SCIENCE CHINA
Mathematics

• ARTICLES •

February 2022 Vol. 65 No. 2: 269–290
https://doi.org/10.1007/s11425-019-1754-0

# Global weak solutions and asymptotics of a singular PDE-ODE chemotaxis system with discontinuous data

Hongyun Peng[1], Zhi-An Wang[2,*] & Changjiang Zhu[3]

[1]*School of Applied Mathematics, Guangdong University of Technology, Guangzhou* 510520*, China;*
[2]*Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong* 999077*, China;*
[3]*School of Mathematics, South China University of Technology, Guangzhou* 510641*, China*

*Email: hypengmath@gdut.edu.cn, mawza@polyu.edu.hk, machjzhu@scut.edu.cn*

Received March 19, 2020; accepted July 31, 2020; published online December 4, 2020

**Abstract** This paper is concerned with the well-posedness and large-time behavior of a two-dimensional PDE-ODE hybrid chemotaxis system describing the initiation of tumor angiogenesis. We first transform the system via a Cole-Hopf type transformation into a parabolic-hyperbolic system and then show that the solution to the transformed system converges to a constant equilibrium state as time tends to infinity. Finally we reverse the Cole-Hopf transformation and obtain the relevant results for the pre-transformed PDE-ODE hybrid system. In contrast to the existing related results, where continuous initial data is imposed, we are able to prove the asymptotic stability for discontinuous initial data with large oscillations. The key ingredient in our proof is the use of the so-called "effective viscous flux", which enables us to obtain the desired energy estimates and regularity. The technique of the "effective viscous flux" turns out to be a very useful tool to study chemotaxis systems with initial data having low regularity and was rarely (if not) used in the literature for chemotaxis models.

**Keywords** chemotaxis, asymptotic stability, discontinuous initial data, effective viscous flux

**MSC(2010)** 35A01, 35B40, 35Q92, 92C17

**Citation:** Peng H Y, Wang Z-A, Zhu C J. Global weak solutions and asymptotics of a singular PDE-ODE chemotaxis system with discontinuous data. Sci China Math, 2022, 65: 269–290, https://doi.org/10.1007/s11425-019-1754-0

## 1 Introduction

In this paper, we study the following PDE-ODE hybrid chemotaxis model:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\xi u \nabla \ln c), \\ c_t = -\mu uc \end{cases} \tag{1.1}$$

which was proposed in [28] to describe the interaction between the signaling molecule vascular endothelial growth factor (VEGF) and vascular endothelial cells during the initiation of tumor angiogenesis (see

* Corresponding author

 

also [5, 6]). Here, $u(x,t)$ denotes the density of vascular endothelial cells and $c(x,t)$ means the concentration of the VEGF. This parameter $\xi > 0$ indicates the chemotactic coefficient measuring the strength of chemotaxis and $\mu$ is the degradation rate of the chemical VEGF. Here, chemical diffusion is ignored because it is far less important than its interaction with endothelial cells (see [28]). On the other hand, this system can also be regarded as a special form of the famous Keller-Segel system proposed in the seminal paper [25] describing the propagation of traveling wave bands formed by bacterial chemotaxis observed in Adler's experiments [2].

The distinguishing feature of the model (1.1) is twofold: (i) the first equation of (1.1) contains a logarithmic sensitivity function $\ln c$ which is singular at $c = 0$ but is a very meaningful sensitivity representation (see [24, 40]); (ii) the second equation of (1.1) is an ODE and hence lacks a spatial structure. Either of these features is the source of challenges for mathematical analysis and numerical computations. To overcome these obstacles, a Cole-Hopf type transformation

$$\boldsymbol{v} = -\frac{1}{\mu}\nabla(\ln c) = -\frac{1}{\mu}\frac{\nabla c}{c} \tag{1.2}$$

has been introduced in [27, 46], which transforms the system (1.1) into a parabolic-hyperbolic system

$$\begin{cases} u_t - \chi\nabla\cdot(u\boldsymbol{v}) = \Delta u, \\ \boldsymbol{v}_t - \nabla u = 0, \end{cases} \tag{1.3}$$

where $\chi = \mu\xi > 0$, $\boldsymbol{v}$ is a gradient vector and $\mathrm{curl}\boldsymbol{v} = 0$. Apparently the transformed system (1.3) is much more manipulable mathematically than the original system (1.1) since the singularity vanishes and the quantity $\boldsymbol{v}$ possesses a spatial structure through $u$. In this paper, we consider the well-posedness and asymptotic behavior of solutions to (1.3) with the initial data

$$(u, \boldsymbol{v})(x, 0) = (u_0, \boldsymbol{v}_0)(x), \quad x \in \mathbb{R}^2 \tag{1.4}$$

subject to the following asymptotic states:

$$(u, \boldsymbol{v})(\pm\infty, t) = (\bar{u}, \mathbf{0}), \tag{1.5}$$

where $(\bar{u}, \mathbf{0})$ with $\bar{u} > 0$ is a constant ground state of (1.3). Then we reverse the Cole-Hopf transformation (1.2) and get the well-posedness and asymptotic behavior of solutions to the original PDE-ODE chemotaxis system (1.1).

A lot of interesting works have been carried out for the transformed system (1.3). Firstly, the one-dimensional problem was extensively studied from various aspects such as the traveling wave solutions [22, 32, 34, 35, 39, 41, 45], and global dynamics of solutions in $\mathbb{R}$ [12, 30, 37, 49] or in bounded intervals [19, 31, 33, 48]. For the multidimensional whole space $\mathbb{R}^2$, the nonlinear stability of the planar traveling wave was recently established in [3]. When the initial data is close to the constant equilibrium $(\bar{u}, \mathbf{0})$, the system (1.3) has achieved numerous results. First, the blowup criteria of classical solutions were established in [8, 29] and the large-time behavior of solutions was obtained in [29], if $(u_0 - \bar{u}, \boldsymbol{v}_0) \in H^s(\mathbb{R}^d)$ for $s > \frac{d}{2} + 1$ and $\|(u_0 - \bar{u}, \boldsymbol{v}_0)\|_{H^s \times H^s}$ is small. Later, in the Chemin-Lerner space framework , Hao [13] obtained the global existence of mild solutions in the critical Besov space $\dot{B}^{-\frac{1}{2}}_{2,1} \times (\dot{B}^{\frac{1}{2}}_{2,1})^d$ with minimal regularity. The global well-posedness of strong solutions to (1.3) in $\mathbb{R}^3$ was established in [7], if $\|(u_0 - \bar{u}, \boldsymbol{v}_0)\|_{L^2 \times H^1}$ is small. If the initial data has a higher regularity such that $\|(u_0 - \bar{u}, \boldsymbol{v}_0)\|_{H^2 \times H^1}$ is small, the algebraic decay of solutions was further derived in [7]. Wang et al. [47] established the global existence and time decay rates of solutions to (1.3) in $\mathbb{R}^d$ for $d = 2, 3$ if $(u_0 - \bar{u}, \boldsymbol{v}_0) \in H^2(\mathbb{R}^d)$ and $\|(u_0 - \bar{u}, \boldsymbol{v}_0)\|_{H^1 \times H^1}$ is small. Recently, Wang et al. [44] established the small energy solution in $\mathbb{R}^d$ $(d = 2, 3)$. In the high-dimensional bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, under Neumann boundary conditions, the global existence and exponential decay rates of solutions were obtained in [33, 42] for small data, and the local existence of classical solutions in two dimensions with Dirichlet boundary conditions was given in [20]. When the Laplace operator in (1.3) was modified to a fractional Laplace operator, the global existence of solutions to (1.3) in a torus with periodic boundary conditions in some dissipation states was established in [10, 11].

In the above-mentioned results, all solutions obtained to (1.3) are classical thanks to the high regularity and smallness of the $H^s$ ($s \geqslant 1$)-norm on the initial data. The goal of this paper is to exploit the parabolic-hyperbolic system (1.3) with rougher initial data which is allowed to be discontinuous and possess large amplitude. Specifically we consider the initial data $(u_0, v_0)$ in some appropriate $L^p$ space and allow $\|u_0 - \bar{u}\|_{L^\infty} + \|\boldsymbol{v}_0\|_{L^\infty}$ to be arbitrarily large. The low regularity of the initial data and large amplitude bring us many new difficulties in analysis compared with the existing works (see [7, 29, 47]). In particular, the integrability of $\nabla \boldsymbol{v}$, which plays a crucial role in the analysis of [7, 29, 47], can be easily attained therein but appears to be unattainable for our less regular initial data due to the hyperbolic effect of the second equation. Hence one can only expect to examine $L^p$-integrability for $\boldsymbol{v}$ and hence obtain weak solutions instead of strong (classical) solutions. The key idea of gaining $L^p$-integrability of $\boldsymbol{v}$ developed in this paper is to rewrite the first equation of (1.3) as $\tilde{u}_t = \nabla \cdot \mathbf{F}$ inspired by the works [15, 36] on the Navier-Stokes equations, where $\mathbf{F}$ is the so-called "effective viscous flux" defined by

$$\mathbf{F} = \nabla u + \chi u \boldsymbol{v}. \tag{1.6}$$

Then we rewrite the second equation of (1.3) as

$$\boldsymbol{v}_t = -\chi u \boldsymbol{v} + \mathbf{F}$$

and try to attain the regularity of $\boldsymbol{v}$ through $\mathbf{F}$ whose regularity is easier than $\boldsymbol{v}$ to obtain. We remark that the dynamics of PDEs with discontinuous data has been an important topic arising from fluid mechanics and gas dynamics in order to understand how the discontinuity evolves in the fluid. For this, Hoff with his collaborators has developed some nice ideas and obtained a series of important results in this aspect (see [14–18]). We refer to [4, 21, 26, 38] and the references therein for further development on the discontinuous data problem. However in our current work the "effective viscous flux" $\mathbf{F}$ exists in its divergence form (i.e., $\nabla \cdot \mathbf{F}$), which is different from that used in the Navier-Stokes equations where the "effective viscous flux" exists in its gradient form and the $L^p$-norm of $\nabla \mathbf{F}$ can be achieved directly (see [15–18, 36]). This implies that besides the estimate of $\nabla \cdot \mathbf{F}$, we have to estimate for the curl of $\mathbf{F}$ (denoted by $\nabla^\perp \mathbf{F}$) in order to get the estimates for $\nabla \mathbf{F}$ and hence the regularity for $\mathbf{F}$, which is a key difference from the Navier-Stokes equation. The "effective viscous flux" technique appears to be a very powerful tool to study chemotaxis systems with low regular initial data and was rarely (if not) used in the literature.

Since the dependence of solutions on $\chi$ and $\bar{u}$ is not the interest of this paper, we henceforth assume $\chi = \bar{u} = 1$ throughout the paper for simplicity without further clarification.

To state our results, we first introduce the definition of weak solutions to (1.3)–(1.5).

**Definition 1.1** (Weak solutions)**.** We say that $(u, \boldsymbol{v})$ is a weak solution to (1.3)–(1.5), if $(u, \boldsymbol{v})$ is suitably integrable and satisfies for all test functions $\Psi \in C_0^\infty(\mathbb{R}^2 \times [0, \infty))$ that

$$\int_{\mathbb{R}^2} u_0 \Psi_0 dx + \int_0^\infty \int_{\mathbb{R}^2} (u \Psi_t - \nabla u \cdot \nabla \Psi)\, dx dt = \int_0^\infty \int_{\mathbb{R}^2} u \boldsymbol{v} \cdot \nabla \Psi dx dt$$

and

$$\int_{\mathbb{R}^2} \boldsymbol{v}_0^j \Psi_0 dx + \int_0^\infty \int_{\mathbb{R}^2} (\boldsymbol{v}^j \Psi_t - u \Psi_{x_j}) dx dt = 0,$$

where $j = 1, 2$ and $\Psi_0(x) = \Psi(x, 0)$.

Then our main results in this paper are given as follows.

**Theorem 1.2.** *Let $4 < p_0 < \infty$ and the initial data satisfy*

$$u_0 - 1 \in L^2(\mathbb{R}^2), \quad \boldsymbol{v}_0 \in L^2(\mathbb{R}^2) \cap L^{p_0}(\mathbb{R}^2), \quad u_0 \geqslant 0, \quad \nabla^\perp \cdot \boldsymbol{v}_0 = 0, \tag{1.7}$$

*where* $\nabla^{\perp} = (\partial_2, -\partial_1)$ *denotes the curl operator. Then for any constant* $M > 0$ *with* $\|\boldsymbol{v}_0\|_{L^{p_0}(\mathbb{R}^2)} \leqslant M$*, there exists a constant* $\varepsilon > 0$ *depending on* $M$*, such that if*

$$\|u_0 - 1\|_{L^2(\mathbb{R}^2)}^2 + \|\boldsymbol{v}_0\|_{L^2(\mathbb{R}^2)}^2 = \theta_0 \leqslant \varepsilon,$$

*then the Cauchy problem* (1.3)–(1.5) *has a global weak solution* $(u, \boldsymbol{v})(x,t)$ *satisfying*

$$\begin{cases} u - 1 \in L^{\infty}([0,\infty); L^2(\mathbb{R}^2)) \cap C((0,\infty); C(\mathbb{R}^2)), \quad \nabla u \in L^2([0,\infty); L^2(\mathbb{R}^2)), \\ \boldsymbol{v} \in L^{\infty}([0,\infty); L^2(\mathbb{R}^2) \cap L^{p_0}(\mathbb{R}^2)) \cap C([0,\infty), H^{-1}(\mathbb{R}^2)) \end{cases} \tag{1.8}$$

*and the following asymptotic convergence*:

$$\|u(x,t) - 1\|_{L^{p_1}(\mathbb{R}^2)} \to 0, \quad \|\boldsymbol{v}(x,t)\|_{L^{p_2}(\mathbb{R}^2)} \to 0 \quad as \quad t \to \infty, \tag{1.9}$$

*where* $2 < p_1 \leqslant \infty$ *and* $2 < p_2 < p_0 < \infty$*.*

**Remark 1.3.** The above results hold true regardless of the strength of initial perturbation from the constant ground state $(1, 0)$, namely the amplitude $\|u_0 - 1\|_{L^\infty} + \|\boldsymbol{v}_0\|_{L^\infty}$ can be arbitrarily large. The initial conditions (1.7) imply that the initial data of the Cauchy problem (1.3)–(1.5) could be discontinuous (such as piecewise constant with arbitrarily large jump discontinuities), which brings great challenges to the analysis. Moreover, the condition $\nabla^{\perp} \cdot \boldsymbol{v}_0 = 0$ is a natural consequence of the Cole-Hopf transformation (1.2).

To prove Theorem 1.2, we first mollify the initial data with a mollifying parameter $\delta$ and obtain the global smooth solution $(u^\delta, \boldsymbol{v}^\delta)$ depending on the mollifying parameter $\delta$. Then we pass to the limit as $\delta \to 0$ and obtain a weak solution to (1.3)–(1.5). The core of the proof is to derive the global *a priori* estimates independent of the mollifying parameter $\delta$. In this connection, the approaches and estimates developed in previous works [29, 47] for small-amplitude and continuous initial data are not adequate for our current problem. In this paper, we introduce the so-called "effective viscous flux" technique and make a full use of the structure of (1.3) to obtain the desired uniform-in-$\delta$ estimates.

Converting the results from $v$ to $c$ by reversing the Cole-Hopf transformation (1.2), we get the results for the original chemotaxis model (1.1).

**Theorem 1.4.** *Let* $4 < p_0 < \infty$ *and the initial data satisfy*

$$u_0 - 1 \in L^2(\mathbb{R}^2), \quad u_0(x) \geqslant 0, \quad \nabla \ln c_0 \in L^2(\mathbb{R}^2) \cap L^{p_0}(\mathbb{R}^2).$$

*Then for any constant* $M > 0$ *with* $\|\nabla \ln c_0\|_{L^{p_0}(\mathbb{R}^2)} \leqslant M$*, there exists a constant* $\varepsilon > 0$ *depending on* $M$*, such that if*

$$\|u_0 - 1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \ln c_0\|_{L^2(\mathbb{R}^2)}^2 \leqslant \varepsilon,$$

*the Cauchy problem* (1.3)–(1.5) *has a global weak solution* $(u, c)(x,t)$ *satisfying*

$$\begin{cases} u - 1 \in L^{\infty}([0,\infty); L^2(\mathbb{R}^2)), \quad \nabla u \in L^2([0,\infty); L^2(\mathbb{R}^2)), \\ u - 1 \in C((0,\infty); C(\mathbb{R}^2)), \quad \nabla \ln c \in C([0,\infty), H^{-1}(\mathbb{R}^2)), \\ \nabla \ln c \in L^{\infty}([0,\infty); L^2(\mathbb{R}^2) \cap L^{p_0}(\mathbb{R}^2)) \cap L^4([0,\infty); L^4(\mathbb{R}^2)) \end{cases}$$

*and*

$$\|u - 1\|_{L^{p_1}(\mathbb{R}^2)} \to 0, \quad \|\nabla \ln c\|_{L^{p_2}(\mathbb{R}^2)} \to 0 \quad as \quad t \to \infty,$$

*where* $2 < p_1 \leqslant \infty$ *and* $2 < p_2 < p_0 < \infty$*.*

*Furthermore, if* $c_0 \in L^\infty(\mathbb{R}^2)$*,* $c_0 > 0$*, then there exists a positive constant* $C$ *independent of* $t$ *such that the solution has the following decay rates in time*:

$$\|c\|_{L^\infty(\mathbb{R}^2)} \leqslant C\mathrm{e}^{-\frac{3}{4}t}. \tag{1.10}$$

The rest of this paper is organized as follows. In Section 2, we collect some elementary facts and inequalities that will be used in later analysis. Section 3 is devoted to deriving the *a priori* estimates on smooth solutions. Then we prove Theorem 1.2 in Section 4. Finally we prove Theorem 1.4 in Section 5.

## 2 Preliminaries

In this section, we recall and prove some basic results that will be used later. Before embarking on this, we first introduce some notations used throughout this paper.

**Notations.** In what follows, $C$ denotes a generic positive constant which may vary in the context.

• $H^k(\mathbb{R}^2)$ denotes the usual $k$-th order Sobolev space on $\mathbb{R}^2$ with the norm

$$\|f\|_{H^k(\mathbb{R}^2)} := \left(\sum_{j=0}^{k}\int_{\mathbb{R}^2}|\partial_x^j f|^2 dx\right)^{1/2}.$$

For simplicity, we denote $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}^2)}$ and $\|\cdot\|_k := \|\cdot\|_{H^k(\mathbb{R}^2)}$, and furthermore $L^2(\mathbb{R}^2)$ will be abbreviated as $L^2$ without confusion.

• We denote the curl operator by

$$\nabla^\perp = (\partial_2, -\partial_1). \tag{2.1}$$

• For simplicity we set

$$\theta_0 = \|u_0 - 1\|^2 + \|\boldsymbol{v}_0\|^2. \tag{2.2}$$

Since $\theta_0$ will be assumed to be small in this paper, we assume that $\theta_0 < 1$ without loss of generality in the sequel.

We start with the local existence and blowup criterion of smooth solutions to the Cauchy problem (1.3)–(1.5) established in [8, 29].

**Lemma 2.1** (See [29])**.** *Let $s > \frac{d}{2}+1$ and $(u_0-1, \boldsymbol{v}_0) \in H^s(\mathbb{R}^d)$. Then there exists a small-time $T_* = T_*(\|u_0-1\|_{H^s}, \|\boldsymbol{v}_0\|_{H^s(\mathbb{R}^d)}) > 0$ such that the Cauchy problem* (1.3)–(1.5) *admits a unique solution $(u-1, \boldsymbol{v}) \in L^\infty((0, T_*], H^s(\mathbb{R}^d))$.*

**Lemma 2.2** (See [8])**.** *Let $s > \frac{d}{2}+1$ and $(u_0-1, \boldsymbol{v}_0) \in H^s(\mathbb{R}^d)$. Let $(u, \boldsymbol{v})$ be the unique local solution in Lemma* 2.1 *with the maximal lifespan $T_* > 0$. If*

$$\int_0^{T_*} \|\boldsymbol{v}\|_{L^q(\mathbb{R}^d)}^{\frac{2q}{q-d}} < \infty \quad \textit{with} \ \ d < q \leqslant \infty, \tag{2.3}$$

*then the solution can be extended beyond $T_* > 0$.*

Next, we introduce a change of variable $\tilde{u} = u - 1$. Thus, the problem (1.3)–(1.4) turns into

$$\begin{cases} \tilde{u}_t - \Delta\tilde{u} = \nabla\cdot(\tilde{u}\boldsymbol{v}) + \nabla\cdot\boldsymbol{v}, \\ \boldsymbol{v}_t - \nabla\tilde{u} = 0, \\ (\tilde{u}, \boldsymbol{v})(x, 0) = (u_0 - 1, \boldsymbol{v}_0)(x). \end{cases} \tag{2.4}$$

Then the "effective viscous flux" $\mathbf{F}$ defined in (1.6) can be written as

$$\mathbf{F} = \nabla\tilde{u} + (\tilde{u}+1)\boldsymbol{v}. \tag{2.5}$$

By the first equation of (2.4), it is easy to see that

$$\nabla\cdot\mathbf{F} = \tilde{u}_t. \tag{2.6}$$

Then $\mathbf{F}$ has the following useful estimate.

**Lemma 2.3.** *Let $(\tilde{u}, \boldsymbol{v})$ be a smooth solution to* (2.4)*. Then there exists a positive constant $C$ such that*

$$\|\nabla\mathbf{F}\|_{L^p} \leqslant C(\|\tilde{u}_t\|_{L^p} + \|\nabla^\perp\tilde{u}\cdot\boldsymbol{v}\|_{L^p}), \tag{2.7}$$

*where $p > 1$.*

*Proof.* First we recall the following inequality (see [23, Lemma 2.4(1)]):

$$\|\nabla\mathbf{F}\|_{L^p}\leqslant C(\|\nabla\cdot\mathbf{F}\|_{L^p}+\|\nabla^{\perp}\cdot\mathbf{F}\|_{L^p}).\tag{2.8}$$

It then follows from (2.1) and (2.5) that

$$\begin{aligned}\nabla^{\perp}\cdot\mathbf{F}&=\nabla^{\perp}\cdot\nabla\tilde{u}+\nabla^{\perp}\cdot(\tilde{u}\boldsymbol{v})+\nabla^{\perp}\cdot\boldsymbol{v}\\&=\nabla^{\perp}\cdot(\tilde{u}\boldsymbol{v})+\nabla^{\perp}\cdot\boldsymbol{v}\\&=(\tilde{u}+1)\nabla^{\perp}\cdot\boldsymbol{v}+\nabla^{\perp}\tilde{u}\cdot\boldsymbol{v},\end{aligned}\tag{2.9}$$

where in the second equality we have used the fact that $\nabla^{\perp}\cdot\nabla=0$. Operating $\nabla^{\perp}$ on the second equation of (2.4) and integrating the result over $(0,t)$, we have from $\nabla^{\perp}\cdot\boldsymbol{v}_0=0$ that $\nabla^{\perp}\cdot\boldsymbol{v}=0$, which updates (2.9) as $\nabla^{\perp}\cdot\mathbf{F}=\nabla^{\perp}\tilde{u}\cdot\boldsymbol{v}$. This together with (2.6) and (2.8) gives

$$\|\nabla\mathbf{F}\|_{L^p}\leqslant C\|\tilde{u}_t\|_{L^p}+C\|\nabla^{\perp}\tilde{u}\cdot\boldsymbol{v}\|_{L^p}.$$

Thus, the proof of Lemma 2.3 is completed. □

The following Gagliardo-Nirenberg inequality will be frequently used in this paper.

**Lemma 2.4** (See [9]). *Let $1\leqslant q,r\leqslant\infty$ and $0<a\leqslant1$ such that*

$$\frac{1}{p}=a\left(\frac{1}{q}-\frac{1}{2}\right)+(1-a)\frac{1}{r}.$$

*Then, for any $u\in W^{1,p}(\mathbb{R}^2)\cap L^r(\mathbb{R}^2)$, there exists a positive constant $C$ depending only on $q,r$ and $n$ such that the following inequality holds*:

$$\|u\|_{L^p(\mathbb{R}^2)}\leqslant C\|\nabla u\|^a_{L^q(\mathbb{R}^2)}\|u\|^{1-a}_{L^r(\mathbb{R}^2)}.\tag{2.10}$$

## 3 The *a priori* estimates for approximate solutions

In this section, we derive some *a priori* estimates for the approximate solutions, by constructing approximate solutions based upon the following mollified initial data:

$$\tilde{u}_0^{\delta}=j^{\delta}*\tilde{u}_0,\quad \boldsymbol{v}_0^{\delta}=j^{\delta}*\boldsymbol{v}_0,$$

where $\tilde{u}_0=u_0-1$ and $j^{\delta}$ is the standard mollifying kernel of width $\delta$ (see, e.g., [1]). Then we consider the following approximate system:

$$\begin{cases}\tilde{u}_t^{\delta}-\Delta\tilde{u}^{\delta}=\nabla\cdot(\tilde{u}^{\delta}\boldsymbol{v}^{\delta})+\nabla\cdot\boldsymbol{v}^{\delta},\\ \boldsymbol{v}_t^{\delta}-\nabla\tilde{u}^{\delta}=0\end{cases}\tag{3.1}$$

with the smooth initial data $(\tilde{u}_0^{\delta},\boldsymbol{v}_0^{\delta})$ which satisfies

$$(\tilde{u}_0^{\delta},\boldsymbol{v}_0^{\delta})\in H^3(\mathbb{R}^2)\tag{3.2}$$

and

$$\|\tilde{u}_0^{\delta}\|^2+\|\boldsymbol{v}_0^{\delta}\|^2\leqslant\|\tilde{u}_0\|^2+\|\boldsymbol{v}_0\|^2=\theta_0.\tag{3.3}$$

By Lemma 2.1, we can obtain the local existence of solutions to the approximate system (3.1) with the initial data $(u_0^{\delta},\boldsymbol{v}_0^{\delta})$ satisfying (3.2)–(3.3). Next, we shall show in a sequence of lemmas that these approximate solutions satisfy some global *a priori* estimates, independent of the mollifying parameter $\delta$. These estimates will then be applied in Section 4 to obtain solutions to Theorem 1.2 as the limits of these approximate solutions.

For the sake of simplicity, we still use $(\tilde{u}, \boldsymbol{v})$ to represent the approximate solution $(\tilde{u}^\delta, \boldsymbol{v}^\delta)$ in this section. Let $T > 0$ be a fixed time and $(\tilde{u}, \boldsymbol{v})$ be the smooth solution to (3.1) on $\mathbb{R}^2 \times (0, T]$. We set $\sigma = \sigma(t) = \min\{1, t\}$ and define

$$\begin{cases} A_1(T) \triangleq \sup\limits_{t\in[0,T]} (\|\tilde{u}\|^2 + \|\boldsymbol{v}\|^2) + \displaystyle\int_0^T \|\nabla\tilde{u}\|^2 dt, \\ A_2(T) \triangleq \sup\limits_{t\in[0,T]} (\sigma\|\nabla\tilde{u}\|^2 + \sigma^2\|\tilde{u}_t\|^2 + \sigma^2\|\boldsymbol{v}_t\|^2) + \displaystyle\int_0^T (\sigma\|\tilde{u}_t\|^2 + \sigma^2\|\nabla\tilde{u}_t\|^2) dt, \\ A_3(T) \triangleq \sup\limits_{t\in[0,T]} \|\boldsymbol{v}\|_{L^4}^4 + \displaystyle\int_0^T \|\boldsymbol{v}\|_{L^4}^4 dt. \end{cases} \tag{3.4}$$

Next, we shall employ the technique of the *a priori* assumption to derive the *a priori* estimates for the smooth solutions to (3.1)–(3.3), i.e., we first assume that the smooth solution $(\tilde{u}, \boldsymbol{v})$ satisfies that for any $t \in [0, T]$,

$$A_1(T) \leqslant 2\theta_0, \quad A_2(T) \leqslant 2\theta_0^{\frac{1}{2}}, \quad A_3(T) \leqslant 2\theta_0^{\eta_0}, \quad \sup_{t\in[0,T]} \|\boldsymbol{v}\|_{L^{p_0}} \leqslant 6M, \tag{3.5}$$

where $M$ and $\theta_0$ are from Theorem 1.2, and $\eta_0$ is defined as

$$\eta_0 \triangleq \frac{p_0 - 4}{2(p_0 - 2)} \in \left(0, \frac{1}{2}\right). \tag{3.6}$$

Then we will derive the *a priori* estimates to obtain global solutions. Finally, we show that the obtained global solutions satisfy the above *a priori* assumption and close our argument.

We depart with the $L^2$-estimate of $(\tilde{u}, \boldsymbol{v})$.

**Lemma 3.1.** *Let the conditions of Theorem* 1.2 *hold and* $(\tilde{u}, \boldsymbol{v})$ *be a smooth solution to* (3.1)–(3.3) *satisfying* (3.5)*. Then it holds that*

$$\|\tilde{u}\|^2 + \|\boldsymbol{v}\|^2 + \int_0^T \|\nabla\tilde{u}\|^2 dt \leqslant \frac{3\theta_0}{2}. \tag{3.7}$$

*Proof.* Multiplying the first equation of (3.1) by $\tilde{u}$ and the second by $v$, adding the results and integrating by parts over $\mathbb{R}^2$, we have

$$\frac{1}{2}\frac{d}{dt}(\|\tilde{u}\|^2 + \|\boldsymbol{v}\|^2) + \|\nabla\tilde{u}\|^2 = \int_{\mathbb{R}^2} \nabla\cdot(\tilde{u}\boldsymbol{v})\tilde{u} dx = -\int_{\mathbb{R}^2} \tilde{u}\boldsymbol{v}\nabla\tilde{u} dx. \tag{3.8}$$

For the term on the right-hand side of (3.8), we use (2.10), the Hölder's inequality and the Cauchy-Schwarz inequality to estimate it as

$$\begin{aligned} -\int_{\mathbb{R}^2} \tilde{u}\boldsymbol{v}\nabla\tilde{u} dx &\leqslant \|\tilde{u}\boldsymbol{v}\|^2 + \frac{1}{4}\|\nabla\tilde{u}\|^2 \\ &\leqslant \|\tilde{u}\|_{L^4}^2 \|\boldsymbol{v}\|_{L^4}^2 + \frac{1}{4}\|\nabla\tilde{u}\|^2 \\ &\leqslant C\|\tilde{u}\|\|\nabla\tilde{u}\|\|\boldsymbol{v}\|_{L^4}^2 + \frac{1}{4}\|\nabla\tilde{u}\|^2 \\ &\leqslant C\|\tilde{u}\|^2\|\boldsymbol{v}\|_{L^4}^4 + \frac{1}{2}\|\nabla\tilde{u}\|^2, \end{aligned}$$

which therefore updates (3.8) as

$$\frac{d}{dt}(\|\tilde{u}\|^2 + \|\boldsymbol{v}\|^2) + \|\nabla\tilde{u}\|^2 \leqslant C\|\tilde{u}\|^2\|\boldsymbol{v}\|_{L^4}^4.$$

Integrating the above result over $[0, t]$ and using (2.2), (3.4) and (3.5), we have

$$\begin{aligned} \|\tilde{u}\|^2 + \|\boldsymbol{v}\|^2 + \int_0^T \|\nabla\tilde{u}\|^2 dt &\leqslant \|\tilde{u}_0\|^2 + \|\boldsymbol{v}_0\|^2 + C\int_0^T \|\tilde{u}\|^2\|\boldsymbol{v}\|_{L^4}^4 dt \\ &\leqslant \|\tilde{u}_0\|^2 + \|\boldsymbol{v}_0\|^2 + CA_1(T)A_3(T) \\ &\leqslant \theta_0 + C\theta_0^{1+\eta_0} \leqslant \frac{3\theta_0}{2}, \end{aligned}$$

provided that $C\theta_0^{\eta_0} \leqslant \frac{1}{2}$. This yields (3.7) and completes the proof of Lemma 3.1. □

**Lemma 3.2.** *Let the conditions of Theorem* 1.2 *hold and* $(\tilde{u}, \boldsymbol{v})$ *be a smooth solution to* (3.1)–(3.3) *satisfying* (3.5)*. Then it holds that*

$$\sigma\|\nabla\tilde{u}\|^2 + \sigma^2\|\tilde{u}_t\|^2 + \sigma^2\|\boldsymbol{v}_t\|^2 + \int_0^T \sigma\|\tilde{u}_t\|^2 dt + \int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2 dt \leqslant \theta_0^{\frac{1}{2}}, \tag{3.9}$$

*where* $\sigma = \sigma(t) = \min\{1, t\}$.

*Proof.* We divide our proof into two steps.

**Step 1.** We first multiply the first equation of (3.1) by $\sigma u_t$ and integrate the resulting equation over $\mathbb{R}^2 \times [0, T]$ to get

$$\begin{aligned} &\frac{1}{2}\sigma\|\nabla\tilde{u}\|^2 + \int_0^T \sigma\|\tilde{u}_t\|^2 dt \\ &\quad = \frac{1}{2}\int_0^{\sigma(T)} \|\nabla\tilde{u}\|^2 dt - \int_0^T \sigma\int_{\mathbb{R}^2} \tilde{u}\boldsymbol{v}\nabla\tilde{u}_t dxdt - \int_0^T \sigma\int_{\mathbb{R}^2} \boldsymbol{v}\nabla\tilde{u}_t dxdt, \end{aligned} \tag{3.10}$$

where we have used the fact that

$$\frac{1}{2}\int_0^T \sigma_t\|\nabla\tilde{u}\|^2 dt = \frac{1}{2}\int_0^{\sigma(T)} \|\nabla\tilde{u}\|^2 dt.$$

For the first term on the right-hand side of (3.10), we have from (3.7)

$$\frac{1}{2}\int_0^{\sigma(T)} \|\nabla\tilde{u}\|^2 dt \leqslant \frac{1}{2}\int_0^T \|\nabla\tilde{u}\|^2 dt \leqslant C\theta_0. \tag{3.11}$$

For the second term on the right-hand side of (3.10), we have from (2.10), the Cauchy-Schwarz inequality, (3.5) and (3.7) that

$$\begin{aligned} -\int_0^T \sigma\int_{\mathbb{R}^2} \tilde{u}\boldsymbol{v}\nabla\tilde{u}_t dxdt &\leqslant \lambda\int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2 dt + C\int_0^T \|\tilde{u}\boldsymbol{v}\|^2 dt \\ &\leqslant \lambda\int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2 dt + C\int_0^T \|\tilde{u}\|_{L^4}^2\|\boldsymbol{v}\|_{L^4}^2 dt \\ &\leqslant \lambda\int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2 dt + C\int_0^T \|\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2}\|\boldsymbol{v}\|_{L^4}^2 dt \\ &\leqslant \lambda\int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2 dt + C\int_0^T \|\nabla\tilde{u}\|^2 dt + C\int_0^T \|\tilde{u}\|^2\|\boldsymbol{v}\|_{L^4}^4 dt \\ &\leqslant \lambda\int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2 dt + C\theta_0 + C\theta_0^{1+\eta_0} \\ &\leqslant \lambda\int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2 dt + C\theta_0, \end{aligned} \tag{3.12}$$

where $\lambda > 0$ is a constant to be determined later. For the last term on the right-hand side of (3.10), we have

$$\begin{aligned} -\int_0^T \sigma\int_{\mathbb{R}^2} \boldsymbol{v}\nabla\tilde{u}_t dxdt = &-\int_0^T \left(\sigma\int_{\mathbb{R}^2} \boldsymbol{v}\nabla\tilde{u} dx\right)_t dt + \int_0^{\sigma(T)}\int_{\mathbb{R}^2} \boldsymbol{v}\nabla\tilde{u} dxdt \\ &+ \int_0^T \sigma\int_{\mathbb{R}^2} \boldsymbol{v}_t\nabla\tilde{u} dxdt =: I_1 + I_2 + I_3. \end{aligned} \tag{3.13}$$

By the Cauchy-Schwarz inequality and (3.7), we have from $0 \leqslant \sigma \leqslant 1$ that

$$
\begin{aligned}
I_1 &= -\sigma \int_{\mathbb{R}^2} \boldsymbol{v} \nabla \tilde{u} dx \leqslant \frac{\sigma}{4}\|\nabla \tilde{u}\|^2 + \sigma \|\boldsymbol{v}\|^2 \leqslant \frac{\sigma}{4}\|\nabla \tilde{u}\|^2 + 2\theta_0, \\
I_2 &= \int_0^{\sigma(T)} \int_{\mathbb{R}^2} \boldsymbol{v} \nabla \tilde{u} dx dt \\
&\leqslant \frac{1}{2} \int_0^{\sigma(T)} \|\nabla \tilde{u}\|^2 dt + \frac{1}{2} \int_0^{\sigma(T)} \|\boldsymbol{v}\|^2 dt \\
&\leqslant \frac{1}{2} \int_0^{T} \|\nabla \tilde{u}\|^2 dt + \frac{1}{2} \sup_{t \in [0,T]} \|\boldsymbol{v}\|^2 \\
&\leqslant 2\theta_0
\end{aligned}
$$

and

$$
I_3 = \int_0^T \sigma \int_{\mathbb{R}^2} \boldsymbol{v}_t \nabla \tilde{u} dx dt = \int_0^T \sigma \int_{\mathbb{R}^2} |\nabla \tilde{u}|^2 dx dt \leqslant \int_0^T \int_{\mathbb{R}^2} |\nabla \tilde{u}|^2 dx dt \leqslant 2\theta_0,
$$

where we have used the second equation of (3.1). Substituting (3.10) with (3.11)–(3.13) and the above estimates for $I_i$ $(i = 1, 2, 3)$, we have

$$
\sigma \|\nabla \tilde{u}\|^2 + 4 \int_0^T \sigma \|\tilde{u}_t\|^2 dt \leqslant 4\lambda \int_0^T \sigma^2 \|\nabla \tilde{u}_t\|^2 dt + C\theta_0. \tag{3.14}
$$

**Step 2.** Next, we need to obtain the estimate of $\int_0^T \sigma^2 \|\nabla \tilde{u}_t\|^2 dt$. To this end, we differentiate (3.1) with respect to time $t$ to get

$$
\begin{cases}
\tilde{u}_{tt} - \Delta \tilde{u}_t = \nabla \cdot (\tilde{u}\boldsymbol{v})_t + \nabla \cdot \boldsymbol{v}_t, \\
\boldsymbol{v}_{tt} - \nabla \tilde{u}_t = 0.
\end{cases} \tag{3.15}
$$

Multiplying the first equation of (3.15) by $\sigma^2 \tilde{u}_t$ and the second by $\sigma^2 \boldsymbol{v}_t$, adding the results followed by an integration over $\mathbb{R}^2 \times [0, t]$, we have

$$
\begin{aligned}
&\frac{\sigma^2}{2}\|\tilde{u}_t\|^2 + \frac{\sigma^2}{2}\|\boldsymbol{v}_t\|^2 + \int_0^T \sigma^2 \|\nabla \tilde{u}_t\|^2 dt \\
&\quad = \int_0^{\sigma(T)} \sigma \|\tilde{u}_t\|^2 dt + \int_0^{\sigma(T)} \sigma \|\boldsymbol{v}_t\|^2 dt - \int_0^T \sigma^2 \int_{\mathbb{R}^2} (\tilde{u}\boldsymbol{v})_t \nabla \tilde{u}_t dx dt \\
&\quad \leqslant \int_0^T \sigma \|\tilde{u}_t\|^2 dt + \int_0^T \sigma \|\nabla \tilde{u}\|^2 dt - \int_0^T \sigma^2 \int_{\mathbb{R}^2} \tilde{u} \boldsymbol{v}_t \nabla \tilde{u}_t dx dt - \int_0^T \sigma^2 \int_{\mathbb{R}^2} \tilde{u}_t \boldsymbol{v} \nabla \tilde{u}_t dx dt \\
&\quad \leqslant C\theta_0 + \int_0^T \sigma \|\tilde{u}_t\|^2 dt - \int_0^T \sigma^2 \int_{\mathbb{R}^2} \tilde{u} \nabla \tilde{u} \nabla \tilde{u}_t dx dt - \int_0^T \sigma^2 \int_{\mathbb{R}^2} \tilde{u}_t \boldsymbol{v} \nabla \tilde{u}_t dx dt,
\end{aligned} \tag{3.16}
$$

where we have used the second equation of (3.1) and (3.7). For the third term on the right-hand of the above inequality, we estimate it by the Cauchy-Schwarz inequality, (2.10) and (3.7) as

$$
\begin{aligned}
\int_0^T \sigma^2 \int_{\mathbb{R}^2} \tilde{u} \nabla \tilde{u} \nabla \tilde{u}_t dx dt &\leqslant \frac{1}{4} \int_0^T \sigma^2 \|\nabla \tilde{u}_t\|^2 dt + \int_0^T \sigma^2 \|\tilde{u} \nabla \tilde{u}\|^2 dt \\
&\leqslant \frac{1}{4} \int_0^T \sigma^2 \|\nabla \tilde{u}_t\|^2 dt + \int_0^T \sigma^2 \|\tilde{u}\|_{L^4}^2 \|\nabla \tilde{u}\|_{L^4}^2 dt \\
&\leqslant \frac{1}{4} \int_0^T \sigma^2 \|\nabla \tilde{u}_t\|^2 dt + C \int_0^T \sigma^2 \|\tilde{u}\| \|\nabla \tilde{u}\| \|\nabla \tilde{u}\|_{L^4}^2 dt \\
&\leqslant \frac{1}{4} \int_0^T \sigma^2 \|\nabla \tilde{u}_t\|^2 dt + C \int_0^T \sigma^2 \|\nabla \tilde{u}\|^2 dt + C \int_0^T \sigma^2 \|\tilde{u}\|^2 \|\nabla \tilde{u}\|_{L^4}^4 dt \\
&\leqslant C\theta_0 + \frac{1}{4} \int_0^T \sigma^2 \|\nabla \tilde{u}_t\|^2 dt + C\theta_0 \int_0^T \sigma^2 \|\nabla \tilde{u}\|_{L^4}^4 dt.
\end{aligned} \tag{3.17}
$$

Furthermore, we have from (2.10) and (2.5) that

$$\begin{aligned}\|\nabla\tilde{u}\|_{L^4} &\leqslant \|\mathbf{F}-\tilde{u}\boldsymbol{v}-\boldsymbol{v}\|_{L^4}\\ &\leqslant \|\mathbf{F}\|_{L^4}+\|\tilde{u}\boldsymbol{v}\|_{L^4}+\|\boldsymbol{v}\|_{L^4}\\ &\leqslant C(\|\mathbf{F}\|^{\frac12}\|\nabla\mathbf{F}\|^{\frac12}+\|\tilde{u}\|_{L^\infty}\|\boldsymbol{v}\|_{L^4}+\|\boldsymbol{v}\|_{L^4})\\ &\leqslant C(\|\mathbf{F}\|^{\frac12}\|\nabla\mathbf{F}\|^{\frac12}+\|\tilde{u}\|_{L^4}^{\frac12}\|\nabla\tilde{u}\|_{L^4}^{\frac12}\|\boldsymbol{v}\|_{L^4}+\|\boldsymbol{v}\|_{L^4}).\end{aligned}\tag{3.18}$$

Thus, by (3.5) and (3.18), we have

$$\begin{aligned}\int_0^T\sigma^2\|\nabla\tilde{u}\|_{L^4}^4dt &\leqslant C\int_0^T\sigma^2(\|\mathbf{F}\|^2\|\nabla\mathbf{F}\|^2+\|\tilde{u}\|_{L^4}^2\|\nabla\tilde{u}\|_{L^4}^2\|\boldsymbol{v}\|_{L^4}^4+\|\boldsymbol{v}\|_{L^4}^4)dt\\ &\leqslant CA_3(T)+C\int_0^T\sigma^2\|\mathbf{F}\|^2\|\nabla\mathbf{F}\|^2dt+C\int_0^T\sigma^2\|\tilde{u}\|_{L^4}^2\|\nabla\tilde{u}\|_{L^4}^2\|\boldsymbol{v}\|_{L^4}^4dt\\ &=: C\theta_0^{\eta_0}+I_4+I_5.\end{aligned}\tag{3.19}$$

To control $I_4$, we need to estimate $\|\mathbf{F}\|$ and $\|\nabla\mathbf{F}\|$ first. By (2.10), (2.5) and the Cauchy-Schwarz inequality, we have from (3.4), (3.5) and (3.7) that

$$\begin{aligned}\|\mathbf{F}\| &\leqslant \|\nabla\tilde{u}+\tilde{u}\boldsymbol{v}+\boldsymbol{v}\|\\ &\leqslant \|\nabla\tilde{u}\|+\|\tilde{u}\boldsymbol{v}\|+\|\boldsymbol{v}\|\\ &\leqslant \|\nabla\tilde{u}\|+\|\tilde{u}\|_{L^4}\|\boldsymbol{v}\|_{L^4}+\|\boldsymbol{v}\|\\ &\leqslant \|\nabla\tilde{u}\|+C\|\tilde{u}\|^{\frac12}\|\nabla\tilde{u}\|^{\frac12}\|\boldsymbol{v}\|_{L^4}+\|\boldsymbol{v}\|\\ &\leqslant \|\nabla\tilde{u}\|+C\|\tilde{u}\|\|\boldsymbol{v}\|_{L^4}^2+\|\boldsymbol{v}\|\\ &\leqslant \|\nabla\tilde{u}\|+C\theta_0^{\frac{\eta_0+1}{2}}+C\theta_0^{\frac12}\\ &\leqslant \|\nabla\tilde{u}\|+C\theta_0^{\frac12},\end{aligned}\tag{3.20}$$

where we have used the assumption that $\theta_0<1$ and the fact that $\eta_0>0$.

By (2.7) and Hölder's inequality, we have

$$\|\nabla\mathbf{F}\|\leqslant C\|\tilde{u}_t\|+C\|\nabla^\perp\tilde{u}\cdot\boldsymbol{v}\|\leqslant C\|\tilde{u}_t\|+C\|\nabla\tilde{u}\|_{L^4}\|\boldsymbol{v}\|_{L^4}.\tag{3.21}$$

Then by (3.4), (3.5), (3.20), (3.21) and the Cauchy-Schwarz inequality, we estimate $I_4$ as

$$\begin{aligned}I_4 &\leqslant C\int_0^T\sigma(\|\nabla\tilde{u}\|^2+\theta_0)\sigma(\|\tilde{u}_t\|^2+\|\nabla\tilde{u}\|_{L^4}^2\|\boldsymbol{v}\|_{L^4}^2)dt\\ &\leqslant C(A_2(T)+\theta_0)\int_0^T\sigma(\|\tilde{u}_t\|^2+\|\nabla\tilde{u}\|_{L^4}^2\|\boldsymbol{v}\|_{L^4}^2)dt\\ &\leqslant C\theta_0^{\frac12}\int_0^T\sigma\|\tilde{u}_t\|^2dt+C\theta_0^{\frac12}\int_0^T\sigma^2\|\nabla\tilde{u}\|_{L^4}^4dt+C\theta_0^{\frac12}\int_0^T\|\boldsymbol{v}\|_{L^4}^4dt\\ &\leqslant C\theta_0^{\frac12}A_2(T)+C\theta_0^{\frac12}\int_0^T\sigma^2\|\nabla\tilde{u}\|_{L^4}^4dt+C\theta_0^{\frac12}A_3(T)\\ &\leqslant C\theta_0+C\theta_0^{\frac12}\int_0^T\sigma^2\|\nabla\tilde{u}\|_{L^4}^4dt+C\theta_0^{\frac12+\eta_0}\\ &\leqslant C\theta_0^{\frac12}+C\theta_0^{\frac12}\int_0^T\sigma^2\|\nabla\tilde{u}\|_{L^4}^4dt.\end{aligned}\tag{3.22}$$

Furthermore, we use (2.10), (3.5) and (3.7) to estimate $I_5$ as

$$I_5=C\int_0^T\sigma^2\|\tilde{u}\|_{L^4}^2\|\nabla\tilde{u}\|_{L^4}^2\|\boldsymbol{v}\|_{L^4}^4dt$$

$$
\begin{aligned}
&\leqslant \frac{1}{4}\int_0^T \sigma^2\|\nabla\tilde{u}\|_{L^4}^4dt + CA_3^2(T)\int_0^T \sigma^2\|\tilde{u}\|_{L^4}^4dt \\
&\leqslant \frac{1}{4}\int_0^T \sigma^2\|\nabla\tilde{u}\|_{L^4}^4dt + CA_3^2(T)\int_0^T \sigma^2\|\tilde{u}\|^2\|\nabla\tilde{u}\|^2dt \\
&\leqslant \frac{1}{4}\int_0^T \sigma^2\|\nabla\tilde{u}\|_{L^4}^4dt + CA_1^2(T)A_3^2(T) \\
&\leqslant \frac{1}{4}\int_0^T \sigma^2\|\nabla\tilde{u}\|_{L^4}^4dt + C\theta_0^{2(1+\eta_0)} \\
&\leqslant \frac{1}{4}\int_0^T \sigma^2\|\nabla\tilde{u}\|_{L^4}^4dt + C\theta_0^{\eta_0}.
\end{aligned} \tag{3.23}
$$

Substituting (3.19) with (3.22) and (3.23) and choosing $\theta_0$ small enough such that $C\theta_0^{\frac{1}{2}} \leqslant \frac{1}{4}$, we get from (3.6) that

$$
\int_0^T \sigma^2\|\nabla\tilde{u}\|_{L^4}^4dt \leqslant C\theta_0^{\eta_0} + C\theta_0^{\frac{1}{2}} \leqslant C\theta_0^{\eta_0}. \tag{3.24}
$$

By (3.17) and (3.24), we get

$$
\begin{aligned}
\left|\int_0^T \sigma^2\int_{\mathbb{R}^2} \tilde{u}\nabla\tilde{u}\nabla\tilde{u}_t dxdt\right| &\leqslant C\theta_0^{1+\eta_0} + \frac{1}{4}\int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2dt \\
&\leqslant C\theta_0 + \frac{1}{4}\int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2dt.
\end{aligned} \tag{3.25}
$$

So far we have finished estimating the third term on the right-hand side of (3.16). Next, we proceed to estimate the last term on the right-hand side of (3.16). To this end, we use the Cauchy-Schwarz inequality, (2.10) and (3.5) to get

$$
\begin{aligned}
\int_0^T \sigma^2\int_{\mathbb{R}^2} \tilde{u}_t\boldsymbol{v}\nabla\tilde{u}_t dxdt &\leqslant \frac{1}{4}\int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2dt + \int_0^T \sigma^2\|\tilde{u}_t\boldsymbol{v}\|^2dt \\
&\leqslant \frac{1}{4}\int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2dt + \int_0^T \sigma^2\|\tilde{u}_t\|_{L^4}^2\|\boldsymbol{v}\|_{L^4}^2dt \\
&\leqslant \frac{1}{4}\int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2dt + C\int_0^T \sigma^2\|\tilde{u}_t\|\|\nabla\tilde{u}_t\|\|\boldsymbol{v}\|_{L^4}^2dt \\
&\leqslant \frac{1}{2}\int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2dt + C\int_0^T \sigma^2\|\tilde{u}_t\|^2\|\boldsymbol{v}\|_{L^4}^4dt \\
&\leqslant \frac{1}{2}\int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2dt + C\theta_0^{\eta_0}\int_0^T \sigma^2\|\tilde{u}_t\|^2dt.
\end{aligned} \tag{3.26}
$$

Substituting (3.16) with (3.25) and (3.26), we get from $0 \leqslant \sigma \leqslant 1$ and $\theta_0 \leqslant 1$ that

$$
\sigma^2\|\tilde{u}_t\|^2 + \sigma^2\|\boldsymbol{v}_t\|^2 + \int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2dt \leqslant C\int_0^T \sigma\|\tilde{u}_t\|^2dt + C\theta_0,
$$

which, along with (3.14), immediately leads to

$$
\sigma^2\|\tilde{u}_t\|^2 + \sigma^2\|\boldsymbol{v}_t\|^2 + \int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2dt \leqslant C\lambda\int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2dt + C\theta_0.
$$

Choosing $\lambda$ small enough such that $C\lambda \leqslant \frac{1}{2}$, we have

$$
\sigma^2\|\tilde{u}_t\|^2 + \sigma^2\|\boldsymbol{v}_t\|^2 + \int_0^T \sigma^2\|\nabla\tilde{u}_t\|^2dt \leqslant C\theta_0,
$$

which, together with (3.14), gives

$$\sigma\|\nabla\tilde{u}\|^2+\sigma^2\|\tilde{u}_t\|^2+\sigma^2\|\boldsymbol{v}_t\|^2+\int_0^T\sigma\|\tilde{u}_t\|^2dt+\int_0^T\sigma^2\|\nabla\tilde{u}_t\|^2dt\leqslant C\theta_0. \tag{3.27}$$

Taking $\theta_0$ small enough such that $C\theta_0^{\frac{1}{2}}\leqslant 1$, we have from (3.27) that

$$\sigma\|\nabla\tilde{u}\|^2+\sigma^2\|\tilde{u}_t\|^2+\sigma^2\|\boldsymbol{v}_t\|^2+\int_0^T\sigma\|\tilde{u}_t\|^2dt+\int_0^T\sigma^2\|\nabla\tilde{u}_t\|^2dt\leqslant \theta_0^{\frac{1}{2}}.$$

Thus, the proof of Lemma 3.2 is completed. □

Next, we derive the *a priori* estimate of $\|\boldsymbol{v}\|_{L^4}$. For this, we first derive some estimates that will be used later.

**Lemma 3.3.** *Let the conditions of Theorem* 1.2 *hold and* $(\tilde{u},\boldsymbol{v})$ *be a smooth solution to* (3.1)–(3.3) *satisfying* (3.5). *Then it holds that*

$$\sigma^4\|\tilde{u}\|_{L^\infty}^4\leqslant C\theta_0.$$

*Furthermore, for any* $t\in(\sigma(T),T]$, *it holds that*

$$-\frac{1}{4}\leqslant\tilde{u}(x,t)\leqslant\frac{1}{4}. \tag{3.28}$$

*Proof.* By the Gagliardo-Nirenberg inequality (2.10), we have from (3.7) and (3.9) that

$$\begin{aligned}\sigma^4\|\tilde{u}\|_{L^\infty}^4&\leqslant C\sigma^4\|\tilde{u}\|_{L^4}^2\|\nabla\tilde{u}\|_{L^4}^2\\&\leqslant C\sigma^4\|\tilde{u}\|\|\nabla\tilde{u}\|\|\nabla\tilde{u}\|_{L^4}^2\\&\leqslant C\sigma^{\frac{3}{2}}\|\tilde{u}\|(\sigma^{\frac{1}{2}}\|\nabla\tilde{u}\|)\sigma^2\|\nabla\tilde{u}\|_{L^4}^2\\&\leqslant C\sigma^{\frac{3}{2}}A_1^{\frac{1}{2}}(T)A_2^{\frac{1}{2}}(T)\sigma^2\|\nabla\tilde{u}\|_{L^4}^2\\&\leqslant C\theta_0^{\frac{3}{4}}\sigma^2\|\nabla\tilde{u}\|_{L^4}^2.\end{aligned} \tag{3.29}$$

With (2.10), (3.6), (3.7) and (3.18), we have the following estimates:

$$\begin{aligned}\|\nabla\tilde{u}\|_{L^4}&\leqslant C(\|\mathbf{F}\|^{\frac{1}{2}}\|\nabla\mathbf{F}\|^{\frac{1}{2}}+\|\tilde{u}\|_{L^4}^{\frac{1}{2}}\|\nabla\tilde{u}\|_{L^4}^{\frac{1}{2}}\|\boldsymbol{v}\|_{L^4}+\|\boldsymbol{v}\|_{L^4})\\&\leqslant\frac{1}{2}\|\nabla\mathbf{F}\|+C\|\mathbf{F}\|+\frac{1}{2}\|\nabla\tilde{u}\|_{L^4}+C\|\tilde{u}\|_{L^4}\|\boldsymbol{v}\|_{L^4}^2+C\|\boldsymbol{v}\|_{L^4}\\&\leqslant\frac{1}{2}\|\nabla\mathbf{F}\|+C\|\mathbf{F}\|+\frac{1}{2}\|\nabla\tilde{u}\|_{L^4}+C\|\tilde{u}\|^{\frac{1}{2}}\|\nabla\tilde{u}\|^{\frac{1}{2}}\|\boldsymbol{v}\|_{L^4}^2+C\|\boldsymbol{v}\|_{L^4}\\&\leqslant\frac{1}{2}\|\nabla\mathbf{F}\|+C\|\mathbf{F}\|+\frac{1}{2}\|\nabla\tilde{u}\|_{L^4}+C\|\nabla\tilde{u}\|+C\|\tilde{u}\|\|\boldsymbol{v}\|_{L^4}^4+C\|\boldsymbol{v}\|_{L^4}\\&\leqslant\frac{1}{2}\|\nabla\mathbf{F}\|+C\|\mathbf{F}\|+\frac{1}{2}\|\nabla\tilde{u}\|_{L^4}+C\|\nabla\tilde{u}\|+CA_1^{\frac{1}{2}}(T)A_3(T)+CA_3^{\frac{1}{4}}(T)\\&\leqslant\frac{1}{2}\|\nabla\mathbf{F}\|+C\|\mathbf{F}\|+\frac{1}{2}\|\nabla\tilde{u}\|_{L^4}+C\|\nabla\tilde{u}\|+C\theta_0^{\frac{1}{2}+\eta_0}+C\theta_0^{\frac{\eta_0}{4}}\\&\leqslant\frac{1}{2}\|\nabla\mathbf{F}\|+C\|\mathbf{F}\|+\frac{1}{2}\|\nabla\tilde{u}\|_{L^4}+C\|\nabla\tilde{u}\|+C\theta_0^{\frac{\eta_0}{4}}.\end{aligned} \tag{3.30}$$

Then by (3.5), (3.20) and (3.21), one has

$$\|\mathbf{F}\|\leqslant C\|\nabla\tilde{u}\|+C\theta_0^{\frac{1}{2}} \tag{3.31}$$

and

$$\|\nabla\mathbf{F}\|\leqslant C\|\tilde{u}_t\|+C\theta_0^{\frac{\eta_0}{4}}\|\nabla\tilde{u}\|_{L^4}. \tag{3.32}$$

Substituting (3.30) with (3.31) and (3.32), we get

$$\|\nabla\tilde{u}\|_{L^4} \leqslant C\theta_0^{\frac{1}{2}} + C\|\nabla\tilde{u}\| + C\|\tilde{u}_t\| + C\theta_0^{\frac{\eta_0}{4}}\|\nabla\tilde{u}\|_{L^4} + C\theta_0^{\frac{\eta_0}{4}}. \tag{3.33}$$

Choosing $\theta_0$ small enough such that $C\theta_0^{\frac{\eta_0}{4}} \leqslant \frac{1}{2}$, using (3.6) and (3.33), we end up with

$$\|\nabla\tilde{u}\|_{L^4} \leqslant C\theta_0^{\frac{\eta_0}{4}} + C\|\nabla\tilde{u}\| + C\|\tilde{u}_t\|, \tag{3.34}$$

which, along with (3.32) and $\theta_0 \leqslant 1$, yields

$$\|\nabla\mathbf{F}\| \leqslant C\theta_0^{\frac{\eta_0}{2}} + C\|\tilde{u}_t\| + C\|\nabla\tilde{u}\|. \tag{3.35}$$

Then by the fact that $0 \leqslant \sigma \leqslant 1$, (3.6), (3.9) and (3.34), it holds that

$$\begin{aligned}
\sigma^2\|\nabla\tilde{u}\|_{L^4}^2 &\leqslant C\sigma^2\theta_0^{\frac{\eta_0}{2}} + C\sigma^2\|\nabla\tilde{u}\|^2 + C\sigma^2\|\tilde{u}_t\|^2 \\
&\leqslant C\theta_0^{\frac{\eta_0}{2}} + CA_2(T) \\
&\leqslant C\theta_0^{\frac{\eta_0}{2}} + C\theta_0^{\frac{1}{2}} \\
&\leqslant C\theta_0^{\frac{\eta_0}{2}}.
\end{aligned} \tag{3.36}$$

With (3.36), (3.29) is updated as

$$\sigma^4\|\tilde{u}\|_{L^\infty}^4 \leqslant C\theta_0^{\frac{3+2\eta_0}{4}} \leqslant C\theta_0^{\frac{3}{4}}.$$

Choosing $\theta_0$ small enough such that $C\theta_0^{\frac{3}{4}} \leqslant 4^{-4}$ and using the fact $\sigma(t) = 1$ for $t \in (\sigma(T), T]$, we have

$$\sup_{t\in[\sigma(T),T]} |\tilde{u}(x,t)| \leqslant \frac{1}{4}, \tag{3.37}$$

which implies

$$-\frac{1}{4} \leqslant \tilde{u}(x,t) \leqslant \frac{1}{4} \quad \text{for } t \in (\sigma(T), T].$$

Thus, the proof of Lemma 3.3 is completed. □

**Lemma 3.4.** *Let the conditions of Theorem* 1.2 *hold and* $(\tilde{u}, \boldsymbol{v})$ *be a smooth solution to* (3.1)–(3.3) *satisfying* (3.5). *Then it holds that*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^2} \boldsymbol{v}^4 dx + \int_0^t \int_{\mathbb{R}^2} \boldsymbol{v}^4 dxdt \leqslant \theta_0^{\eta_0},$$

*where* $\eta_0 \triangleq \frac{p_0-4}{2(p_0-2)}$.

*Proof.* First, it follows from $\boldsymbol{v}_t = \nabla\tilde{u}$ and (2.5) that

$$\boldsymbol{v}_t + (\tilde{u}+1)\boldsymbol{v} = \mathbf{F}. \tag{3.38}$$

Multiplying (3.38) by $|\boldsymbol{v}|^2\mathbf{v}$ and integrating the resulting equality over $\mathbb{R}^2$, one has

$$\frac{1}{4}\left(\int_{\mathbb{R}^2} \boldsymbol{v}^4 dx\right)_t + \int_{\mathbb{R}^2} (\tilde{u}+1)\boldsymbol{v}^4 dx = \int_{\mathbb{R}^2} \mathbf{F}|\boldsymbol{v}|^2\boldsymbol{v} dx. \tag{3.39}$$

Integrating the above equality over $[\sigma(T), T]$ and using (3.28), we have

$$\frac{1}{4}\int_{\mathbb{R}^2} \boldsymbol{v}^4 dx + \frac{3}{4}\int_{\sigma(T)}^T \int_{\mathbb{R}^2} \boldsymbol{v}^4 dxdt \leqslant \sup_{t\in[0,\sigma(T)]} \left(\frac{1}{4}\int_{\mathbb{R}^2} \boldsymbol{v}^4 dx\right) + \int_{\sigma(T)}^T \int_{\mathbb{R}^2} |\mathbf{F}||\boldsymbol{v}|^3 dxdt. \tag{3.40}$$

By the interpolation inequality and (3.5), for any $2 < p < p_0$, we have

$$\|\boldsymbol{v}\|_{L^p} \leqslant \|\boldsymbol{v}\|^{\frac{2(p_0-p)}{p(p_0-2)}} \|\boldsymbol{v}\|_{L^{p_0}}^{\frac{p_0(p-2)}{p(p_0-2)}} \leqslant C\theta_0^{\frac{(p_0-p)}{p(p_0-2)}} M^{\frac{p_0(p-2)}{p(p_0-2)}},$$

which implies that

$$\|\boldsymbol{v}\|_{L^4} \leqslant C\theta_0^{\frac{(p_0-4)}{2(p_0-2)}} M^{\frac{p_0}{2(p_0-2)}} \leqslant C(M)\theta_0^{\frac{(p_0-4)}{2(p_0-2)}} \leqslant C(M)\theta_0^{\eta_0}. \tag{3.41}$$

We need to further estimate the last term in (3.40). By Young's inequality, we have

$$\int_{\sigma(T)}^T \int_{\mathbb{R}^2} |\mathbf{F}||\boldsymbol{v}|^3 dxdt \leqslant \frac{1}{4} \int_{\sigma(T)}^T \int_{\mathbb{R}^2} \boldsymbol{v}^4 dxdt + C \int_{\sigma(T)}^T \int_{\mathbb{R}^2} |\mathbf{F}|^4 dxdt,$$

which updates (3.40) as

$$\begin{aligned} \frac{1}{4} \int_{\mathbb{R}^2} \boldsymbol{v}^4 dx + \frac{1}{2} \int_{\sigma(T)}^T \int_{\mathbb{R}^2} \boldsymbol{v}^4 dxdt &\leqslant \sup_{t \in [0,\sigma(T)]} \left( \frac{1}{4} \int_{\mathbb{R}^2} \boldsymbol{v}^4 dx \right) + C \int_{\sigma(T)}^T \int_{\mathbb{R}^2} |\mathbf{F}|^4 dxdt \\ &\leqslant C(M)\theta_0^{4\eta_0} + C \int_{\sigma(T)}^T \int_{\mathbb{R}^2} |\mathbf{F}|^4 dxdt, \end{aligned} \tag{3.42}$$

where we have used (3.41). By the Gagliardo-Nirenberg inequality (2.10), one has

$$\begin{aligned} \int_{\sigma(T)}^T \|\mathbf{F}\|_{L^4}^4 dt &\leqslant C \int_{\sigma(T)}^T \|\mathbf{F}\|^2 \|\nabla \mathbf{F}\|^2 dt \\ &\leqslant C \Big( \sup_{\sigma(T) \leqslant t \leqslant T} \|\mathbf{F}\|^2 \Big) \int_{\sigma(T)}^T \|\nabla \mathbf{F}\|^2 dt. \end{aligned} \tag{3.43}$$

It follows from (3.9) and (3.20) that

$$\sup_{\sigma(T) \leqslant t \leqslant T} \|\mathbf{F}\|^2 \leqslant C \sup_{\sigma(T) \leqslant t \leqslant T} \sigma \|\nabla \tilde{u}\|^2 + C\theta_0 \leqslant C\theta_0^{\frac{1}{2}} + C\theta_0 \leqslant C\theta_0^{\frac{1}{2}}, \tag{3.44}$$

where the fact that $\sigma(t) = 1$ for $t \geqslant \sigma(T)$ has been used. By (3.9) and (3.21), we have

$$\begin{aligned} \int_{\sigma(T)}^T \|\nabla \mathbf{F}\|^2 dt &\leqslant C \int_{\sigma(T)}^T \sigma \|\tilde{u}_t\|^2 dt + C \int_{\sigma(T)}^T \sigma \|\nabla \tilde{u}\|_{L^4}^2 \|\boldsymbol{v}\|_{L^4}^2 dt \\ &\leqslant C\theta_0^{\frac{1}{2}} + C \int_{\sigma(T)}^T \sigma \|\nabla \tilde{u}\|_{L^4}^2 \|\boldsymbol{v}\|_{L^4}^2 dt. \end{aligned} \tag{3.45}$$

Then with (3.24), the Cauchy-Schwarz inequality and (3.5), we get

$$C \int_{\sigma(T)}^T \sigma \|\nabla \tilde{u}\|_{L^4}^2 \|\boldsymbol{v}\|_{L^4}^2 dt \leqslant C \int_{\sigma(T)}^T \sigma^2 \|\nabla \tilde{u}\|_{L^4}^4 dt + C \int_{\sigma(T)}^T \|\boldsymbol{v}\|_{L^4}^4 dt \leqslant C\theta_0^{\eta_0},$$

which, along with (3.45) gives

$$\int_{\sigma(T)}^T \|\nabla \mathbf{F}\|^2 dt \leqslant C\theta_0^{\eta_0}.$$

This together with (3.43) and (3.44) leads to

$$\int_{\sigma(T)}^T \|\mathbf{F}\|_{L^4}^4 dt \leqslant C\theta_0^{\frac{1}{2}+\eta_0}. \tag{3.46}$$

Substituting (3.42) with (3.46) yields

$$\sup_{t \in [\sigma(T), T]} \int_{\mathbb{R}^2} \boldsymbol{v}^4 dx + \int_{\sigma(T)}^T \int_{\mathbb{R}^2} \boldsymbol{v}^4 dxdt \leqslant C(M)\theta_0^{4\eta_0} + C\theta_0^{\frac{1}{2}+\eta_0}.$$

For $0 \leqslant t \leqslant \sigma(T)$, we have from (3.41) that

$$\sup_{t\in[0,\sigma(T)]}\int_{\mathbb{R}^2}\boldsymbol{v}^4dx+\int_0^{\sigma(T)}\int_{\mathbb{R}^2}\boldsymbol{v}^4dxdt\leqslant 2\sup_{t\in[0,\sigma(T)]}\int_{\mathbb{R}^2}\boldsymbol{v}^4dx\leqslant C(M)\theta_0^{4\eta_0}.$$

Coupling the above two inequalities together, we arrive at

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^2}\boldsymbol{v}^4dx+\int_0^{t}\int_{\mathbb{R}^2}\boldsymbol{v}^4dxdt\leqslant C(M)\theta_0^{4\eta_0}+C\theta_0^{\frac{1}{2}+\eta_0}\leqslant\theta_0^{\eta_0}, \tag{3.47}$$

provided that $C(M)\theta_0^{3\eta_0}\leqslant\frac{1}{2}$ and $C\theta_0^{\frac{1}{2}}\leqslant\frac{1}{2}$. This completes the proof of Lemma 3.4. □

**Lemma 3.5.** *Let the conditions of Theorem* 1.2 *hold and* $(\tilde{u},\boldsymbol{v})$ *be a smooth solution to* (3.1)–(3.3) *satisfying* (3.5). *Then it holds that*

$$\sup_{t\in[0,T]}\|\boldsymbol{v}\|_{L^{p_0}}\leqslant 3M. \tag{3.48}$$

*Proof.* Multiplying (3.38) by $|\boldsymbol{v}|^{p_0-2}\boldsymbol{v}$ $(4<p_0<\infty)$ and integrating the resulting equality over $\mathbb{R}^2$, one has

$$\frac{1}{p_0}\left(\int_{\mathbb{R}^2}|\boldsymbol{v}|^{p_0}dx\right)_t+\int_{\mathbb{R}^2}(\tilde{u}+1)|\boldsymbol{v}|^{p_0}dx=\int_{\mathbb{R}^2}\mathbf{F}|\boldsymbol{v}|^{p_0-2}\boldsymbol{v}dx. \tag{3.49}$$

For $t\in[0,\sigma(T)]$, it is known that $u=\tilde{u}+1\geqslant 0$. Then Hölder's inequality yields that $\frac{1}{p_0}(\|\boldsymbol{v}\|_{L^{p_0}}^{p_0})_t\leqslant\|\mathbf{F}\|_{L^{p_0}}\|\boldsymbol{v}\|_{L^{p_0}}^{p_0-1}$, which hence leads to

$$(\|\boldsymbol{v}\|_{L^{p_0}})_t\leqslant\|\mathbf{F}\|_{L^{p_0}}.$$

Integrating the above equality over $[0,\sigma(T)]$, we have

$$\sup_{t\in[0,\sigma(T)]}\|\boldsymbol{v}\|_{L^{p_0}}\leqslant\|\boldsymbol{v}_0\|_{L^{p_0}}+\int_0^{\sigma(T)}\|\mathbf{F}\|_{L^{p_0}}dt. \tag{3.50}$$

The last term in (3.50) can be estimated by the Gagliardo-Nirenberg inequality (2.10) as

$$\int_0^{\sigma(T)}\|\mathbf{F}\|_{L^{p_0}}dt\leqslant C\int_0^{\sigma(T)}\|\mathbf{F}\|^{\frac{2}{p_0}}\|\nabla\mathbf{F}\|^{\frac{p_0-2}{p_0}}dt. \tag{3.51}$$

By (3.9) and (3.31), one gets

$$\sup_{t\in[0,\sigma(T)]}\sigma^{\frac{1}{2}}\|\mathbf{F}\|\leqslant C\sigma^{\frac{1}{2}}\|\nabla\tilde{u}\|+C\sigma^{\frac{1}{2}}\theta_0^{\frac{1}{2}}\leqslant C\theta_0^{\frac{1}{4}}. \tag{3.52}$$

It then follows from (3.6), (3.7), (3.9), (3.21) and (3.24) that

$$\begin{aligned}\int_0^T\sigma\|\nabla\mathbf{F}\|^2dt&\leqslant C\int_0^T\sigma\|\tilde{u}_t\|^2dt+C\int_0^T\sigma\|\nabla\tilde{u}\|_{L^4}^2\|\boldsymbol{v}\|_{L^4}^2dt\\&\leqslant C\int_0^T\sigma\|\tilde{u}_t\|^2dt+C\int_0^T(\sigma^2\|\nabla\tilde{u}\|_{L^4}^4+\|\boldsymbol{v}\|_{L^4}^4)dt\\&\leqslant C\theta_0^{\frac{1}{2}}+C\theta_0^{\eta_0}\leqslant C\theta_0^{\eta_0}.\end{aligned} \tag{3.53}$$

Substituting (3.51) with (3.52)–(3.53) and using (3.9) with Hölder's inequality, we end up with

$$\begin{aligned}\int_0^{\sigma(T)}\|\mathbf{F}\|_{L^{p_0}}dt&\leqslant C\int_0^{\sigma(T)}\|\mathbf{F}\|^{\frac{2}{p_0}}\|\nabla\mathbf{F}\|^{\frac{p_0-2}{p_0}}dt\\&\leqslant\int_0^{\sigma(T)}(\sigma^{\frac{1}{2}}\|\mathbf{F}\|)^{\frac{2}{p_0}}(\sigma\|\nabla\mathbf{F}\|^2)^{\frac{p_0-2}{2p_0}}\sigma^{-\frac{1}{2}}dt\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\theta_0^{\frac{1}{2p_0}}\int_0^{\sigma(T)}(\sigma\|\nabla\mathbf{F}\|^2)^{\frac{p_0-2}{2p_0}}\sigma^{-\frac12}dt\\
&\leqslant C\theta_0^{\frac{1}{2p_0}}\left(\int_0^{\sigma(T)}\sigma\|\nabla\mathbf{F}\|^2dt\right)^{\frac{p_0-2}{2p_0}}\left(\int_0^{\sigma(T)}\sigma^{-\frac12\times\frac{2p_0}{p_0+2}}dt\right)^{\frac{p_0+2}{2p_0}}\\
&\leqslant C\theta_0^{\frac{1+\eta_0(p_0-2)}{2p_0}}\left(\int_0^{\sigma(T)}t^{-\frac{p_0}{p_0+2}}dt\right)^{\frac{p_0+2}{2p_0}}\\
&\leqslant C\theta_0^{\frac{p_0-2}{4p_0}},
\end{aligned}
$$

which, along with (3.50), gives

$$
\sup_{t\in[0,\sigma(T)]}\|\boldsymbol{v}\|_{L^{p_0}}\leqslant M+C\theta_0^{\frac{p_0-2}{4p_0}}\leqslant\frac{3M}{2} \tag{3.54}
$$

provided that $C\theta_0^{\frac{p_0-2}{4p_0}}\leqslant\frac{M}{2}$.

Next, consider the estimate of $\|\boldsymbol{v}\|_{L^{p_0}}$ for $t\in[\sigma(T),T]$. Indeed integrating (3.49) over $[\sigma(T),T]$ and using $\tilde{u}\geqslant-\frac14$, we have

$$
\frac{1}{p_0}\int_{\mathbb{R}^2}|\boldsymbol{v}|^{p_0}dx+\frac34\int_{\sigma(T)}^T\int_{\mathbb{R}^2}|\boldsymbol{v}|^{p_0}dxdt\leqslant\sup_{t\in[0,\sigma(T)]}\left(\frac{1}{p_0}\int_{\mathbb{R}^2}|\boldsymbol{v}|^{p_0}dx\right)+\int_{\sigma(T)}^T\int_{\mathbb{R}^2}|\mathbf{F}||\boldsymbol{v}|^{p_0-1}dxdt. \tag{3.55}
$$

We proceed to estimate the last term in (3.55). By Young's inequality

$$
ab\leqslant\epsilon a^q+(\epsilon q)^{-\frac{r}{q}}r^{-1}b^r,\quad a,b\geqslant0,\quad\epsilon>0,\quad q,r>0,\quad\frac1q+\frac1r=1,
$$

we see that

$$
\int_{\sigma(T)}^T\int_{\mathbb{R}^2}|\mathbf{F}||\boldsymbol{v}|^{p_0-1}dxdt\leqslant\frac12\int_{\sigma(T)}^T\int_{\mathbb{R}^2}|\boldsymbol{v}|^{p_0}dxdt+\frac{1}{p_0}\left(\frac{2p_0-2}{p_0}\right)^{p_0-1}\int_{\sigma(T)}^T\int_{\mathbb{R}^2}|\mathbf{F}|^{p_0}dxdt,
$$

which updates (3.55) as

$$
\begin{aligned}
&\int_{\mathbb{R}^2}|\boldsymbol{v}|^{p_0}dx+\frac{p_0}{4}\int_{\sigma(T)}^T\int_{\mathbb{R}^2}|\boldsymbol{v}|^{p_0}dxdt\\
&\quad\leqslant\sup_{t\in[0,\sigma(T)]}\left(\int_{\mathbb{R}^2}|\boldsymbol{v}|^{p_0}dx\right)+\left(\frac{2p_0-2}{p_0}\right)^{p_0-1}\int_{\sigma(T)}^T\int_{\mathbb{R}^2}|\mathbf{F}|^{p_0}dxdt.
\end{aligned} \tag{3.56}
$$

On the other hand, from (2.10), (2.6) and (3.43), one has

$$
\int_{\sigma(T)}^T\|\mathbf{F}\|_{L^{p_0}}^{p_0}dt\leqslant C\int_{\sigma(T)}^T\|\mathbf{F}\|^2\|\nabla\mathbf{F}\|^{p_0-2}dt\leqslant C\sup_{t\in[\sigma(T),T]}\|\mathbf{F}\|^2\|\nabla\mathbf{F}\|^{p_0-4}\int_{\sigma(T)}^T\|\nabla\mathbf{F}\|^2dt. \tag{3.57}
$$

Then we have from (3.52) and (3.53) that

$$
\sup_{t\in[\sigma(T),T]}\|\mathbf{F}\|^2\leqslant\sup_{t\in[\sigma(T),T]}(\sigma\|\mathbf{F}\|^2)\leqslant C\theta_0^{\frac12}
$$

and

$$
\int_{\sigma(T)}^T\|\nabla\mathbf{F}\|^2dt=\int_{\sigma(T)}^T\sigma\|\nabla\mathbf{F}\|^2dt\leqslant C\theta_0^{\eta_0},
$$

where the fact that $\sigma(t)=1$ for $t\in[\sigma(T),T]$ has been used. By (3.6), (3.9) and (3.35), we get

$$
\sup_{t\in[\sigma(T),T]}\|\nabla\mathbf{F}\|\leqslant C\theta_0^{\frac{\eta_0}{2}}+C\sigma\|\tilde{u}_t\|+C\sigma^{\frac12}\|\nabla\tilde{u}\|\leqslant C\theta_0^{\frac{\eta_0}{2}}+\theta_0^{\frac14}\leqslant C\theta_0^{\frac{\eta_0}{2}},
$$

which implies

$$\sup_{t\in[\sigma(T),T]} \|\nabla \mathbf{F}\|^{p_0-4} \leqslant C\theta_0^{\frac{\eta_0(p_0-4)}{2}}.$$

Substituting (3.57) with the above inequalities, we have from (3.6) that

$$\int_{\sigma(T)}^{T} \|\mathbf{F}\|_{L^{p_0}}^{p_0} dt \leqslant C\theta_0^{\frac{\eta_0(p_0-2)+1}{2}} = C\theta_0^{\frac{p_0-2}{4}}. \tag{3.58}$$

It follows from (3.54), (3.56) and (3.58) that

$$\begin{aligned}\sup_{t\in[\sigma(T),T]} \int_{\mathbb{R}^2} \boldsymbol{v}^{p_0} dx + \frac{p_0}{4}\int_{\sigma(T)}^{T}\int_{\mathbb{R}^2} \boldsymbol{v}^{p_0} dxdt &\leqslant \left(\frac{3M}{2}\right)^{p_0} + C\left(\frac{2p_0-2}{p_0}\right)^{p_0-1}\theta_0^{\frac{p_0-2}{4}} \\ &\leqslant \left(\frac{3M}{2}\right)^{p_0} + C\theta_0^{\frac{p_0-2}{4}}2^{p_0-1} \\ &\leqslant (3M)^{p_0} \end{aligned} \tag{3.59}$$

provided that $\theta_0^{\frac{p_0-2}{4p_0}} \leqslant \frac{3M}{2}$. Then, raising the power $\frac{1}{p_0}$ to both sides of (3.59), we obtain that

$$\sup_{t\in[\sigma(T),T])} \|\boldsymbol{v}\|_{L^{p_0}} \leqslant 3M,$$

which combined with (3.54) yields that

$$\sup_{t\in[0,T]} \|\boldsymbol{v}\|_{L^{p_0}} \leqslant 3M.$$

Thus, the proof of Lemma 3.5 is finished. □

With the help of Lemmas 3.1–3.5, we have

$$A_1(T) \leqslant \frac{3}{2}\theta_0, \quad A_2(T) \leqslant \theta_0^{\frac{1}{2}}, \quad A_3(T) \leqslant \theta_0^{\eta_0}, \quad \sup_{t\in[0,T]} \|\boldsymbol{v}\|_{L^{p_0}} \leqslant 3M, \tag{3.60}$$

which closes the *a priori* assumption (3.5).

## 4 Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by constructing weak solutions as limits of approximate smooth solutions. We start with the global-in-time existence of smooth solutions to (3.1).

**Proposition 4.1.** *Assume that the initial data satisfies* $(\tilde{u}_0^\delta, \boldsymbol{v}_0^\delta) \in H^3(\mathbb{R}^2)$ *and* $\theta_0 \leqslant \varepsilon$. *Then there exists a unique solution to the system* (3.1) *such that* $(\tilde{u}^\delta, \boldsymbol{v}^\delta) \in L^\infty([0,\infty), H^3)$.

*Proof.* By Lemma 2.1, there exists a $T_* > 0$ such that the Cauchy problem (3.1)–(3.3) has a unique solution on $\mathbb{R}^2 \times (0, T_*]$. First, it follows from (1.7), (3.3) and (3.4) that

$$A_1(0) \leqslant \theta_0, \quad A_2(0) = 0, \quad \|\boldsymbol{v}_0^\delta\|_{L^{p_0}} \leqslant M.$$

By (3.41), we have

$$A_3(0) = \|\boldsymbol{v}_0^\delta\|_{L^4}^4 \leqslant C(M)\theta_0^{4\eta_0} \leqslant \theta_0^{\eta_0}$$

provided that $C(M)\theta_0^{3\eta_0} \leqslant 1$. Therefore, there exists a $T_1 \in (0, T_*]$ such that (3.5) holds for $T = T_1$.

Now, we set

$$T^* = \sup\{T \mid (3.5) \text{ holds}\}. \tag{4.1}$$

Then, $T^* > T_1 > 0$. Next, we claim that

$$T^* = \infty. \tag{4.2}$$

Otherwise, if $T^* < \infty$, by the blowup criterion (2.3) with $d = 2$ and $q = 4$, we have that if

$$\int_0^{T^*} \|\boldsymbol{v}^\delta\|_{L^4}^4 < \infty, \tag{4.3}$$

then the solution can be extended beyond $T^*$. From Lemmas 3.1–3.5 and (4.1), we see that (3.60) holds for $T = T^*$, which immediately implies (4.3). This means that there exists a $T^{**} > T^*$ such that

$$(u-1, \boldsymbol{v}) \in L^\infty([0, T^{**}], H^3)$$

and (3.5) holds for $T = T^{**}$, which contradicts (4.1). Hence (4.2) holds. Using (4.2) and the blowup criterion (2.3) again, we complete the proof of Proposition 4.1. □

Note that Proposition 4.1 holds for any given positive constant $\delta$ (where $T_*, T_1$ and $T^*$ may depend on $\delta$). We cannot obtain the $\delta$-independent estimates of the global solution $(u^\delta - 1,\ \boldsymbol{v}^\delta)$ in the $H^3$-norm. However, we can pass to the limit as $\delta \to 0$ in the proper functional space in which the $\delta$-independent estimates of the solution are obtained, as shown below.

From (3.60), we have the uniform-in-$\delta$ estimates as follows:

$$\begin{cases} \sup\limits_{t\in[0,T]} (\|u^\delta - 1\|^2 + \|\boldsymbol{v}^\delta\|^2) + \displaystyle\int_0^T \|\nabla u^\delta\|^2 dt \leqslant \frac{3\theta_0}{2}, \\ \sup\limits_{t\in[0,T]} (\sigma\|\nabla u^\delta\|^2 + \sigma^2\|u_t^\delta\|^2 + \sigma^2\|\boldsymbol{v}_t^\delta\|^2) + \displaystyle\int_0^T \sigma\|u_t^\delta\|^2 dt + \int_0^T \sigma^2\|\nabla u_t^\delta\|^2 dt \leqslant \theta_0^{\frac{1}{2}}, \\ \sup\limits_{t\in[0,T]} \|\boldsymbol{v}^\delta\|_{L^4}^4 + \displaystyle\int_0^t \|\boldsymbol{v}^\delta\|_{L^4}^4 dt \leqslant \theta_0^{\eta_0}, \\ \sup\limits_{t\in[0,T]} \|\boldsymbol{v}^\delta\|_{L^{p_0}} \leqslant 3M. \end{cases} \tag{4.4}$$

On the other hand, by (2.10), (4.4) and (3.36), one has

$$\sigma\|u^\delta - 1\|_{L^4}^2 \leqslant C\|u^\delta - 1\|\sigma\|\nabla u^\delta\| \leqslant C, \quad \sigma^2\|\nabla u^\delta\|_{L^4}^2 \leqslant C. \tag{4.5}$$

Noticing that $\sigma = \min\{1, t\}$, we have from (4.4) and (4.5) that

$$\begin{cases} u^\delta - 1 \in L^\infty([0,\infty), L^2(\mathbb{R}^2)), \quad (v_t^\delta, \nabla u^\delta) \in L^2([0,\infty), L^2(\mathbb{R}^2)), \\ u^\delta - 1 \in L^\infty((0,\infty), W^{1,4}(\mathbb{R}^2)), \quad u_t^\delta \in L^2((0,\infty), H^1(\mathbb{R}^2)), \\ \boldsymbol{v}^\delta \in L^\infty([0,\infty); L^2(\mathbb{R}^2) \cap L^{p_0}(\mathbb{R}^2)) \cap L^4([0,\infty); L^4(\mathbb{R}^2)). \end{cases} \tag{4.6}$$

By (4.6) and the Aubin-Lions-Simon lemma (see [43]), we can extract a subsequence, still denoted by $(u^\delta, \boldsymbol{v}^\delta)$, such that the following convergence holds as $\delta \to 0$:

$$\begin{cases} \boldsymbol{v}^\delta(\cdot,t) \to \boldsymbol{v} \text{ strongly in } C([0,\infty), H^{-1}(\mathbb{R}^2)), \\ u^\delta(\cdot,t) \to u(\cdot,t) \text{ strongly in } C((0,\infty), C(\mathbb{R}^2)), \\ \nabla u^\delta(\cdot,t) \to \nabla u(\cdot,t) \text{ weakly in } L^2([0,\infty), L^2(\mathbb{R}^2)). \end{cases}$$

Thus, the limit function $(u, \boldsymbol{v})$ is indeed a weak solution to the system (1.3)–(1.5) and inherits all the bounds of (4.4) which yield (1.8) and

$$\begin{cases} \sup\limits_{t\in[0,T]}(\|u-1\|^2+\|\boldsymbol{v}\|^2)+\displaystyle\int_0^T\|\nabla u\|^2dt\leqslant \frac{3\theta_0}{2}, \\ \sup\limits_{t\in[0,T]}(\sigma\|\nabla u\|^2+\sigma^2\|u_t\|^2+\sigma^2\|\boldsymbol{v}_t\|^2)+\displaystyle\int_0^T\sigma\|u_t\|^2dt+\int_0^T\sigma^2\|\nabla u_t\|^2dt\leqslant\theta_0^{\frac{1}{2}}, \\ \sup\limits_{t\in[0,T]}\|\boldsymbol{v}\|_{L^4}^4+\displaystyle\int_0^t\|\boldsymbol{v}\|_{L^4}^4dt\leqslant\theta_0^{\eta_0}, \\ \sup\limits_{t\in[0,T]}\|\boldsymbol{v}\|_{L^{p_0}}\leqslant 3M. \end{cases} \tag{4.7}$$

To complete the proof of Theorem 1.2, we only need to prove (1.9). It follows from (4.7) and $\sigma=1$ for $t\geqslant 1$ that

$$\int_1^\infty\|\nabla\tilde{u}\|^2dt\leqslant C$$

and

$$\begin{aligned}\int_1^\infty|(\|\nabla\tilde{u}\|^2)_t|dt&\leqslant 2\int_1^\infty\|\nabla\tilde{u}\|\|\nabla\tilde{u}_t\|dt\\ &\leqslant\left(\int_1^\infty\|\nabla\tilde{u}\|^2dt\right)^{\frac{1}{2}}\left(\int_1^\infty\sigma^2\|\nabla\tilde{u}_t\|^2dt\right)^{\frac{1}{2}}\leqslant C.\end{aligned}$$

Thus,

$$\|\nabla\tilde{u}\|\to 0\quad\text{as}\ t\to\infty. \tag{4.8}$$

Using the Gagliardo-Nirenberg inequality (2.10) and $\sigma=1$ for $t\geqslant 1$, we have

$$\sup_{t\geqslant1}\|\tilde{u}\|_{L^\infty}^4\leqslant C\sup_{t\geqslant1}\sigma^{\frac{5}{2}}\|\tilde{u}\|_{L^4}^2\|\nabla\tilde{u}\|_{L^4}^2\leqslant C\sup_{t\geqslant1}\|\tilde{u}\|_{L^2}\sigma^{\frac{1}{2}}\|\nabla\tilde{u}\|_{L^2}\sigma^2\|\nabla\tilde{u}\|_{L^4}^2,$$

which together with (3.36), (4.7) and (4.8) gives

$$\|u-1\|_{L^\infty}\to 0\quad\text{as}\ t\to\infty. \tag{4.9}$$

By the interpolation inequality, (4.7) and (4.9), for any $2<p_1\leqslant\infty$, we have

$$\|u-1\|_{L^{p_1}}\to 0\quad\text{as}\ t\to\infty.$$

On the other hand, by (3.39), we have from (3.46) and (3.47) that

$$\begin{aligned}\int_1^\infty|(\|\boldsymbol{v}\|_{L^4}^4)_t|dt&\leqslant C\int_1^\infty\int_{\mathbb{R}^2}|\tilde{u}+1|\boldsymbol{v}^4dxdt+C\int_1^\infty\int_{\mathbb{R}^2}|\mathbf{F}||\boldsymbol{v}|^3dxdt\\ &\leqslant C\int_1^\infty\int_{\mathbb{R}^2}\boldsymbol{v}^4dxdt+C\int_1^\infty\int_{\mathbb{R}^2}|\mathbf{F}||\boldsymbol{v}|^3dxdt\\ &\leqslant C\int_1^\infty\int_{\mathbb{R}^2}\boldsymbol{v}^4dxdt+C\int_1^\infty\int_{\mathbb{R}^2}|\mathbf{F}|^4dxdt\\ &\leqslant C.\end{aligned}$$

Combining the above inequality with (3.47), we have

$$\|\boldsymbol{v}\|_{L^4}\to 0\quad\text{as}\ t\to\infty,$$

which together with the interpolation inequality, (3.7) and (3.48) implies

$$\|\boldsymbol{v}\|_{L^{p_2}}\to 0\quad\text{as}\ t\to\infty,\quad 2<p_2<p_0.$$

Hence (1.9) is proved and the proof of Theorem 1.2 is thus completed.

## 5 Proof of Theorem 1.4

In this section, we prove Theorem 1.4. This requires us to transfer the results of the transformed chemotaxis model (1.3) to the original system (1.1). Noting that the converted system and the original system have the same quantity $u$, we only need to prove the result of $c$. We start with the proof of (1.10). From the equation $(1.1)_2$ and the Cole-Hopf transformation (1.2), we have

$$(\ln c)_t = -u, \tag{5.1}$$

which together with $c_0 > 0$ and $u \geqslant 0$ leads to

$$0 \leqslant c(x,t) \leqslant c(x,0) = c_0. \tag{5.2}$$

Integrating (5.1) over $[1,t)$ and using (5.2), we get

$$\begin{aligned} c(x,t) &= c(x,1)\exp\bigg(-(t-1) - \int_1^t (u-1)d\tau\bigg) \\ &\leqslant c_0 \exp\bigg(-(t-1) - \int_1^t (u-1)d\tau\bigg). \end{aligned} \tag{5.3}$$

By (3.37), we have

$$\int_1^t \|u-1\|_{L^\infty} d\tau \leqslant \frac{1}{4}(t-1),$$

which, along with (5.3) yields

$$\|c\|_{L^\infty} \leqslant C\mathrm{e}^{-\frac{3}{4}(t-1)} \leqslant C\mathrm{e}^{-\frac{3}{4}t}.$$

This completes the proof of Theorem 1.4.

**Acknowledgements** The first author was supported by Academy of Mathematics and Systems Science, Chinese Academy of Sciences and the Joint Laboratory of Applied Mathematics in the Hong Kong Polytechnic University where he was a postdoctoral fellow, National Natural Science Foundation of China (Grant No. 11901115), Natural Science Foundation of Guangdong Province (Grant No. 2019A1515010706) and Guangdong University of Technology (Grant No. 220413228). The second author was supported by the Hong Kong Research Grant Council General Research Fund (Grant No. PolyU 153031/17P) and the Hong Kong Polytechnic University (Grant No. ZZHY). The third author was supported by National Natural Science Foundation of China (Grant Nos. 11771150, 11831003 and 11926346) and Guangdong Basic and Applied Basic Research Foundation (Grant No. 2020B1515310015).

## References

1 Adams R, Fournier J. Sobolev Spaces, 2nd ed. Pure and Applied Mathematics, vol. 140. Boston: Academic Press, 2003
2 Adler J. Chemotaxis in bacteria. Science, 1966, 153: 708–716
3 Chae M, Choi K, Kang K, et al. Stability of planar traveling waves in a Keller-Segel equation on an infinite strip domain. J Differential Equations, 2018, 265: 237–279
4 Chen M T, Xu X Y, Zhang J W. Global weak solutions of 3D compressible micropolar fluids with discontinuous initial data and vacuum. Commun Math Sci, 2015, 13: 225–247
5 Corrias L, Perthame B, Zaag H. A chemotaxis model motivated by angiogenesis. C R Math Acad Sci Paris, 2003, 2: 141–146
6 Corrias L, Perthame B, Zaag H. Global solutions of some chemotaxis and angiogenesis systems in high space dimensions. Milan J Math, 2004, 72: 1–28
7 Deng C, Li T. Well-posedness of a 3D parabolic-hyperbolic Keller-Segel system in the Sobolev space framework. J Differential Equations, 2014, 257: 1311–1332

8 Fan J, Zhao K. Blow up criteria for a hyperbolic-parabolic system arising from chemotaxis. J Math Anal Appl, 2012, 394: 687–695
9 Friedman A. Partial Differential Equations. New York: Academic Press, 1969
10 Granero-Belinchón R. On the fractional Fisher information with applications to a hyperbolic-parabolic system of chemotaxis. J Differential Equations, 2017, 262: 3250–3283
11 Granero-Belinchón R. Global solutions for a hyperbolic-parabolic system of chemotaxis. J Math Anal Appl, 2017, 449: 872–883
12 Guo J, Xiao J X, Zhao H J, et al. Global solutions to a hyperbolic-parabolic coupled system with large initial data. Acta Math Sci Ser B Engl Ed, 2009, 29: 629–641
13 Hao C. Global well-posedness for a multidimensional chemotaxis model in critical Besov spaces. Z Angew Math Phys, 2012, 63: 825–834
14 Hoff D. Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. J Differential Equations, 1995, 120: 215–254
15 Hoff D. Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data. Arch Ration Mech Anal, 1995, 132: 1–14
16 Hoff D. Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids. Arch Ration Mech Anal, 1997, 139: 303–354
17 Hoff D. Dynamics of singularity surfaces for compressible, viscous flows in two space dimensions. Comm Pure Appl Math, 2002, 55: 1365–1407
18 Hoff D, Santos M M. Lagrangean structure and propagation of singularities in multidimensional compressible flow. Arch Ration Mech Anal, 2008, 188: 509–543
19 Hou Q Q, Liu C J, Wang Y G, et al. Stability of boundary layers for a viscous hyperbolic system arising from chemotaxis: One-dimensional case. SIAM J Math Anal, 2018, 50: 3058–3091
20 Hou Q Q, Wang Z A. Convergence of boundary layers for the Keller-Segel system with singular sensitivity in the half-plane. J Math Pures Appl (9), 2019, 130: 251–287
21 Hu X P, Lin F H. Global solutions of two-dimensional incompressible viscoelastic flows with discontinuous initial data. Comm Pure Appl Math, 2016, 69: 372–404
22 Jin H Y, Li J Y, Wang Z A. Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity. J Differential Equations, 2013, 255: 193–219
23 Jiu Q S, Wang Y, Xin Z P. Global well-posedness of the Cauchy problem of two-dimensional compressible Navier-Stokes equations in weighted spaces. J Differential Equations, 2013, 255: 351–404
24 Kalinin Y V, Jiang L, Tu Y, et al. Logarithmic sensing in *Escherichia coli* bacterial chemotaxis. Biophys J, 2009, 96: 2439–2448
25 Keller E F, Segel L A. Traveling bands of chemotactic bacteria: A theoretical analysis. J Theor Biol, 1971, 26: 235–248
26 Kong H H, Li H L, Liang C C, et al. Global existence and exponential stability for the compressible Navier-Stokes equations with discontinuous data. J Differential Equations, 2017, 263: 4267–4323
27 Levine H A, Sleeman B D. A system of reaction diffusion equations arising in the theory of reinforced random walks. SIAM J Appl Math, 1997, 57: 683–730
28 Levine H A, Sleeman B D, Nilsen-Hamilton M. A mathematical model for the roles of pericytes and macrophages in the initiation of angiogenesis. I. The role of protease inhibitors in preventing angiogenesis. Math Biosci, 2000, 168: 77–115
29 Li D, Li T, Zhao K. On a hyperbolic-parabolic system modeling chemotaxis. Math Models Methods Appl Sci, 2011, 21: 1631–1650
30 Li D, Pan R H, Zhao K. Quantitative decay of a hybrid type chemotaxis model with large data. Nonlinearity, 2015, 28: 2181–2210
31 Li H C, Zhao K. Initial-boundary value problems for a system of hyperbolic balance laws arising from chemotaxis. J Differential Equations, 2015, 258: 302–308
32 Li J Y, Wang L N, Zhang K J. Asymptotic stability of a composite wave of two traveling waves to a hyperbolic-parabolic system modeling chemotaxis. Math Methods Appl Sci, 2013, 36: 1862–1877
33 Li T, Pan R H, Zhao K. Global dynamics of a chemotaxis model on bounded domains with large data. SIAM J Appl Math, 2012, 72: 417–443
34 Li T, Wang Z A. Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis. SIAM J Appl Math, 2009, 70: 1522–1541
35 Li T, Wang Z A. Nonlinear stability of large amplitude viscous shock waves of a generalized hyperbolic-parabolic system arising in chemotaxis. Math Models Methods Appl Sci, 2010, 20: 1967–1998
36 Lions P L. Mathematical Topics in Fluid Mechanics. Volume II: Compressible Models. Oxford: Clarendon Press, 1998
37 Martinez V, Wang Z A, Zhao K. Asymptotic and viscous stability of large-amplitude solutions of a hyperbolic system arising from biology. Indiana Univ Math J, 2018, 67: 1383–1424

38 Masmoudi N. Global existence of weak solutions to the FENE dumbbell model of polymeric flows. Invent Math, 2013, 191: 427–500

39 Mei M, Peng H Y, Wang Z A. Asymptotic profile of a parabolic-hyperbolic system with boundary effect arising from tumor angiogenesis. J Differential Equations, 2015, 259: 5168–5191

40 Othmer H G, Stevens A. Aggregation, blowup, and collapse: The ABC's of taxis in reinforced random walks. SIAM J Appl Math, 1997, 57: 1044–1081

41 Peng H Y, Wang Z A. On a parabolic-hyperbolic chemotaxis system with discontinuous data: Well-posedness, stability and regularity. J Differential Equations, 2020, 268: 4374–4415

42 Rebholz L G, Wang D H, Wang Z A, et al. Initial boundary value problems for a system of parabolic conservation laws arising from chemotaxis in multi-dimensions. Discrete Contin Dyn Syst, 2019, 39: 3789–3838

43 Roubíček T. Nonlinear Partial Differential Equations with Applications, 2nd ed. Basel: Birkhäuser, 2013

44 Wang D H, Wang Z A, Zhao K. Cauchy problem of a system of parabolic conservation laws arising from a Keller-Segel type chemotaxis model in multi-dimensions. Indiana Univ Math J, 2021, 70: 1–47

45 Wang Z A. Mathematics of traveling waves in chemotaxis. Discrete Contin Dyn Syst Ser B, 2013, 18: 601–641

46 Wang Z A, Hillen T. Shock formation in a chemotaxis model. Math Methods Appl Sci, 2008, 31: 45–70

47 Wang Z A, Xiang Z Y, Yu P. Asymptotic dynamics on a singular chemotaxis system modeling onset of tumor angiogenesis. J Differential Equations, 2016, 260: 2225–2258

48 Zhang M, Zhu C J. Global existence of solutions to a hyperbolic-parabolic system. Proc Amer Math Soc, 2007, 135: 1017–1027

49 Zhang Y H, Tan Z, Sun M B. Global existence and asymptotic behavior of smooth solutions to a coupled hyperbolic-parabolic system. Nonlinear Anal Real World Appl, 2013, 14: 465–482