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ScienceDirect
J. Differential Equations 265 (2018) 2577–2613
Journal of Differential Equations
www.elsevier.com/locate/jde

Nonlinear stability of strong traveling waves for the singular Keller–Segel system with large perturbations

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Received 13 February 2018
Available online 27 April 2018

Abstract

This paper is concerned with the nonlinear stability of traveling wave solutions for a conserved system of parabolic equations derived from a singular chemotaxis model describing the initiation of tumor angiogenesis. When the initial datum is a continuous small perturbation with zero integral from the spatially shifted traveling wave, the asymptotic stability of the large-amplitude (strong) traveling waves has been established in a series of works [29,34,35] by the second author with his collaborators. In this paper, we shall show that similar stability results indeed hold true for large and discontinuous initial data (i.e. the initial perturbation from the traveling wave could be discontinuous and has large oscillations) such as Riemann data with large jumps. To the best of our knowledge, this paper provides a first result on the asymptotic stability of large-amplitude traveling waves with large initial perturbation for a system of conservation laws, although similar results have been available for the scalar equations (cf. [8,42]). We also extend existing results to the initial data with lower regularity.

MSC: 35A01; 35B40; 35B44; 35K57; 35Q92; 92C17

Keywords: Chemotaxis; Traveling wave solutions; Nonlinear stability; Logarithmic sensitivity; Large perturbation; Discontinuous data

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https://doi.org/10.1016/j.jde.2018.04.041
0022-0396/

1. Introduction

It is well known that chemotaxis, the movement of organism towards higher concentration of chemical substance, can produce rich wave patterns in different circumstances, such as traveling band of bacterial toward the oxygen [2], the outward propagation of concentric ring waves by *E. coli* [4], the spiral wave patterns during the aggregation of *Dictyostelium discoideum* [9] and the migration of *Myxococcus xanthus* in the early stage of starvation-induced fruiting body development [55]. The mathematical study of chemotactic traveling waves was started by Keller and Segel in their seminal paper [22] wherein the following model

$$\begin{cases} u_t = [Du_x - \chi u(\ln c)_x]_x, \\ c_t = \varepsilon c_{xx} - uc^m \end{cases} \tag{1.1}$$

was proposed to describe the propagation of traveling bands of chemotactic bacteria observed in the celebrated experiment of Adler [2], where $u(x,t)$ denotes the bacterial density and $c(x,t)$ the oxygen concentration. $D > 0$ and $\varepsilon \geq 0$ are bacterial and chemical diffusion coefficients, respectively, $\chi > 0$ is the chemotactic coefficient and $m \geq 0$ is the oxygen consumption rate.

When $0 \leq m < 1$, Keller and Segel [22] managed to use the model (1.1) with $\varepsilon = 0$ to interpret the traveling bands observed in the experiment of [2], followed with a serious of works for the case $\varepsilon \geq 0$ (cf. [21,37,41,44,47]). When $m > 1$, the model (1.1) does not admit traveling wave solutions (e.g., see [47,53]). In the borderline case $m = 1$, the model (1.1) with $\varepsilon > 0$ was first used by Rosen [45,46] to describe the chemotactic movement of motile aerobic bacterial toward oxygen, and later was employed to describe the directed movement of endothelial cells toward the signaling molecule vascular endothelial growth factor (VEGF) during the initiation of angiogenesis (cf. [6,7,24,25]).

While the existence of traveling wave solutions of the Keller–Segel model (1.1) with $\varepsilon \geq 0$ and $m \geq 0$ has been well established (see a review paper [53]), the stability of traveling wave solutions still remains as a very challenging question due to the singularity caused by the logarithmic sensitivity $\ln c$ whose mathematical derivation and biological relevance have been later presented in [20,43]. For the case $0 \leq m < 1$, expect some instability result [41] and classification of essential spectrum (cf. [38,39]) based on spectral analysis, no stability results on traveling wave solutions are available so far. However, in the case $m = 1$, the stability of traveling wave solutions to (1.1) with small $\varepsilon > 0$ (or $\varepsilon = 0$) has been gradually obtained (cf. [26,29,31–35]) by the (weighted) energy estimates. The success of these results heavily reply on the following Cole–Hope type transformation (cf. [23,34])

$$v = -(\ln c)_x = -\frac{c_x}{c},$$

which converts (1.1) with $m = 1$ into a parabolic system of conservation laws without singularity

$$\begin{cases} u_t - \chi(uv)_x = Du_{xx}, \\ v_t + (\varepsilon v^2 - u)_x = \varepsilon v_{xx}. \end{cases} \tag{1.2}$$

The transformation (1.2) significantly clears the obstruction caused by the logarithmic singularity in the original Keller–Segel system (1.1). Consequently a great deal of interesting results have been carried out for the transformed system (1.2) from various perspectives. For the global dynamics of classical solutions and nonlinear stability of traveling wave solutions of (1.2), we refer readers to [3,7,10,19,26–28,30–33,57] for $\varepsilon = 0$, and [5,28,29,34,35,40,49] for $\varepsilon > 0$. The diffusion limit and boundary layer problem of (1.2) as $\varepsilon \to 0$ were investigated in [17,18,28,49,54]. In addition, the well-posedness of system (1.1) has been studied recently in [56] by a different transformation $v = \ln c$ in a bounded domain with Neumann boundary conditions.

The main purpose of this paper will be to establish the stability of large-amplitude traveling waves of (1.2) with large and discontinuous initial data in $\mathbb{R}$:

$$(u, v)(x, 0) = (u_0, v_0)(x) \to (u_\pm, v_\pm), \text{ as } x \to \pm\infty \tag{1.3}$$

where $u_\pm \geq 0$ since u represents the density of biological species. Our work is motivated in the following ways. In one dimensional whole space $\mathbb{R}$, the nonlinear asymptotic stability of large-amplitude traveling wave solutions to (1.2) has been established in [29,34,35] when the initial datum $(u_0, v_0) \in H^1(\mathbb{R})$ is a small perturbation around the background traveling waves. However the numerical simulations in [29,35] have illustrated that traveling waves are still asymptotically stable under large initial perturbations, but rigorous justification still remains open. Though the nonlinear stability of traveling wave solutions of the scalar (viscous) conservation laws under large initial perturbations has been established (cf. [8,42]), no results have been available for a system of conservation laws as far as we know. In this paper, we shall fully exploit the peculiar structure of the system (1.2) and establish the nonlinear stability of traveling waves of (1.2) with initial data $(u_0, v_0) \in L^2(\mathbb{R})$ which allows large oscillations and discontinuity such as Riemann initial data with arbitrarily large jumps. Hence our present work will not only provide a first result for the asymptotic stability of large-amplitude traveling waves with large initial perturbation for a system of conservation laws, but also extend previous results with lower regularity on initial data. The problem of global dynamics with discontinuous data is an important topic of PDEs arising from fluid mechanics and gas dynamics. Hoff with his collaborators [11–15] has developed a series of important results in this topic (see [16,58] for further development). Some ideas in these works will be employed to establish our results.

We remark that this paper will be focused on the transformed chemotaxis system (1.2) only. The transfer of the results from (1.2) to the original Keller–Segel system (1.1) with $m = 1$ has been standard (cf. [29,35] for details) and hence will not be detailed in this paper for brevity.

The rest of paper is organized as follows. In section 2, the existence and properties of traveling wave solutions of (1.2) in the whole space $\mathbb{R}$ will be studied first. Then we state our main results. In section 3, we show the nonlinear stability of traveling wave solutions of (1.2)–(1.3) and prove our main results.

2. Statement of main results

In this section, we shall state our main results on the asymptotic stability of traveling wave solutions of the Cauchy problem (1.2)–(1.3). We depart with the existence of traveling wave solutions of (1.2), which is a non-constant special solution $(U, V) \in C^\infty(\mathbb{R})$ in the form of

$$(u, v)(x, t) = (U, V)(z), \ z = x - st,$$

which, upon a substitution onto (1.2), satisfies

$$\begin{cases} -sU' - \chi(UV)' = DU'', \\ -sV' + (\varepsilon V^2 - U)' = \varepsilon V'', \end{cases} \tag{2.1}$$

where $' = \frac{d}{dz}$ and s is called the wave speed. The traveling wave profile (U, V) satisfies the following asymptotic conditions at far field from (1.3)

$$U(\pm\infty) = u_\pm, \ \ V(\pm\infty) = v_\pm.$$

Integrating (2.1) in moving coordinate z over $\mathbb{R}$ with the above asymptotic conditions yields

$$\begin{cases} DU' = -sU - \chi UV + \varrho_1, \\ \varepsilon V' = -sV + \varepsilon V^2 - U + \varrho_2, \end{cases} \tag{2.2}$$

where

$$\begin{cases} \varrho_1 = su_- + \chi u_- v_- = su_+ + \chi u_+ v_+, \\ \varrho_2 = sv_- - \varepsilon(v_-)^2 + u_- = sv_+ - \varepsilon(v_+)^2 + u_+. \end{cases} \tag{2.3}$$

The wave speed s is uniquely determined by the Rankine–Hugoniot condition (cf. [51])

$$\begin{cases} -s(u_+ - u_-) - \chi(u_+ v_+ - u_- v_-) = 0, \\ -s(v_+ - v_-) + [\varepsilon(v_+)^2 - u_+ - \varepsilon(v_-)^2 + u_-] = 0. \end{cases}$$

The traveling wave solution (U, V) exists for any asymptotic states $v_+ \in \mathbb{R}$ and $u_+ \geq 0$ (cf. [34]). However it has been shown in [35] that $v_+ = u_+ = 0$ was the only biologically meaningful case for which the results of transformed system (1.2) can be converted to the original chemotaxis system (1.1). In this paper, we shall consider this meaningful case only which, along with (2.3), gives rise to $\varrho_1 = \varrho_2 = 0$ and

$$\begin{cases} s + \chi v_- = 0, \\ u_- = (\chi + \varepsilon) v_-^2. \end{cases} \tag{2.4}$$

Then the existence of traveling wave solutions of (1.2) is given below (cf. [29]).

Proposition 2.1. *Let $\varepsilon > 0$ and $u_+ = v_+ = 0$. Then the system (2.1) admits a unique (up to a translation) monotone traveling wave solution $(U, V)(x - st)$ satisfying*

$$U' < 0, \ \ V' > 0,$$

and

$$|V'| \leq C,$$

where the wave speed $s = -\chi v_-$ and $C > 0$ is a constant independent of ε.

Our main purpose is to exploit the nonlinear asymptotic stability of traveling wave solutions to the Cauchy problem (1.2)–(1.3) with discontinuous initial data having large oscillations. Roughly speaking, the stability means that the solution of (1.2)–(1.3) approaches the traveling wave solution $(U, V)(x - st)$, properly translated by an amount x_0, i.e.,

$$\sup_{x\in\mathbb{R}} |(u,v)(x,t)-(U,V)(x+x_0-st)| \to 0, \quad \text{as } \ t\to+\infty,$$

where x_0 satisfies an identity derived from the "conservation of mass" principle (cf. [51])

$$\int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x)-U(x) \\ v_0(x)-V(x) \end{pmatrix} dx = x_0 \begin{pmatrix} u_+-u_- \\ v_+-v_- \end{pmatrix} + \beta r_1(u_-,v_-),$$

with $r_1(u_-,v_-)$ denoting the first right eigenvector of the Jacobian matrix of (1.2) in the absence of viscous terms evaluated at (u_-,v_-). The coefficient β yields the diffusion wave in general. Both β and x_0 are uniquely determined by the initial data (u_0,v_0). For the stability of small-amplitude shock waves of conservation laws with diffusion wave (i.e. $\beta\neq 0$), we refer to [36,52] for details. In the present paper, we will not consider the diffusion wave by assuming $\beta=0$, but consider the stability of large-amplitude waves with large discontinuous data. Then by the conservation property of (1.2), we derive that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \begin{pmatrix} u(x,t)-U(x+x_0-st) \\ v(x,t)-V(x+x_0-st) \end{pmatrix} dx = \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x)-U(x+x_0) \\ v_0(x)-V(x+x_0) \end{pmatrix} dx \\
&= \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x)-U(x) \\ v_0(x)-V(x) \end{pmatrix} dx + \int_{-\infty}^{+\infty} \begin{pmatrix} U(x)-U(x+x_0) \\ V(x)-V(x+x_0) \end{pmatrix} dx \\
&= \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x)-U(x) \\ v_0(x)-V(x) \end{pmatrix} dx - x_0 \begin{pmatrix} u_+-u_- \\ v_+-v_- \end{pmatrix}.
\end{aligned} \tag{2.5}$$

This, along with $\beta=0$, implies the zero integral of the initial perturbation

$$\int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x)-U(x+x_0) \\ v_0(x)-V(x+x_0) \end{pmatrix} dx = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.6}$$

Then we employ the technique of taking anti-derivative to decompose the solution of (1.2)–(1.3) as

$$(u,v)(x,t) = (U,V)(x+x_0-st) + (\phi_x,\psi_x)(x,t). \tag{2.7}$$

That is

$$(\phi(x,t),\psi(x,t)) = \int_{-\infty}^{x} (u(y,t)-U(y+x_0-st), v(y,t)-V(y+x_0-st))dy$$

for $(x,t)\in\mathbb{R}\times\mathbb{R}_+$. The asymptotic states of the perturbation function (ϕ,ψ) are given from (2.5) as

$$\phi(\pm\infty, t) = \psi(\pm\infty, t) = 0, \quad \text{for all } \ t > 0.$$

The initial perturbation $(\phi_0, \psi_0)(x) = (\phi(x, 0), \psi(x, 0))$ is thus given by

$$(\phi_0, \psi_0)(x) = \int_{-\infty}^{x} (u_0(y) - U(y + x_0), v_0(y) - V(y + x_0))dy, \tag{2.8}$$

with $(\phi_0, \psi_0)(\pm\infty) = 0$ due to (2.6).

In the proof of our main results, we find that in the energy estimates (see the proof of Lemma 3.3) there is a singularity caused by $u_+ = 0$ (i.e. vacuum). To resolve this singularity, we invoke the ideas of works [19,29] to introduce an unbounded weight function and apply the weighted energy estimates, where the weight function $w(z)$ is defined by

$$w(z) := 1 + e^{\lambda z}, \quad \text{with } \ \lambda = \frac{s}{D} > 0, \ z \in \mathbb{R}. \tag{2.9}$$

It has been shown in [29] that there exist two constants $C_2 > C_1 > 0$ such that

$$C_1 w(z) \le \frac{1}{U(z)} \le C_2 w(z) \ \text{ for all } z \in \mathbb{R}. \tag{2.10}$$

To state our main result, we introduce some notations for the convenience of statement.

Notations. In what follows, C denotes a generic positive constant which may vary in the context. $H^k(\mathbb{R})$ denotes the usual k-th order Sobolev space on $\mathbb{R}$ with norm $\|f\|_{H^k(\mathbb{R})} := (\sum_{j=0}^{k} \int_{\mathbb{R}} |\partial_x^j f|^2 dx)^{1/2}$ and $H_w^k(\mathbb{R})$ denotes the weighted Sobolev space of measurable functions f so that $\sqrt{w}\partial_x^j f \in L^2(\mathbb{R})$ for $0 \le j \le k$ with norm $\|f\|_{H_w^k(\mathbb{R})} := (\sum_{j=0}^{k} \int_{\mathbb{R}} w(x)|\partial_x^j f|^2 dx)^{1/2}$. For simplicity, we denote $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R})}$, $\|\cdot\|_k := \|\cdot\|_{H^k(\mathbb{R})}$ and $\|\cdot\|_{k,w} := \|\cdot\|_{H_w^k(\mathbb{R})}$. Furthermore we use $\|\cdot\|_w$ to denote $\|\cdot\|_{L_w^2}$.

Then our main results are stated in the following theorem.

Theorem 2.2 *(Stability of traveling waves). Let $u_+ = v_+ = 0$ and $(U, V)(x - st)$ be a traveling wave solution of (2.1) obtained in Proposition 2.1. Assume that there exists a constant x_0 such that the initial perturbation from the spatially shifted traveling waves with shift x_0 is of zero mass, namely $\phi_0(\infty) = \psi_0(\infty) = 0$, where $(\phi_0, \psi_0)(x)$ is defined in (2.8). If $\varepsilon > 0$ is small, then there exists a constant $\eta > 0$, such that if*

$$\|\phi_0\|_w^2 + \|\psi_0\|^2 + \|u_0 - U\|_w^2 + \|v_0 - V\|_w^2 \le \eta, \tag{2.11}$$

the Cauchy problem (1.2)–(1.3) has a global solution $(u, v)(x, t)$ satisfying

$$u(x,t) - U(x - st) \in L^\infty([0,\infty); L_w^2) \cap L^2([0,\infty); H_w^1),$$
$$v(x,t) - V(x - st) \in L^\infty([0,\infty); L_w^2) \cap L^2([0,\infty); H_w^1)$$

and the following asymptotic stability:

$$\sup_{x\in\mathbb{R}} |(u, v)(x, t) - (U, V)(x + x_0 - st)| \to 0 \ \ as \ \ t \to \infty.$$

Remark 2.1 *(Relaxation on initial data).* The above nonlinear stability results hold true regardless of the size of amplitude of wave profiles and initial perturbations. In particular, either of the initial oscillations $\|u_0 - U_0\|_{L^\infty}$ and $\|v_0 - V_0\|_{L^\infty}$ can be arbitrarily large in Theorem 2.2, which is a substantial improvement of previous works (cf. [19,29,30,32–34]). From the initial condition (2.11), we see that the initial datum (u_0, v_0) is allowed to be discontinuous. In particular, it could be piecewise constant with arbitrarily large jump discontinuities such as Riemann data. The property of large oscillation and discontinuity on the initial data brings various difficulties to the analysis and make the present work distinct from the existing ones.

Remark 2.2 *(New ingredient in the proof).* The proofs of Theorem 2.2 differs from those in the existing literatures (cf. [19,29,30,32–34]) in the following two ways. First in the existing results, the initial perturbation has small oscillation (i.e. $\|u_0 - U\|_{L^\infty}$ and $\|v_0 - V\|_{L^\infty}$ are small) which was essentially used to estimate higher-order nonlinear terms. In our present paper, we have to devise some new refined estimates to estimates these terms (see the proof of Lemma 3.6). Second, the initial data $(u_0 - U, v_0 - V) \in L^2_w(\mathbb{R})$ considered presently has lower regularity than those in the existing works wherein $(u_0 - U, v_0 - V) \in H^1_w(\mathbb{R})$. The old ideas on the second-order estimates reply heavily on this higher regularity but fail in our present work. In this paper, we invoke the idea of Hoff [14] on the Navier–Stokes equations by introducing a time-dependent weight function combined with the parabolic smoothing effect to gain the desired second-order estimates (see the proof of Lemma 3.7).

3. Proof of Theorem 2.2

3.1. Reformulation of the problem

Substituting (2.7) into (1.2), using (2.1) and integrating the system with respect to x, we find that $(\phi, \psi)(x, t)$ satisfies

$$\begin{cases} \phi_t = D\phi_{xx} + \chi V\phi_x + \chi U\psi_x + \chi\phi_x\psi_x, \ t > 0, \ \ x \in \mathbb{R} \\ \psi_t = \varepsilon\psi_{xx} - 2\varepsilon V\psi_x + \phi_x - \varepsilon\psi_x^2, \end{cases} \tag{3.1}$$

with initial perturbation

$$(\phi_0, \psi_0)(x) = \int_{-\infty}^{x} (u_0(y) - U(y + x_0), v_0(y) - V(y + x_0))dy,$$

and

$$\phi_0(x) \in H^1_w(\mathbb{R}), \ \psi_0(x) \in L^2(\mathbb{R}), \ \psi_{0x}(x) \in L^2_w(\mathbb{R}). \tag{3.2}$$

We denote

$$m_0 := \|\phi_0\|^2_{1,w} + \|\psi_0\|^2 + \|\psi_{0x}\|^2_w. \tag{3.3}$$

For the reformulated problem (3.1)–(3.2), we have the following results.

Theorem 3.1. *If $\varepsilon > 0$ is small, then there exists a constant $\eta > 0$, such that if $m_0 \le \eta$, the problem (3.1)–(3.2) has a global strong solution (ϕ, ψ) satisfying for any $0 < T < \infty$*

$$\|\phi\|_{1,w}^2 + \|\psi\|_1^2 + \|\psi_x\|_w^2 + \int_0^T \left(\|\phi_x\|_{1,w}^2 + \|\psi_x\|_w^2 + \varepsilon\|\psi_x\|_1^2 + \varepsilon\|\psi_{xx}\|_w^2\right) dt \le C, \tag{3.4}$$

$$\begin{aligned}&\int_{\mathbb{R}} \sigma \left(\|\phi_t\|^2 + \|\psi_t\|^2 + \|\phi_{xx}\|^2 + \varepsilon\|\psi_{xx}\|^2\right) dx \\ &\qquad + \int_0^T \int_{\mathbb{R}} \sigma \left(\|\phi_{xt}\|^2 + \varepsilon\|\psi_{xt}\|^2 + \varepsilon\|\phi_{xxx}\|^2 + \varepsilon^2\|\psi_{xxx}\|^2\right) dx dt \le C,\end{aligned} \tag{3.5}$$

where $\sigma = \sigma(t) = \min\{1, t\}$ and C is a positive constant independent of t and ε. Moreover, it follows that

$$\sup_{x\in\mathbb{R}} |\phi_x(x,t),\ \psi_x(x,t)| \to 0 \ \ as \ \ t \to \infty. \tag{3.6}$$

In view of (2.7), Theorem 2.2 is a consequence of Theorem 3.1. We now outline the main procedures for the proof of Theorem 3.1. First, we mollify the (coarse) initial data (ϕ_0, ψ_0) as follows:

$$\phi_0^\delta = j^\delta * \phi_0, \quad \psi_0^\delta = j^\delta * \psi_0,$$

where j^δ is the standard mollifying kernel of width δ (e.g. see [1]). Then we consider the following augmented system

$$\begin{cases}\phi_t^\delta = D\phi_{xx}^\delta + \chi V\phi_x^\delta + \chi U\psi_x^\delta + \chi\phi_x^\delta\psi_x^\delta, \ t > 0, \quad x \in \mathbb{R}, \\ \psi_t^\delta = \varepsilon\psi_{xx}^\delta - 2\varepsilon V\psi_x^\delta + \phi_x^\delta - \varepsilon(\psi_x^\delta)^2,\end{cases} \tag{3.7}$$

with smooth initial perturbation functions $(\phi_0^\delta, \psi_0^\delta)$ which satisfies

$$\psi_0^\delta(x) \in L^2(\mathbb{R}),\ \psi_{0x}^\delta(x) \in H_w^1(\mathbb{R}),\ \phi_0^\delta(x) \in H_w^2(\mathbb{R}), \tag{3.8}$$

and

$$\|\phi_0^\delta\|_{1,w}^2 + \|\psi_0^\delta\|^2 + \|\psi_{0x}^\delta\|_w^2 \le \|\phi_0\|_{1,w}^2 + \|\psi_0\|^2 + \|\psi_{0x}\|_w^2 = m_0, \tag{3.9}$$

where we have used (3.3) and the following properties:

$$\|\partial_k\phi_0^\delta\|_w \le \|\partial_k\phi_0\|_w,\ \|\partial_k\psi_0^\delta\|_w \le \|\partial_k\psi_0\|_w \text{ for every } k = 0, 1,\ \delta > 0. \tag{3.10}$$

Next, by standard approaches, we prove the local existence of solutions to the system (3.7) with initial data $(\phi_0^\delta, \ \psi_0^\delta)$ satisfying (3.8)–(3.9). Then by the continuation argument, the global existence of $(\phi^\delta, \psi^\delta)$ follows from the *a priori* estimates. Finally, we show that the limit of $(\phi^\delta, \psi^\delta)$ as $\delta \to 0$ is a global strong solution of the Cauchy problem (3.1)–(3.2), and thus Theorem 3.1 is proved.

Lemma 3.2 *(Local existence). Assume that* $(\phi_0^\delta, \psi_0^\delta) \in H_w^2(\mathbb{R})$. *Then there exist a time* $T_0 = T_0\left(\|\phi_0^\delta\|_{H_w^2(\mathbb{R})}, \|\psi_0^\delta\|_{H_w^2(\mathbb{R})}\right) > 0$ *such that the system* (3.7) *has a unique solution* $(\phi^\delta, \psi^\delta) \in C([0, T_0), H_w^2(\mathbb{R}))$.

3.2. A priori estimates

In this subsection, we shall employ the technique of *a priori* assumption to derive the *a priori* estimates for the smooth solutions of (3.7)–(3.8). To this end, we first assume that the solution $(\phi^\delta, \psi^\delta)$ satisfies for any $t \in [0, T]$ that

$$\|\phi^\delta\|_{1,w}^2 + \|\psi^\delta\|^2 + \|\psi_x^\delta\|_w^2 \le 2\kappa_0, \tag{3.11}$$

where κ_0 is a positive constant. Then we derive the *a priori* estimates to obtain global solutions. Finally, we show the obtained global solutions in turn satisfy the above *a priori* assumption and close our argument.

We depart with the L^2-estimate of $(\phi^\delta, \psi^\delta)$. The main procedures for the proof are similar to those in the existing works (cf. [29]). For clarity and completeness, we present some details below.

Lemma 3.3 *(L^2-estimates). Let the conditions of Theorem 3.1 hold and* $(\phi^\delta, \psi^\delta)$ *be a smooth solution of* (3.7)–(3.8) *satisfying* (3.11). *Then there exists a constant* $C > 0$ *independent of* t, ε *and* δ *such that*

$$\|\phi^\delta\|_w^2 + \|\psi^\delta\|^2 + \int_0^T \|\phi_x^\delta\|_w^2 dt + \varepsilon \int_0^T \|\psi_x^\delta\|^2 dt \le Cm_0 + C\kappa_0 \int_0^T \|\psi_x^\delta\|_w^2 dt. \tag{3.12}$$

Proof. Multiplying the first and second equation of (3.7) by ϕ^δ/U and $\chi\psi^\delta$, respectively, and adding the resulting equalities, we obtain

$$\begin{aligned} &\frac{1}{2}\left(\frac{(\phi^\delta)^2}{U}\right)_t - \frac{(\phi^\delta)^2}{2}\left(\frac{1}{U}\right)_t + \left(\frac{\chi(\psi^\delta)^2}{2}\right)_t \\ &= \frac{D\phi^\delta\phi_{xx}^\delta}{U} + \chi\left(\phi^\delta\psi^\delta\right)_x + \frac{\chi V\phi^\delta\phi_x^\delta}{U} + \frac{\chi\phi^\delta\phi_x^\delta\psi_x^\delta}{U} \\ &\quad - 2\chi\varepsilon V\psi_x^\delta\psi^\delta - \chi\varepsilon(\psi_x^\delta)^2\psi^\delta + \chi\varepsilon\psi_{xx}^\delta\psi^\delta. \end{aligned} \tag{3.13}$$

Noting that

$$\frac{(\phi^\delta)^2}{2}\left(\frac{1}{U}\right)_t = -\frac{s(\phi^\delta)^2}{2}\left(\frac{1}{U}\right)_x,$$

$$\frac{\phi^\delta\phi^\delta_{xx}}{U} = \left(\frac{\phi^\delta\phi^\delta_x}{U}\right)_x - \frac{(\phi^\delta_x)^2}{U} - \phi^\delta\phi^\delta_x\left(\frac{1}{U}\right)_x = \left(\frac{\phi^\delta\phi^\delta_x}{U}\right)_x - \frac{(\phi^\delta_x)^2}{U} + \frac{U_x\phi^\delta\phi^\delta_x}{U^2},$$

$$\frac{V\phi^\delta\phi^\delta_x}{U} = \frac{1}{2}\left(\frac{V(\phi^\delta)^2}{U}\right)_x - \frac{(\phi^\delta)^2}{2}\left(\frac{V}{U}\right)_x,$$

$$-2\chi\varepsilon V\psi^\delta_x\psi^\delta = -\chi\varepsilon(V(\psi^\delta)^2)_x + \chi\varepsilon V_x(\psi^\delta)^2,$$

$$\chi\varepsilon\psi^\delta_{xx}\psi^\delta = \chi\varepsilon(\psi^\delta_x\psi^\delta)_x - \chi\varepsilon(\psi^\delta_x)^2,$$

we get from (3.13) that

$$\begin{aligned}&\frac{1}{2}\left(\frac{(\phi^\delta)^2}{U} + \chi(\psi^\delta)^2\right)_t + \frac{D(\phi^\delta_x)^2}{U} + \chi\varepsilon(\psi^\delta_x)^2 + \frac{(\phi^\delta)^2}{2}\left(\frac{s+\chi V}{U}\right)_x \\ &= \left(\chi\phi^\delta\psi^\delta + \frac{D\phi^\delta\phi^\delta_x}{U} + \frac{\chi V(\phi^\delta)^2}{2U} - \chi\varepsilon V(\psi^\delta)^2 + \chi\varepsilon\psi^\delta_x\psi^\delta\right)_x \\ &\quad + \frac{DU_x\phi^\delta\phi^\delta_x}{U^2} + \frac{\chi\phi^\delta\phi^\delta_x\psi^\delta_x}{U} + \chi\varepsilon V_x(\psi^\delta)^2 - \chi\varepsilon(\psi^\delta_x)^2\psi^\delta.\end{aligned} \tag{3.14}$$

A direct calculation along with $\frac{DU_x}{U} = -s - \chi V$ due to (2.2) gives

$$\left(\frac{s+\chi V}{U}\right)_x = \frac{\chi V_x}{U} - \frac{(s+\chi V)U_x}{U^2} = \frac{\chi V_x}{U} + \frac{DU_x^2}{U^3}. \tag{3.15}$$

Substituting (3.15) into (3.14) and integrating the resulting equation over $\mathbb{R}\times[0,T]$, we have that

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}}\left(\frac{(\phi^\delta)^2}{U} + \chi(\psi^\delta)^2\right)dx + D\int_0^T\int_{\mathbb{R}}\frac{(\phi^\delta_x)^2}{U}dxdt + \chi\varepsilon\int_0^T\int_{\mathbb{R}}(\psi^\delta_x)^2dxdt \\ &+ \frac{\chi}{2}\int_0^T\int_{\mathbb{R}}\frac{V_x(\phi^\delta)^2}{U}dxdt + \frac{D}{2}\int_0^T\int_{\mathbb{R}}\frac{U_x^2(\phi^\delta)^2}{U^3}dxdt \\ &= \frac{1}{2}\int_{\mathbb{R}}\left(\frac{(\phi^\delta_0)^2}{U} + \chi(\psi^\delta_0)^2\right)dx + D\int_0^T\int_{\mathbb{R}}\frac{U_x\phi^\delta\phi^\delta_x}{U^2}dxdt + \chi\int_0^T\int_{\mathbb{R}}\frac{\phi^\delta\phi^\delta_x\psi^\delta_x}{U}dxdt \\ &\quad - \chi\varepsilon\int_0^T\int_{\mathbb{R}}(\psi^\delta_x)^2\psi^\delta dxdt + \chi\varepsilon\int_0^T\int_{\mathbb{R}}V_x(\psi^\delta)^2dxdt\end{aligned} \tag{3.16}$$

$$= J_0 + J_1 + J_2 + J_3 + J_4.$$

We proceed to estimate J_i $(i = 0, \cdots, 3)$. We first have from (2.10) and (3.9) that

$$J_0 \le C\left(\|\phi_0^\delta\|_w^2 + \|\psi_0^\delta\|^2\right) \le Cm_0.$$

For J_1, by the Cauchy–Schwarz inequality, we have the following estimate:

$$J_1 \le \frac{3D}{4}\int_0^T\int_{\mathbb{R}} \frac{(\phi_x^\delta)^2}{U}dxdt + \frac{D}{3}\int_0^T\int_{\mathbb{R}} \frac{U_x^2(\phi^\delta)^2}{U^3}dxdt.$$

For J_2, by the Cauchy–Schwarz inequality and the Sobolev inequality $\|f\|_{L^\infty}^2 \le 2\|f\|\|f_x\|$, we derive from (3.11) and $\|\phi^\delta\|_1 \le C\|\phi^\delta\|_{1,w}$ that

$$\begin{aligned} J_2 \le& \frac{D}{8}\int_0^T\int_{\mathbb{R}} \frac{(\phi_x^\delta)^2}{U}dxdt + \frac{2\chi^2}{D}\int_0^T\int_{\mathbb{R}} \frac{(\phi^\delta)^2(\psi_x^\delta)^2}{U}dxdt \\ \le& \frac{D}{8}\int_0^T\int_{\mathbb{R}} \frac{(\phi_x^\delta)^2}{U}dxdt + \frac{2\chi^2}{D}\int_0^T \|\phi^\delta\|_{L^\infty}^2\int_{\mathbb{R}} \frac{(\psi_x^\delta)^2}{U}dxdt \\ \le& \frac{D}{8}\int_0^T\int_{\mathbb{R}} \frac{(\phi_x^\delta)^2}{U}dxdt + \frac{4\chi^2}{D}\int_0^T \|\phi^\delta\|\|\phi_x^\delta\|\int_{\mathbb{R}} \frac{(\psi_x^\delta)^2}{U}dxdt \\ \le& \frac{D}{8}\int_0^T\int_{\mathbb{R}} \frac{(\phi_x^\delta)^2}{U}dxdt + C\kappa_0\int_0^T\int_{\mathbb{R}} \frac{(\psi_x^\delta)^2}{U}dxdt. \end{aligned}$$

Similarly, we have

$$\begin{aligned} J_3 \le& \chi\varepsilon\int_0^T \|\psi^\delta\|_{L^\infty}\int_{\mathbb{R}} (\psi_x^\delta)^2dxdt \\ \le& \sqrt{2}\chi\varepsilon\int_0^T \|\psi^\delta\|^{\frac{1}{2}}\|\psi_x^\delta\|^{\frac{1}{2}}\int_{\mathbb{R}} (\psi_x^\delta)^2dxdt \\ \le& 2\kappa_0^{\frac{1}{2}}\chi\varepsilon\int_0^T\int_{\mathbb{R}} (\psi_x^\delta)^2dxdt \\ \le& \frac{\chi\varepsilon}{2}\int_0^T\int_{\mathbb{R}} (\psi_x^\delta)^2dxdt, \end{aligned}$$

provided that $\kappa_0 \le \frac{1}{16}$. Substituting the above estimates of $J_0 - J_3$ into (3.16), we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\frac{(\phi^\delta)^2}{U} + \chi(\psi^\delta)^2 \right) dx + \frac{D}{4} \int_0^T \int_{\mathbb{R}} \frac{(\phi_x^\delta)^2}{U} dxdt + \chi\varepsilon \int_0^T \int_{\mathbb{R}} (\psi_x^\delta)^2 dxdt \\
&+ \chi \int_0^T \int_{\mathbb{R}} \frac{V_x(\phi^\delta)^2}{U} dxdt + \frac{D}{3} \int_0^T \int_{\mathbb{R}} \frac{U_x^2(\phi^\delta)^2}{U^3} dxdt \\
\le& Cm_0 + 2\chi\varepsilon \int_0^T \int_{\mathbb{R}} V_x(\psi^\delta)^2 dxdt + C\kappa_0 \int_0^T \int_{\mathbb{R}} \frac{(\psi_x^\delta)^2}{U} dxdt.
\end{aligned} \tag{3.17}$$

Next, we need to estimate $\int_0^T \int_{\mathbb{R}} V_x(\psi^\delta)^2 dxdt$. Multiplying the first equation of (3.7) by $V\phi^\delta/U$ and the second one by $\chi V\psi^\delta$, and then adding the results to get

$$\begin{aligned}
&\frac{1}{2}\left(\frac{V(\phi^\delta)^2}{U}\right)_t + \frac{(\phi^\delta)^2}{2}\left(\frac{sV}{U}\right)_x + \left(\frac{\chi V(\psi^\delta)^2}{2}\right)_t + \frac{s\chi V_x(\psi^\delta)^2}{2} \\
=& \frac{DV\phi^\delta\phi_{xx}^\delta}{U} + \frac{\chi V^2\phi^\delta\phi_x^\delta}{U} + \chi\left(V\phi^\delta\psi^\delta\right)_x - \chi V_x\phi^\delta\psi^\delta + \frac{\chi V\phi^\delta\phi_x^\delta\psi_x^\delta}{U} \\
&- 2\chi\varepsilon V^2\psi_x^\delta\psi^\delta - \chi\varepsilon V(\psi_x^\delta)^2\psi^\delta + \chi\varepsilon V\psi_{xx}^\delta\psi^\delta.
\end{aligned} \tag{3.18}$$

A direct calculation leads to

$$\begin{aligned}
\frac{V\phi^\delta\phi_{xx}^\delta}{U} =& \left(\frac{V\phi^\delta\phi_x^\delta}{U}\right)_x - \frac{V(\phi_x^\delta)^2}{U} - \phi^\delta\phi_x^\delta\left(\frac{V}{U}\right)_x, \\
\frac{V^2\phi^\delta\phi_x^\delta}{U} =& \frac{1}{2}\left(\frac{V^2(\phi^\delta)^2}{U}\right)_x - \frac{(\phi^\delta)^2}{2}\left(\frac{V^2}{U}\right)_x, \\
-2\chi\varepsilon V^2\psi_x^\delta\psi^\delta =& -\chi\varepsilon(V^2(\psi^\delta)^2)_x + \chi\varepsilon(V^2)_x(\psi^\delta)^2, \\
\chi\varepsilon V\psi_{xx}^\delta\psi^\delta =& \chi\varepsilon(V\psi_x^\delta\psi^\delta)_x - \chi\varepsilon V(\psi_x^\delta)^2 - \chi\varepsilon V_x\psi^\delta\psi_x^\delta.
\end{aligned}$$

Substituting the above equalities into (3.18) and integrating the resultant equation over $\mathbb{R} \times [0, T]$, we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\left(\frac{V(\phi^\delta)^2}{U}+\chi V(\psi^\delta)^2\right)dx+D\int_0^T\int_{\mathbb{R}}\frac{V(\phi_x^\delta)^2}{U}dxdt\\
&+\chi\varepsilon\int_0^T\int_{\mathbb{R}}V(\psi_x^\delta)^2dxdt+\frac{s\chi}{2}\int_0^T\int_{\mathbb{R}}V_x(\psi^\delta)^2dxdt-\chi\varepsilon\int_0^T\int_{\mathbb{R}}\varepsilon(V^2)_x(\psi^\delta)^2dxdt\\
=&\frac{1}{2}\int_{\mathbb{R}}\left(\frac{V(\phi_0^\delta)^2}{U}+\chi V(\psi_0^\delta)^2\right)dx-\frac{1}{2}\int_0^T\int_{\mathbb{R}}(\phi^\delta)^2\left(\frac{sV}{U}\right)_x dxdt\\
&-\frac{1}{2}\int_0^T\int_{\mathbb{R}}(\phi^\delta)^2\left(\frac{\chi V^2}{U}\right)_x dxdt-D\int_0^T\int_{\mathbb{R}}\phi^\delta\phi_x^\delta\left(\frac{V}{U}\right)_x dxdt-\chi\int_0^T\int_{\mathbb{R}}V_x\phi^\delta\psi^\delta dxdt\\
&+\chi\int_0^T\int_{\mathbb{R}}\frac{V\phi^\delta\phi_x^\delta\psi_x^\delta}{U}dxdt-\chi\varepsilon\int_0^T\int_{\mathbb{R}}V(\psi_x^\delta)^2\psi^\delta dxdt-\chi\varepsilon\int_0^T\int_{\mathbb{R}}V_x\psi^\delta\psi_x^\delta dxdt.
\end{aligned}
\tag{3.19}
$$

By a direct computation, we have

$$
\left(\frac{sV}{U}\right)_x+\left(\frac{\chi V^2}{U}\right)_x=\left[\frac{1}{U}(s+\chi V)V\right]_x=\frac{\chi V_x(V-v_-)}{U}+\frac{\chi V_xV}{U}+\frac{DVU_x^2}{U^3},
$$

where we have used the fact that $s=-\chi v_-$ and $s+\chi V=-\frac{DU_x}{U}$ due to (2.2) and (2.4). Thus,

$$
\begin{aligned}
&-\frac{1}{2}\int_0^T\int_{\mathbb{R}}(\phi^\delta)^2\left(\frac{sV}{U}\right)_x dxdt-\frac{1}{2}\int_0^T\int_{\mathbb{R}}(\phi^\delta)^2\left(\frac{\chi V^2}{U}\right)_x dxdt\\
=&-\frac{1}{2}\int_0^T\int_{\mathbb{R}}(\phi^\delta)^2\left[\frac{1}{U}(s+\chi V)V\right]_x dxdt\\
=&-\chi\int_0^T\int_{\mathbb{R}}\frac{V_x(V-v_-)}{2U}(\phi^\delta)^2dxdt-\frac{\chi}{2}\int_0^T\int_{\mathbb{R}}\frac{V_xV}{U}(\phi^\delta)^2dxdt-\frac{D}{2}\int_0^T\int_{\mathbb{R}}\frac{VU_x^2(\phi^\delta)^2}{U^3}dxdt
\end{aligned}
$$

and

$$
\frac{s\chi}{2}\int_0^T\int_{\mathbb{R}}V_x(\psi^\delta)^2dxdt-\chi\varepsilon\int_0^T\int_{\mathbb{R}}\varepsilon(V^2)_x(\psi^\delta)^2dxdt=\chi\int_0^T\int_{\mathbb{R}}\left(\frac{s}{2}+2\varepsilon|V|\right)V_x(\psi^\delta)^2dxdt.
$$

By the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
&D\int_0^T\int_{\mathbb{R}}\left(\frac{V}{U}\right)_x\phi^\delta\phi_x^\delta dxdt\\
&=D\int_0^T\int_{\mathbb{R}}\left(\frac{V_x}{U}-\frac{VU_x}{U^2}\right)\phi^\delta\phi_x^\delta dxdt\\
&\le\frac{D}{2}\int_0^T\int_{\mathbb{R}}\left(\frac{V_x(\phi^\delta)^2}{U}+\frac{V_x(\phi_x^\delta)^2}{U}\right)dxdt+\frac{D}{2}\int_0^T\int_{\mathbb{R}}\left(\frac{|V|(\phi_x^\delta)^2}{U}+\frac{|V|U_x^2(\phi^\delta)^2}{U^3}\right)dxdt,
\end{aligned}
$$

$$
\chi\int_0^T\int_{\mathbb{R}}V_x\phi^\delta\psi^\delta dxdt\le\frac{\chi}{s}\int_0^T\int_{\mathbb{R}}V_x(\phi^\delta)^2dxdt+\frac{\chi s}{4}\int_0^T\int_{\mathbb{R}}V_x(\psi^\delta)^2dxdt,
$$

and

$$
-\chi\varepsilon\int_0^T\int_{\mathbb{R}}V_x\psi^\delta\psi_x^\delta dxdt\le\frac{\varepsilon\chi}{2}\int_0^T\int_{\mathbb{R}}\left((\psi_x^\delta)^2+V_x^2(\psi^\delta)^2\right)dxdt.
$$

Similar to the estimates of J_2 and J_3, we have from $|V|\le -v_-$ that

$$
\chi\int_0^T\int_{\mathbb{R}}\frac{V\phi^\delta\phi_x^\delta\psi_x^\delta}{U}dxdt\le\frac{D}{8}\int_0^T\int_{\mathbb{R}}\frac{(\phi_x^\delta)^2}{U}dxdt+C\kappa_0\int_0^T\int_{\mathbb{R}}\frac{(\psi_x^\delta)^2}{U}dxdt,
$$

and

$$
\varepsilon\chi\int_0^T\int_{\mathbb{R}}V(\psi_x^\delta)^2\psi^\delta dxdt\le\frac{\varepsilon\chi}{2}\int_0^T\int_{\mathbb{R}}(\psi_x^\delta)^2dxdt.
$$

Substituting the above estimates into (3.19), we get from $0<U<u_-, v_-<V<0$, $V_x>0, |V_x|\le C$ that

$$
\begin{aligned}
&\chi\int_0^T\int_{\mathbb{R}}\left(\frac{s}{4}+2\varepsilon|V|\right)V_x(\psi^\delta)^2dxdt+\chi\int_0^T\int_{\mathbb{R}}\frac{V_x(V-v_-)}{2U}(\phi^\delta)^2dxdt\\
&\le\frac{1}{2}\int_{\mathbb{R}}\left(\frac{V(\phi^\delta)^2}{U}+\chi V(\psi^\delta)^2-\frac{V(\phi_0^\delta)^2}{U}-\chi V(\psi_0^\delta)^2\right)dx\\
&\quad+\left(\frac{\chi|V|}{2}+\frac{D}{2}+\frac{\chi|U|}{s}\right)\int_0^T\int_{\mathbb{R}}\frac{V_x(\phi^\delta)^2}{U}dxdt
\end{aligned}
$$

$$+ D|V|\int_0^T\int_{\mathbb{R}} \frac{U_x^2(\phi^\delta)^2}{U^3}dxdt + \left(\frac{D}{8}+\frac{|V_x|}{2}+\frac{3|V|}{2}\right)\int_0^T\int_{\mathbb{R}} \frac{(\phi_x^\delta)^2}{U}dxdt$$

$$+ \varepsilon\chi(1+|V|)\int_0^T\int_{\mathbb{R}} (\psi_x^\delta)^2dxdt + \frac{\varepsilon\chi}{2}|V_x|\int_0^T\int_{\mathbb{R}} V_x(\psi^\delta)^2dxdt + C\kappa_0\int_0^T\int_{\mathbb{R}} \frac{(\psi_x^\delta)^2}{U}dxdt,$$

which, along with (3.17), implies that

$$\chi\int_0^T\int_{\mathbb{R}} \left(\frac{s}{4}+2\varepsilon|V|\right)V_x(\psi^\delta)^2dxdt \le Cm_0 + C\varepsilon\int_0^T\int_{\mathbb{R}} V_x(\psi^\delta)^2dxdt + C\kappa_0\int_0^T\int_{\mathbb{R}} \frac{(\psi_x^\delta)^2}{U}dxdt.$$

Then choosing ε small enough such that $C\varepsilon < \frac{s}{8}$, we have

$$\int_0^T\int_{\mathbb{R}} V_x(\psi^\delta)^2dxdt \le Cm_0 + C\kappa_0\int_0^T\int_{\mathbb{R}} \frac{(\psi_x^\delta)^2}{U}dxdt,$$

which, combined with (3.17), gives that

$$\int_{\mathbb{R}} \left(\frac{(\phi^\delta)^2}{U}+\chi(\psi^\delta)^2\right)dx + D\int_0^T\int_{\mathbb{R}} \frac{(\phi_x^\delta)^2}{U}dxdt + \chi\varepsilon\int_0^T\int_{\mathbb{R}} (\psi_x^\delta)^2dxdt$$

$$\le Cm_0 + C\kappa_0\int_0^T\int_{\mathbb{R}} \frac{(\psi_x^\delta)^2}{U}dxdt.$$

This completes the proof of Lemma 3.3 by using (2.10). □

Next we shall derive the *a priori* estimates of the first order derivatives of $(\phi^\delta, \psi^\delta)$. To this end, we first derive some estimates that will be used later.

Lemma 3.4. *Under the conditions of Theorem 3.1, the solution of (3.7) satisfies for any $0 < T < \infty$ that*

$$\int_0^T\int_{\mathbb{R}} U(\psi_x^\delta)^2dxdt \le Cm_0 + C\kappa_0\int_0^T \|\psi_x^\delta\|_w^2dt + C\kappa_0\int_0^T \|\phi_{xx}^\delta\|_w^2dt, \tag{3.20}$$

where C is a positive constant independent of t, ε and δ.

Proof. Multiplying the first equation of (3.7) by ψ_x^δ, we get

$$\chi U(\psi_x^\delta)^2 = \phi_t^\delta\psi_x^\delta - D\phi_{xx}^\delta\psi_x^\delta - \chi V\phi_x^\delta\psi_x^\delta - \chi\phi_x^\delta(\psi_x^\delta)^2. \tag{3.21}$$

Integrating (3.21) over $\mathbb{R}\times[0,T]$, using the second equation of (3.7) and following results

$$\phi_t^\delta\psi_x^\delta=(\phi^\delta\psi_x^\delta)_t-\phi^\delta\psi_{xt}^\delta=(\phi^\delta\psi_x^\delta)_t-\phi^\delta\left[\varepsilon\psi_{xxx}^\delta-2\varepsilon(V\psi_x^\delta)_x+\phi_{xx}^\delta-\varepsilon((\psi_x^\delta)^2)_x\right]$$

$$=(\phi^\delta\psi_x^\delta)_t-\varepsilon(\phi^\delta\psi_{xx}^\delta)_x+\varepsilon\phi_x^\delta\psi_{xx}^\delta+2\varepsilon(V\phi^\delta\psi_x^\delta)_x-2\varepsilon V\phi_x^\delta\psi_x^\delta-(\phi^\delta\phi_x^\delta)_x$$
$$+(\phi_x^\delta)^2+\varepsilon(\phi^\delta(\psi_x^\delta)^2)_x-\varepsilon\phi_x^\delta(\psi_x^\delta)^2,$$

$$-D\phi_{xx}^\delta\psi_x^\delta=D\psi_x^\delta\left[-\psi_{xt}^\delta+\varepsilon\psi_{xxx}^\delta-2\varepsilon(V\psi_x^\delta)_x-\varepsilon((\psi_x^\delta)^2)_x\right]$$

$$=D\left[-\frac{1}{2}((\psi_x^\delta)^2)_t+\varepsilon(\psi_x^\delta\psi_{xx}^\delta)_x-\varepsilon(\psi_{xx}^\delta)^2-\varepsilon(V(\psi_x^\delta)^2)_x-\varepsilon V_x(\psi_x^\delta)^2\right.$$
$$\left.-\frac{2\varepsilon}{3}((\psi_x^\delta)^3)_x\right],$$

we obtain that

$$\frac{D}{2}\int_{\mathbb{R}}(\psi_x^\delta)^2dx+\chi\int_0^T\int_{\mathbb{R}}U(\psi_x^\delta)^2dxdt+D\varepsilon\int_0^T\int_{\mathbb{R}}(\psi_{xx}^\delta)^2dxdt+D\varepsilon\int_0^T\int_{\mathbb{R}}V_x(\psi_x^\delta)^2dxdt$$
$$=\frac{D}{2}\int_0^\infty(\psi_{0x}^\delta)^2dx+\int_{\mathbb{R}}\phi^\delta\psi_x^\delta dx-\int_{\mathbb{R}}\phi_0^\delta\psi_{0x}^\delta dx+\varepsilon\int_0^T\int_{\mathbb{R}}\phi_x^\delta\psi_{xx}^\delta dxdt$$
$$-2\varepsilon\int_0^T\int_{\mathbb{R}}V\phi_x^\delta\psi_x^\delta dxdt+\int_0^T\int_{\mathbb{R}}(\phi_x^\delta)^2dxdt-\varepsilon\int_0^T\int_{\mathbb{R}}\phi_x^\delta(\psi_x^\delta)^2dxdt$$
$$-\chi\int_0^T\int_{\mathbb{R}}V\phi_x^\delta\psi_x^\delta dxdt-\chi\int_0^T\int_{\mathbb{R}}\phi_x^\delta(\psi_x^\delta)^2dxdt. \tag{3.22}$$

By the Cauchy–Schwarz inequality and the fact ε and V are bounded, we have the following estimates:

$$\int_{\mathbb{R}}\phi^\delta\psi_x^\delta dx-\int_{\mathbb{R}}\phi_0^\delta\psi_{0x}^\delta dx\le\frac{1}{2}\int_{\mathbb{R}}(\psi_{0x}^\delta)^2dx+\frac{1}{2}\int(\phi_0^\delta)^2dx+\frac{1}{D}\int_{\mathbb{R}}(\phi^\delta)^2dx+\frac{D}{4}\int_{\mathbb{R}}(\psi_x^\delta)^2dx,$$

$$\varepsilon\int_0^T\int_{\mathbb{R}}\phi_x^\delta\psi_{xx}^\delta dxdt\le\frac{D\varepsilon}{2}\int_0^T\int_{\mathbb{R}}(\psi_{xx}^\delta)^2dxdt+\frac{\varepsilon}{2D}\int_0^T\int_{\mathbb{R}}(\phi_x^\delta)^2dxdt,$$

$$2\varepsilon\int_0^T\int_{\mathbb{R}}V\phi_x^\delta\psi_x^\delta dxdt\le C\varepsilon\int_0^T\int_{\mathbb{R}}(\phi_x^\delta)^2dxdt+C\varepsilon\int_0^T\int_{\mathbb{R}}(\psi_x^\delta)^2dxdt,$$

$$(\varepsilon+\chi)\int_0^T\int_{\mathbb{R}}\phi_x^\delta(\psi_x^\delta)^2dxdt \le \frac{\chi}{4}\int_0^T\int_{\mathbb{R}}U(\psi_x^\delta)^2dxdt + C\int_0^T\int_{\mathbb{R}}\frac{(\phi_x^\delta)^2(\psi_x^\delta)^2}{U}dxdt,$$

and

$$\chi\int_0^T\int_{\mathbb{R}}V\phi_x^\delta\psi_x^\delta dxdt \le \frac{\chi}{4}\int_0^T\int_{\mathbb{R}}U(\psi_x^\delta)^2dxdt + C\int_0^T\int_{\mathbb{R}}\frac{V^2(\phi_x^\delta)^2}{U}dxdt.$$

Substituting the above estimates into (3.22), we have

$$\begin{aligned}
&\frac{D}{2}\int_{\mathbb{R}}(\psi_x^\delta)^2dx + \chi\int_0^T\int_{\mathbb{R}}U(\psi_x^\delta)^2dxdt + D\varepsilon\int_0^T\int_{\mathbb{R}}(\psi_{xx}^\delta)^2dxdt + D\varepsilon\int_0^T\int_{\mathbb{R}}V_x(\psi_x^\delta)^2dxdt\\
\le&\frac{D+1}{2}\int_{\mathbb{R}}(\psi_{0x}^\delta)^2dx + \frac{1}{2}\int(\phi_0^\delta)^2dx + \frac{1}{D}\int_{\mathbb{R}}(\phi^\delta)^2dx + \frac{D}{4}\int_{\mathbb{R}}(\psi_x^\delta)^2dx\\
&+\frac{D\varepsilon}{2}\int_0^T\int_{\mathbb{R}}(\psi_{xx}^\delta)^2dxdt + C\int_0^T\int_{\mathbb{R}}(\phi_x^\delta)^2dxdt + C\varepsilon\int_0^T\int_{\mathbb{R}}(\phi_x^\delta)^2dxdt\\
&+C\varepsilon\int_0^T\int_{\mathbb{R}}(\psi_x^\delta)^2dxdt + \frac{\chi}{2}\int_0^T\int_{\mathbb{R}}U(\psi_x^\delta)^2dxdt + C\int_0^T\int_{\mathbb{R}}\frac{V^2(\phi_x^\delta)^2}{U}dxdt\\
&+C\int_0^T\int_{\mathbb{R}}\frac{(\phi_x^\delta)^2(\psi_x^\delta)^2}{U}dxdt. \qquad (3.23)
\end{aligned}$$

For the last term on the right-hand side of the above inequality, by the Sobolev inequality $\|f\|_{L^\infty}^2 \le 2\|f\|\|f_x\|$, (3.11) and (3.12), we have from $1 \le \frac{u_-}{U}$ that

$$\begin{aligned}
C\int_0^T\int_{\mathbb{R}}\frac{(\phi_x^\delta)^2(\psi_x^\delta)^2}{U}dxdt &\le C\int_0^T\|\phi_x^\delta\|_{L^\infty}^2\int_{\mathbb{R}}\frac{(\psi_x^\delta)^2}{U}dxdt \le C\kappa_0\int_0^T\|\phi_x^\delta\|\|\phi_{xx}^\delta\|dt\\
&\le C\kappa_0 + C\kappa_0\int_0^T\int_{\mathbb{R}}\frac{(\phi_x^\delta)^2}{U}dxdt + C\kappa_0\int_0^T\int_{\mathbb{R}}\frac{(\phi_{xx}^\delta)^2}{U}dxdt \qquad (3.24)\\
&\le Cm_0 + C\kappa_0\int_0^T\|\psi_x^\delta\|_w^2dt + C\kappa_0\int_0^T\int_{\mathbb{R}}\frac{(\phi_{xx}^\delta)^2}{U}dxdt.
\end{aligned}$$

From the fact $V_x > 0$, $|V| \le C$, $1 \le \frac{u_-}{U}$, (3.12), (3.23) and (3.24), we have that

$$\begin{aligned}
&\int_{\mathbb{R}} (\psi_x^\delta)^2 dx + \int_0^T \int_{\mathbb{R}} U(\psi_x^\delta)^2 dxdt + \varepsilon \int_0^T \int_{\mathbb{R}} (\psi_{xx}^\delta)^2 dxdt \\
&\le C\left(\int_{\mathbb{R}} (\psi_{0x}^\delta)^2 dx + \int_{\mathbb{R}} (\phi_{0x}^\delta)^2 dx\right) + C\int_{\mathbb{R}} (\phi^\delta)^2 dx + C\int_0^T \int_{\mathbb{R}} \frac{(\phi_x^\delta)^2}{U} dxdt \\
&\quad + C\varepsilon \int_0^T \int_{\mathbb{R}} (\psi_x^\delta)^2 dxdt + C\int_0^T \int_{\mathbb{R}} \frac{(\phi_x^\delta)^2(\psi_x^\delta)^2}{U} dxdt \\
&\le Cm_0 + C\kappa_0 \int_0^T \|\psi_x^\delta\|_w^2 dt + C\kappa_0 \int_0^T \int_{\mathbb{R}} \frac{(\phi_{xx}^\delta)^2}{U} dxdt.
\end{aligned} \tag{3.25}$$

This immediately leads to (3.20) and completes the proof. □

Lemma 3.5. *Under the conditions of Theorem 3.1, the solution of* (3.7) *satisfies for any* $0 < T < \infty$ *that*

$$\begin{aligned}
&\int_{\mathbb{R}} w(\psi_x^\delta)^2 dx + \int_0^T \int_{\mathbb{R}} w(\psi_x^\delta)^2 dxdt + \varepsilon \int_0^T \int_{\mathbb{R}} w(\psi_{xx}^\delta)^2 dxdt \\
&\le Cm_0 + C\kappa_0 \left(\int_0^T \|\psi_x^\delta\|_w^2 dt + \int_0^T \|\phi_{xx}^\delta\|_w^2 dt + \varepsilon \int_0^T \|\psi_{xx}^\delta\|^2 dt\right),
\end{aligned} \tag{3.26}$$

where C *is a positive constant independent of* t, ε *and* δ.

Proof. Note that U is monotone decreasing in $(-\infty, \infty)$ and hence $0 = u_+ < U(0) < U(z) < u_-$. From (2.9), we see that $1 < w(z) < 2$ for all $z \in (-\infty, 0]$. Then we have $U(z) > \dfrac{U(0)}{2} w(z)$ for all $z \in (-\infty, 0]$. This means $U(z) > \dfrac{U(0)}{2} w(z)$ hold for all $x \in (-\infty, st]$. Then from (3.25), it follows that

$$\begin{aligned}
&\int_{-\infty}^{st} w(\psi_x^\delta)^2 dx + \int_0^T \int_{-\infty}^{st} w(\psi_x^\delta)^2 dxdt + \varepsilon \int_0^T \int_{-\infty}^{st} w(\psi_{xx}^\delta)^2 dxdt \\
&\le Cm_0 + C\kappa_0 \int_0^T \|\psi_x^\delta\|_w^2 dt + C\kappa_0 \int_0^T \|\phi_{xx}^\delta\|_w^2 dt.
\end{aligned} \tag{3.27}$$

Now, we multiply the second equation of (3.31) by $e^{\lambda z}\psi_x^\delta$ and obtain that

$$e^{\lambda z}\psi^\delta_{xt}\psi^\delta_x = \varepsilon e^{\lambda z}\psi^\delta_x\psi^\delta_{xxx} - 2\varepsilon e^{\lambda z}(V\psi^\delta_x)_x\psi^\delta_x + e^{\lambda z}\psi^\delta_x\phi^\delta_{xx} - \varepsilon e^{\lambda z}((\psi^\delta_x)^2)_x\psi^\delta_x,$$

which gives

$$\begin{aligned}&\left(\frac{e^{\lambda z}(\psi^\delta_x)^2}{2}\right)_t + \left(\frac{s\lambda}{2} + 2\varepsilon V_x\right)e^{\lambda z}(\psi^\delta_x)^2 + \varepsilon e^{\lambda z}(\psi^\delta_{xx})^2\\ &= \varepsilon(e^{\lambda z}\psi^\delta_x\psi^\delta_{xx})_x - \varepsilon\lambda e^{\lambda z}\psi^\delta_x\psi^\delta_{xx} - 2\varepsilon V e^{\lambda z}\psi^\delta_x\psi^\delta_{xx} + e^{\lambda z}\psi^\delta_x\phi^\delta_{xx} - 2\varepsilon e^{\lambda z}(\psi^\delta_x)^2\psi^\delta_{xx}.\end{aligned}$$

Integrating the above equation over $\mathbb{R}\times[0,T]$, we get

$$\begin{aligned}&\int_{\mathbb{R}} \frac{e^{\lambda z}(\psi^\delta_x)^2}{2}dx + \int_0^T\int_{\mathbb{R}}\left(\frac{s\lambda}{2} + 2\varepsilon V_x\right)e^{\lambda z}(\psi^\delta_x)^2dxdt + \varepsilon\int_0^T\int_{\mathbb{R}} e^{\lambda z}(\psi^\delta_{xx})^2dxdt\\ &= \int_{\mathbb{R}} \frac{e^{\lambda x}(\psi^\delta_{0x})^2}{2}dx - \varepsilon\lambda\int_0^T\int_{\mathbb{R}} e^{\lambda z}\psi^\delta_x\psi^\delta_{xx}dxdt + 2\varepsilon\int_0^T\int_{\mathbb{R}} V e^{\lambda z}\psi^\delta_x\psi^\delta_{xx}dxdt\\ &\quad + \int_0^T\int_{\mathbb{R}} e^{\lambda z}\psi^\delta_x\phi^\delta_{xx}dxdt - 2\varepsilon\int_0^T\int_{\mathbb{R}} e^{\lambda z}(\psi^\delta_x)^2\psi^\delta_{xx}dxdt\\ &= \int_{\mathbb{R}} \frac{e^{\lambda x}(\psi^\delta_{0x})^2}{2}dx + R_1 + R_2 + R_3 + R_4.\end{aligned} \tag{3.28}$$

By the Cauchy–Schwarz inequality, we have

$$R_1 \le \frac{\varepsilon}{4}\int_0^T\int_{\mathbb{R}} e^{\lambda z}(\psi^\delta_{xx})^2dxdt + \varepsilon\lambda^2\int_0^T\int_{\mathbb{R}} e^{\lambda z}(\psi^\delta_x)^2dxdt,$$

$$R_2 \le \frac{\varepsilon}{4}\int_0^T\int_{\mathbb{R}} e^{\lambda z}(\psi^\delta_{xx})^2dxdt + 4\varepsilon v_-^2\int_0^T\int_{\mathbb{R}} e^{\lambda z}(\psi^\delta_x)^2dxdt$$

and

$$R_3 \le \frac{s\lambda}{8}\int_0^T\int_{\mathbb{R}} e^{\lambda z}(\psi^\delta_x)^2dxdt + \frac{2}{s\lambda}\int_0^T\int_{\mathbb{R}} e^{\lambda z}(\phi^\delta_x)^2dxdt.$$

Using the Sobolev and Cauchy–Schwarz inequalities, (3.11) and the fact $e^{\lambda z} \le w \le \frac{1}{C_1 U}$ due to (2.10), we have from (3.12) that

$$
\begin{aligned}
R_4 \leq & \frac{\varepsilon}{4}\int_0^T\int_{\mathbb{R}} e^{\lambda z}(\psi_{xx}^\delta)^2 dxdt + 4\varepsilon\int_0^T\int_{\mathbb{R}} e^{\lambda z}(\psi_x^\delta)^4 dxdt \\
\leq & \frac{\varepsilon}{4}\int_0^T\int_{\mathbb{R}} e^{\lambda z}(\psi_{xx}^\delta)^2 dxdt + 4\varepsilon\int_0^T \|\psi_x^\delta\|_{L^\infty}^2 \int_{\mathbb{R}} e^{\lambda z}(\psi_x^\delta)^2 dxdt \\
\leq & \frac{\varepsilon}{4}\int_0^T\int_{\mathbb{R}} e^{\lambda z}(\psi_{xx}^\delta)^2 dxdt + C\varepsilon\int_0^T \|\psi_x^\delta\|\|\psi_{xx}^\delta\|\|\psi_x^\delta\|_w^2 dt \\
\leq & \frac{\varepsilon}{4}\int_0^T\int_{\mathbb{R}} e^{\lambda z}(\psi_{xx}^\delta)^2 dxdt + C\kappa_0\varepsilon\int_0^T \|\psi_x^\delta\|^2 dt + C\kappa_0\varepsilon\int_0^T \|\psi_{xx}^\delta\|^2 dt \\
\leq & \frac{\varepsilon}{4}\int_0^T\int_{\mathbb{R}} e^{\lambda z}(\psi_{xx}^\delta)^2 dxdt + Cm_0 + C\kappa_0\int_0^T \|\psi_x^\delta\|_w^2 dt + C\kappa_0\varepsilon\int_0^T \|\psi_{xx}^\delta\|^2 dt.
\end{aligned}
$$

Substituting the estimates of $R_1 - R_4$ into (3.28) and choosing $\varepsilon > 0$ is small enough such that $\varepsilon \leq \frac{s\lambda}{4(\lambda^2+4v_-^2)}$, we have from (3.9) and (3.12) that

$$
\begin{aligned}
& \int_{\mathbb{R}} e^{\lambda z}(\psi_x^\delta)^2 dx + \int_0^T\int_{\mathbb{R}} e^{\lambda z}(\psi_x^\delta)^2 dxdt + \varepsilon\int_0^T\int_{\mathbb{R}} e^{\lambda z}(\psi_{xx}^\delta)^2 dxdt \\
\leq & C\left(\|\psi_{0x}^\delta\|_w^2 + m_0 + \kappa_0\int_0^T \|\psi_x^\delta\|_w^2 dt + \int_0^T\int_{\mathbb{R}} w(\phi_x^\delta)^2 dxdt + \kappa_0\varepsilon\int_0^T \|\psi_{xx}^\delta\|^2 dt\right) \qquad (3.29) \\
\leq & Cm_0 + C\kappa_0\int_0^T \|\psi_x^\delta\|_w^2 dt + C\kappa_0\varepsilon\int_0^T \|\psi_{xx}^\delta\|^2 dt,
\end{aligned}
$$

where we have used $e^{\lambda z} \leq w$ and $V_x > 0$. Recalling that $w = 1 + e^{\lambda z}$, we have $e^{\lambda z} \geq \frac{w}{2}$ in $z \in [0, \infty)$ (i.e. $x \in [st, \infty)$). Then, it follows from (3.29) that

$$
\begin{aligned}
& \int_{st}^\infty w(\psi_x^\delta)^2 dx + \int_0^T\int_{st}^\infty w(\psi_x^\delta)^2 dxdt + \varepsilon\int_0^T\int_{st}^\infty w(\psi_{xx}^\delta)^2 dxdt \\
\leq & Cm_0 + C\kappa_0\int_0^T \|\psi_x^\delta\|_w^2 dt + C\kappa_0\varepsilon\int_0^T \|\psi_{xx}^\delta\|^2 dt,
\end{aligned}
$$

which, in combination with (3.27) gives

$$\int_{\mathbb{R}} w(\psi_x^\delta)^2 dx + \int_0^T \int_{\mathbb{R}} w(\psi_x^\delta)^2 dxdt + \varepsilon \int_0^T \int_{\mathbb{R}} w(\psi_{xx}^\delta)^2 dxdt$$

$$\le Cm_0 + C\kappa_0 \int_0^T \|\psi_x^\delta\|_w^2 dt + C\kappa_0 \int_0^T \|\phi_{xx}^\delta\|_w^2 dt + C\kappa_0\varepsilon \int_0^T \|\psi_{xx}^\delta\|^2 dt.$$

This completes the proof. □

Lemma 3.6 *(H^1-estimates). Assume the conditions of Theorem 3.1 hold and let $(\phi^\delta, \psi^\delta)$ be a smooth solution of (3.7) satisfying (3.11). Then it holds that*

$$\|\phi^\delta\|_{1,w}^2 + \|\psi^\delta\|_1^2 + \|\psi_x^\delta\|_w^2 + \int_0^T \left(\|\phi_x^\delta\|_{1,w}^2 + \|\psi_x^\delta\|_w^2 + \varepsilon\|\psi_x^\delta\|_1^2 + \varepsilon\|\psi_{xx}^\delta\|_w^2 \right) dt \le Cm_0, \quad (3.30)$$

where C is a positive constant independent of t, ε and δ.

Proof. Differentiating (3.7) with respect to x yields

$$\begin{cases} \phi_{xt}^\delta = D\phi_{xxx}^\delta + \chi V \phi_{xx}^\delta + \chi V_x \phi_x^\delta + \chi U \psi_{xx}^\delta + \chi U_x \psi_x^\delta + \chi(\phi_x^\delta \psi_x^\delta)_x, \\ \psi_{xt}^\delta = \varepsilon \psi_{xxx}^\delta - 2\varepsilon(V\psi_x^\delta)_x + \phi_{xx}^\delta - \varepsilon((\psi_x^\delta)^2)_x. \end{cases} \quad (3.31)$$

Multiplying the first equation of (3.31) by ϕ_x^δ/U and the second by $\chi\psi_x^\delta$ and adding these equalities, we obtain

$$\frac{\phi_x^\delta \phi_{xt}^\delta}{U} + \chi\psi_x^\delta\psi_{xt}^\delta = \frac{D\phi_{xxx}^\delta\phi_x^\delta}{U} + \frac{\chi V\phi_{xx}^\delta\phi_x^\delta}{U} + \frac{\chi V_x(\phi_x^\delta)^2}{U} + \chi\left(\phi_x^\delta\psi_x^\delta\right)_x$$
$$+ \frac{\chi U_x\phi_x^\delta\psi_x^\delta}{U} + \frac{\chi(\phi_x^\delta\psi_x^\delta)_x\phi_x^\delta}{U}$$
$$+ \chi\varepsilon\psi_{xxx}^\delta\psi_x^\delta - 2\chi\varepsilon(V\psi_x^\delta)_x\psi_x^\delta - \chi\varepsilon((\psi_x^\delta)^2)_x\psi_x^\delta.$$

Simple calculations give us that

$$\frac{\phi_x^\delta\phi_{xt}^\delta}{U} = \left(\frac{(\phi_x^\delta)^2}{2U}\right)_t + \frac{(\phi_x^\delta)^2}{2}\left(\frac{s}{U}\right)_x,$$

$$\frac{D\phi_{xxx}^\delta\phi_x^\delta}{U} = \left(\frac{D\phi_{xx}^\delta\phi_x^\delta}{U}\right)_x - \frac{D(\phi_{xx}^\delta)^2}{U} - \left(\frac{D(\phi_x^\delta)^2}{2}\left(\frac{1}{U}\right)_x\right)_x + \frac{D(\phi_x^\delta)^2}{2}\left(\frac{1}{U}\right)_{xx},$$

$$\frac{\chi V\phi_x^\delta\phi_{xx}^\delta}{U} = \left(\frac{\chi V(\phi_x^\delta)^2}{2U}\right)_x - \frac{(\phi_x^\delta)^2}{2}\left(\frac{\chi V}{U}\right)_x,$$

$$\frac{\chi(\phi_x^\delta\psi_x^\delta)_x\phi_x^\delta}{U}=\left(\frac{\chi(\phi_x^\delta)^2\psi_x^\delta}{U}\right)_x-\frac{\chi\phi_x^\delta\psi_x^\delta\phi_{xx}^\delta}{U}+\frac{\chi U_x(\phi_x^\delta)^2\psi_x^\delta}{U^2},$$

$$-2\chi\varepsilon(V\psi_x^\delta)_x\psi_x^\delta=-2\chi\varepsilon\left(V(\psi_x^\delta)^2\right)_x+2\chi\varepsilon V\psi_x^\delta\psi_{xx}^\delta,$$

$$-\chi\varepsilon((\psi_x^\delta)^2)_x\psi_x^\delta=-\left(\frac{2\chi\varepsilon}{3}(\psi_x^\delta)^3\right)_x.$$

Thus we get from above equalities that

$$\begin{aligned}&\frac{1}{2}\left(\frac{(\phi_x^\delta)^2}{U}+\chi(\psi_x^\delta)^2\right)_t+\frac{D(\phi_{xx}^\delta)^2}{U}+\chi\varepsilon(\psi_{xx}^\delta)^2\\&=\left(\chi\phi_x^\delta\psi_x^\delta+\frac{D\phi_{xx}^\delta\phi_x^\delta}{U}-\frac{D(\phi_x^\delta)^2}{2}\left(\frac{1}{U}\right)_x+\frac{\chi V(\phi_x^\delta)^2}{2U}+\frac{\chi(\phi_x^\delta)^2\psi_x^\delta}{U}\right)_x\\&\quad+\left(\chi\varepsilon\psi_{xx}^\delta\psi_x^\delta-2\chi\varepsilon V(\psi_x^\delta)^2-\frac{2\chi\varepsilon}{3}(\psi_x^\delta)^3\right)_x+\frac{(\phi_x^\delta)^2}{2}\left[\left(\frac{D}{U}\right)_{xx}-\left(\frac{s+\chi V}{U}\right)_x\right]\\&\quad+\frac{\chi V_x(\phi_x^\delta)^2}{U}+\frac{\chi U_x\phi_x^\delta\psi_x^\delta}{U}-\frac{\chi\phi_x^\delta\psi_x^\delta\phi_{xx}^\delta}{U}+\frac{\chi U_x(\phi_x^\delta)^2\psi_x^\delta}{U^2}+2\chi\varepsilon V\psi_x^\delta\psi_{xx}^\delta.\end{aligned}\tag{3.32}$$

By using (2.1) and the fact that $u_+=0$, it can be checked that

$$\left(\frac{D}{U}\right)_{xx}-\left(\frac{s+\chi V}{U}\right)_x=\frac{2u_+}{U^3}(s+\chi v_+)\cdot U_x=0.\tag{3.33}$$

Integrating (3.32) over $\mathbb{R}\times[0,T]$ and using (3.33), we obtain

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}}\left(\frac{(\phi_x^\delta)^2}{U}+\chi(\psi_x^\delta)^2\right)dx+D\int_0^T\int_{\mathbb{R}}\frac{(\phi_{xx}^\delta)^2}{U}dxdt+\chi\varepsilon\int_0^T\int_{\mathbb{R}}(\psi_{xx}^\delta)^2dxdt\\&=\frac{1}{2}\int_{\mathbb{R}}\left(\frac{(\phi_{0x}^\delta)^2}{U}+\chi(\psi_{0x}^\delta)^2\right)dx+\chi\int_0^T\int_{\mathbb{R}}\frac{V_x(\phi_x^\delta)^2}{U}dxdt+\chi\int_0^T\int_{\mathbb{R}}\frac{U_x\phi_x^\delta\psi_x^\delta}{U}dxdt\\&\quad-\chi\int_0^T\int_{\mathbb{R}}\frac{\phi_x^\delta\psi_x^\delta\phi_{xx}^\delta}{U}dxdt+\chi\int_0^T\int_{\mathbb{R}}\frac{U_x(\phi_x^\delta)^2\psi_x^\delta}{U^2}dxdt+2\chi\varepsilon\int_0^T\int_{\mathbb{R}}V\psi_x^\delta\psi_{xx}^\delta dxdt\\&=I_0+I_1+I_2+I_3+I_4+I_5.\end{aligned}\tag{3.34}$$

For I_0, we have from (2.10) and (3.9) that

$$I_0 \le C\left(\|\phi_0^\delta\|_{1,w}^2 + \|\psi_0^\delta\|_1^2\right) \le Cm_0.$$

For I_1, by $|V_x| \le C$ and (3.12), we have

$$I_1 \le C\int_0^T\int_{\mathbb{R}} \frac{(\phi_x^\delta)^2}{U} dxdt \le Cm_0 + C\kappa_0\int_0^T \|\psi_x^\delta\|_w^2 dt.$$

Using the Cauchy–Schwarz inequality and (3.12), we can estimate I_2 as

$$\begin{aligned}
I_2 \le & C\int_0^T\int_{\mathbb{R}} |\phi_x^\delta\psi_x^\delta| dxdt \\
\le & C\int_0^T\int_{\mathbb{R}} \frac{(\phi_x^\delta)^2}{U} dxdt + C\int_0^T\int_{\mathbb{R}} U(\psi_x^\delta)^2 dxdt \\
\le & Cm_0 + C\kappa_0\int_0^T \|\psi_x^\delta\|_w^2 dt + C\int_0^T\int_{\mathbb{R}} U(\psi_x^\delta)^2 dxdt,
\end{aligned}$$

where in the first inequality we have used the fact $\left|\frac{U_x}{U}\right| = -\frac{U_x}{U} = \frac{s+\chi V}{D} \le \frac{s}{D}$ due to (2.2). Similarly, we have

$$I_3 \le \frac{D}{2}\int_0^T\int_{\mathbb{R}} \frac{(\phi_{xx}^\delta)^2}{U} dxdt + C\int_0^T\int_{\mathbb{R}} \frac{|\phi_x^\delta\psi_x^\delta|^2}{U} dxdt,$$

$$\begin{aligned}
I_4 \le & C\int_0^T\int_{\mathbb{R}} \frac{(\phi_x^\delta)^2\psi_x^\delta}{U} dxdt \\
\le & C\int_0^T\int_{\mathbb{R}} \frac{(\phi_x^\delta)^2}{U} dxdt + C\int_0^T\int_{\mathbb{R}} \frac{|\phi_x^\delta\psi_x^\delta|^2}{U} dxdt \\
\le & Cm_0 + C\kappa_0\int_0^T \|\psi_x^\delta\|_w^2 dt + C\int_0^T\int_{\mathbb{R}} \frac{|\phi_x^\delta\psi_x^\delta|^2}{U} dxdt,
\end{aligned}$$

and

$$I_5 \le \frac{\chi\varepsilon}{2}\int_0^T\int_{\mathbb{R}} (\psi_{xx}^\delta)^2 dxdt + C\varepsilon\int_0^T\int_{\mathbb{R}} V^2(\psi_x^\delta)^2 dxdt$$

$$\le \frac{\chi\varepsilon}{2}\int_0^T\int_{\mathbb{R}} (\psi_{xx}^\delta)^2 dxdt + Cm_0 + C\kappa_0 \int_0^T \|\psi_x^\delta\|_w^2 dt.$$

Substituting the above estimates into (3.34) to get

$$\begin{aligned}&\int_{\mathbb{R}} \left(\frac{(\phi_x^\delta)^2}{U} + \chi(\psi_x^\delta)^2\right) dx + D\int_0^T\int_{\mathbb{R}} \frac{(\phi_{xx}^\delta)^2}{U} dxdt + \chi\varepsilon\int_0^T\int_{\mathbb{R}} (\psi_{xx}^\delta)^2 dxdt \\ &\le Cm_0 + C\kappa_0\int_0^T \|\psi_x^\delta\|_w^2 dt + C\int_0^T\int_{\mathbb{R}} U(\psi_x^\delta)^2 dxdt + C\int_0^T\int_{\mathbb{R}} \frac{|\phi_x^\delta\psi_x^\delta|^2}{U} dxdt.\end{aligned} \tag{3.35}$$

We have from (3.24) and (3.35) that

$$\begin{aligned}&\int_{\mathbb{R}} \left(\frac{(\phi_x^\delta)^2}{U} + \chi(\psi_x^\delta)^2\right) dx + D\int_0^T\int_{\mathbb{R}} \frac{(\phi_{xx}^\delta)^2}{U} dxdt + \chi\varepsilon\int_0^T\int_{\mathbb{R}} (\psi_{xx}^\delta)^2 dxdt \\ &\le Cm_0 + C\kappa_0\int_0^T \|\psi_x^\delta\|_w^2 dt + C\int_0^T\int_{\mathbb{R}} U(\psi_x^\delta)^2 dxdt + C\kappa_0\int_0^T\int_{\mathbb{R}} \frac{(\phi_{xx}^\delta)^2}{U} dxdt.\end{aligned} \tag{3.36}$$

Then substituting (3.20) into (3.36) and choosing κ_0 small enough such that $C\kappa_0 \le \frac{D}{2}$, we have

$$\begin{aligned}&\int_{\mathbb{R}} \left(\frac{(\phi_x^\delta)^2}{U} + (\psi_x^\delta)^2\right) dx + \int_0^T\int_{\mathbb{R}} \frac{(\phi_{xx}^\delta)^2}{U} dxdt + \varepsilon\int_0^T\int_{\mathbb{R}} (\psi_{xx}^\delta)^2 dxdt \\ &\le Cm_0 + C\kappa_0\int_0^T \|\psi_x^\delta\|_w^2 dt.\end{aligned} \tag{3.37}$$

It follows from (3.12), (3.26) and (3.37) that

$$\begin{aligned}&\|\phi^\delta\|_{1,w}^2 + \|\psi^\delta\|_1^2 + \|\psi_x^\delta\|_w^2 + \int_0^T \left(\|\phi_x^\delta\|_{1,w}^2 + \|\psi_x^\delta\|_w^2 + \varepsilon\|\psi_x^\delta\|_1^2 + \varepsilon\|\psi_{xx}^\delta\|_w^2\right) dt \\ &\le Cm_0 + C\kappa_0\left(\int_0^T \|\psi_x^\delta\|_w^2 dt + \int_0^T \|\phi_{xx}^\delta\|_w^2 dt + \varepsilon\int_0^T \|\psi_{xx}^\delta\|^2 dt\right),\end{aligned}$$

where C is independent of t and ε. Choosing $C\kappa_0 \le \frac{1}{2}$, we have

$$\|\phi^\delta\|_{1,w}^2 + \|\psi^\delta\|_1^2 + \|\psi_x^\delta\|_w^2 + \int_0^T \left(\|\phi_x^\delta\|_{1,w}^2 + \|\psi_x^\delta\|_w^2 + \varepsilon\|\psi_x^\delta\|_1^2 + \varepsilon\|\psi_{xx}^\delta\|_w^2\right) dt \le Cm_0.$$

Thus, the proof of Lemma 3.6 is completed. □

Now, taking m_0 sufficiently small such that $Cm_0 \le \kappa_0$, we have from (3.30) that

$$\|\phi^\delta\|_{1,w}^2 + \|\psi^\delta\|_1^2 + \|\psi_x^\delta\|_w^2 + \int_0^T \left(\|\phi_x^\delta\|_{1,w}^2 + \|\psi_x^\delta\|_w^2 + \varepsilon\|\psi_x^\delta\|_1^2 + \varepsilon\|\psi_{xx}^\delta\|_w^2\right) dt \le \kappa_0,$$

which closes the *a priori* assumption (3.11).

Next, we derive appropriate estimates for the second order derivative of $(\phi^\delta, \psi^\delta)$. Since we plan to use the limit of the mollified function $(\phi^\delta, \psi^\delta)$ as $\delta \to 0$ to obtain the solution (ϕ, ψ) of our target system (3.1)–(3.2), the estimates of the second order derivative of $(\phi^\delta, \psi^\delta)$ need to be independent of δ. If we employ the similar energy estimates method for H^1-estimates in Lemma 3.6, we shall encounter the term $\int_{\mathbb{R}}(|\phi_{0xx}^\delta|^2 + |\psi_{0xx}^\delta|^2)dx$ which is out of control since the boundedness of initial data $(\phi_0^\delta, \psi_0^\delta)$ is assumed up to $H^1(\mathbb{R})$ only, see (3.8)–(3.10). Indeed in general the bound of $\int_{\mathbb{R}}(|\phi_{0xx}^\delta|^2 + |\psi_{0xx}^\delta|^2)dx$ is of order $\frac{1}{\delta}$ given that $H^1(\mathbb{R})$-norm is bounded (see [48, Lemma 1.2]). Hence we have to find an idea to avoid the estimates of second-order derivative of $(\phi_0^\delta, \psi_0^\delta)$ to attain the uniform boundedness of second-order estimates in δ. Inspired by the brilliant idea of Hoff [13,14] of treating discontinuous data, we introduce a weight function $\sigma = \sigma(t) = \min\{1, t\}$ to resolve this obstacle. The price paid by this idea is that the solution behavior sufficiently close to time $t = 0$ is unclear. However this is sufficient to study the large-time behavior as we seek in this paper.

Lemma 3.7 *(H^2-estimates). Let the conditions of Theorem 3.1 hold, and let $(\phi^\delta, \psi^\delta)$ be a smooth solution of (3.7) satisfying (3.11). Then it holds that*

$$\begin{aligned}&\int_{\mathbb{R}} \sigma\left(\|\phi_t^\delta\|^2 + \|\psi_t^\delta\|^2 + \|\phi_{xx}^\delta\|^2 + \varepsilon\|\psi_{xx}^\delta\|^2\right) dx \\ &\qquad + \int_0^T\int_{\mathbb{R}} \sigma\left(\|\phi_{xt}^\delta\|^2 + \varepsilon\|\psi_{xt}^\delta\|^2 + \varepsilon\|\phi_{xxx}^\delta\|^2 + \varepsilon^2\|\psi_{xxx}^\delta\|^2\right) dxdt \le C,\end{aligned} \tag{3.38}$$

where $\sigma = \sigma(t) = \min\{1, t\}$ and C is a positive constant independent of t, ε and δ.

Proof. We differentiate (3.7) with respect to t to get

$$\begin{cases}\phi_{tt}^\delta = D\phi_{xxt}^\delta - \chi s V_x \phi_x^\delta + \chi V\phi_{xt}^\delta - \chi s U_x \psi_x^\delta + \chi U \psi_{xt}^\delta + \chi\phi_{xt}^\delta\psi_x^\delta + \chi\phi_x^\delta\psi_{xt}^\delta, \\ \psi_{tt}^\delta = \varepsilon\psi_{xxt}^\delta + 2\varepsilon s V_x \psi_x^\delta - 2\varepsilon V\psi_{xt}^\delta + \phi_{xt}^\delta - \varepsilon((\psi_x^\delta)^2)_t.\end{cases} \tag{3.39}$$

Multiplying the first equation of (3.39) by $\sigma\phi_t^\delta$ and the second by $\sigma\psi_t^\delta$ and adding these equalities, we obtain

$$
\begin{aligned}
&\sigma\phi^\delta_{tt}\phi^\delta_t+\sigma\psi^\delta_{tt}\psi^\delta_t\\
&=D\sigma\phi^\delta_{xxt}\phi^\delta_t+\varepsilon\sigma\psi^\delta_{xxt}\psi^\delta_t+\chi\sigma\phi^\delta_t(-sV_x\phi^\delta_x+V\phi^\delta_{xt}-sU_x\psi^\delta_x+\phi^\delta_{xt}\psi^\delta_x) \qquad (3.40)\\
&\quad+\chi\sigma U\psi^\delta_{xt}\phi^\delta_t+\chi\sigma\phi^\delta_t\phi^\delta_x\psi^\delta_{xt}+\sigma\psi^\delta_t(2\varepsilon sV_x\psi^\delta_x-2\varepsilon V\psi^\delta_{xt}+\phi^\delta_{xt}-2\varepsilon\psi^\delta_x\psi^\delta_{xt}).
\end{aligned}
$$

Integrating (3.40) over $\mathbb{R}\times[0,T]$ and rearranging the resulting equation, we get

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\sigma\left((\phi^\delta_t)^2+(\psi^\delta_t)^2\right)dx+D\int_0^T\int_{\mathbb{R}}\sigma(\phi^\delta_{xt})^2dxdt+\varepsilon\int_0^T\int_{\mathbb{R}}\sigma(\psi^\delta_{xt})^2dxdt\\
&=\frac{1}{2}\int_0^{\sigma(T)}\int_{\mathbb{R}}((\phi^\delta_t)^2+(\psi^\delta_t)^2)dxdt+\chi\int_0^T\int_{\mathbb{R}}\sigma\phi^\delta_t\left(-sV_x\phi^\delta_x+V\phi^\delta_{xt}-sU_x\psi^\delta_x+\phi^\delta_{xt}\psi^\delta_x\right)dxdt\\
&\quad+\int_0^T\int_{\mathbb{R}}\sigma\psi^\delta_t\left(2\varepsilon sV_x\psi^\delta_x-2\varepsilon V\psi^\delta_{xt}+\phi^\delta_{xt}-2\varepsilon\psi^\delta_x\psi^\delta_{xt}\right)dxdt+\chi\int_0^T\int_{\mathbb{R}}\sigma U\psi^\delta_{xt}\phi^\delta_tdxdt\\
&\quad+\chi\int_0^T\int_{\mathbb{R}}\sigma\phi^\delta_t\phi^\delta_x\psi^\delta_{xt}dxdt\\
&=K_1+K_2+K_3+K_4+K_5, \qquad (3.41)
\end{aligned}
$$

where we have used the fact that

$$
\frac{1}{2}\int_0^T\int_{\mathbb{R}}\sigma_t\left((\phi^\delta_t)^2+(\psi^\delta_t)^2\right)dxdt=\frac{1}{2}\int_0^{\sigma(T)}\int_{\mathbb{R}}\left((\phi^\delta_t)^2+(\psi^\delta_t)^2\right)dxdt.
$$

Because ε, $|V|$ and $|U|$ are all bounded, we get from (3.7), (3.30) and (3.24) that

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}}(\phi^\delta_t)^2dxdt&\le C\int_0^T\int_{\mathbb{R}}\left((\phi^\delta_{xx})^2+V^2(\phi^\delta_x)^2+U^2(\psi^\delta_x)^2\right)dxdt+C\int_0^T\int_{\mathbb{R}}(\phi^\delta_x)^2(\psi^\delta_x)^2dxdt\\
&\le C\int_0^T\int_{\mathbb{R}}\left((\phi^\delta_{xx})^2+(\phi^\delta_x)^2+(\psi^\delta_x)^2\right)dxdt+C\\
&\le C \qquad (3.42)
\end{aligned}
$$

and

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} (\psi_t^\delta)^2 dxdt \leq & C\int_0^T\int_{\mathbb{R}} \left(\varepsilon^2(\psi_{xx}^\delta)^2 + V^2(\psi_x^\delta)^2 + (\phi_x^\delta)^2\right) dxdt + C\varepsilon^2 \int_0^T\int_{\mathbb{R}} (\psi_x^\delta)^4 dxdt \\ & \leq C\int_0^T\int_{\mathbb{R}} \left(\varepsilon(\psi_{xx}^\delta)^2 + (\phi_x^\delta)^2 + (\psi_x^\delta)^2\right) dxdt + C\varepsilon \int_0^T\int_{\mathbb{R}} (\psi_x^\delta)^4 dxdt \\ & \leq C, \end{aligned} \tag{3.43}$$

where we have used the following estimate

$$\begin{aligned} C\varepsilon \int_0^T\int_{\mathbb{R}} (\psi_x^\delta)^4 dxdt \leq & C\varepsilon \int_0^T \|\psi_x^\delta\|^3 \|\psi_{xx}^\delta\| dt \leq C\kappa_0\varepsilon \int_0^T \|\psi_x^\delta\| \|\psi_{xx}^\delta\| dt \\ \leq & C\kappa_0\varepsilon \int_0^T \|\psi_{xx}^\delta\|^2 dt + C\kappa_0\varepsilon \int_0^T \|\psi_x^\delta\|^2 dt \\ \leq & C. \end{aligned}$$

Then, K_1 can be bounded as

$$K_1 \leq \int_0^T\int_{\mathbb{R}} \left((\phi_t^\delta)^2 + (\psi_t^\delta)^2\right) dxdt \leq C.$$

By the Cauchy–Schwarz inequality, $|U_x| \leq \left|\frac{U_x}{U}\right| |U| \leq C, \ |V_x| \leq C, \ |V| \leq C$ and (3.30), we have

$$\begin{aligned} K_2 \leq & \frac{D}{8}\int_0^T\int_{\mathbb{R}} \sigma(\phi_{xt}^\delta)^2 dxdt + C\int_0^T \sigma(\phi_t^\delta)^2(\psi_x^\delta)^2 dt + C\int_0^T \sigma\left(\|\phi_t^\delta\|^2 + \|\phi_x^\delta\|^2 + \|\psi_x^\delta\|^2\right) dt \\ \leq & \frac{D}{8}\int_0^T\int_{\mathbb{R}} \sigma(\phi_{xt}^\delta)^2 dxdt + C\int_0^T \sigma(\phi_t^\delta)^2(\psi_x^\delta)^2 dt + C \end{aligned}$$

and

$$\begin{aligned} K_3 \leq & \frac{D}{8}\int_0^T\int_{\mathbb{R}} \sigma(\phi_{xt}^\delta)^2 dxdt + \frac{\varepsilon}{4}\int_0^T\int_{\mathbb{R}} \sigma(\psi_{xt}^\delta)^2 dxdt \\ & + C\int_0^T \sigma\left(\|\psi_t^\delta\|^2 + \|\psi_x^\delta\|^2\right) dt + C\varepsilon\int_0^T \sigma(\psi_t^\delta)^2(\psi_x^\delta)^2 dt \end{aligned}$$

$$\leq \frac{D}{8}\int_0^T\int_{\mathbb{R}}\sigma(\phi_{xt}^\delta)^2dxdt+\frac{\varepsilon}{4}\int_0^T\int_{\mathbb{R}}\sigma(\psi_{xt}^\delta)^2dxdt+C\varepsilon\int_0^T\sigma(\psi_t^\delta)^2(\psi_x^\delta)^2dt+C.$$

For K_4, using the integration by parts and Cauchy–Schwarz inequality, (3.30), (3.42) and (3.43), we have from $|U_x|\leq C,\ |U|\leq C$ that

$$\begin{aligned}K_4 =&\chi\int_0^T\int_{\mathbb{R}}\sigma U\psi_{xt}^\delta\phi_t^\delta dxdt\\ =&-\chi\int_0^T\int_{\mathbb{R}}\sigma U\psi_t^\delta\phi_{xt}^\delta dxdt-\chi\int_0^T\int_{\mathbb{R}}\sigma U_x\psi_t^\delta\phi_t^\delta dxdt\\ \leq&\frac{D}{8}\int_0^T\int_{\mathbb{R}}\sigma(\phi_{xt}^\delta)^2dxdt+C\int_0^T\sigma\left(\|\phi_t^\delta\|^2+\|\psi_t^\delta\|^2\right)dt\\ \leq&\frac{D}{8}\int_0^T\int_{\mathbb{R}}\sigma(\phi_{xt}^\delta)^2dxdt+C.\end{aligned}$$

Since ε, $|V|$ and $|V_x|$ are all bounded, we get by the second equation of (3.31), the Cauchy–Schwarz inequality and (3.30) that

$$\begin{aligned}K_5 =&\chi\int_0^T\int_{\mathbb{R}}\sigma\phi_t^\delta\phi_x^\delta\psi_{xt}^\delta dxdt\\ =&\chi\int_0^T\int_{\mathbb{R}}\sigma\phi_t^\delta\phi_x^\delta\left(\varepsilon\psi_{xxx}^\delta-2\varepsilon(V\psi_x^\delta)_x+\phi_{xx}^\delta-\varepsilon((\psi_x^\delta)^2)_x\right)dxdt\\ \leq&\varepsilon^2\int_0^T\int_{\mathbb{R}}\sigma(\psi_{xxx}^\delta)^2dxdt+C\int_0^T\int_{\mathbb{R}}\sigma(\phi_t^\delta)^2(\phi_x^\delta)^2dxdt+C\varepsilon^2\int_0^T\int_{\mathbb{R}}\sigma V_x^2(\psi_x^\delta)^2dxdt\\ &+C\varepsilon^2\int_0^T\int_{\mathbb{R}}\sigma V^2(\psi_{xx}^\delta)^2dxdt+C\int_0^T\int_{\mathbb{R}}\sigma(\phi_{xx}^\delta)^2dxdt+C\varepsilon^2\int_0^T\int_{\mathbb{R}}\sigma(\psi_x^\delta)^2(\psi_{xx}^\delta)^2dxdt\\ \leq&\varepsilon^2\int_0^T\int_{\mathbb{R}}\sigma(\psi_{xxx}^\delta)^2dxdt+C\int_0^T\int_{\mathbb{R}}\sigma(\phi_t^\delta)^2(\phi_x^\delta)^2dxdt+C\varepsilon^2\int_0^T\int_{\mathbb{R}}\sigma(\psi_x^\delta)^2(\psi_{xx}^\delta)^2dxdt+C.\end{aligned}$$

Substituting the estimates of K_1-K_5 into (3.41), one has

$$\begin{aligned}
&\int_{\mathbb{R}} \sigma\left((\phi_t^\delta)^2+(\psi_t^\delta)^2\right)dx + D\int_0^T\int_{\mathbb{R}} \sigma(\phi_{xt}^\delta)^2 dxdt + \varepsilon\int_0^T\int_{\mathbb{R}} \sigma(\psi_{xt}^\delta)^2 dxdt \\
&\le C + C\int_0^T \sigma(\phi_t^\delta)^2(\psi_x^\delta)^2 dt + C\int_0^T\int_{\mathbb{R}} \sigma(\phi_t^\delta)^2(\phi_x^\delta)^2 dxdt + C\varepsilon\int_0^T \sigma(\psi_t^\delta)^2(\psi_x^\delta)^2 dt \\
&\quad + C\varepsilon^2\int_0^T\int_{\mathbb{R}} \sigma(\psi_x^\delta)^2(\psi_{xx}^\delta)^2 dxdt + \varepsilon^2\int_0^T\int_{\mathbb{R}} \sigma(\psi_{xxx}^\delta)^2 dxdt \\
&= C + K_6 + K_7 + K_8 + K_9 + K_{10}.
\end{aligned} \tag{3.44}$$

By Sobolev inequality $\|f\|_{L^\infty}^2 \le 2\|f\|\|f_x\|$, (3.30), (3.42), we have

$$\begin{aligned}
K_6 &= C\int_0^T\int_{\mathbb{R}} \sigma(\phi_t^\delta)^2(\psi_x^\delta)^2 dxdt \\
&\le C\int_0^T \sigma\|\phi_t^\delta\|_{L^\infty}^2 \int_{\mathbb{R}} (\psi_x^\delta)^2 dxdt \\
&\le C\int_0^T \sigma\|\phi_t^\delta\|\|\phi_{xt}^\delta\| dt \\
&\le \frac{D}{4}\int_0^T \sigma\|\phi_{xt}^\delta\|^2 dt + C\int_0^T \sigma\|\phi_t^\delta\|^2 dt \\
&\le \frac{D}{4}\int_0^T \sigma\|\phi_{xt}^\delta\|^2 dt + C
\end{aligned}$$

and

$$\begin{aligned}
K_7 &= C\int_0^T\int_{\mathbb{R}} \sigma(\phi_t^\delta)^2(\phi_x^\delta)^2 dxdt \\
&\le C\int_0^T \sigma\|\phi_t^\delta\|_{L^\infty}^2 \int_{\mathbb{R}} (\phi_x^\delta)^2 dxdt \\
&\le C\int_0^T \sigma\|\phi_t^\delta\|\|\phi_{xt}^\delta\| dt
\end{aligned}$$

$$\leq \frac{D}{4}\int_0^T \sigma\|\phi^\delta_{xt}\|^2 dt + C.$$

Using Sobolev inequality, (3.30), (3.43), we have

$$\begin{aligned}
K_8 &= C\varepsilon \int_0^T \int_{\mathbb{R}} \sigma(\psi^\delta_t)^2(\psi^\delta_x)^2 dxdt \\
&\leq C\varepsilon \int_0^T \sigma\|\psi^\delta_t\|^2_{L^\infty}\int_{\mathbb{R}}(\psi^\delta_x)^2 dxdt \\
&\leq C\varepsilon \int_0^T \sigma\|\psi^\delta_t\|\|\psi^\delta_{xt}\| dt \\
&\leq \frac{\varepsilon}{4}\int_0^T \sigma\|\psi^\delta_{xt}\|^2 dt + C\int_0^T \sigma\|\psi^\delta_t\|^2 dt \\
&\leq \frac{\varepsilon}{4}\int_0^T \sigma\|\psi^\delta_{xt}\|^2 dt + C
\end{aligned}$$

and

$$\begin{aligned}
K_9 &= C\varepsilon^2 \int_0^T \int_{\mathbb{R}} \sigma(\psi^\delta_x)^2(\psi^\delta_{xx})^2 dxdt \\
&\leq C\varepsilon^2 \int_0^T \sigma\|\psi^\delta_{xx}\|^2_{L^\infty}\int_{\mathbb{R}}(\psi^\delta_x)^2 dxdt \\
&\leq C\varepsilon^2 \int_0^T \sigma\|\psi^\delta_{xx}\|\|\psi^\delta_{xxx}\| dt \qquad (3.45)\\
&\leq \frac{\varepsilon^2}{4}\int_0^T \sigma\|\psi^\delta_{xxx}\|^2 dt + C\varepsilon^2\int_0^T \sigma\|\psi^\delta_{xx}\|^2 dt \\
&\leq \frac{\varepsilon^2}{4}\int_0^T \sigma\|\psi^\delta_{xxx}\|^2 dt + C,
\end{aligned}$$

where we have used the smallness of ε.

We are left to estimate the term $\varepsilon^2 \int_0^T \sigma \|\psi_{xxx}^\delta\|^2 dt$. Indeed multiplying the second equation of (3.31) by $-\varepsilon\sigma\psi_{xxx}^\delta$ and integrating the result over $\mathbb{R} \times [0, T]$, we get by $|V| \le C$, $|V_x| \le C$, $\sigma \le 1$, Cauchy–Schwarz inequality, (3.30) and (3.45) that

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_{\mathbb{R}} \sigma(\psi_{xx}^\delta)^2 dx + \varepsilon^2 \int_0^T\int_{\mathbb{R}} \sigma(\psi_{xxx}^\delta)^2 dxdt \\
&= \frac{\varepsilon}{2}\int_0^{\sigma(T)}\int_{\mathbb{R}} \sigma(\psi_{xx}^\delta)^2 dxdt + 2\varepsilon^2 \int_0^T\int_{\mathbb{R}} \sigma(V\psi_x^\delta)_x \psi_{xxx}^\delta dxdt, \\
&\quad - \varepsilon \int_0^T\int_{\mathbb{R}} \sigma\phi_{xx}^\delta \psi_{xxx}^\delta dxdt + \varepsilon^2 \int_0^T\int_{\mathbb{R}} \sigma((\psi_x^\delta)^2)_x \psi_{xxx}^\delta dxdt \\
&\le \frac{\varepsilon}{2}\int_0^T\int_{\mathbb{R}} (\psi_{xx}^\delta)^2 dxdt + \frac{\varepsilon^2}{4}\int_0^T\int_{\mathbb{R}} \sigma(\psi_{xxx}^\delta)^2 dxdt + C\varepsilon^2 \int_0^T\int_{\mathbb{R}} \sigma V_x^2 (\psi_x^\delta)^2 dxdt \\
&\quad + C\varepsilon^2 \int_0^T\int_{\mathbb{R}} \sigma V^2 (\psi_{xx}^\delta)^2 dxdt + C\int_0^T\int_{\mathbb{R}} \sigma(\phi_{xx}^\delta)^2 dxdt + C\varepsilon^2 \int_0^T\int_{\mathbb{R}} \sigma(\psi_x^\delta)^2(\psi_{xx}^\delta)^2 dxdt \\
&\le \frac{\varepsilon^2}{4}\int_0^T\int_{\mathbb{R}} \sigma(\psi_{xxx}^\delta)^2 dxdt + C\varepsilon^2 \int_0^T\int_{\mathbb{R}} \sigma(\psi_x^\delta)^2(\psi_{xx}^\delta)^2 dxdt + C \\
&\le \frac{\varepsilon^2}{2}\int_0^T\int_{\mathbb{R}} \sigma(\psi_{xxx}^\delta)^2 dxdt + C,
\end{aligned}$$

which leads to

$$\varepsilon \int_{\mathbb{R}} \sigma(\psi_{xx}^\delta)^2 dx + \varepsilon^2 \int_0^T\int_{\mathbb{R}} \sigma(\psi_{xxx}^\delta)^2 dxdt \le C. \tag{3.46}$$

It follows from (3.45) and (3.46) that

$$K_9 + K_{10} \le C.$$

Substituting the estimates of $K_6 - K_{10}$ into (3.44), we have

$$\int_{\mathbb{R}} \sigma\left((\phi_t^\delta)^2+(\psi_t^\delta)^2\right)dx + D\int_0^T\int_{\mathbb{R}} \sigma(\phi_{xt}^\delta)^2 dxdt + \varepsilon\int_0^T\int_{\mathbb{R}} \sigma(\psi_{xt}^\delta)^2 dxdt \le C, \tag{3.47}$$

which, combined with (3.7), (3.30) and (3.24) gives

$$\begin{aligned}\sigma\int_{\mathbb{R}}(\phi_{xx}^\delta)^2dx &\le C\sigma\int_{\mathbb{R}}\left((\phi_t^\delta)^2+V^2(\phi_x^\delta)^2+U^2(\psi_x^\delta)^2\right)dx + C\sigma\int_{\mathbb{R}}(\phi_x^\delta)^2(\psi_x^\delta)^2dx\\ &\le C\sigma\int_{\mathbb{R}}\left((\phi_t^\delta)^2+(\phi_x^\delta)^2+(\psi_x^\delta)^2\right)dx + C\\ &\le C.\end{aligned} \tag{3.48}$$

It follows from the Sobolev inequality, (3.30), (3.31) and (3.46)–(3.48) that

$$\begin{aligned}&\varepsilon\int_0^T\int_{\mathbb{R}}\sigma(\phi_{xxx}^\delta)^2dxdt\\ &\le C\varepsilon\int_0^T\int_{\mathbb{R}}\sigma\left((\phi_{xt}^\delta)^2+V^2(\phi_{xx}^\delta)^2+V_x^2(\phi_x^\delta)^2+U^2(\psi_{xx}^\delta)^2+U_x^2(\psi_x^\delta)^2\right)dxdt\\ &\quad + C\varepsilon\int_0^T\int_{\mathbb{R}}\sigma(\phi_{xx}^\delta)^2(\psi_x^\delta)^2dxdt + C\varepsilon\int_0^T\int_{\mathbb{R}}\sigma(\phi_x^\delta)^2(\psi_{xx}^\delta)^2dxdt\\ &\le C + C\varepsilon\int_0^T\sigma\|\phi_{xx}^\delta\|^2\|\psi_x^\delta\|\|\psi_{xx}^\delta\|dt + C\varepsilon\int_0^T\sigma\|\psi_{xx}^\delta\|^2\|\phi_x^\delta\|\|\phi_{xx}^\delta\|dt\\ &\le C + C\varepsilon\int_0^T\left(\|\psi_x^\delta\|^2+\|\psi_{xx}^\delta\|^2\right)dt\\ &\le C,\end{aligned} \tag{3.49}$$

where we have used the fact that ε, $|U|$, $|V|$, $|U_x|$ and $|V_x|$ are all bounded. By (3.46), (3.47) and (3.49), we have

$$\begin{aligned}&\int_{\mathbb{R}}\sigma\left((\phi_t^\delta)^2+(\psi_t^\delta)^2+(\phi_{xx}^\delta)^2+\varepsilon(\psi_{xx}^\delta)^2\right)dx\\ &\qquad + \int_0^T\int_{\mathbb{R}}\sigma\left((\phi_{xt}^\delta)^2+\varepsilon(\psi_{xt}^\delta)^2+\varepsilon(\phi_{xxx}^\delta)^2+\varepsilon^2(\psi_{xxx}^\delta)^2\right)dxdt \le C.\end{aligned}$$

Thus, the proof of Lemma 3.7 is completed. □

Finally, we turn to prove Theorem 3.1.

3.3. *Proof of Theorem 3.1*

To prove Theorem 3.1, we shall invoke the Aubin–Lions–Simon lemma (cf. [50]). For convenience, we state it below.

Lemma 3.8 *(Aubin–Lions–Simon lemma). Let X_0, X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. For $1 \le p, q \le \infty$, let*

$$W = \{f \in L^p([0,T]; X_0) | \partial_t f \in L^q([0,T]; X_1)\}.$$

(i) If $p < \infty$, then the embedding of W into $L^p([0,T]; X)$ is compact (that is W is relatively compact in $L^p([0,T]; X)$);
(ii) If $p = \infty$ and $q > 1$, then the embedding of W into $C([0,T]; X)$ is compact.

Next we prove Theorem 3.1. It first follows from (3.30) and (3.38) that

$$\begin{cases} \|\phi^\delta\|_{1,w}^2 + \|\psi^\delta\|_1^2 + \|\psi_x^\delta\|_w^2 \\ \quad + \int_0^T \left(\|\phi_x^\delta\|_{1,w}^2 + \|\psi_x^\delta\|_w^2 + \varepsilon\|\psi_x^\delta\|_1^2 + \varepsilon\|\psi_{xx}^\delta\|_w^2 \right) dt \le C, \\ \int_{\mathbb{R}} \sigma \left(\|\phi_t^\delta\|^2 + \|\psi_t^\delta\|^2 + \|\phi_{xx}^\delta\|^2 + \varepsilon\|\psi_{xx}^\delta\|^2 \right) dx \\ \quad + \int_0^T \int_{\mathbb{R}} \sigma \left(\|\phi_{xt}^\delta\|^2 + \varepsilon\|\psi_{xt}^\delta\|^2 + \varepsilon\|\phi_{xxx}^\delta\|^2 + \varepsilon^2\|\psi_{xxx}^\delta\|^2 \right) dxdt \le C. \end{cases} \tag{3.50}$$

On the other hand, by (3.39) and (3.50), one has

$$\begin{aligned} \int_0^T \sigma^2 \int_{\mathbb{R}} (\phi_{tt}^\delta)^2 dxdt \le & C \int_0^T \sigma^2 \int_{\mathbb{R}} \left((\phi_{xxt}^\delta)^2 + (\phi_x^\delta)^2 + (\phi_{xt}^\delta)^2 + (\psi_x^\delta)^2 + (\psi_{xt}^\delta)^2 \right) dxdt \\ & + C \int_0^T \sigma^2 \int_{\mathbb{R}} (\phi_{xt}^\delta)^2 (\psi_x^\delta)^2 dxdt + C \int_0^T \sigma^2 \int_{\mathbb{R}} (\phi_x^\delta)^2 (\psi_{xt}^\delta)^2 dxdt \\ \le & C \int_0^T \sigma^2 \int_{\mathbb{R}} (\phi_{xxt}^\delta)^2 dxdt + C. \end{aligned} \tag{3.51}$$

Similarly, we have

$$\int_0^T \sigma^2 \int_{\mathbb{R}} (\psi_{tt}^\delta)^2 dxdt \le C \int_0^T \sigma^2 \int_{\mathbb{R}} (\psi_{xxt}^\delta)^2 dxdt + C. \tag{3.52}$$

It follows from (3.42), (3.43), (3.50), (3.51) and (3.52) that

$$\begin{cases} (\phi^\delta, \psi^\delta) \in L^\infty([0,\infty), H^1(\mathbb{R})), & (\phi_t^\delta, \psi_t^\delta) \in L^2([0,\infty), L^2(\mathbb{R})), \\ (\phi_x^\delta, \psi_x^\delta) \in L^\infty((0,\infty), H^1(\mathbb{R})), & (\phi_{xt}^\delta, \psi_{xt}^\delta) \in L^2((0,\infty), L^2(\mathbb{R})), \\ (\phi_{xx}^\delta, \psi_{xx}^\delta) \in L^2((0,\infty), H^1(\mathbb{R})), & (\phi_{xxt}^\delta, \psi_{xxt}^\delta) \in L^2((0,\infty), H^{-1}(\mathbb{R})), \\ (\phi_t^\delta, \psi_t^\delta) \in L^2((0,\infty), H^1(\mathbb{R})), & (\phi_{tt}^\delta, \psi_{tt}^\delta) \in L^2((0,\infty), H^{-1}(\mathbb{R})). \end{cases} \tag{3.53}$$

By (3.53) and the Aubin–Lions–Simon lemma, we can extract a subsequence, still denoted by $(\phi^\delta, \psi^\delta)$, such that the following convergence hold as $\delta \to 0$

$$\begin{cases} (\phi^\delta, \psi^\delta)(\cdot,t) \to (\phi,\psi)(\cdot,t) \text{ strongly in } C([0,\infty), C(\mathbb{R})), \\ (\phi_x^\delta, \psi_x^\delta)(\cdot,t) \to (\phi_x,\psi_x)(\cdot,t) \text{ strongly in } C((0,\infty), C(\mathbb{R})), \\ (\phi_{xx}^\delta, \psi_{xx}^\delta)(\cdot,t) \to (\phi_{xx},\psi_{xx})(\cdot,t) \text{ strongly in } L^2((0,\infty), L^2(\mathbb{R})), \\ (\phi_t^\delta, \psi_t^\delta)(\cdot,t) \to (\phi_t,\psi_t)(\cdot,t) \text{ strongly in } L^2((0,\infty), L^2(\mathbb{R})). \end{cases}$$

Thus, it is easy to show that the limit function (ϕ, ψ) is indeed a strong solution of the system (3.7)–(3.8) and inherits all the bounds of (3.50) which yield (3.4) and (3.5).

To complete the proof of Theorem 3.1, it remains to prove (3.6). From $\sigma = 1$ for $t \ge 1$, (3.4) and (3.5), we have

$$\int_1^\infty \left(\|\phi_x\|^2 + \|\phi_{xt}\|^2 + \|\psi_x\|^2 + \|\psi_{xt}\|^2 \right) dt \le C,$$

which implies that

$$\|\phi_x(\cdot,t), \psi_x(\cdot,t)\| \to 0 \text{ as } t \to \infty.$$

Hence, for all $x \in \mathbb{R}$, $t > 1$, it follows that

$$\begin{aligned} \phi_x^2(x,t) &= 2 \left| \int_x^\infty \phi_x \phi_{xx}(y,t) dy \right| \\ &\le 2 \left(\int_{\mathbb{R}} \phi_x^2 dy \right)^{1/2} \left(\int_{\mathbb{R}} \phi_{xx}^2 dy \right)^{1/2} \end{aligned}$$

$$= 2\left(\int_{\mathbb{R}} \phi_x^2 dy\right)^{1/2} \left(\int_{\mathbb{R}} \sigma \phi_{xx}^2 dy\right)^{1/2}$$

$$\le C\|\phi_x(\cdot,t)\| \to 0 \text{ as } t \to \infty,$$

where we have used (3.5) and $\sigma(t) = 1$ for $t > 1$. Thus,

$$\sup_{x\in\mathbb{R}} |\phi_x(x,t)| \to 0 \text{ as } t \to \infty.$$

The same procedure applied to ψ_x leads to

$$\sup_{x\in\mathbb{R}} |\psi_x(x,t)| \to 0 \text{ as } t \to \infty.$$

Hence (3.6) is proved and the proof of Theorem 3.1 is completed.

Acknowledgments

The work was completed when the first author was a postdoctoral follow of the Department of Applied Mathematics in the Hong Kong Polytechnic University partially supported from AMSS-PolyU Joint Research Institute. H. Peng was also supported by the Fundamental Research Funds for the Central Universities No. 2015ZM192 and 2017BQ008. Z.-A. Wang was supported by the Hong Kong RGC GRF grant No. PolyU 153032/15P.

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