SIAM J. MATH. ANAL.
Vol. 44, No. 5, pp. 3502–3525




# GLOBAL REGULARITY VERSUS INFINITE-TIME SINGULARITY FORMATION IN A CHEMOTAXIS MODEL WITH VOLUME-FILLING EFFECT AND DEGENERATE DIFFUSION*

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**Abstract.** A system of quasi-linear parabolic and elliptic-parabolic equations describing chemotaxis is studied. Due to the assumed presence of a volume-filling effect it is assumed that there is an impassable threshold for the density of cells. This assumption leads to singular or degenerate operators in both the diffusive and the chemotactic components of the flux of cells. We improve results from earlier works and find critical conditions which reflect the interplay between diffusion and chemotaxis and warrant that classical solutions are global in time and separated uniformly from the threshold. In the case of degenerate diffusion for the elliptic-parabolic version of the model we prove the existence of radially symmetric solutions which exhibit a phenomenon of *infinite-time singularity formation* in that they are global and smooth but attain the threshold in the large time limit.

**Key words.** chemotaxis, volume filling, singular diffusion, degenerate diffusion, singularity formation

**AMS subject classifications.** 35K55, 35K65, 34B15, 34C25, 92C17

**DOI.** 10.1137/110853972

## 1. Introduction.

The movement of biological cells or organisms in response to a chemical gradient is called chemotaxis. Focusing on the understanding of corresponding processes of self-organization detectable in certain cell populations, many theoretical studies of this phenomenon concentrate on the situation when the latter chemical is secreted by the cells themselves. Since the pioneering works of Patlak [26] in 1953 and Keller and Segel [20] in 1970, a number of particularized models have been proposed to describe the aggregation phase of such processes. In most of these works the formation of a cell aggregate is interpreted as a finite-time blow-up of the cell density [14, 13]. In contrast to this, some models derived in the last decade do not treat cells as point masses and take into account their positive sizes. In such a description, arbitrarily high cell densities can be precluded and a threshold value for cell density, for convenience normalized to $u = 1$ corresponding to the tight packing state, can a priori be assumed. Concepts of this type, in the context of chemotaxis named the *volume-filling effect*, were first introduced by Painter and Hillen [25] and further developed in [31].

Different modeling approaches based on this assumption lead to a class of quasi-linear parabolic systems with singularities as $u$ approaches the threshold 1 in either the diffusion coefficient or the chemotactic sensitivity. For instance, this is the case

*Received by the editors November 4, 2011; accepted for publication (in revised form) July 12, 2012; published electronically October 16, 2012.
http://www.siam.org/journals/sima/44-5/85397.html

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in both the model in [24], recently derived as a macroscopic limit of a cellular Potts model, and the system in [31], obtained on taking limits in a reinforced random walk approach on a discrete grid. Surveys of mathematical results on chemotaxis equations are available in [14, 13, 37], the last one particularly focusing on various models of chemotaxis with the density threshold. Accordingly, in this paper we shall consider the system

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u - uh(u)\nabla v), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

and its parabolic-elliptic simplification of Jäger–Luckhaus type [17] given by

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - uh(u)\nabla v), & x\in\Omega,\ t>0,\\ 0 = \Delta v - m + u, & x\in\Omega,\ t>0,\\ \int_\Omega v(x,t)dx = 0, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.2}$$

with

$$m = m(u_0) = \bar u_0 := \frac{1}{|\Omega|}\int_\Omega u_0(x)dx. \tag{1.3}$$

For example, the modeling approach in [25, 31] suggests that in the presence of a volume-filling effect, accounting for the finite size of cells, the diffusion coefficient $D$ and the chemotactic sensitivity $h$ are of the form

$$D(u) = d(q(u) - uq'(u)),\ h(u) = \chi q(u), \tag{1.4}$$

where $d$ and $\chi$ are positive constants and $q(u)$ denotes the probability for cells, located at a point with cell density $u$, to move to some neighboring site. The precise form of this probability function $q(u)$ is basically unknown and not directly accessible to experiments. An important class of functions $q(u)$ is obtained by assuming that there exists a known maximal number of cells that can be accommodated at any site of unit volume. Then a prototypical choice of $q(u)$ (see [25, 30]) is

$$q(u) = \begin{cases} (1-u)^r, & 0\le u\le 1,\\ 0, & u>1, \end{cases} \tag{1.5}$$

where $r>0$ and we have assumed for convenience that this maximal cell density is $u=1$. In this situation, (1.4) yields the precise formulae

$$\begin{aligned} D(u) &= d(1-u)^{r-1}[1-u(1-r)]\\ &= d[(1-u)^r + ru(1-u)^{r-1}],\ h(u) = \chi(1-u)^r, \end{aligned} \tag{1.6}$$

whence already in this simplified setting we may encounter different types of coefficient behavior: Namely, for instance, we see that then

$$D(u) \text{ is } \begin{cases} \text{degenerate and } \ge d(1-u)^{r-1} & \text{if } r>1,\\ \text{linear and } = d & \text{if } r=1,\\ \text{singular and } \ge rd(1-u)^{r-1} & \text{if } r<1. \end{cases} \tag{1.7}$$

Guided by (1.6)–(1.7), we shall suppose that the diffusivity $D$ and the cross-diffusivity $h$ generalize the prototypes

$$D(u) = c_D(1-u)^{-\alpha}, \quad h(u) = c_h(1-u)^{\beta}, \qquad u \in [0,1), \tag{1.8}$$

with some $c_D > 0$ and $c_h > 0$, where we admit $\alpha$ and $\beta$ to attain any real value, not necessarily linked through (1.6).

Within this framework, in which existence and uniqueness of global weak solutions for (1.1) have been asserted in [23], a natural question concerning the qualitative solution behavior is the one posed in [36]:

(Q) Will all solutions, initially satisfying $u < 1$, remain separated from the threshold value $u = 1$ uniformly for all times, or may there exist solutions which approach this singular value either in finite or in infinite time?

Clearly, this problem is similar to that of blow-up vs. the existence of global solutions for the classical reaction-diffusion equations, and partial answers have already been given in [32]. Namely, it has been shown there that if

$$\alpha + \beta < 1 \tag{1.9}$$

and $\Omega$ is a ball with radius $R$, under the assumption that $\frac{c_h}{c_D}$ is sufficiently large, depending on $R$, one can find smooth initial data $u_0$ such that $0 \le u_0 < 1$ in $\Omega$ but such that the corresponding solution of (1.2) attains the value $u = 1$ in either finite or infinite time.

As for results on the opposite type of behavior, it was shown in [4] that the global classical solution exists if $\alpha \ge 2, \beta = 0$, where the solutions are bounded away from $u = 1$ for any finite time. A similar result was subsequently extended in [36] to a general parameter regime

$$\beta \ge 1 - \frac{\alpha}{2} \qquad \text{and} \qquad \alpha > 0 \tag{1.10}$$

but only under the stronger requirement that $\beta > 2$ solutions are known to remain bounded away from $u = 1$ uniformly for all times [36].

It is the goal of the present paper to close the apparently remaining gap between the above parameter regimes, at the same time allowing also for negative $\alpha$. In this respect, the results presented here generalize and significantly improve those from [36] by employing a different approach to derive uniform-in-time $L^p$ bounds on $\frac{1}{1-u}$. Indeed, our technique will enable us to prove that the sole condition $\alpha + \beta > 1$ ensures the existence of global-in-time classical solutions separated from 1 uniformly with respect to time.

In order to make this more precise, let us now be more specific about the technical framework: We shall study (1.1) and (1.2) under the assumptions that the coefficient functions $D$ and $h$ are smooth in $[0,1)$ and such that

$$D(u) \ge c_D(1-u)^{-\alpha} \qquad \text{for all } u \in [0,1) \tag{1.11}$$

as well as

$$h(u) \le c_h(1-u)^{\beta} \qquad \text{for all } u \in [0,1) \tag{1.12}$$

hold with some $\alpha \in \mathbb{R}, \beta \in \mathbb{R}$, $c_D > 0$, and $c_h > 0$. Moreover, we suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\partial\Omega$ and outward normal unit vector field $\nu$, and that $m := \frac{1}{|\Omega|}\int_\Omega u_0$, where

$$u_0 \in W^{1,\infty}(\Omega) \qquad \text{and} \qquad 0 \leq u_0(x) < 1 \quad \text{for all } x \in \bar{\Omega}. \tag{1.13}$$

In the parabolic-parabolic system (1.1) we shall additionally assume that

$$v_0 \in W^{1,\infty}(\Omega) \qquad \text{and} \qquad 0 \leq v_0(x) \quad \text{for all } x \in \bar{\Omega}. \tag{1.14}$$

Then the first of our main results reads as follows.

THEOREM 1.1. *Suppose that $\Omega$ is convex and that* (1.11) *and* (1.12) *are satisfied with some $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ fulfilling*

$$\beta > 1 - \alpha. \tag{1.15}$$

(i) *For all $(u_0, v_0)$ satisfying* (1.13)–(1.14)*, the problem* (1.1) *has a global classical solution $(u, v)$ which remains uniformly regular in the sense that there exists $\delta > 0$, possibly depending on $\|u_0\|_\infty \, \|v_0\|_\infty \, , c_D \, , c_h \, , \alpha \, , \beta$, such that*

$$u(x,t) \leq 1 - \delta \qquad \textit{for all } (x,t) \in \Omega \times (0,\infty). \tag{1.16}$$

(ii) *For any $u_0$ fulfilling* (1.13)*, the problem* (1.2) *admits a global classical solution $(u, v)$ satisfying* (1.16).

In light of the mentioned possibility of singularity formation asserted under the assumption (1.9), this evidently is optimal and thereby completes the answer to (Q), up to equality in (1.9), at least for the parabolic-elliptic system (1.2), a case for which singularity formation does occur when (1.9) holds (cf. [32])

Let us note here that the above convexity assumption is required as a hypothesis in a variant of the Poincaré inequality, which will be a technical cornerstone of our analysis (see Lemma 3.3 below and [18, Corollary 8.1.4]).

Next, it turns out that the mere existence of a global smooth solution can be asserted under a condition on $\alpha$ and $\beta$ which is different from (1.15).

THEOREM 1.2. *Let* (1.11) *and* (1.12) *hold with some $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that*

$$\beta \geq 1 - \frac{\alpha}{2}. \tag{1.17}$$

(i) *For any $(u_0, v_0)$ satisfying* (1.13)–(1.14)*, the problem* (1.1) *has a global classical solution $(u, v)$. Moreover, for all $T > 0$ there exists $\delta(T) > 0$ such that for this solution we have*

$$u(x,t) \leq 1 - \delta(T) \qquad \textit{for all } (x,t) \in \Omega \times (0,T). \tag{1.18}$$

(ii) *For each $u_0$ such that* (1.13) *holds, there exists a global classical solution $(u, v)$ of* (1.2)*. Furthermore, for all $T > 0$ one can pick $\delta(T) > 0$ such that* (1.18) *holds.*

As a by-product, utilizing a blow-up result obtained in [32], we finally detect some global classical solutions of (1.2) exhibiting a singularity formation in infinite time, provided that $\alpha$ is negative and $\beta$ lies in the intermediate range $[1 - \frac{\alpha}{2}, 1 - \alpha)$. In formulating this result in a precise way, besides (1.11) and (1.12) we shall refer to the complementary conditions

$$D(u) \leq \tilde{c}_D(1-u)^{-\alpha} \qquad \text{for all } u \in [0,1) \tag{1.19}$$

and

$$h(u) \geq \tilde{c}_h (1-u)^\beta \qquad \text{for all } u \in [0,1) \tag{1.20}$$

for some $\alpha \in \mathbb{R}, \beta \in \mathbb{R}$, $\tilde{c}_D > 0$, and $\tilde{c}_h > 0$.

THEOREM 1.3. *Let $\Omega$ be a ball, and assume that* (1.11), (1.12), (1.19), *and* (1.20) *are satisfied with some $\alpha < 0$ and*

$$\beta \in \Big[1 - \frac{\alpha}{2}, 1-\alpha\Big). \tag{1.21}$$

*Then if $\frac{\tilde{c}_h}{\tilde{c}_D}$ is large enough, there exists $u_0 \in C^\infty(\bar\Omega)$ such that $0 < u_0 < 1$ in $\bar\Omega$ and such that* (1.2) *possesses a global classical solution $(u,v)$ which fulfills $0 < u < 1$ in $\bar\Omega \times [0,\infty)$ but*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to 1 \qquad \text{as } t \to \infty. \tag{1.22}$$

We remark here that instead of assuming (1.19) and (1.20), one might alternatively require that

$$\frac{h(u)}{D(u)} \geq c_{hD}(1-u)^{\alpha+\beta}$$

hold with $\alpha$ and $\beta$ as above and some appropriately large $c_{hD} > 0$.

In conjunction with the outcome from [32], the above results form an essentially complete picture with regard to singularity formation in (1.2) at least in the situation when both $D(u)$ and $h(u)$ exhibit an algebraic behavior near $u = 1$. A brief summary thereof can be found in Figure 1 and Table 1.

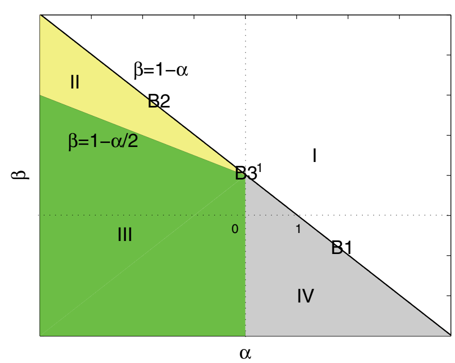



FIG. 1. *An illustration of parameter regimes for $\alpha$ and $\beta$, which includes the four regions* I $:= \{(\alpha,\beta) \mid \alpha+\beta > 1\}$, II $:= \{(\alpha,\beta) \mid \alpha < 0, 1-\frac{\alpha}{2} \leq \beta < 1-\alpha\}$, III $:= \{(\alpha,\beta) \mid \alpha < 0, \beta < 1-\frac{\alpha}{2}\}$, *and* IV $:= \{(\alpha,\beta) \mid \alpha \geq 0, \alpha+\beta < 1\}$. *The lines* B1 $:= \{(\alpha,\beta) \mid \alpha > 0, \alpha+\beta = 1\}$, B2 $:= \{(\alpha,\beta) \mid \alpha < 0, \alpha+\beta = 1\}$, *and* B3 $:= \{(\alpha,\beta) \mid \alpha = 0, \alpha+\beta = 1\}$ *appear as boundaries, whereas two more boundaries have been included in regions* II *and* IV. *An overview over the occurrence of singular solution behavior in the respective ranges can be found in Table* 1.

From a mathematical point of view, we find it worth underlining that the solution behavior detected in Theorem 1.3 might be surprising in itself: As far as we know, in nonlinear parabolic equations and systems not many situations have been previously identified in which solutions are global and smooth but develop a singularity in the

Table 1
*Solution behavior in the parameter ranges from Figure 1 in the case when* $\frac{h(u)}{D(u)} = \sigma(1-u)^{\alpha+\beta}$ *for some* $\sigma > 0$. *"None" means that the solution reaches* 1 *neither in finite nor in infinite time.*

| Parameter regime | Possible occurrence of singularity | References |
|---|---|---|
| I | None for both (1.1) and (1.2) | Present paper |
| II | Yes (in infinite time for (1.2)) if $\sigma$ is large | Present paper |
| III | Yes (in infinite time for (1.2)) if $\sigma$ is large | [32] |
| IV | Yes (in finite time for (1.2)) | [32] |
| B1 | Unknown | |
| B2 | No finite time singularity for both (1.1) and (1.2) | Present paper |
| B3 | None for both (1.1) and (1.2) | Present paper |

large time limit. Indeed, some semilinear and quasi-linear parabolic equations are known to allow for phenomena of this type (cf., e.g., [27, 9, 33, 34]), but in most examples this kind of behavior seems to be unstable with respect to either the initial data or parameters in the equation. This also applies to some related results addressing the standard parabolic-elliptic Keller–Segel system, where global unbounded solutions are known to exist if the total mass of cells precisely attains some critical value (cf., e.g., [19] for a detailed analysis of the time asymptotics of such solutions).

Before going into details, let us finally mention that different variants of the Keller–Segel system are discussed as possible candidates to model volume-filling effects in chemotaxis processes. Unlike the approach pursued here, the class of models studied in [3] does not impose a fixed threshold value that cannot be exceeded by the cell density, but rather assumes that the diffusivity $D(u)$ and the cross-diffusion parameter $h(u)$, though positive for all $u$, decay as $u \to \infty$. In the probabilistic picture of random walks, this corresponds to positive but decaying probabilities $q(u)$ of cells to move towards some neighboring site (see [25]). A considerable literature is concerned with models resulting from approaches of this type, the essential outcome being that again the asymptotic behavior, now as $u \to \infty$, of the ratio of cross-diffusion and diffusion decides whether or not a singularity formation may occur; namely, if

$$\frac{uh(u)}{D(u)} \le Cu^{\frac{2}{n}-\varepsilon} \qquad \text{for all } u \ge 1$$

with positive constants $\varepsilon$ and $C$, then all solutions of both (1.1) and (1.2) are global and uniformly bounded (cf. [29, 8] and also [15, 21, 6] for some precedents), whereas if

$$\frac{uh(u)}{D(u)} \ge cu^{\frac{2}{n}+\varepsilon} \qquad \text{for all } u \ge 1$$

with some $\varepsilon > 0$ and $c > 0$, then unbounded solutions exist (see [12, 35, 5, 7, 8]). Results of a similar flavor have also been derived for associate Cauchy problems in $\Omega = \mathbb{R}^n$ (see [28, 16]).

As contrasted to this, in conjunction with [32] the present work shows that the critical relationship between $h$ and $D$ does *not* depend on the space dimension when an a priori threshold of the above type is built in the model.

**2. Local existence and uniqueness.** Let us first ensure local solvability of (1.1) and (1.2), along with useful extensibility criteria.

LEMMA 2.1. *Let* $(u_0, v_0)$ *satisfy* (1.13)–(1.14). *Then there exist* $T_{max} \in (0, \infty]$ *and a unique pair* $(u, v)$ *of functions* $(u, v) \in \mathcal{C}([0, T_{max}) \times \bar{\Omega}; \mathbb{R}^2) \cap \mathcal{C}^{1,2}((0, T_{max}) \times \bar{\Omega}; \mathbb{R}^2)$ *such that* $(u, v)$ *solves* (1.1) *in the classical sense in* $\Omega \times (0, T_{max})$. *Moreover, if* $T_{max} < \infty$, *then*

$$\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = 1. \tag{2.1}$$

*Proof.* First we may note that the Neumann boundary condition in (1.1) is equivalent to the no-flux boundary condition for $u$, i.e.,

$$\langle D(u)\nabla u - uh(u)\nabla v | \nu \rangle = 0 \quad \text{on} \quad (0, \infty) \times \partial\Omega,$$

as long as $u < 1$; therefore we may use Amann's theory of quasi-linear parabolic equations [2] and proceed as in the proof of [36, Proposition 1]. ☐

Our counterpart of this for (1.2) reads as follows.

LEMMA 2.2. *Suppose that* $u_0$ *satisfies* (1.13). *Then there exist* $T_{max} \in (0, \infty]$ *and a unique pair* $(u, v)$ *of functions* $(u, v) \in \mathcal{C}([0, T_{max}) \times \bar{\Omega}; \mathbb{R}^2) \cap \mathcal{C}^{1,2}((0, T_{max}) \times \bar{\Omega}; \mathbb{R}^2)$ *such that* $(u, v)$ *solves* (1.2) *in the classical sense in* $\Omega \times (0, T_{max})$. *Moreover, if* $T_{max} < \infty$, *then*

$$\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = 1. \tag{2.2}$$

*Proof.* Let $M = \|u_0\|_{L^\infty(\Omega)}$ and $\eta \in (0, 1 - M)$. We define a set

$$X_T := \left\{ w \in C^0(\bar{\Omega} \times [0, T]) \;\middle|\; 0 \le w \le M + \eta < 1 \right. \\ \left. \text{and } \frac{1}{|\Omega|} \int_\Omega w(x, t) dx = m \text{ for all } t \in (0, T) \right\}$$

and a mapping $\Phi : X_T \mapsto X_T$ such that given $\tilde{u} \in X_T$, $\Phi(\tilde{u}) = u$, where $u$ is an $L^2$-weak solution to

$$\begin{cases} u_t = \nabla \cdot (D(\tilde{u})\nabla u - uh(\tilde{u})\nabla v), & x \in \Omega,\ t \in (0, T), \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.3}$$

with $v$ defined to be the solution of

$$\begin{cases} -\Delta v = -m + \tilde{u}, & x \in \Omega,\ t \in (0, T), \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0, T), \end{cases} \tag{2.4}$$

along with the condition

$$\int_\Omega v(x, t)\, dx = 0 \quad \text{for any } t \in [0, T]. \tag{2.5}$$

We shall show that for $T$ small enough $\Phi$ has a fixed point. Notice that by [11, Theorem 8.34] there is a unique solution $v(\cdot, t) \in C^{1+s}(\Omega)$ to (2.4) for some $s \in (0, 1)$. It is easy to see that in fact $v \in L^\infty(0, T : C^{1+s}(\bar{\Omega}))$ for each $s \in (0, 1)$. Then $u \in C^0([0, T] : L^2(\Omega)) \cap L^2(0, T : H^1(\Omega))$ is the unique weak solution to the linear parabolic equation (2.3). Since both $D(\tilde{u})$ and $h(u)\nabla v$ belong to $L^\infty(\Omega \times (0,T))$, and since $u_0$ was assumed to be Hölder continuous in $\bar{\Omega}$, we may apply [22, Theorem V1.1] to conclude that for some $\gamma \in (0,1)$ and $K > 0$ we have $u \in C^{\gamma,\frac{\gamma}{2}}(\bar{\Omega} \times [0,T])$ and

$$\|u\|_{C^{\gamma,\frac{\gamma}{2}}(\bar{\Omega}\times[0,T])} \leq K, \tag{2.6}$$

where $K$ depends on $\min_{\xi\in[0,M+\eta]} D(\xi)$, $\max_{\xi\in[0,M+\eta]} h(\xi)$, and $\|\nabla v\|_{L^\infty(0,T:C^\alpha(\bar{\Omega}))}$, the last quantity being controlled by $\|u\|_{L^\infty(\Omega)} \leq M+\eta$. It follows that

$$\|u(\cdot,t) - u_0\|_{L^\infty(\Omega)} \leq Kt^{\frac{\gamma}{2}}$$

and hence

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq KT^{\frac{\gamma}{2}} + M\,.$$

From this we deduce that if we fix $T = T_0 < (\frac{\eta}{K})^{\frac{2}{\gamma}}$, then we have

$$\sup_{t\in[0,T_0]} \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq M+\eta\,.$$

Notice also that (2.3) implies $\frac{1}{|\Omega|}\int_\Omega u(x,t)dx = m$ for all $t \in (0,T_0)$. Thus, $u \in X_{T_0}$ and we proceed to show using the Schauder fixed point theorem that $\Phi : X_{T_0} \mapsto X_{T_0}$ has a fixed point. To this end observe that $X_{T_0}$ is a convex subset of $C^0(\bar{\Omega} \times [0,T_0])$. From (2.6) we infer that $\Phi$ is a compact mapping. It remains to prove that $\Phi$ is a continuous mapping. Let $\tilde{u}_k \to \tilde{u}$ as $k \to \infty$ in $C^0(\bar{\Omega} \times [0,T_0])$. First notice that (2.5), the Poincaré inequality, and Young's inequality with $\varepsilon$ entail

$$\int_\Omega |\nabla v - \nabla v_k|^2 dx \leq C_1 \int_\Omega |\tilde{u} - \tilde{u}_k|^2 dx \qquad \text{for all } t \in (0,T_0), \tag{2.7}$$

where $C_1$ is a positive constant. Next, using Young's inequality and (2.7) we obtain

$$\begin{aligned} &\int_\Omega |u-u_k|^2 dx + \underline{D}\int_0^t\int_\Omega |\nabla u - \nabla u_k|^2 dxds \\ &\leq \int_0^t\int_\Omega |\nabla u_k|(|D(\tilde{u}) - D(\tilde{u}_k)||\nabla u - \nabla u_k|)dxds \\ &\quad + C_2\left(\int_0^t\int_\Omega |\tilde{u}-\tilde{u}_k|^2 dxds + \int_0^t\int_\Omega |u-u_k|^2 dxds\right) \qquad \text{for all } t \in (0,T_0), \end{aligned} \tag{2.8}$$

where $\underline{D} = \min_{\xi\in[0,M+\eta]} D(\xi)$ and $C_2$ depends on $M+\eta$ and $\max_{\xi\in[0,M+\eta]} |h(\xi) + h'(\xi)|$ and $C_4$. The usual energy estimate for (2.3) yields

$$\sup_{t\in[0,T_0]} \int_\Omega |u_k(x,t)|^2 dx + \int_0^t\int_\Omega |\nabla u_k(x,s)|^2 dxds \leq E,$$

where $E$ is a constant independent of $k$. Hence, using Young's inequality and the local Lipschitz continuity of $D$ and $h$ and (2.7), we obtain

$$\begin{aligned} &\int_\Omega |(u-u_k)(x,t)|^2 dx \leq C_2\int_0^t\int_\Omega |u-u_k|^2 dxds \\ &+ C_3 \sup_{\Omega_{T_0}} |D(\tilde{u}) - D(\tilde{u}_k)|^2 \int_0^t\int_\Omega |\nabla u_k|^2 dxds \\ &+ C_4\int_0^t\int_\Omega |(\tilde{u}-\tilde{u}_k)|^2 dxds, \end{aligned} \tag{2.9}$$

with $C_3$ depending on $K$, $M+\eta$, and $\max_{\xi\in[0\,,M+\eta]}|h(\xi)+h'(\xi)|$ and $C_4$ depending also on $E$. By Gronwall's lemma we deduce that for any $t\in[0\,,T_0]$

$$\begin{aligned}(2.10)\qquad &\int_\Omega |(u-u_k)(x,t)|^2dx\\ &\le \left(C_3\sup_{\Omega_{T_0}}|D(\tilde u)-D(\tilde u_k)|^2\int_0^t\int_\Omega|\nabla u_k|^2dxds\right.\\ &\qquad\left.+\,C_4\int_0^t\int_\Omega|(\tilde u-\tilde u_k)|^2dxds\right)e^{C_2T_0}\\ &\le C_5e^{C_2T_0}\left(\sup_{\Omega_{T_0}}|\tilde u-\tilde u_k)|^2+\int_0^t\int_\Omega|(\tilde u-\tilde u_k)(x,s)|^2dxds\right),\end{aligned}$$

where $C_5$ depends on $C_3\,,C_4\,,E$, and $\max_{\xi\in[0\,,M+\eta]}|D'(\xi)|$. Thus,

$$u_k\to u\quad\text{in}\;\;L^\infty(0\,,T_0:L^2(\Omega)),$$

and using (2.6) we deduce that

$$u_k\to u\quad\text{in}\;\;C^0(\bar\Omega\times[0,T_0])\,.$$

By the Schauder theorem there exists a pair $(u\,,v)$ which solves (1.2) in a weak sense. In fact the solution is more regular: By the classical regularity theory of elliptic equations, for any $t\in(0\,,T_0]$ it follows that

$$v(\cdot\,,t)\in C^{2+\gamma}(\bar\Omega)$$

for some $\gamma\in(0\,,1)$. Then it is easy to check using (2.6) that

$$v\in C^{2+\gamma,\frac{\gamma}{2}}(\bar\Omega\times[\tau,T_0])\qquad\text{for all }\tau\in(0,T_0).$$

The regularity theory for parabolic equations [22, Theorem V.6.1] thus entails

$$u\in C^{2+\gamma,1+\frac{\gamma}{2}}(\bar\Omega\times[\tau,T_0])\qquad\text{for all }\tau\in(0,T_0).$$

The solution may be prolonged in the interval $[0\,,T_{max})$ and either $T_{max}=\infty$ or $T_{max}<\infty$, where in the latter case

$$\|u(\cdot\,,t)\|_\infty\to 1\quad\text{when}\;\;t\to T_{max}\,.$$

To prove the uniqueness of solutions to problem (1.2) let us assume that there are two distinct solutions $(u_1\,,v_1)$ and $(u_2\,,v_2)$. Then we may perform similar estimates to those which led to (2.10) with the substitution $u_1$ in the place of $u$ and $\tilde u$, and with $u_2$ in the place of $u_k$ and $\tilde u_k$. The only difference is to use in (2.8) the fact that $\|\nabla u_2\|_\infty<\infty$. Consequently, we deduce that for any $\tau\in[0\,,T_0]$,

$$\sup_{t\in[0\,,\tau]}\int_\Omega|u_1(x,t)-u_2(x,t)|^2dx\le C_5\tau e^{C_2T_0}\sup_{s\in[0\,,\tau]}\int_\Omega|u_1(x,s)-u_2(x,s)|^2dx\,,$$

which leads to a contradiction. Finally the nonnegativity of $u$ follows from the classical maximum principle if we rewrite the first equation in (1.2) in the nondivergence form.

The above uniqueness statement entails that the assumed radial symmetry of $u_0$ is inherited by both solution components $u$ and $v$. Accordingly, without any danger of confusion we may write $u_0=u_0(r)$ and $u=u(r,t)$ whenever this appears to be convenient in what follows. ∎

**3. Preliminary estimates.** Before deriving a priori estimates, let us provide some preliminary material. We begin by stating a lower bound for the size of the set where a given nonnegative function from $L^1(\Omega)$ remains conveniently small.

Lemma 3.1. *Let $\varphi \in L^1(\Omega)$ be nonnegative. Then for all $a > 1$ the inequality*

$$\left|\left\{\varphi \le \frac{aM}{|\Omega|}\right\}\right| \ge \frac{a-1}{a}|\Omega| \tag{3.1}$$

*holds with $M := \int_\Omega \varphi$.*

*Proof.* Using that $\varphi \ge 0$, we estimate

$$M = \int_\Omega \varphi \ge \int_{\{\varphi > \frac{aM}{|\Omega|}\}} \varphi \ge \left|\left\{\varphi > \frac{aM}{|\Omega|}\right\}\right| \cdot \frac{aM}{|\Omega|}$$

and hence obtain

$$\left|\left\{\varphi > \frac{aM}{|\Omega|}\right\}\right| \le \frac{|\Omega|}{a}.$$

Since

$$\left|\left\{\varphi \le \frac{aM}{|\Omega|}\right\}\right| = |\Omega| - \left|\left\{\varphi > \frac{aM}{|\Omega|}\right\}\right|,$$

this yields (3.1). ◻

We next check the independence of the constants in a Gagliardo–Nirenberg inequality within a certain parameter range.

Lemma 3.2. *Let $\overline{q} > 1$ be such that $\overline{q} < \frac{2n}{(n-2)_+}$. Then there exists $C > 0$ such that whenever $q \in [1, \overline{q}]$ we have*

$$\|z\|_{L^q(\Omega)} \le C\|\nabla z\|^a_{L^2(\Omega)} \cdot \|z\|^{1-a}_{L^1(\Omega)} + C\|z\|_{L^1(\Omega)} \qquad \textit{for all } z \in W^{1,2}(\Omega), \tag{3.2}$$

*where $a := \frac{2n(q-1)}{(n+2)q}$.*

*Proof.* By the Gagliardo–Nirenberg inequality (see [10]),

$$\|z\|_{L^{\overline{q}}(\Omega)} \le c_1\|\nabla z\|^b_{L^2(\Omega)} \cdot \|z\|^{1-b}_{L^1(\Omega)} + c_1\|z\|_{L^1(\Omega)} \qquad \text{for all } z \in W^{1,2}(\Omega)$$

is valid with $b := \frac{2n(\overline{q}-1)}{(n+2)\overline{q}}$ and some $c_1 > 0$. Now the Hölder inequality says that

$$\|z\|_{L^q(\Omega)} \le \|z\|^d_{L^{\overline{q}}(\Omega)} \cdot \|z\|^{1-d}_{L^1(\Omega)}$$

holds for all such $z$ with $d := \frac{\overline{q}(q-1)}{(\overline{q}-1)q}$. Since $(X+Y)^d \le 2^d(X^d + Y^d)$ for all $X \ge 0$ and $Y \ge 0$, we obtain

$$\|z\|_{L^q(\Omega)} \le 2^d \cdot \left(c_1^d\|\nabla z\|^{bd}_{L^2(\Omega)} \cdot \|z\|^{(1-b)d}_{L^1(\Omega)} + c_1^d\|z\|^d_{L^1(\Omega)}\right) \cdot \|z\|^{1-d}_{L^1(\Omega)},$$

which results in (3.2) due to the fact that $bd = a$. ◻

A last preliminary result provides a Poincaré–Sobolev-type inequality for functions remaining suitably small in sets of appropriately large measure. It is an immediate consequence of a corresponding Poincaré inequality for functions with large zero set.

LEMMA 3.3. *Suppose that $\Omega$ is a convex domain. Let $\kappa > 0$, $\delta > 0$, and $q \geq 1$ be such that $q \leq \frac{2n}{(n-2)_+}$. Then there exists $C = C(\delta) > 0$ such that if $z \in W^{1,2}(\Omega)$ is nonnegative with*

$$\Big|\{z \leq \kappa\}\Big| \geq \delta, \tag{3.3}$$

*then*

$$\int_\Omega z^q \leq C \cdot \left\{1 + \Big(\int_\Omega |\nabla z|^2\Big)^{\frac{q}{2}}\right\}. \tag{3.4}$$

*Proof.* Since $q \leq \frac{2n}{(n-2)_+}$, we have $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$, whence there exists $c_1 > 0$ such that

$$\|\varphi\|_{L^q(\Omega)} \leq c_1 \cdot \Big(\|\nabla\varphi\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)}\Big) \qquad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{3.5}$$

Next, we shall use the following variant of the Poincaré inequality which follows from [18, Corollary 8.1.4]. It ensures that in the case of convex $\Omega$ there is constant $c_2(\delta) > 0$ such that

$$\|\varphi\|_{L^2(\Omega)} \leq c_2(\delta) \cdot \|\nabla\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega) \text{ such that } \Big|\{\varphi = 0\}\Big| \geq \delta. \tag{3.6}$$

Applying this to $\varphi := (z-\kappa)_+$ shows that

$$\begin{aligned}
\|z\|_{L^q(\Omega)} &\leq \|(z-\kappa)_+\|_{L^q(\Omega)} + \kappa|\Omega|^{\frac{1}{q}} \\
&\leq c_1(1+c_2(\delta)) \cdot \|\nabla(z-k)_+\|_{L^2(\Omega)} + \kappa|\Omega|^{\frac{1}{q}} \\
&\leq c_1(1+c_2(\delta)) \cdot \|\nabla z\|_{L^2(\Omega)} + \kappa|\Omega|^{\frac{1}{q}},
\end{aligned}$$

which easily yields (3.4). ∎

**4. Basic a priori estimates for an auxiliary problem.** In what follows we shall derive various a priori estimates for solutions of the scalar parabolic problem

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(uh(u)\nabla v), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{4.1}$$

where $v = v(x,t)$ is considered to be a given sufficiently regular function.

Our first observation concerning such solutions is essentially the same as in [36, Lemma 4].

LEMMA 4.1. *Assume that* (1.11) *and* (1.12) *are satisfied with some $\alpha \in \mathbb{R}, \beta \in \mathbb{R}, c_D > 0$, and $c_h > 0$. Let $T > 0$ and suppose that $u$ is a classical solution of* (4.1)*, where $v \in L^\infty((0,T); C^1(\bar\Omega))$ satisfies $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$ and*

$$|\nabla v| \leq K \qquad a.e.\ in\ \Omega \times (0,T) \tag{4.2}$$

*with some $K > 0$. Then for all $p \geq 1$ the function $w := 1 - u$ satisfies the inequality*

$$\frac{d}{dt}\int_\Omega w^{-p} + \frac{p^2 c_D}{2}\int_\Omega w^{-p-\alpha-2}|\nabla w|^2 \leq \frac{p^2 c_h^2 K^2 \|u(\cdot,t)\|^2_{L^\infty(\Omega)}}{c_D}\int_\Omega w^{-p+\alpha+2\beta-2} \tag{4.3}$$

*for all $t \in (0,T)$.*

*Proof.* Testing (4.1) against $w^{-p-1}$ and using (1.11) and (1.12) yields

$$\frac{1}{p}\frac{d}{dt}\int_\Omega w^{-p} + (p+1)c_D\int_\Omega w^{-p-\alpha-2}|\nabla w|^2 \le (p+1)c_h\int_\Omega uw^{-p+\beta-2}|\nabla v\cdot\nabla w| \tag{4.4}$$

for all $t\in(0,T)$. Here, according to Young's inequality we have

$$\begin{aligned}(p+1)c_h\int_\Omega uw^{-p+\beta-2}|\nabla v\cdot\nabla w| &\le \frac{(p+1)c_D}{2}\int_\Omega w^{-p-\alpha-2}|\nabla w|^2\\ &\quad + \frac{(p+1)c_h^2}{2c_D}\int_\Omega u^2w^{-p+\alpha+2\beta-2}|\nabla v|^2,\end{aligned} \tag{4.5}$$

and thanks to (4.2) we obtain

$$\begin{aligned}&\frac{(p+1)c_h^2}{2c_D}\int_\Omega u^2w^{-p+\alpha+2\beta-2}|\nabla v|^2\\ &\qquad\le \frac{(p+1)c_h^2K^2\|u(\cdot,t)\|^2_{L^\infty(\Omega)}}{2c_D}\int_\Omega w^{-p+\alpha+2\beta-2}.\end{aligned} \tag{4.6}$$

Since $p\le p+1\le 2p$ due to the fact that $p\ge 1$, collecting (4.4)–(4.6) we end up with (4.3). ∎

We shall secondly provide a statement ensuring that bounds for $\frac{1}{1-u}$ in spaces $L^\infty((0,T);L^p(\Omega))$ with suitably large $p>1$ already imply pointwise boundedness of $\frac{1}{1-u}$. This lemma will be referred to twice in what follows (cf. Lemmas 5.2 and 6.2), and its proof is based on a Moser–Alikakos-type iteration procedure (see [1]).

LEMMA 4.2. *Assume that* (1.13), (1.11), *and* (1.12) *are valid with some* $c_D>0, c_h>0$, $\alpha\in\mathbb{R}$, *and* $\beta\in\mathbb{R}$. *Then there exists* $p_0>1$ *such that for all* $K>0$ *and* $L>0$ *one can find* $C(K,L)>0$ *with the following property: If, for some* $T\in(0,\infty]$, $v\in L^\infty((0,T);C^1(\bar\Omega))$ *satisfies* $\frac{\partial v}{\partial\nu}=0$ *on* $\partial\Omega$ *and* $|\nabla v|\le K$ *a.e. in* $\Omega\times(0,T)$, *and if* $u$ *is a classical solution of* (4.1) *in* $\Omega\times(0,T)$ *such that* $0\le u<1$ *and*

$$\int_\Omega(1-u)^{-p_0}(x,t)dx\le L \qquad \textit{for all } t\in(0,T), \tag{4.7}$$

*then*

$$\frac{1}{1-u}\le C(K,L) \qquad \textit{in } \Omega\times(0,T). \tag{4.8}$$

*Proof.* Let us fix $p_0>1$ large enough such that

$$p_0>\frac{n|\alpha|}{2} \tag{4.9}$$

and

$$p_0>4(\alpha+\beta-1)-\alpha, \tag{4.10}$$

and such that

$$\eta_0:=\frac{4|\alpha+\beta-1|}{p_0-|\alpha|}$$

satisfies

$$2 + \eta_0 < \frac{2n}{(n-2)_+} \tag{4.11}$$

as well as

$$\eta_0 < \frac{2}{n}. \tag{4.12}$$

Then defining $(p_k)_{k\in\mathbb{N}}$ recursively by setting

$$p_k := 2p_{k-1} - \alpha, \qquad k \ge 1, \tag{4.13}$$

we can easily check using (4.9) that $(p_k)_{k\in\mathbb{N}}$ is strictly increasing, and that there exist $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \cdot 2^k \le p_k \le c_2 \cdot 2^k \qquad \text{for all } k \ge 0. \tag{4.14}$$

Hence, according to (4.10) and the monotonicity of $(p_k)_{k\in\mathbb{N}}$,

$$q_k := \frac{2(p_k - \alpha - 2\beta + 2)}{p_k + \alpha} \equiv 2 - \frac{4(\alpha+\beta-1)}{p_k+\alpha}, \qquad k \ge 1,$$

satisfies

$$1 < q_k \le 2 + \eta_k \qquad \text{for all } k \ge 1 \tag{4.15}$$

with

$$\eta_k := \frac{4|\alpha+\beta-1|}{p_k - |\alpha|}, \qquad k \ge 1.$$

Then (4.12) guarantees that

$$\eta_k \le \eta_0 < \frac{2}{n} \qquad \text{for all } k \ge 1, \tag{4.16}$$

and (4.15) in conjunction with (4.11) entails that

$$q_k \le Q := 2 + \eta_0 < \frac{2n}{(n-2)_+} \qquad \text{for all } k \ge 1. \tag{4.17}$$

Moreover, from (4.14) we see that

$$\eta_k \le \frac{1}{p_k} \cdot \frac{4|\alpha+\beta-1|}{1 - \frac{|\alpha|}{p_0}} \le c_3 \cdot 2^{-k} \qquad \text{for all } k \ge 1 \tag{4.18}$$

holds with $c_3 := \frac{p_0\eta_0}{c_1}$. Likewise, for

$$\tilde{q}_k := \frac{2p_k}{p_k + \alpha}, \qquad k \ge 1, \tag{4.19}$$

we have

$$1 < \tilde{q}_k \le \overline{q} := \frac{2p_0}{p_0 - |\alpha|} \qquad \text{for all } k \ge 1, \tag{4.20}$$

where $\overline{q} < \frac{2n}{(n-2)_+}$ due to (4.9).

Our goal is to derive upper bounds for

$$A_k := \max\Big\{1\,,\ \sup_{t\in(0,T)} \int_\Omega w^{-p_k}(x,t)dx\Big\}, \qquad k \ge 0,$$

where again $w := 1-u$. To this end, we recall Lemma 4.1, which implies that for all $k \ge 1$,

$$\frac{d}{dt}\int_\Omega w^{-p_k} + c_4 \int_\Omega |\nabla w^{-\frac{p_k+\alpha}{2}}|^2 \le c_5 p_k^2 \int_\Omega w^{-p_k+\alpha+2\beta-2} \qquad \text{for all } t\in(0,T) \tag{4.21}$$

is valid with some constants $c_4 \in (0,1]$ and $c_5 > 0$ which, like $c_6, c_7, \dots$ below, may depend on $K$ but not on $t, T$, or $k$.

In view of (4.17), Lemma 3.2 allows us to interpolate

$$\begin{aligned} c_5p_k^2 \int_\Omega w^{-p_k+\alpha+2\beta-2} &= c_5 p_k^2 \|w^{-\frac{p_k+\alpha}{2}}\|_{L^{q_k}(\Omega)}^{q_k} \\ &\le c_6 p_k^2 \|\nabla w^{-\frac{p_k+\alpha}{2}}\|_{L^2(\Omega)}^{\frac{2n(q_k-1)}{n+2}} \cdot \|w^{-\frac{p_k+\alpha}{2}}\|_{L^1(\Omega)}^{\frac{2n-(n-2)q_k}{n+2}} \\ &\quad + c_6 p_k^2 \|w^{-\frac{p_k+\alpha}{2}}\|_{L^1(\Omega)}^{q_k} \end{aligned} \tag{4.22}$$

with some $c_6 > 0$. Here we note that thanks to (4.13),

$$\|w^{-\frac{p_k+\alpha}{2}}(\cdot,t)\|_{L^1(\Omega)} = \int_\Omega w^{-p_{k-1}}(\cdot,t) \le A_{k-1} \qquad \text{for all } t\in(0,T), \tag{4.23}$$

and that from (4.15) and (4.16) we know that

$$\frac{n(q_k-1)}{n+2} \le \frac{n(1+\eta_k)}{n+2} \le \theta := \frac{n(1+\eta_0)}{n+2}, \tag{4.24}$$

where

$$\theta < \frac{n(1+\frac{2}{n})}{n+2} = 1. \tag{4.25}$$

Therefore, upon an application of Young's inequality in the form

$$XY \le \frac{(\varepsilon X)^s}{s} + \frac{(\varepsilon^{-1}Y)^{\frac{s}{s-1}}}{\frac{s}{s-1}}, \qquad X\ge 0,\ Y \ge 0,\ s>1,\ \varepsilon>0, \tag{4.26}$$

to $s := \frac{n+2}{n(q_k-1)}, X := \|\nabla w^{-\frac{p_k+\alpha}{2}}\|_{L^2(\Omega)}^{\frac{2n(q_k-1)}{n+2}}$, $Y := c_6 p_k^2 \|w^{-\frac{p_k+\alpha}{2}}\|_{L^1(\Omega)}^{\frac{2n-(n-2)q_k}{n+2}}$, and $\varepsilon := c_3^{\frac{1}{s}}$, (4.22) turns into the inequality

$$\begin{aligned} c_5 p_k^2 \int_\Omega w^{-p_k+\alpha+2\beta-2} &\le \frac{n(q_k-1)}{n+2} c_4 \cdot \int_\Omega |\nabla w^{-\frac{p_k+\alpha}{2}}|^2 \\ &\quad + \frac{2n+2-nq_k}{n+2}\cdot c_4^{-\frac{n(q_k-1)}{2n+2-nq_k}} \cdot \Big\{c_6 p_k^2 A_{k-1}^{\frac{2n-(n-2)q_k}{n+2}}\Big\}^{\frac{n+2}{2n+2-nq_k}} \\ &\quad + c_6 p_k^2 A_{k-1}^{q_k}. \end{aligned} \tag{4.27}$$

Here clearly

$$\frac{2n+2-nq_k}{n+2} \le \frac{2n+2}{n+2},$$

and thanks to (4.15) and (4.16) we have

$$\frac{n(q_k-1)}{2n+2-nq_k} \le \frac{n(1+\eta_k)}{2-n\eta_k} \le \frac{n(1+\eta_0)}{2-n\eta_0}$$

as well as

$$2 \cdot \frac{n+2}{2n+2-nq_k} \le \gamma,$$

where

$$\gamma := \frac{2(n+2)}{2-n\eta_0}.$$

Finally, from (4.15), (4.16), and (4.18) we obtain that

$$\begin{aligned}\Big(\frac{2n-(n-2)q_k}{n+2} \cdot \frac{n+2}{2n+2-nq_k}\Big) - 2 &= \frac{(n+2)(q_k-2)}{2n+2-nq_k} \\ &\le \frac{(n+2)\eta_k}{2-n\eta_k} \\ &\le \frac{(n+2)\eta_k}{2-n\eta_0} \\ &\le c_7 \cdot 2^{-k}\end{aligned}$$

for all $k \ge 1$, where $c_7 := \frac{(n+2)c_3}{2-n\eta_0}$. Since $c_4 \le 1, p_k \ge 1$, and $A_{k-1} \ge 1$, (4.27) in conjunction with (4.14), (4.15), (4.24), and (4.25) thus implies that

$$c_5 p_k^2 \int_\Omega w^{-p_k+\alpha+2\beta-2} \le (1-\theta)c_4 \int_\Omega |\nabla w^{-\frac{p_k+\alpha}{2}}|^2 + c_8 \cdot 2^{\gamma k} \cdot A_{k-1}^{2+\hat\eta_k}$$

for some $c_8 > 0$ and $\hat\eta_k > 0$ satisfying

$$\hat\eta_k \le c_9 \cdot 2^{-k} \qquad \text{for all } k \ge 1 \tag{4.28}$$

with a certain $c_9 > 0$. Accordingly, (4.21) yields

$$\frac{d}{dt}\int_\Omega w^{-p_k} + (1-\theta)\int_\Omega |\nabla w^{-\frac{p_k+\alpha}{2}}|^2 \le c_8 \cdot 2^{\gamma k} \cdot A_{k-1}^{2+\hat\eta_k} \qquad \text{for all } t \in (0,T). \tag{4.29}$$

We next apply Lemma 3.2 to $q := \tilde q_k$ to find $c_{10} > 1$ fulfilling

$$\int_\Omega w^{-p_k} \le c_{10}\|\nabla w^{-\frac{p_k+\alpha}{2}}\|_{L^2(\Omega)}^{\tilde q_k \cdot a} \cdot \|w^{-\frac{p_k+\alpha}{2}}\|_{L^1(\Omega)}^{\tilde q_k\cdot(1-a)} + c_{10}\|w^{-\frac{p_k+\alpha}{2}}\|_{L^1(\Omega)}^{\tilde q_k}$$

with $a := \frac{2n(\tilde q_k - 1)}{(n+2)\tilde q_k}$. Applying (4.26) to $s := \frac{1}{a}$ and $\varepsilon := 1$ and invoking (4.23), we thus obtain

$$\begin{aligned}\int_\Omega w^{-p_k} &\le c_{10} \cdot \Big\{\|\nabla w^{-\frac{p_k+\alpha}{2}}\|_{L^2(\Omega)}^{\tilde q_k} + \|w^{-\frac{p_k+\alpha}{2}}\|_{L^1(\Omega)}^{\tilde q_k}\Big\} \\ &\quad + c_{10}\|w^{-\frac{p_k+\alpha}{2}}\|_{L^1(\Omega)}^{\tilde q_k} \\ &\le c_{10}\|\nabla w^{-\frac{p_k+\alpha}{2}}\|_{L^2(\Omega)}^{\tilde q_k} + 2c_{10}A_{k-1}^{\tilde q_k},\end{aligned}$$

where we also have used that $s > 1$ and $\frac{s}{s-1} > 1$.

Now thanks to the easily verified elementary inequality $(X-Y)^\sigma \ge 2^{-\sigma}X^\sigma - Y^\sigma$, valid whenever $\sigma > 0$ and $0 \le Y \le X$, we infer that

$$\begin{aligned}\int_\Omega |\nabla w^{-\frac{p_k+\alpha}{2}}|^2 &\ge \Big(\frac{1}{c_{10}}\int_\Omega w^{-p_k} - 2A_{k-1}^{\tilde{q}_k}\Big)^{\frac{2}{\tilde{q}_k}} \\ &\ge (2c_{10})^{-\frac{2}{\tilde{q}_k}} \cdot \Big(\int_\Omega w^{-p_k}\Big)^{\frac{2}{\tilde{q}_k}} - 2^{\frac{2}{\tilde{q}_k}} \cdot A_{k-1}^2 \\ &\ge \frac{1}{4c_{10}^2} \cdot \Big(\int_\Omega w^{-p_k}\Big)^{\frac{p_k+\alpha}{p_k}} - 4A_{k-1}^2\end{aligned}$$

in view of (4.20), (4.19) and the fact that $2c_{10} > 1$. Accordingly, (4.29) leads to the ordinary differential inequality

$$\begin{aligned}\frac{d}{dt}\int_\Omega w^{-p_k} &\le -\frac{1-\theta}{4c_{10}^2}\Big(\int_\Omega w^{-p_k}\Big)^{\frac{p_k+\alpha}{p_k}} + 4(1-\theta)A_{k-1}^2 + c_8 \cdot 2^{\gamma_k} A_{k-1}^{2+\hat{\eta}_k} \\ &\le -c_{11}\Big(\int_\Omega w^{-p_k}\Big)^{\frac{p_k+\alpha}{p_k}} + c_{12}\cdot 2^{\gamma k} \cdot A_{k-1}^{2+\hat{\eta}_k} \qquad \text{for all } t \in (0,T)\end{aligned}$$

for the function $t \mapsto \int_\Omega w^{-p_k}(x,t)dx$, where $c_{11} := \frac{1-\theta}{4c_{10}^2}$ and $c_{12} := 4(1-\theta) + c_8$.

By integration, we see that

$$\begin{aligned}&\int_\Omega w^{-p_k}(x,t)dx \\ &\le \max\left\{\int_\Omega (1-u_0)^{-p_k},\ \Big(\frac{c_{12}\cdot 2^{\gamma k}\cdot A_{k-1}^{2+\hat{\eta}_k}}{c_{11}}\Big)^{\frac{p_k}{p_k+\alpha}}\right\} \qquad \text{for all } t \in (0,T).\end{aligned} \tag{4.30}$$

Since (4.20) entails that $\frac{p_k}{p_k+\alpha} \le \frac{\overline{q}}{2}$ for all $k \ge 1$, we see that

$$\begin{aligned}\frac{(2+\hat{\eta}_k)\frac{p_k}{p_k+\alpha}}{2} - 1 &= \frac{-\alpha}{p_k+\alpha} + \frac{\hat{\eta}_k}{2}\cdot\frac{p_k}{p_k+\alpha} \\ &\le \delta_k := \Big|\frac{-\alpha}{p_k+\alpha}\Big| + \frac{\overline{q}\hat{\eta}_k}{4},\end{aligned}$$

and hence (4.30) in particular implies that for all $k \ge 1$ we have

$$A_k \le \max\left\{1,\ \int_\Omega (1-u_0)^{-p_k},\ b^k \cdot A_{k-1}^{2(1+\delta_k)}\right\}$$

with some $b > 1$ independent of $k$. Now in the case when $A_k \le \max\{1, \int_\Omega (1-u_0)^{-p_k}\}$ for infinitely many $k \in \mathbb{N}$, we immediately conclude that (4.7) holds. Otherwise we may assume upon increasing $p_0$ if necessary that

$$A_k \le b^k A_{k-1}^{2(1+\delta_k)} \qquad \text{for all } k \ge 1.$$

Upon a straightforward induction, using (4.14) we see that

$$\begin{aligned}A_k^{\frac{1}{p_k}} &\le \left\{b^{\sum_{j=1}^k j\cdot\prod_{i=j+1}^k 2(1+\delta_i)} \cdot A_0^{\prod_{i=1}^k 2(1+\delta_i)}\right\}^{\frac{1}{c_1\cdot 2^k}} \\ &= b^{\frac{1}{c_1}\cdot\sum_{j=1}^k j\cdot 2^{-j}\cdot\prod_{i=j+1}^k(1+\delta_i)} \cdot A_0^{\prod_{i=1}^k(1+\delta_i)} \qquad \text{for all } k \ge 1.\end{aligned}$$

Since by (4.14) and (4.28)

$$\begin{aligned}\delta_k &\le \frac{1}{p_k}\cdot\frac{|\alpha|}{1-\frac{|\alpha|}{p_0}}+\frac{\overline{q}\hat{\eta}_k}{4}\\ &\le \frac{1}{c_1\cdot 2^k}\cdot\frac{p_0|\alpha|}{p_0-|\alpha|}+\frac{\overline{q}\cdot c_9\cdot 2^{-k}}{4}\\ &\le c_{13}\cdot 2^{-k} \qquad \text{for all } k\ge 1\end{aligned}$$

holds with $c_{13}:=\frac{p_0|\alpha|}{c_1(p_0-|\alpha|)}+\frac{\overline{q}\cdot c_9}{4}$, it follows that $\prod_{i=1}^{\infty}(1+\delta_i)$ is finite due to the fact that $\sum_{i=1}^{\infty}\delta_i$ converges. Since, moreover, $\sum_{j=1}^{\infty} j\cdot 2^{-j}<\infty$, and since $A_0$ is finite according to (4.7), from this we conclude that (4.8) also holds in this case. The proof is complete. ∎

**5. Global uniformly regular solutions for $\alpha+\beta>1$.** In order to derive Theorem 1.1, we first exploit the estimate from Lemma 4.1 in conjunction with the preparations provided in section 3 to establish bounds, uniform with respect to time, for $\frac{1}{1-u}$ in any space $L^p(\Omega)$, provided that (1.15) and (4.2) hold.

LEMMA 5.1. *Assume that* (1.11) *and* (1.12) *are valid with some* $c_D>0, c_h>0$, $\alpha\in\mathbb{R}$, *and* $\beta\in\mathbb{R}$ *such that*

$$\beta>1-\alpha,$$

*and let* $u_0$ *comply with* (1.13). *Then for each* $K>0$ *and any* $p>1$ *there exists* $C(K,p)>0$ *with the following property: If* $T>0$ *and* $v\in L^\infty((0,T);C^1(\bar\Omega))$ *satisfies* $\frac{\partial v}{\partial\nu}=0$ *on* $\partial\Omega$ *and* $|\nabla v|\le K$ *a.e. in* $\Omega\times(0,T)$, *and if* $u$ *is a classical solution of* (4.1) *in* $\Omega\times(0,T)$ *such that* $0\le u<1$, *then*

$$\int_\Omega (1-u)^{-p}(x,t)dx\le C(K,p) \qquad \text{for all } t\in(0,T). \tag{5.1}$$

*Remark.* We once more underline that the constant in (5.1) does not depend on $T$.

*Proof.* It is evidently sufficient to consider $p>1$ large enough fulfilling

$$\frac{p-\alpha-2\beta+2}{p+\alpha}\ge\frac{1}{2},\quad p>|\alpha|,\quad\text{and}\quad p>-\frac{n\alpha}{2}. \tag{5.2}$$

For such $p$, from Lemma 4.1 and the fact that $u\le 1$, we see that $w:=1-u$ satisfies

$$\begin{aligned}&\frac{d}{dt}\int_\Omega w^{-p}+\frac{2p^2c_D}{(p+\alpha)^2}\int_\Omega|\nabla w^{-\frac{p+\alpha}{2}}|^2\\ &\qquad\le\frac{p^2c_h^2K^2}{c_D}\int_\Omega w^{-p+\alpha+2\beta-2}\qquad\text{for all } t\in(0,T).\end{aligned} \tag{5.3}$$

Now an integration of (4.1) shows that $\int_\Omega u(x,t)dx\equiv\int_\Omega u_0=:M$ for all $t\in(0,T)$, where our assumption $u_0<1$ in $\bar\Omega$ entails that $M<|\Omega|$. It is therefore possible to pick some $a\in(1,\frac{|\Omega|}{M})$ and apply Lemma 3.1 to infer that

$$\left|\left\{u(\cdot,t)\le\frac{aM}{|\Omega|}\right\}\right|\ge\frac{a-1}{a}|\Omega|\qquad\text{for all } t\in(0,T).$$

Since $\frac{aM}{|\Omega|} < 1$, this means that $z := w^{-\frac{p+\alpha}{2}}$ satisfies

$$\Big|\Big\{z(\cdot,t) \le c_1\Big\}\Big| \ge c_2 \qquad \text{for all } t \in (0,T)$$

with the positive constants $c_1 := (1 - \frac{aM}{|\Omega|})^{-\frac{p+\alpha}{2}}$ and $c_2 := \frac{a-1}{a}|\Omega|$. Consequently, Lemma 3.3 says that for some $c_3 > 0$, as all constants $c_4, c_5, \dots$ below possibly depending on $K$ and $p$ but not on $t$ or on $T$, we have

$$\frac{p^2 c_h^2 K^2}{c_D} \int_\Omega w^{-p+\alpha+2\beta-2} = \frac{p^2 c_h^2 K^2}{c_D} \int_\Omega z^{\frac{2(p-\alpha-2\beta+2)}{p+\alpha}} \le c_3 \cdot \Big\{ 1 + \Big( \int_\Omega |\nabla z|^2 \Big)^{\frac{p-\alpha-2\beta+2}{p+\alpha}} \Big\} \quad \text{for all } t \in (0,T), \tag{5.4}$$

because $q := \frac{2(p-\alpha-2\beta+2)}{p+\alpha}$ satisfies $q \ge 1$ and

$$q = 2 - \frac{4(\alpha+\beta-1)}{p+\alpha} \le 2 < \frac{2n}{(n-2)_+}$$

according to our hypotheses (5.2) on $p$ and the fact that $\alpha + \beta \ge 1$. Using that even the strict inequality $\alpha + \beta > 1$ holds, we find that $\frac{p-\alpha-2\beta+2}{p+\alpha} < 1$, whence by means of Young's inequality we can find $c_4 > 0$ such that

$$c_3 \Big( \int_\Omega |\nabla z|^2 \Big)^{\frac{p-\alpha-2\beta+2}{p+\alpha}} \le \frac{p^2 c_D}{(p+\alpha)^2} \int_\Omega |\nabla z|^2 + c_4 \qquad \text{for all } t \in (0,T),$$

and thus obtain from (5.3) and (5.4) that

$$\frac{d}{dt} \int_\Omega w^{-p} + \frac{p^2 c_D}{(p+\alpha)^2} \int_\Omega |\nabla w^{-\frac{p+\alpha}{2}}|^2 \le c_3 + c_4 \qquad \text{for all } t \in (0,T). \tag{5.5}$$

In order to turn the dissipative integral into an appropriate absorptive term that can be used to control the large time behavior, we once more apply Lemma 3.3, but this time to $\tilde{q} := \frac{2p}{p+\alpha}$, which, again by (5.2), also satisfies $1 \le \tilde{q} \le \frac{2n}{(n-2)_+}$. As a consequence, we infer that

$$\int_\Omega w^{-p} = \int_\Omega z^{\frac{2p}{p+\alpha}} \le c_5 \cdot \Big\{ 1 + \Big( \int_\Omega |\nabla z|^2 \Big)^{\frac{p}{p+\alpha}} \Big\} \qquad \text{for all } t \in (0,T)$$

holds with some $c_5 > 0$. From (5.5) we can therefore derive the inequality

$$\frac{d}{dt} \int_\Omega w^{-p} \le c_3 + c_4 - \frac{p^2 c_D}{(p+\alpha)^2} \cdot \Big\{ \frac{1}{c_5} \cdot \int_\Omega w^{-p} - 1 \Big\}^{\frac{p+\alpha}{p}} \qquad \text{for all } t \in (0,T).$$

Upon a straightforward ODE comparison argument, this entails the estimate

$$\int_\Omega w^{-p}(x,t)dx \le \max \Big\{ \int_\Omega (1-u_0)^{-p} \,,\, c_5 \cdot \Big[ \Big( \frac{(p+\alpha)^2 (c_3+c_4)}{p^2 c_D} \Big)^{\frac{p}{p+\alpha}} + 1 \Big] \Big\}$$

for all $t \in (0,T)$, and thereby proves the lemma. ∎

In conjunction with this, an application of Lemma 4.2 now immediately yields the following.

LEMMA 5.2. *Let* (1.13), (1.11), *and* (1.12) *hold with some* $c_D > 0, c_h > 0$, $\alpha \in \mathbb{R}$, *and* $\beta \in \mathbb{R}$ *fulfilling*

$$\beta > 1 - \alpha.$$

*Then for each* $K > 0$ *one can find* $C(K) > 0$ *such that whenever* $T > 0$ *and* $v \in L^\infty((0,T); C^1(\bar{\Omega}))$ *satisfies* $\frac{\partial v}{\partial \nu} = 0$ *on* $\partial\Omega$ *and* $|\nabla v| \le K$ *a.e. in* $\Omega \times (0,T)$, *and if* $u$ *is a classical solution of* (4.1) *in* $\Omega \times (0,T)$ *such that* $0 \le u < 1$, *then*

$$\frac{1}{1-u} \le C(K) \qquad \text{in } \Omega \times (0,T). \tag{5.6}$$

*Remark*. Again, the constant appearing on the right-hand side of (5.6) does not depend on $T$.

We can now directly pass to the proof of our main results concerning the case $\alpha + \beta > 1$.

*Proof of Theorem* 1.1. (i) Since $0 \le u \le 1$ and $\nabla v_0$ is bounded, parabolic regularity applied to the second equation in (1.1) guarantees that there exists $K > 0$ such that $|\nabla v| \le K$ in $\Omega \times (0, T_{max})$ (cf. [36, Proposition 1] for details). By Lemma 5.1 we have

$$\sup_{T \in [0, T_{max})} A_0(T) < \infty,$$

and therefore the claim immediately results from Lemma 5.2 and the extensibility criterion (2.1) provided by Lemma 2.1.

(ii) The proof is similar to the one above, relying on Lemma 2.2 instead of Lemma 2.1. The only difference consists of a usage of elliptic [11] rather than parabolic regularity arguments here in order to make sure that again $\nabla v$ is bounded in $\Omega \times (0, T_{max})$. ☐

## 6. Global solutions for $\beta \ge 1 - \frac{\alpha}{2}$ and singularity formation in infinite time.

**6.1. Global solvability.** We first go back to Lemma 4.1, but proceed from (4.3) by pursuing a strategy different from that used in Lemma 5.1. Namely, we shall no longer rely on the dissipative term in (4.3) here. A similar reasoning has been applied in [36, Corollary 5] for $\alpha > 0$.

LEMMA 6.1. *Suppose that* (1.11) *and* (1.12) *hold with some* $c_D > 0, c_h > 0$, $\alpha \in \mathbb{R}$, *and* $\beta \in \mathbb{R}$ *such that*

$$\beta \ge 1 - \frac{\alpha}{2}, \tag{6.1}$$

*and let* $u_0$ *satisfy* (1.13). *Then for each* $K > 0$, $p > 1$, *and* $T > 0$ *there exists* $C(K,p,T) > 0$ *with the following property: If* $v \in L^\infty((0,T); C^1(\bar{\Omega}))$ *satisfies* $\frac{\partial v}{\partial \nu} = 0$ *on* $\partial\Omega$ *and* $|\nabla v| \le K$ *a.e. in* $\Omega \times (0,T)$, *and if* $u$ *is a classical solution of* (4.1) *in* $\Omega \times (0,T)$ *such that* $0 \le u < 1$, *then*

$$\int_\Omega (1-u)^{-p}(x,t)dx \le C(K,p,T) \qquad \text{for all } t \in (0,T). \tag{6.2}$$

*Proof.* According to (4.3) and the fact that $0 \le u < 1$, the function $w := 1 - u$ satisfies

$$\frac{d}{dt}\int_\Omega w^{-p} \le \frac{p^2 c_h^2 K^2}{c_D} \int_\Omega w^{-p+\alpha+2\beta-2} \qquad \text{for all } t \in (0,T).$$

Since $w \le 1$ and hence $w^{\alpha+2\beta-2} \le 1$ due to the fact that $\alpha + 2\beta \ge 2$, this implies that

$$\frac{d}{dt}\int_\Omega w^{-p} \le \frac{p^2 c_h^2 K^2}{c_D}\int_\Omega w^{-p} \qquad \text{for all } t \in (0,T).$$

Upon integration, this yields

$$\int_\Omega w^{-p}(x,t)dx \le \Big(\int_\Omega w^{-p}(x,0)dx\Big)\cdot e^{\frac{p^2 c_h^2 K^2}{c_D}\cdot t} \qquad \text{for all } t \in (0,T)$$

and thereby proves (6.2). ◻

Based on the above result, Lemma 4.2 now provides a time-dependent pointwise upper bound for $\frac{1}{1-u}$.

LEMMA 6.2. *Assume that* (1.13), (1.11), *and* (1.12) *are valid with some* $c_D > 0, c_h > 0$, $\alpha \in \mathbb{R}$, *and* $\beta \in \mathbb{R}$ *fulfilling*

$$\beta \ge 1 - \frac{\alpha}{2}.$$

*Then for each* $K > 0$ *and any* $T > 0$ *there exists* $C(K,T) > 0$ *such that if* $v \in L^\infty((0,T);C^1(\bar\Omega))$ *satisfies* $\frac{\partial v}{\partial \nu} = 0$ *on* $\partial\Omega$ *and* $|\nabla v| \le K$ *a.e. in* $\Omega \times (0,T)$, *and if* $u$ *is a classical solution of* (4.1) *in* $\Omega \times (0,T)$ *such that* $0 \le u < 1$, *then*

$$\frac{1}{1-u} \le C(K,T) \qquad \text{in } \Omega \times (0,T). \tag{6.3}$$

Now as in the previous section, the main result on global existence in the case $\beta \ge 1 - \frac{\alpha}{2}$ actually reduces to a corollary.

*Proof of Theorem* 1.2. As in the proof of Theorem 1.1, we use parabolic and elliptic regularity arguments to derive a uniform bound for $\nabla v$ and then apply Lemma 6.2 along with Lemmas 2.1 and 2.2. ◻

*Remark* 6.1. We may notice that in the case of $\alpha = 0$ and $\beta = 1$ the strong maximum principle can be applied to the first equation in (1.1) after the change of variable $w := 1 - u$. It then follows that $u(x,t) < 1$ for all $x \in \Omega$ and $t > 0$.

**6.2. Singularity formation in infinite time.**

*Proof of Theorem* 1.3. Writing $\lambda := \alpha + \beta$, we have $\lambda < 1$, and hence from [32, Theorem 1.1] we know that there exists $c_1 > 0$ with the property that if

$$\frac{h(s)}{D(s)} \ge c_1(1-s)^\lambda \qquad \text{for all } s \in (0,1), \tag{6.4}$$

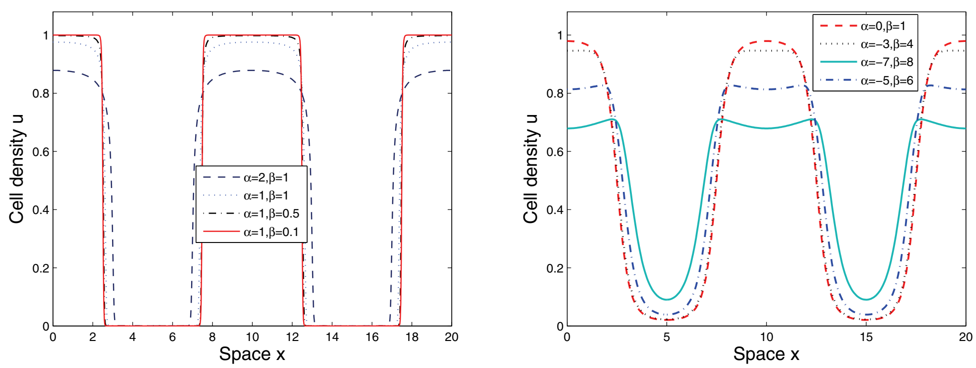
then there exists $u_0 \in C^\infty(\bar\Omega)$ such that $0 < u_0 < 1$ in $\bar\Omega$ and such that the corresponding maximally extended local-in-time solution $(u,v)$ of (1.2) satisfies $0 < u < 1$ in $\bar\Omega \times [0,T_{max})$ and

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to 1 \qquad \text{as } t \nearrow T_{max}, \tag{6.5}$$

where $T_{max} \in (0,\infty]$ denotes its maximal existence time. However, evidently (6.4) is implied by the validity of (1.19) and (1.20) when $\frac{\tilde c_D}{\tilde c_h} \le \frac{1}{c_1}$. We therefore only need to notice that on the other hand any such solution must be global in time, which is an immediate consequence of Theorem 1.2 (ii) in view of our assumption that (1.11) and (1.12) are valid with $\beta \ge 1 - \frac{\alpha}{2}$. ◻

**7. Numerical simulations.** In this section, we shall explore some numerical solutions of the model (1.1) in different parameter regimes to illustrate how the parameter values affect the solution profiles, and make predictions for undiscussed cases. The finite-element-based software COMSOL has been implemented for computation, and for simplicity we restrict our attention to a one-dimensional interval $\Omega = (0, 20)$.

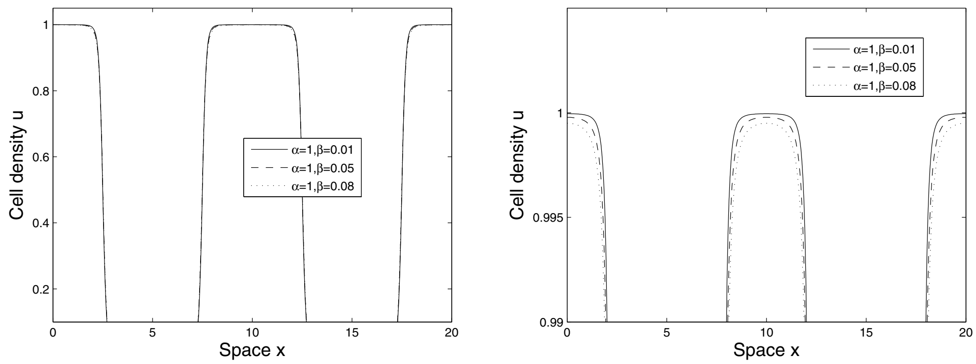
The most interesting component pertaining to our analytical results is to see in which parameter regimes the solution blows up (i.e., $u$ reaches the singular value 1) or is strictly less than 1. Theorem 1.1 makes significant progress toward the complete answer, and we show the relevant numerical solutions in Figure 2 (a), where the solution profile $u$ is plotted at time $t = 800$ for different values of $\alpha$ and $\beta$ such that $\alpha+\beta > 1$ fulfills the conditions of Theorem 1.1. We observe that the maximum of the solution decreases with respect to the value of sum $\alpha + \beta$. In particular the profile of the solution becomes flatter when $\alpha + \beta$ gets smaller.




(a) (b)

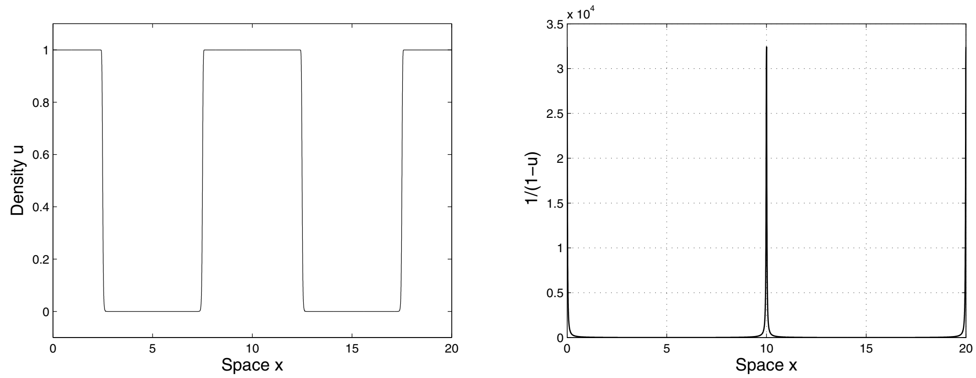
FIG. 2. *Numerical solution $u$ to the system* (1.1) *with initial data $u_0 = v_0 = 0.5+0.4\cos(\pi x/5)$, where $D(u) = d(1-u)^{-\alpha}$ and $h(u) = \chi(1-u)^{\beta}$. Other parameter values are* (a) $\alpha+\beta > 1$, $d = 1, \chi = 100$; (b) $\alpha+\beta = 1$ *with $\alpha < 0$, $d = 1, \chi = 20$. The exact values of $\alpha$ and $\beta$ are indicated in the legend.*

To recall, it was shown in our previous paper [32] that the solution may blow up in finite time if $\alpha > 0, \beta \geq 0$, and $\alpha + \beta < 1$. Theorem 1.1 shows that the solution is classical if $\alpha+\beta > 1$. This indicates that the line $\alpha+\beta = 1$ might become important, and that so does the region where $\alpha + \beta < 1$ and $\alpha \leq 0$. Theorem 1.2 has shown that the solution of (1.1) is classical and bounded away from 1 for any finite time if $\alpha+\beta = 1$ and $\alpha \leq 0$, as shown in Figure 2 (b), where we see the solution is strictly less than the singular value 1 and observe that the maximum of the solution $u$ is getting closer to 1 when $\alpha$ approaches 0 from below. However, in light of Remark 6.1 we know that for the critical case $\alpha = 0$, $u$ will be strictly less than 1 in the interior of the domain for any time $t > 0$. This fact was also numerically illustrated in Figure 2 (b). Now a natural question, left open by our analysis, is whether the solution may blow up in the critical case $\alpha + \beta = 1$ if $\alpha > 0$. Corresponding numerical solutions can be found in Figure 3, which indicates that the distance between the maximum of the large-time profile of $u$ and 1 seems to decrease to zero with respect to the sum $\alpha + \beta$ when this sum decreases to 1. According to this observation we believe that in the limit case $\alpha+\beta = 1$ when $\alpha > 0$, the solution $u$ indeed may attain the threshold value 1 either in finite or infinite time.




(a) (b)

FIG. 3. *The numerical simulation of the solution $u$ of system* (1.1) *with $D(u) = d(1-u)^{-\alpha}$ and $h(u) = \chi(1-u)^{\beta}$ for the critical case $\alpha + \beta = 1$, where $d = 1, \chi = 25$ and initial data are $u_0 = v_0 = 0.5 + 0.4\cos(\pi x/5)$. The process of the solution $u$ converging to the critical value 1 as the sum $\alpha + \beta$ approaches the critical value 1 is illustrated, where* (a) *plots the solution at $t = 400$ for three different values of $\alpha + \beta$ which are close to 1. Since the variation of the solutions $u$ plotted in* (a) *is too small to distinguish by eyes, a correspondingly amplified picture is given in* (b).




(a) (b)

FIG. 4. *The plot of the numerical solution $u$ of system* (1.1) *with $D(u) = d(1-u)^{-\alpha}$ and $h(u) = \chi(1-u)^{\beta}$ for the subcritical case $\alpha + \beta < 1$, where $d = 1, \chi = 100$. The initial data are* (a) *$u_0 = v_0 = 0.5 + 0.4\cos(\pi x/5)$ and* (b) *$u_0 = v_0 = 0.5 + 0.498\cos(\pi x/5)$. The values of $\alpha$ and $\beta$ in* (a) *are $\alpha = -1, \beta = 1.8$ such that $1 - \alpha/2 < \beta < 1 - \alpha$, and in* (b) *they are $\alpha = -1, \beta = 1.4$ such that $\beta < 1 - \alpha/2 < 1 - \alpha$.*

Theorem 1.3 partially addresses what happens to the case $\alpha + \beta < 1$ if $\alpha < 0$. It particularly asserts that if $\alpha$ and $\beta$ fall within the region II as plotted in Figure 1, then the solution may reach 1 in infinite time, provided that $\frac{\tilde{c}_h}{\tilde{c}_D}$ is large. This is numerically supported in Figure 4 (a), which plots the solution profile $u$ at the numerically terminal time step $t = 112$, just before the computation stops due to the degeneracy of both diffusion ($\alpha < 0$) and chemotaxis ($\beta > 0$). The simulation shown in Figure 4 (b) explores the case $\beta < 1 - \alpha/2, \alpha < 0$, for which it still remains open analytically whether the solution reaches singularity. However, Figure 4 (b) seems to indicate accumulative behavior of the graph of $\frac{1}{1-u}$, plotted here at $t = 0.12$; we thus may speculate that the solution $u$ will reach 1 in finite time.

## REFERENCES


[1] N. D. Alikakos, *An application of invariance principle to reaction-diffusion equations*, J. Differential Equations, 33 (1979), pp. 201–225.

[2] H. Amann, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, in Function Spaces, Differential Operators and Nonlinear Analysis, H. Triebel and H. J. Schmeisser, eds., Teubner-Texte Math. 133, Teubner, Stuttgart, 1993, pp. 9–126.

[3] V. Calvez and J. A. Carrillo, *Volume effects in the Keller-Segel model: Energy estimates preventing blow-up*, J. Math. Pures Appl. (9), 86 (2006), pp. 155–175.

[4] Y. S. Choi and Z. A. Wang, *Prevention of blow up by fast diffusion in chemotaxis*, J. Math. Anal. Appl., 362 (2010), pp. 553–564.

[5] T. Cieślak and P. Laurençot, *Finite time blow-up for radially symmetric solutions to a critical quasilinear Smoluchowski-Poisson system*, C. R. Math. Acad. Sci. Paris, 347 (2009), pp. 237–242.

[6] T. Cieślak and C. Morales-Rodrigo, *Quasilinear non-uniformly parabolic-elliptic system modelling chemotaxis with volume filling effect. Existence and uniqueness of global-in-time solutions*, Topol. Methods Nonlinear Anal., 29 (2007), pp. 361–381.

[7] T. Cieślak and M. Winkler, *Finite-time blow-up in a quasilinear system of chemotaxis*, Nonlinearity, 21 (2008), pp. 1057–1076.

[8] K. Djie and M. Winkler, *Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect*, Nonlinear Anal., 72 (2010), pp. 1044–1064.

[9] J. W. Dold, V. A. Galaktionov, A. A. Lacey, and J. L. Vázquez, *Rate of approach to a singular steady state in quasilinear reaction-diffusion equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 26 (1998), pp. 663–687.

[10] A. Friedman, *Partial Differential Equations*, Holt, Rinehart & Winston, New York, 1969.

[11] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 2001.

[12] M. A. Herrero and J. J. L. Velázquez, *A blow-up mechanism for a chemotaxis model*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 24 (1997), pp. 633–683.

[13] T. Hillen and K. J. Painter, *A user's guide to PDE models for chemotaxis*, J. Math. Biol., 58 (2009), pp. 183–217.

[14] D. Horstmann, *From* 1970 *until present: The Keller-Segel model in chemotaxis and its consequences*. I, Jahresber. Deutsch. Math.-Verein., 105 (2003), pp. 103–165.

[15] D. Horstmann and M. Winkler, *Boundedness vs. blow-up in a chemotaxis system*, J. Differential Equations, 215 (2005), pp. 52–107.

[16] S. Ishida and T. Yokota, *Global existence of weak solutions to quasilinear degenerate Keller-Segel systems of parabolic-parabolic type*, J. Differential Equations, 252 (2012), pp. 1421–1440.

[17] W. Jäger and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc., 329 (1992), pp. 819–824.

[18] J. Jost, *Partial Differential Equations*, Springer-Verlag, New York, 2002.

[19] N. I. Kavallaris and Ph. Souplet, *Grow-up rate and refined asymptotics for a two-dimensional Patlak–Keller–Segel model in a disk*, SIAM J. Math. Anal., 40 (2009), pp. 1852–1881.

[20] E. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theoret. Biol., 26 (1970), pp. 399–415.

[21] R. Kowalczyk and Z. Szymańska, *On the global existence of solutions to an aggregation model*, J. Math. Anal. Appl., 343 (2008), pp. 379–398.

[22] O. A. Ladyzenskaya, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasi-linear Equations of Parabolic Type*, Amer. Math. Soc. Transl. 23, AMS, Providence, RI, 1968.

[23] Ph. Laurençot and D. Wrzosek, *A chemotaxis model with threshold density and degenerate diffusion*, in Nonlinear Elliptic and Parabolic Problems, Progr. Nonlinear Differential Equations Appl. 64, Birkhäuser, Basel, 2005, pp. 273–290.

[24] P. M. Lushnikov, N. Chen, and M. Alber, *Macroscopic dynamics of biological cells interacting via chemotaxis and direct contact*, Phys. Rev. E., 78 (2008), 061904.

[25] K. J. Painter and T. Hillen, *Volume-filling and quorum-sensing in models for chemosensitive movement*, Can. Appl. Math. Q., 10 (2002), pp. 501–543.

[26] C. S. Patlak, *Random walk with persistence and external bias*, Bull. Math. Biophys., 15 (1953), pp. 311–338.

[27] P. Poláčik and E. Yanagida, *On bounded and unbounded global solutions of a supercritical semilinear heat equation*, Math. Ann., 327 (2003), pp. 745–771.

[28] Y. Sugiyama and H. Kunii, *Global existence and decay properties for a degenerate Keller-Segel model with a power factor in drift term*, J. Differential Equations, 227 (2006), pp. 333–364.

[29] Y. Tao and M. Winkler, *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity*, J. Differential Equations, 252 (2012), pp. 692–715.
[30] Z. A. Wang, *On chemotaxis models with cell population interactions*, Math. Model. Nat. Phenom., 5 (2010), pp. 173–190.
[31] Z. A. Wang and T. Hillen, *Classical solutions and pattern formation for a volume filling chemotaxis model*, Chaos, 17 (2007), 037108.
[32] Z. A. Wang, M. Winkler, and D. Wrzosek, *Singularity formation in chemotaxis systems with volume-filling effect*, Nonlinearity, 24 (2011), pp. 3279–3297.
[33] M. Winkler, *A critical exponent in a degenerate parabolic equation*, Math. Methods Appl. Sci., 25 (2002), pp. 911–925.
[34] M. Winkler, *A doubly critical degenerate parabolic problem*, Math. Methods Appl. Sci., 27 (2004), pp. 1619–1627.
[35] M. Winkler, *Does a "volume-filling effect" always prevent chemotactic collapse?*, Math. Methods Appl. Sci., 33 (2010), pp. 12–24.
[36] D. Wrzosek, *Model of chemotaxis with threshold density and singular diffusion*, Nonlinear Anal., 73 (2010), pp. 338–349.
[37] D. Wrzosek, *Volume filling effect in modelling chemotaxis*, Math. Model. Nat. Phenom., 5 (2010), pp. 123–147.