds}(p^-)^2 - \dfrac{1}{Ds}(\varrho_1 - s^2)p^- \end{cases} \tag{63}$$

where $\varrho_1 = p^- + sq^- = p^+ + cq^+$.

Proof
From the above analysis, it only remains to prove that the travelling wave speed c is identical to shock speed s. Note that $\varrho_1 = p^- + cq^-$. Feeding this expression into (62) yields that

$$c^2q^- - c^2q^+ - q^+q^- - q^{+2} + q^-p^- - q^+p^- = 0 \tag{64}$$

which is an agreement with the reformulated Rankine–Hugonoit jump condition (21). This implies that the travelling speed c and the left state $u^- = (p^-, q^-)$ as well as the right state $u^+ = (p^+, q^+)$ agree with the Rankine–Hugoniot jump condition (17). Hence $c = s$ and (63) is obtained directly from (46), (55) and (61). □

3.2. Travelling wave for $\alpha>0$

In this section, we shall consider the travelling wave solutions of system (41) for the repulsive case, which is an opposite case compared to the preceding subsection. Here we have $B(u) = \tilde{A}(u)$, where $\tilde{A}(u)$ is as defined in Section 2 for the repulsive case. In this subsection, many details will be omitted since they are analogous to the analysis of the preceding subsection. We denote the travelling speed by $\tilde{c}$ to distinguish it with the travelling speed c used for the case $\alpha<0$. Then, we use relation (42) to derive that

$$\begin{cases} -qp' - pq' - \tilde{c}p' = Dp'' \\ -p' - \tilde{c}q' = 0 \end{cases} \tag{65}$$

By the second equation of (65), we get that $p + \tilde{c}q = \tilde{\varrho}_1$ with a constant $\tilde{\varrho}_1$ determined by the two end states (p^-, q^-) and (p^+, q^+)

$$\tilde{\varrho}_1 = p^- + \tilde{c}q^- = p^+ + \tilde{c}q^+ \tag{66}$$

Then using the invariant of motion, we obtain from (65) that

$$Dp'' = \frac{2}{\tilde{c}}\left[p - \left(\frac{\tilde{\varrho}_1}{2} + \frac{\tilde{c}}{2}\right)\right]p' \tag{67}$$

Denoting $v = p'$, $\tilde{\sigma} = 2/D\tilde{c}$ and $\tilde{\beta} = \tilde{\varrho}_1/2 + \tilde{c}^2/2$, we convert (67) into a system

$$\begin{cases} p' = v \\ v' = \tilde{\sigma}v(p - \tilde{\beta}) \end{cases} \tag{68}$$

Clearly system (68) has a continuum of steady states $(\tilde{\theta}, 0)$ with $\tilde{\theta}>0$ and the eigenvalues of linearized system about equilibria $(\tilde{\theta}, 0)$ are

$$\tilde{\lambda}_1 = 0, \quad \tilde{\lambda}_2 = \sigma(\tilde{\theta} - \tilde{\beta})$$

We have the following observation due to $\tilde{\sigma}>0$:

$$\begin{aligned}
&\tilde{\theta}<\tilde{\beta}\Rightarrow\tilde{\lambda}_2<0\\
&\tilde{\theta}=\tilde{\beta}\Rightarrow\tilde{\lambda}_2=0\\
&\tilde{\theta}>\tilde{\beta}\Rightarrow\tilde{\lambda}_2>0
\end{aligned} \tag{69}$$

Due to the biological relevance, it is required that $\tilde{\beta}>0$ to obtain a heteroclinic connection. Then we have

$$\tilde{c}^2+\tilde{\varrho}_1=\tilde{c}^2+\tilde{c}q^+ + p^+>0 \tag{70}$$

Note that $p^+>0$. Hence (70) holds for all $\tilde{c}\geqslant 0$ if $q^+>0$. For $q^+<0$, by solving (70), we find a sufficient condition $(q^+)^2-4p^+<0$ to obtain a non-negative travelling speed $\tilde{c}$ satisfying (70). So we assume that

$$q^+<0, \quad (q^+)^2-4p^+<0 \tag{71}$$

and solve (70), to obtain that

$$\underline{c}\leqslant\tilde{c}\leqslant\overline{c} \tag{72}$$

where

$$\underline{c}=\frac{-q^+-\sqrt{(q^+)^2-4p^+}}{2}, \quad \overline{c}=\frac{-q^+ +\sqrt{(q^+)^2-4p^+}}{2} \tag{73}$$

Then by the very routine argument as we used in Section 3.1, we easily obtain the following results for the corresponding linearized system of (68).

Lemma 3.5
Let either $q^+>0, \tilde{c}\geqslant 0$ or (71)–(73) hold. Then the linearized system of (68) is locally stable for $\tilde{\theta}<\tilde{\beta}$ and unstable $\tilde{\theta}>\tilde{\beta}$.

Next, we study the stability of the nonlinear system (68). As before, we separate $p-v$ space into two regions: $v>0$ and $v\leqslant 0$. We first look at the case $v\leqslant 0$ and give the following theorem.

Lemma 3.6
Let either $q^+>0, \tilde{c}\geqslant 0$ or (71)–(73) hold. Assume that $v\leqslant 0$ and $p^->p^+$. Then system (68) is asymptotically stable and all flows of system (68), as $z\to+\infty$, converge to the following set:

$$\mathscr{V}=\{(p,v)|\ v=0, 0<p<\tilde{\beta}\}$$

and converge, as $z\to-\infty$, to the set

$$\mathscr{W}=\{(p,v)|\ v=0,\ p>\tilde{\beta}\}$$

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Proof

By introducing a function $\phi(p, v)$ by $\phi(p, v) = p$, we can easily prove that $\phi(p, v)$ is a Lyapunov function of system (68) and confirms the global stability. Applying LaSalle's invariant principle, we can obtain the asymptotic behaviour of the trajectories of system (68). The details are quite similar to the proof of Theorem 3.2 and hence omitted. □

With Lemma 3.6 in hand, we shall show the existence of a non-increasing travelling solution for system (68).

Lemma 3.7

Let the assumptions in Lemma 3.6 hold. Then there exists a uniformly bounded, negative heteroclinic orbit (for v) connecting equilibria $(e_1, 0)$ and equilibria $(e_2, 0)$ for system (68), where $e_1 < \tilde{\beta}$, $e_2 > \tilde{\beta}$. As a consequence, there exists a travelling pulse in v and a non-increasing travelling front in p.

Proof

According to Lemma 3.7, we only need to prove that the solution v as a function of p is bounded. To this end, we first write (68) as

$$\frac{\mathrm{d}v}{\mathrm{d}p} = \tilde{\sigma}(p - \tilde{\beta})$$

Integrating the equation gives rise to

$$v(p) = \frac{1}{D\tilde{c}} p^2 - \frac{1}{D\tilde{c}}(\tilde{\varrho}_1 + \tilde{c}^2) p + \tilde{\varrho}_2 \tag{74}$$

where $\tilde{\sigma}$ and $\tilde{\beta}$ has been recovered and $\tilde{\varrho}_2$ is a constant of integration which can be determined by the boundary conditions of p and q,

$$\tilde{\varrho}_2 = -\frac{1}{D\tilde{c}}(p^+)^2 + \frac{1}{D\tilde{c}}(\tilde{\varrho}_1 + \tilde{c}^2) p^+ \tag{75}$$

Since $p' = v \leqslant 0$, p is decreasing. By the boundary condition $p(-\infty) = p^-$, $p(+\infty) = p^+$, it follows that $p^+ \leqslant p \leqslant p^-$ and hence p is uniformly bounded. Therefore v, as a quadratic form of p, is uniformly bounded as well. This finishes the proof. □

Connecting the travelling wave solutions obtained above with the shock solution obtained in Section 2, we have the following existence theorem for a travelling wave with shock profile $u(x - \tilde{c}t)$, i.e. $\tilde{c} = \tilde{s}$, where $\tilde{s}$ is the shock speed discussed in Section 2 for case $\alpha > 0$..

Theorem 3.8

Let $\alpha = 1/D > 0$ and the assumptions in Lemma 3.7 hold, then there exists a non-increasing travelling wave solution $u(z) = u(x - \tilde{c}t)$ for system (41), where $\tilde{s}$ is the shock speed and $\tilde{c} = \tilde{s}$. The conditions that p^- and p^+ lie on the same parabola $v(p)$ given by (74) is identical to the Rankine–Hugoniot condition. The travelling wave connects left state u^- and right state u^+ if and only if $u^+ \in \tilde{S}_i(u^-)$, where $u^+ \in \tilde{S}_i(u^-)$ denotes the Hugoniot locus for left state u^- obtained in Section 2. Furthermore,

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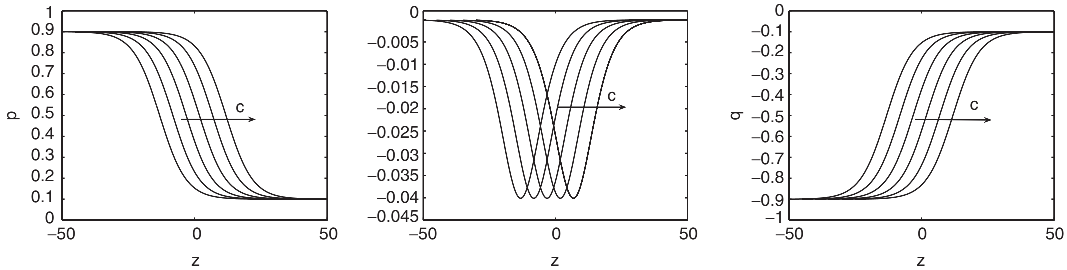

Figure 5. The travelling wave (p,q) determined by (76) for the case $\alpha = 1/D > 0$, where we choose $s=1$, $D=4$, $p^+=2$, $q^+=-2$ and time $t=0, 5, 10, 15, 20, 25$. The wave moves from left to right.

the shock structure near $z=0$, i.e. $x=\tilde{s}t$, is given by

$$\begin{cases} \tilde{s}q = \tilde{\varrho}_1 - p \\ p' = \dfrac{1}{D\tilde{s}}p^2 - \dfrac{1}{D\tilde{s}}(\tilde{\varrho}_1+\tilde{s}^2)p - \dfrac{1}{D\tilde{s}}(p^+)^2 + \dfrac{1}{D\tilde{s}}(\tilde{\varrho}_1+\tilde{s}^2)p^+ \end{cases} \tag{76}$$

where $\tilde{\varrho}_1 = p^- + \tilde{s}q^- = p^+ + \tilde{s}q^+$.

Proof
System (76) is obtained from (68), (74) and (75) directly if we replace $\tilde{c}$ by $\tilde{s}$. So by Lemma 3.7, it only remains to prove that the travelling speed $\tilde{c}$ is identical to the shock speed $\tilde{s}$. Since $v(p^-)=v(p^+)=0$, we have from (74) that

$$-\frac{1}{D\tilde{c}}(p^-)^2 + \frac{1}{D\tilde{c}}(\tilde{\varrho}_1+\tilde{c}^2)p^- = -\frac{1}{D\tilde{c}}(p^+)^2 + \frac{1}{D\tilde{c}}(\tilde{\varrho}_1+\tilde{c}^2)p^+ \tag{77}$$

Hence substituting $\tilde{\varrho}_1 = p^- + \tilde{c}q^-$ into (77) and cancelling D out yields that

$$\tilde{c}(q^+)^2 - p^-q^+ - \tilde{c}q^-q^+ + \tilde{c}^2q^+ + p^-q^- - \tilde{c}^2q^- = 0 \tag{78}$$

which is the same as the reformulated Ranking–Hugoniot jump condition (36). Hence $\tilde{c}=\tilde{s}$, as required. □

A phase portrait of system (68) is given in Figure 3(b) and a numerical travelling solution is plotted in Figure 5. In the remainder of this section, we will investigate the travelling solutions for the other case $v>0$. It turns out that travelling wave solutions do not exist for $v>0$. To prove this, we first derive from (68) that

$$p' = \tfrac{1}{2}\tilde{\sigma}p^2 - \tilde{\sigma}\tilde{\beta}p + \tilde{\varrho}_2 \tag{79}$$

Note that

$$\tilde{\beta}^2 - \frac{2\tilde{\varrho}_2}{\tilde{\sigma}} = \frac{1}{4}(\tilde{\varrho}_1 + \tilde{c}^2 - 2p^-)^2 \geqslant 0$$

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Then we solve Equation (79) to get that

$$p = \tilde{a}_2 + \frac{\tilde{a}_2 - \tilde{a}_1}{\tilde{C}_0 \exp\left(\dfrac{\tilde{\sigma}}{2}(\tilde{a}_1 - \tilde{a}_2)z\right) - 1} \tag{80}$$

where $\tilde{a}_1 = \tilde{\beta} - \sqrt{\tilde{\beta}^2 - 2\tilde{\varrho}_2/\tilde{\sigma}}$, $\tilde{a}_2 = \tilde{\beta} + \sqrt{\tilde{\beta}^2 - 2\tilde{\varrho}_2/\tilde{\sigma}}$.

It is clear that $\tilde{a}_1 - \tilde{a}_2 \leqslant 0$. Taking the limits for (80), we have that

$$\begin{aligned} p(z) &\to \tilde{a}_2 \quad \text{as } z \to -\infty \\ p(z) &\to \tilde{a}_1 \quad \text{as } z \to +\infty \end{aligned} \tag{81}$$

Using (81), we can show the non-existence of travelling solutions of system (68) for $v>0$.

Theorem 3.9
Assume that either $p^+>0, \tilde{c}\geqslant 0$ or (71)–(73) hold. Then there is no travelling wave solution to system (68) for $v>0$.

4. ENTROPY SOLUTION

As is well known, weak solutions of the Cauchy problem of a system of conservation laws are generally non-unique and a so-called 'entropy condition' is required to pick out the physical relevant viscosity solution [9]. One condition which picks a physical solution is that it should be the limiting solution of the viscous equation as the viscosity coefficient tends to zero [14]. Another approach to the 'entropy condition' is to define an entropy pair for which an additional conservation law holds for smooth solutions that becomes an inequality for discontinuous solutions. In this section, we are devoted to developing a convex entropy and an entropy flux pair (η, ρ) for the case of $\alpha = 1/D > 0$. Toward this end, we first rewrite the conservation law (32) in the form

$$\begin{cases} p_t - (pq)_x = 0 \\ q_t - p_x = 0 \end{cases} \tag{82}$$

or

$$u_t + \tilde{f}(u)_x = 0 \tag{83}$$

where $u = (p, q)$ and $\tilde{f}(u) = (-pq, -p)$.

From Definition 1.4 given in the Introduction, we know that the entropy pair (η, ρ) satisfies an additional conservation law for any smooth solution $u = (p, q)$ to system (82)

$$\eta(u)_t + \rho(u)_x = 0 \tag{84}$$

Substituting (83) into (84), we end up with

$$\eta'(u) \cdot f'(u) = \rho'(u) \tag{85}$$

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where $'$ denotes the derivative with respect to vector $u=(p,q)$. Expanding (85) gives the following relation:

$$\begin{cases} \rho_p = -\eta_q - q\eta_p \\ \rho_q = -p\eta_p \end{cases} \tag{86}$$

Eliminating η from (86) gives that

$$\eta_{qq} + q\eta_{pq} - p\eta_{pp} = 0 \tag{87}$$

We assume that the entropy $\eta(u)$ of the conservation law (83) has the following form:

$$\eta(p,q) = \tfrac{1}{2}q^2 + g(p) \tag{88}$$

where $g(p)$ is expected to be a convex function.

Substituting (88) into (87) yields that

$$1 - pg''(p) = 0 \tag{89}$$

Solving (89) gives

$$g(p) = p\ln p - p + k_1 p + k_2 \tag{90}$$

where k_1, k_2 are arbitrary constants.

Then substituting (88) and (90) into the first equation of (86) enables us to find $\rho(p,q)$ as

$$\rho(p,q) = -pq\ln p - k_1 pq + k_3 \tag{91}$$

where k_3 is an arbitrary constant.

If we particularly choose $k_1 = k_2 = k_3 = 0$, we obtain an entropy–entropy flux pair (η, ρ) which reads

$$\begin{cases} \eta(p,q) = \tfrac{1}{2}q^2 + p\ln p - p \\ \rho(p,q) = -pq\ln p \end{cases} \tag{92}$$

Accordingly, $g(p) = p\ln p - p$ and it is easy to verify that $g''(p) = 1/p > 0$ due to $p > 0$. As a consequence, the second derivative of $\eta(u)$ is a positive-definite quadratic form. That is $\eta(u)$ is a convex function.

The entropy $\eta(u)$ is conserved for smooth solutions of (83) by its definition. For discontinuous solutions (shock solutions), however, the manipulations performed above in general are not valid, i.e. $\eta(u)$ is not conserved. Since we are particularly interested in how the entropy behaves for the vanishing viscosity weak solution, we look at the related viscous problem

$$u_t + \tilde{f}(u)_x = \bar{D}u_{xx} \tag{93}$$

and let the viscosity coefficient $\bar{D}$ tends to zero.

Since the solutions of Equation (93) are always smooth, we can derive the corresponding evolution equation for the entropy following the same procedure applied for smooth solutions for the inviscid equation (83). Therefore, we multiply (93) by $\eta'(u)$ to obtain from (85) that

$$\eta(u)_t + \rho(u)_x = \bar{D}\eta'(u)u_{xx} \tag{94}$$

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That is

$$\eta(u)_t + \rho(u)_x = \bar{D}\eta(u)_{xx} - \bar{D}\eta''(u)u_x^2 \tag{95}$$

Applying a standard argument (e.g. in [14, pp. 604–606]), we end up with the following inequality:

$$\int_0^T \int_0^l \eta(u)_t + \rho(u)_x \,\mathrm{d}x\,\mathrm{d}t \leqslant 0 \tag{96}$$

The fact that inequality (96) holds for any l and T is summarized by saying that $\eta(u)_t + \rho(u)_x \leqslant 0$ almost everywhere. We are led to the following theorem.

Theorem 4.1 (*Entropy solution*)
Any solution (p, q) of (83) which is the limit of the viscosity equation (93) satisfies

$$\eta(p,q)_t + \rho(p,q)_x \leqslant 0 \tag{97}$$

in the weak sense, where $\eta(p,q)$ and $\rho(p,q)$ are given by (92).

5. DISCUSSION

In this work, we establish the existence of shock solutions for a simplified version of a chemotaxis model (1)–(4) for both attractive ($\alpha>0$) and repulsive ($\alpha<0$) cases. The shock curves are given in parameterized forms. The requirements on the choice of the parameter σ are different for the cases of $\alpha>0$ and $\alpha<0$. Moreover, we discuss the general Riemann problem. By studying the travelling wave of the system, we examine the shock structures of the corresponding shock waves for both cases. We prove that the travelling wave speed is identical to the shock speed. We show that, for the attractive case, there exists only non-decreasing shock travelling waves and for the repulsive case, there exists only non-increasing shock travelling waves. For the uniqueness of the weak solutions (shock solutions), we also find an entropy–entropy pair for the repulsive case. For the attractive case, we are unable to find the corresponding entropy–entropy pair.

When $\alpha=-1$ (or <0), it has been proven by Levine and Sleeman [3] that there are solutions (p, w) for which $p>0$ blows up in finite time and an explicit family of such blow-up solutions has been constructed in Section 3 of [3]. But there are no results available about the global existence or non-existence of solutions for $\alpha>0$. To show the global existence for model (5)–(7) is not an easy problem. Below we give a reformulation of the problem that leads to Dirichlet boundary conditions. We hope it can provide some useful clues for the global existence for the solution.

The boundary condition (2) or (6) seems like a Neumann boundary condition. But it is not the standard form of a Neumann boundary condition. We will first reformulate the form of boundary conditions. A direct calculation shows that the boundary condition (2) is weaker than the non-flux boundary condition $p_x(0,t)=p_x(l,t)=0$. However, for those solutions of the simplified problem (5)–(7) for which this stronger condition holds, one might be able to apply the argument in [15] to obtain the local-in-time existence and the uniqueness of solution as well. Furthermore, from the second equation of (5), it follows that $(\ln w)_t = \lambda p - \mu$. Hence at the domain boundaries $x=0, l$, we have that $(\ln w)_{tx} = \lambda p_x = -\lambda\alpha p(w_x/w) = -\lambda\alpha(\ln w)_x$, i.e. $(\ln w)_{tx} + \lambda\alpha(\ln w)_x = 0$.

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Solving this equation gives the solution

$$(\ln w)_x = \frac{w_x}{w} = \frac{w_x(x,0)}{w(x,0)} \exp\left(-\lambda\alpha \int_0^t p(x,\tau)\, d\tau\right) \quad \text{at } x = 0, l$$

From this point, we know that if $w_x = 0$ initially at the domain boundaries, both p and w have zero flux on the boundary in the entire existence time interval. We therefore always have the local existence and uniqueness for problem (5)–(7) such that either p or w initially satisfies the zero flux boundary condition. And consequently we get the zero flux boundary condition for either p or w. Therefore, it is plausible to suppose that $w_x(0,t) = w_x(l,t) = 0$ for all $t>0$.

In addition, we can simplify system (8)–(10) as well. As usual we let $q = (\ln w)_x$. Then from the second equation of (8) it follows that $q_t = (\ln w)_{xt} = (w_t/w)_x = p_x$. Together with the boundary condition discussed above, we translate system (8)–(10) into the following nicer form:

$$\begin{cases} p_t = p_{xx} + \alpha(pq)_x, & (x,t) \in (0,l) \times (0,\infty) \\ q_t = p_x \end{cases} \tag{98}$$

with boundary condition

$$p(0,t) = M, \quad q(0,t) = q(l,t) = 0 \tag{99}$$

and initial data

$$p(x,0) = p_0(x) > 0, \quad q(x,0) = q_0(x) \quad \text{for } 0 \leqslant x \leqslant l \tag{100}$$

where M is a positive constant.

If $\alpha>0$, we introduce the new variables by $\bar{t} = t/(1/\alpha M)$, $\bar{x} = x/l\sqrt{1/\alpha M}$, $\bar{p} = p/M$, $\bar{q} = q/\sqrt{\alpha/M}$ and redefine initial data by $\bar{p}_0(x) = p_0(x)/M$, $\bar{q}_0(x) = q_0(x)/\sqrt{\alpha/M}$. Substituting these transformations into (98)–(100) and dropping the bar for clarity, we obtain the following non-dimensional form:

$$\begin{cases} p_t - (pq)_x = p_{xx}, & (x,t) \in (0,1) \times (0,\infty) \\ q_t - p_x = 0 \end{cases} \tag{101}$$

with boundary condition

$$p(0,t) = 1, \quad q(0,t) = q(1,t) = 0 \quad \text{for } t \geqslant 0 \tag{102}$$

and initial data

$$p(x,0) = p_0(x) > 0, \quad q(x,0) = q_0(x) \quad \text{for } 0 \leqslant x \leqslant 1 \tag{103}$$

It is worth pointing out that the spatial domain has been rescaled to [0, 1]. By ignoring the diffusion term in (101), we obtain

$$\begin{cases} p_t - (pq)_x = 0, & (x,t) \in (0,1) \times (0,\infty) \\ q_t - p_x = 0 \end{cases} \tag{104}$$

It is easy to get the characteristic equation of system (104) which reads $\lambda^2 - \lambda q - p = 0$ where λ denotes the eigenvalues. So the discriminant $\triangle = q^2 + 4p$ is positive due to $p>0$. Hence, system (104) is a hyperbolic system which is consistent with the discussion in Section 1.1.

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Problem (8)–(10) now is reduced to non-dimensional system (101)–(103). One of the difficulties to consider the global existence of the solution to (101)–(103) is that the second equation of (101) missed the diffusion component. Novel ideas need to be developed to deal with such an issue.

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REFERENCES

1. Patlak CS. Random walk with persistence and external bias. *Bulletin of Mathematical Biophysics* 1953; **15**: 388–414.
2. Othmer HG, Stevens A. Aggregation, blowup, and collapse: the ABC's of taxis and reinforced random walks. *SIAM Journal on Applied Mathematics* 1997; **57**:1044–1081.
3. Levine HA, Sleeman BD. A system of reaction diffusion equations arising in the theory of reinforced random walks. *SIAM Journal of Applied Mathematics* 1997; **57**:683–730.
4. Yang Y, Chen H, Liu W. On existence of global solutions and blow-up to a system of the reaction–diffusion equations modelling chemotaxis. *SIAM Journal on Mathematical Analysis* 2001; **33**(4):763–785.
5. Yang Y, Chen H, Liu W, Sleeman BD. The solvability of some chemotaxis systems. *Journal of Differential Equations* 2005; **212**(2):432–451.
6. Ladyžhenskaya OA, Solonnikov VA, Ural'ceva NN. *Linear and Quasilinear Equations of Parabolic Type*. AMS: Providence, RI, 1968.
7. Sleeman BD, Ward MJ, Wei JC. The existence and stability of spike patterns in a chemotaxis model. *SIAM Journal on Applied Mathematics* 2005; **65**(3):790–817.
8. LeFloch PG. *Hyperbolic System of Conservation Laws: the Theory of Classical and Nonclassical Shock Waves*. Lectures in Mathematics: ETH Zürich. Birkhäuser: Basel, 2002.
9. Bressan A. *Hyperbolic System of Conservation Laws: The One-dimensional Cauchy Problem*. Oxford University Press: New York, 2001.
10. Smoller J. *Shock Waves and Reaction–Diffusion Equations*. Springer: Berlin, 1983.
11. LeVeque RJ. *Numerical Methods for Conservational Laws*. Lectures in Mathematics: ETH Zürich. Birkhäuser: Basel, 1992.
12. Hespanha JP. Uniform stability of switched linear systems: extensions of LaSalle's invariant principle. *IEEE Transactions on Automatic Control* 2004; **49**(4):470–482.
13. LaSalle JP. *The Stability of Dynamical Systems*. Regional Conference Series in Applied Mathematics, vol. 25. SIAM: Philadelphia, PA, 1976.
14. Evans LC. *Partial Differential Equations*. AMS: Providence, RI, 1997.
15. Rascle W. Sur une eq́uation integro-differentielle non lineaire issue de la biologie. *Journal of Differential Equations* 1979; **32**:420–453.

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