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Infinitely many self-similar blow-up profiles for the Keller-Segel system in dimensions 3 to 9

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Abstract

Based on the method of matched asymptotic expansions and Banach fixed point theorem, we rigorously construct infinitely many self-similar blow-up profiles for the parabolic-elliptic Keller-Segel system

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \\ 0 = \Delta \Phi_u + u, \\ u(\cdot, 0) = u_0 \geq 0 \end{cases} \quad \text{in } \mathbb{R}^d,$$

where $d \in \{3, \cdots, 9\}$. Our findings demonstrate that the infinitely many backward self-similar profiles approximate the rescaling radial steady-state near the origin (i.e. $0 < |x| \ll 1$) and $\frac{2(d-2)}{|x|^2}$ at spatial infinity (i.e. $|x| \gg 1$). We also establish the convergence of the self-similar blow-up solutions as time tends to the blow-up time $T > 0$. Our results can give a refined description of backward self-similar profiles for all $|x| \geq 0$ rather than for $0 < |x| \ll 1$ or $|x| \gg 1$, indicating that the blow-up point is the origin and

$$u(x,t) \sim \frac{1}{|x|^2}, \quad x \neq 0, \text{ as } t \to T.$$

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1. Introduction

This paper is concerned with the parabolic-elliptic Keller-Segel system

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \\ 0 = \Delta \Phi_u + u, \end{cases} \quad \text{in } \mathbb{R}^d, \tag{1.1}$$

equipped with an initial data $u(\cdot, 0) = u_0$, where $d \in \{3, \cdots, 9\}$. The system (1.1) is the so-called minimal chemotaxis used to describe the chemotactic motion of mono-cellular organisms, where $u(x,t)$ represents the cell density and Φ_u stands for the concentration of the chemoattractant [36]. System (1.1) also models the self-gravitating matter in stellar dynamics in astrophysical fields [54]. This system has been extensively studied due to its rich biological and physical backgrounds and lot of interesting results have been obtained, e.g., see [6,15,19,22,33–35,39,52,55] and references therein.

For any radial initial data $u_0 \in L^\infty(\mathbb{R}^d)$, there exists a maximal time of existence $T > 0$ such that (1.1) admits a unique smooth solution on $(0, T) \times \mathbb{R}^d$, see [27]. One may refer to [2,3] for other local well-posedness spaces. Due to the quadratic nature of the convective term in (1.1), the solutions may blow up in finite time $T < +\infty$ in the sense that

$$\limsup_{t \to T} ||u(t)||_{L^\infty(\mathbb{R}^d)} = +\infty.$$

If blow-up occurs, then it holds that

$$||u(t)||_{L^\infty(\mathbb{R}^d)} \geq (T - t)^{-1}, \quad 0 < t < T,$$

by a comparison principle. We say that the blow-up is of type I if

$$\limsup_{t \to T} (T - t) ||u(t)||_{L^\infty(\mathbb{R}^d)} < \infty,$$

otherwise, the blow-up is of type II. The blow-up set $B(u_0)$ is defined by

$$B(u_0) := \{x_0 \in \mathbb{R}^d : |u(x_j, t_j)| \to \infty \text{ for some sequence } (x_j, t_j) \to (x_0, T)\},$$

and we call x_0 the blow-up point. Thanks to the divergence structure of (1.1), the total mass of the solution is conserved in the following sense:

$$M(u_0) := \int_{\mathbb{R}^d} u_0(x) dx = \int_{\mathbb{R}^d} u(x,t) dx, \quad 0 \leq t < T.$$

Problem (1.1) admits the following scaling invariance: for all $a \in \mathbb{R}^d$ and $\lambda > 0$, the function

$$u_{\lambda,a}(x,t) = \frac{1}{\lambda^2} u\left(\frac{x-a}{\lambda}, \frac{t}{\lambda^2}\right) \tag{1.2}$$

also solves (1.1). This scaling invariance gives rise to the notion of the mass-criticality in the sense that

$$\|u_{\lambda,a}\|_{L^1(\mathbb{R}^d)} = \lambda^{d-2}\|u\|_{L^1(\mathbb{R}^d)},$$

by which $d = 2$ is referred to as the mass critical case, while $d = 1$ and $d \geq 3$ the mass sub-critical and the mass super-critical cases, respectively.

The solution of (1.1) exists globally for $d = 1$ as proved in [13,44]. The critical mass threshold 8π acts as a sharp criterion separating the global existence from finite-time blow-up in the case of $d = 2$, see [5,7,13,14,23]. The 8π mass threshold implies that supposing

$$u_0 \geq 0, \quad (1 + x^2 + |\ln u_0|)u_0 \in L^1(\mathbb{R}^2),$$

the positive solution of (1.1) blows up in finite time for $M > 8\pi$ [37,49] and exists globally in time for $M < 8\pi$ [5,24]. If $M = 8\pi$, radial solutions exist globally in time [4] but infinite-time blow-up solutions with 8π mass may exist as constructed in [6,22,26]. For $M > 8\pi$, a refined finite time blow-up profile was obtained with the form

$$u(x,t) \sim \frac{1}{\lambda^2(t)} U\left(\frac{x}{\lambda(t)}\right), \quad \lambda(t) \sim \sqrt{T-t}\, e^{-\sqrt{\frac{|\log(T-t)|}{2}}}, \tag{1.3}$$

where $U(x) = \frac{8}{(1+|x|^2)^2}$ is a steady-state solution of (1.1), see [11,15,31,49,53]. The form (1.3) is the unique finite time blow-up behavior for radial non-negative solutions of (1.1) [42]. An interesting phenomenon that two steady-state solutions are simultaneously collapsing and colliding is recently constructed in [16]. It is remarkable that any blow-up solutions are of type II for $d = 2$, see [45,51].

For $d \geq 3$, we note that the system (1.1) is referred to as the L^1-supercritical and $L^{d/2}$-critical since the scaling transformation (1.2) preserves the $L^{d/2}$−norm, i.e., $||u_{\lambda,a}||_{L^{\frac{d}{2}}(\mathbb{R}^d)} = ||u||_{L^{\frac{d}{2}}(\mathbb{R}^d)}$. Initial data with small $L^{d/2}$−norm lead to solutions that exist globally in time [20]. Subsequently, this result was improved in [12] by showing that if the $L^{d/2}$−norm of initial data is less than a sharp constant derived from the Gagliardo-Nirenberg inequality, then the solution exists globally. Large initial data give rise to finite-time blow-up [12,20,44]. In contrast to dimension $d = 2$, the solutions of (1.1) with $d \geq 3$ may blow up in finite time for an arbitrary mass since $M(u_{\lambda,a}) = \lambda^{d-2}M(u)$.

Singularity formation of blow-up solutions to system (1.1) for $d \geq 3$ exhibits rich dynamical behavior. When the initial data are nonnegative and radially non-increasing, it was shown in [43] that all blow-up solutions of (1.1) are of type I for $d \in [3, 9]$. A family of type I self-similar blow-up solutions was obtained by the shooting method in [8,30,46]. Remarkably, it was shown in [27] that all radial and non-negative type I blow-up solutions are asymptotically backward self-similar near the origin as $t \to T$, which signifies the significance of backward self-similar profiles for understanding the structure of singularities. A new type I-log blow-up solution of

(1.1) in dimensions 3 and 4 was constructed in [47]. There are also type II blow-up solutions for $d \geq 3$ [10,17,29,41]. The authors of [17] showed the existence and radial stability of type II blow-up solutions, characterized by mass concentrating near a sphere that shrinks to a point. This pattern, known as collapsing-ring blow-up, also emerges in the nonlinear Schrödinger equation [25,40]. For $d \geq 11$, type II solutions concentrating at a steady-state solution are constructed in [41]. This paper is concerned with type I blow-up solutions.

Backward self-similar solutions of (1.1) are of the form

$$u(x,t) = \frac{1}{T-t} U(y), \quad y = \frac{x}{\sqrt{T-t}}, \tag{1.4}$$

where $U(y)$ is the backward self-similar profile satisfying

$$\Delta U - \frac{y \cdot \nabla U}{2} - U - \nabla \cdot (U \nabla \Phi_U) = 0, \ \ \Delta \Phi_U + U = 0. \tag{1.5}$$

We denote $r = |y|$. In the radial case, for $d \geq 1$, there holds

$$\partial_r \Phi_U(r) = -\frac{1}{r^{d-1}} \int_0^r U(s) s^{d-1} ds.$$

Then the equation (1.5) can be written in the radial form

$$\partial_{rr} U + \frac{d-1}{r} \partial_r U - \frac{1}{2} r \partial_r U - U + U^2 + \left(\frac{1}{r^{d-1}} \int_0^r U(s) s^{d-1} ds \right) \partial_r U = 0. \tag{1.6}$$

There are four known classes of solutions of (1.6):

- For $d \geq 1$, the constant solutions

$$\bar{U}_0 = 0, \quad \bar{U}_1 = 1. \tag{1.7}$$

- For $d \geq 3$, the solution singular at the origin

$$\bar{U}_2 = \frac{2(d-2)}{r^2}. \tag{1.8}$$

- For $d \geq 3$, the explicit smooth positive solution [8]

$$\bar{U}_3 = \frac{4(d-2)(2d + r^2)}{(2(d-2) + r^2)^2}. \tag{1.9}$$

- For $d \in [3, 9]$, there exists a countable family of positive smooth radially symmetric solutions $\{\bar{U}_n\}_{n \geq 4}$ [8,30,46], where

$$\bar{U}_n \sim \frac{1}{r^2}, \quad \text{as} \ \ r \to +\infty. \tag{1.10}$$

With the shooting method, a family of radially symmetric solutions $\{\bar{U}_n\}_{n\geq 4}$ has been constructed in [30] for $d=3$ and in [8,46] for $3 \leq d \leq 9$. For $d=3$, it was shown in [28] that $\bar{U}_3$ is a stable self-similar profile based on the semigroup approach. Very recently, the non-radial stability of $\bar{U}_3$ was proved in [38]. For $d \geq 3$, it was proved in [19] that all the fundamental self-similar profiles $\{\bar{U}_n\}_{n\geq 3}$ are conditionally stable (of finite co-dimension).

Backward self-similar profiles of (1.1) (i.e. the solutions of (1.5)) are still not completely classified, even in the radial setting. Accurately describing the self-similar profiles is a crucial step in classifying all possible blow-up profiles for (1.1) (at least in the radial case).

This paper aims to construct more precise backward self-similar profiles by using different approaches. We recall some results below in connection with our work. For $d=3$, the authors of [30] showed that there exists a sequence of self-similar profiles (i.e. solutions of (1.6)), denoted by $\{G_n(r)\}_{n\geq 1}$, which satisfy

$$G_n(r) \sim K_n \text{ as } r \to 0, \quad \lim_{r\to\infty} G_n(r) = \frac{A_n}{r^2},$$

where $K_n > 0$, A_n are constants, and $\lim_{n\to+\infty} K_n = \infty$. Subsequently, for $3 \leq d \leq 9$, it was shown in [8] that there exists a countable number of self-similar profiles $\{\bar{G}_n\}_{n\geq 1}$ satisfying

$$\bar{G}_n(r) \lesssim 1 \text{ as } r \to 0, \quad \lim_{r\to\infty} \bar{G}_n(r) = \frac{c_n}{r^2}, \quad \text{for some constant } c_n \in (0,2].$$

The works [8,30] discovered two essential common properties for the family of self-similar profiles for fixed n, that is they are bounded as $0 < r \ll 1$ and behave like $\frac{1}{r^2}$ as $r \gg 1$. In another work [46], for $3 \leq d \leq 9$, the authors proved that there exist a countable number of self-similar profiles $\{\tilde{G}_n(r)\}_{n\geq 1}$ which are bounded near the origin for every $n \geq 1$ and

$$\lim_{n\to\infty} \tilde{G}_n(0) = +\infty, \quad \lim_{n\to\infty} \tilde{G}_n(r) = \frac{2(d-2)}{r^2} \text{ for } r > 0. \tag{1.11}$$

The work [46] gave an asymptotic description of self-similar profiles as $n \to \infty$. For fixed $n \geq 1$, the self-similar profiles were precisely described only for $r \gg 1$ in [8,30], while the precise descriptions of self-similar profiles for $r > 0$ not large are unavailable. Recently, for $d \geq 3$, self-similar profiles of blow-up solutions to (1.1) were shown to behave like $\frac{1}{r^2}$ for $0 < r \ll 1$ for a certain class of radially non-increasing initial data in [1] by the zero number argument, answering an open question in [50]. In this paper, by using a different approach, namely the method of matched asymptotic expansions and the Banach fixed point theorem, we obtain a precise description of self-similar profiles $U_n(r)$ for all $r \in [0,\infty)$, as described in (1.16) below.

To state our result, we first present the asymptotic behavior of steady-state solution of (1.1). Let $Q(r)$ be the unique solution to

$$\begin{cases} \partial_{rr} Q + \dfrac{d-1}{r}\partial_r Q + Q^2 + \partial_r Q \dfrac{1}{r^{d-1}} \displaystyle\int_0^r Q(s) s^{d-1} ds = 0, \\ Q(0) = 1, \quad Q'(0) = 0. \end{cases} \tag{1.12}$$

It is clear that $Q(r)$ is a radial steady-state solution of (1.1) with $r = |x|$. It will be shown in Section 2 that the asymptotic behavior of Q is

$$Q(r) = \frac{2(d-2)}{r^2} + O(r^{-\frac{5}{2}}), \quad \text{as} \quad r \to +\infty,$$

where $Q = 2d\bar{Q} + 2r\partial_r\bar{Q}$ and the asymptotic profile of $\bar{Q}$ as $r \to \infty$ is given in (2.46). Our main results are stated as follows.

Theorem 1.1. *For $3 \le d \le 9$, there exist infinitely many smooth radially symmetric solutions $U_n(y)$ ($n \in \mathbb{N}$) to the self-similar equation (1.5). Moreover, there exists a sufficiently small constant $r_0 > 0$ independent of n such that the following results hold.*

1. *(Profiles near the origin.) There exists a sequence $\mu_n > 0$ with $\lim_{n\to+\infty} \mu_n = 0$ such that*

$$\lim_{n\to+\infty} \sup_{r\le r_0} \left| U_n(r) - \frac{1}{\mu_n^2} Q\left(\frac{r}{\mu_n}\right) \right| = 0. \tag{1.13}$$

2. *(Profiles away from the origin.) As $r \ge r_0$, $U_n(r)$ satisfies*

$$\lim_{n\to+\infty} \sup_{r\ge r_0} (1+r^2) \left| U_n(r) - \frac{2(d-2)}{r^2} \right| = 0. \tag{1.14}$$

For any $0 < T < +\infty$, the solution of (1.1) with initial data $u_0 = \frac{1}{T} U_n(\frac{x}{\sqrt{T}})$ blows up at time T with

$$u(x,t) = \frac{1}{T-t} U_n\left(\frac{x}{\sqrt{T-t}}\right),$$

where the blow-up is of type I and $B(u_0) = 0$. Moreover, there exists a function $u^*(x) \sim \frac{1}{|x|^2}$ such that $\lim_{t\to T} u(x,t) = u^*(x)$ for all $|x| > 0$ and

$$\lim_{t\to T} ||u(\cdot,t) - u^*(\cdot)||_{L^p(\mathbb{R}^d)} = 0, \ \forall \ p \in [1, \frac{d}{2}). \tag{1.15}$$

Remark 1.2. Based on the proof of Theorem 1.1, the profile of the solutions U_n of (1.5), as constructed in Theorem 1.1, can be more precisely described as follows. First, we define

$$\mathcal{U} = 2d\tilde{u}_1 + 2r\partial_r\tilde{u}_1$$

where $\tilde{u}_1 := u_1$ is a known function for $d = 3$ (see Lemma 2.2[1]). Then there exist

$$0 < r_0 \ll 1, \ 0 < \mu_n < r_0, \ 0 < \varepsilon(\mu_n) \ll r_0^{\frac{1}{2}}$$

with $\lim_{n\to+\infty} \mu_n = 0$, $\lim_{n\to+\infty} \varepsilon(\mu_n) = 0$, and

[1] The definitions of $\tilde{u}_1$ for $d \in [4, 9]$ are obtained by the same process as in Lemma 2.2.

$$\tilde{\mathcal{U}} \in \tilde{X}_{r_0}, \quad \tilde{Q} \in \tilde{Y}_{\frac{r_0}{\mu_n}}$$

where the definitions of the spaces $\tilde{X}_{r_0}$, $\tilde{Y}_r$ are given in (2.9) and (2.51) for $d = 3$, respectively,[2] such that

$$U_n(r) := \begin{cases} \left(\frac{Q}{\mu_n^2} + \mu_n^2 \tilde{Q}\right)\left(\frac{r}{\mu_n}\right) & \text{for } 0 \le r \le r_0, \\ \frac{2(d-2)}{r^2} + \varepsilon(\mu_n)(\mathcal{U} + \tilde{\mathcal{U}})(r) & \text{for } r > r_0, \end{cases} \tag{1.16}$$

solves (1.6).

By (1.16) we obtain a precise description of self-similar profiles $U_n(r)$ for all $r \in [0, \infty)$. In particular, we show that $U_n(r)$ behaves like the rescaled steady-state solutions $\frac{1}{\mu_n^2} Q(\frac{r}{\mu_n})$ for $0 \le r \ll 1$ and $U_n(r) \sim \frac{2(d-2)}{r^2}$ for $r \gg 1$. For $3 < d \le 9$, we know from (1.16) that the profiles obtained in this paper are different from those in [8] since $2(d-2) > 2$, but have the same asymptotic properties as in (1.11) as $n \to \infty$. Whether the self-similar profiles constructed in [30,46] and in Theorem 1.1 are equivalent is an interesting open question.

For $d = 2$, the limiting spatial profile of radial blow-up solutions to (1.1) resembles a Dirac mass perturbed by a L^1 function, i.e.,

$$u(\cdot, t) \rightharpoonup 8\pi\delta_0 + f \ \text{ in } C_0(\mathbb{R}^2)^* \ \text{ as } t \to T, \tag{1.17}$$

where $0 \le f \in L^1(\mathbb{R}^2)$, see [31,32]. In contrast, for $d \in [3, 9]$, as seen from (1.16), our result shows that there exist radial solutions of (1.1) that satisfy

$$u(x, t) \sim 1/|x|^2, \quad x \ne 0, \ \text{as } t \to T,$$

which is quite different from the case $d = 2$ in (1.17).

Remark 1.3 *(Finite codimensional radial stability).* The stability of self-similar blow-up profiles constructed in [8,30] was established in [19,28]. Using the same ideas of [19], one can also show that the profiles constructed in Theorem 1.1 are stable along a set of radial initial data with finite Lipschitz codimension equal to the number of unstable eigenmodes. The non-radial stability of self-similar profiles is still an open problem as far as we know.

Organization of the paper. In Section 2, we first introduce a key transformation which converts (1.5) into a local elliptic equation in $\mathbb{R}^{d+2}$. Then using the method of matched asymptotic expansions, we rigorously derive a sequence of smooth self-similar profiles. In Section 3, we give a complete proof for Theorem 1.1.

[2] The definitions of the spaces $\tilde{X}_{r_0}$, $\tilde{Y}_r$ for $d \in [4, 9]$ are similar by the same process of the proof for $d = 3$.

2. Construction of self-similar profiles

We start by introducing some notations.

Notation. We write $a \lesssim b$, if there exists $c > 0$ such that $a \le cb$, and $a \sim b$ if simultaneously $a \lesssim b$ and $b \lesssim a$. If the inequality $|f| \le C|g|$ holds for some constant $C > 0$, then we write $f = O(g)$.

2.1. Key results

Our main goal is to derive the radial self-similar profile $U(r) := U(|y|)$ which satisfies (1.6). To study the nonlocal equation (1.6), we introduce the following so-called reduced mass (cf. [8]),

$$\Phi(r) = \frac{1}{2r^d}\int_0^r U(s)s^{d-1}ds, \tag{2.1}$$

and transform (1.6) into a local equation for $\Phi(r)$ satisfying

$$\partial_{rr}\Phi + \frac{d+1}{r}\partial_r\Phi - \Phi - \frac{r\partial_r\Phi}{2} + 2d\Phi^2 + 2r\Phi\Phi_r = 0.$$

Clearly, $\Phi(r)$ is the radially symmetric solution of

$$\Delta\Phi - \frac{1}{2}\Lambda\Phi + 2d\Phi^2 + y\cdot\nabla(\Phi^2) = 0, \quad y \in \mathbb{R}^{d+2}, \tag{2.2}$$

with Λ being a differential operator defined by

$$\Lambda u := 2u + y\cdot\nabla u.$$

By (1.7), for $d \ge 1$, (2.2) admits constant solutions $\bar\Phi_0 = 0$, $\bar\Phi_1 = \frac{1}{2d}$. By (1.8) and (1.9), for $d \ge 3$, (2.2) admits explicit radial solutions

$$\bar\Phi_2 = \frac{1}{|y|^2}, \quad \bar\Phi_3 = \frac{2}{2(d-2)+|y|^2}. \tag{2.3}$$

From (1.10), for $d \in [3, 9]$, there exists a countable family of positive smooth radially symmetric solutions $\{\bar\Phi_n\}_{n\ge 4}$ of (2.2) such that

$$\bar\Phi_n \sim \frac{1}{|y|^2} \text{ as } |y| \to +\infty. \tag{2.4}$$

The main result of this paper, as stated in Theorem 1.1 along with Remark 1.2, consists of the construction of a class of more general solutions than those given in (2.3) and (2.4), which share some similar properties when $0 < |y| \ll 1$ or $|y| \gg 1$.

The rest of this paper is focused on the case $d = 3$ for the simplicity of presentation. The extension of the result to $d \in [4, 9]$[3] is straightforward since the oscillating behavior of the radial steady-state profile $Q = 2d\bar{Q} + 2r\partial_r\bar{Q}$ for $d = 3$ (see (2.46) for the definition of $\bar{Q}$) also exists for $d \in [4, 9]$. As in [9,18,21], the matching of exterior solutions with interior solutions can be obtained by this oscillating behavior.

When $d = 3$, equation (2.2) is reduced to

$$\Delta\Phi - \frac{1}{2}\Lambda\Phi + 6\Phi^2 + y\cdot\nabla(\Phi^2) = 0, \quad y \in \mathbb{R}^5. \tag{2.5}$$

Applying the transformation (2.1), we then obtain the radially symmetric solution of (1.5) as follows

$$U = 6\Phi + 2r\partial_r\Phi.$$

We define

$$\Phi_* := \bar{\Phi}_2 = \frac{1}{r^2}, \quad \bar{Q}(r) = \frac{1}{2r^3}\int_0^r Q(s)s^2 ds,$$

where Q is given by (1.12).

The following is the key proposition of this paper, from which Theorem 1.1 directly follows.

Proposition 2.1. *There exist infinitely many smooth radially symmetric solutions Φ_n ($n \in \mathbb{N}$) to equation (2.5). Moreover, there exists a sufficiently small constant $r_0 > 0$ which is independent of n such that the following results hold.*

1. (Behavior near the origin.) There exists a sequence $\mu_n > 0$ with $\lim_{n\to+\infty}\mu_n = 0$ such that

$$\lim_{n\to+\infty}\sup_{r\le r_0}\left|\Phi_n - \frac{1}{\mu_n^2}\bar{Q}\left(\frac{r}{\mu_n}\right)\right| = 0. \tag{2.6}$$

2. (Behavior away from the origin.) As $r \ge r_0$, $\Phi_n(r)$ satisfies

$$\lim_{n\to+\infty}\sup_{r\ge r_0}(1+r^2)|\Phi_n - \Phi_*| = 0. \tag{2.7}$$

The remainder of this section is devoted to proving the above proposition.

2.2. *Exterior profiles*

The aim of this subsection is to construct a radial solution to (2.5) on $[r_0, +\infty)$, where $0 < r_0 < 1$. We are initially concerned with the asymptotic behavior of the fundamental solutions

[3] This oscillating behavior exists when the differential equation $x^2 + (d+2)x + 4(d-1) = 0$ has complex roots, which holds in the case $d \in [3, 9]$.

for the equation $L(u) = 0$ on $(0, +\infty)$, where L is the linearized operator of (2.5) around Φ_*, defined as

$$L = -\Delta + \frac{1}{2}\Lambda - 2y \cdot \nabla(\Phi_* \cdot) - 12\Phi_*. \tag{2.8}$$

Given $0 < r_0 < 1$, we define X_{r_0} as the space of continuous functions on $[r_0, +\infty)$ such that the following norm is finite

$$||w||_{X_{r_0}} = \sup_{r_0 \le r \le 1} (r^{\frac{5}{2}}|w| + r^{\frac{7}{2}}|\partial_r w|) + \sup_{r \ge 1} \left(r^4|w| + r^5|\partial_r w| \right). \tag{2.9}$$

Lemma 2.2. *Let L be defined in (2.8). Then the following results hold.*
1. The basis of the fundamental solutions: The equation

$$L(u) = 0 \ \ on \ \ (0, +\infty)$$

has two fundamental solutions u_i $(i = 1, 2)$ with the following asymptotic behavior as $r \to \infty$:

$$u_1(r) = r^{-2}(1 + O(r^{-2})) \quad and \quad u_2(r) = r^{-5}e^{\frac{r^2}{4}}(1 + O(r^{-2})), \tag{2.10}$$

and as $r \to 0$:

$$u_1(r) = \frac{c_1 \sin(\frac{\sqrt{7}}{2}\log(r) + c_3)}{r^{\frac{5}{2}}} + O(r^{-\frac{1}{2}}) \quad and \quad u_2(r) = \frac{c_2 \sin(\frac{\sqrt{7}}{2}\log(r) + c_4)}{r^{\frac{5}{2}}} + O(r^{-\frac{1}{2}}), \tag{2.11}$$

where c_1, $c_2 \neq 0$, c_3, $c_4 \in \mathbb{R}$.
2. The continuity of the resolvent: The inverse

$$\tau(f) = \left(\int_r^{+\infty} f u_2 s^6 e^{-\frac{s^2}{4}} ds \right) u_1 - \left(\int_r^{+\infty} f u_1 s^6 e^{-\frac{s^2}{4}} ds \right) u_2 \tag{2.12}$$

satisfies $L(\tau(f)) = f$ and

$$||\tau(f)||_{X_{r_0}} \lesssim \int_{r_0}^{1} |f| s^{\frac{7}{2}} ds + \sup_{r \ge 1} r^4 |f|. \tag{2.13}$$

Proof. Step 1. Basis of homogeneous solutions. We define the changing of variable

$$u(r) = \frac{1}{z^{\frac{\gamma}{2}}}\phi(z), \qquad z = r^2, \tag{2.14}$$

where γ satisfies $-\gamma^2 + 5\gamma - 8 = 0$. From

$$\partial_r = 2r\partial_z, \ \partial_{rr} = 4z\partial_{zz} + 2\partial_z, \ r\partial_r = 2z\partial_z,$$

one has

$$
\begin{aligned}
L(u) &= (-4z\partial_{zz} - 2\partial_z - 8\partial_z + z\partial_z + 1 - 8\Phi_* - 4\Phi_* z\partial_z)\left(\frac{1}{z^{\frac{\gamma}{2}}}\phi(z)\right) \\
&= \frac{1}{z^{\frac{\gamma}{2}}}\left\{ -4z\phi''(z) + (4\gamma - 14 + z)\phi'(z) + \left[1 - \frac{\gamma}{2} + \frac{1}{z}(-\gamma^2 + 5\gamma - 8)\right]\phi\right\} \\
&= \frac{1}{z^{\frac{\gamma}{2}}}\left\{ -4z\phi''(z) + (4\gamma - 14 + z)\phi'(z) + (1 - \frac{\gamma}{2})\phi\right\}.
\end{aligned}
$$

Let $\phi(z) = \nu(\xi)$ and $\xi = \frac{z}{4}$. Then,

$$
L(u) = -\frac{1}{z^{\frac{\gamma}{2}}}\left\{\xi\nu''(\xi) + \left(-\gamma + \frac{7}{2} - \xi\right)\nu'(\xi) + (\frac{\gamma}{2} - 1)\nu(\xi)\right\}.
$$

Therefore, $L(u) = 0$ if and only if

$$
\xi\frac{d^2\nu}{d\xi^2} + (b - \xi)\frac{d\nu}{d\xi} - a\nu = 0, \tag{2.15}
$$

where

$$
b = \frac{7}{2} - \gamma,\ a = 1 - \frac{\gamma}{2}.
$$

The equation (2.15) is known as the well studied Kummer's equation (see [48]). If the parameter a is not a negative integer (which holds in particular for our case), then the fundamental solutions to Kummer's equation consist of the Kummer function $M(a, b, \xi)$ and the Tricomi function $U(a, b, \xi)$. Therefore, $\nu(\xi)$ is a linear combination of the special functions $M(a, b, \xi)$ and $U(a, b, \xi)$, whose asymptotic profiles at infinity are given by

$$
M(a, b, \xi) = \frac{\Gamma(b)}{\Gamma(a)}\xi^{a-b}e^{\xi}(1 + O(\xi^{-1})),\ \ U(a, b, \xi) = \xi^{-a}(1 + O(\xi^{-1}))\ \text{ as } \xi \to +\infty. \tag{2.16}
$$

Then by (2.14) and (2.16), one obtains (2.10).

For the behavior near the origin, we have

$$
M(a, b, \xi) = 1 + O(\xi)\ \text{ as } \xi \to 0. \tag{2.17}
$$

It is easy to check that the real part of b satisfies $\mathcal{R}(b) = 1$ $(b \neq 1)$. Then it follows that

$$
U(a, b, \xi) = \frac{\Gamma(b-1)}{\Gamma(a)}\xi^{1-b} + \frac{\Gamma(1-b)}{\Gamma(a-b+1)} + O(\xi)\ \text{ as } \xi \to 0. \tag{2.18}
$$

Since the polynomial $\gamma^2 - 5\gamma + 8 = 0$ has complex roots $\gamma = \frac{5}{2} \pm \frac{\sqrt{7}i}{2}$, then combining (2.14), (2.17) and (2.18), one obtains (2.11).

Step 2. Estimate on the resolvent. The Wronskian $W := u_1'u_2 - u_2'u_1$ satisfies $W' = \left(\frac{r}{2} - \frac{6}{r}\right)W$, and $W = \frac{C}{r^6} e^{\frac{r^2}{4}}$. We may assume $C = 1$ without loss of generality. Next, we solve $L(w) = f$. By the variation of constants, we obtain

$$w = \left(a_1 + \int_r^{+\infty} f u_2 s^6 e^{-\frac{s^2}{4}} ds \right) u_1 + \left(a_2 - \int_r^{+\infty} f u_1 s^6 e^{-\frac{s^2}{4}} ds \right) u_2, \quad a_1, \; a_2 \in \mathbb{R}.$$

Then, $\tau(f)$ satisfies $L(\tau(f)) = f$ by choosing $a_1 = a_2 = 0$ in the above.

Next, we estimate the asymptotic behavior of $\tau(f)$. For $r \geq 1$, we have

$$\begin{aligned} r^4 |\tau(f)| &= r^4 \left| \left(\int_r^{+\infty} f u_2 s^6 e^{-\frac{s^2}{4}} ds \right) u_1 - \left(\int_r^{+\infty} f u_1 s^6 e^{-\frac{s^2}{4}} ds \right) u_2 \right| \\ &\lesssim r^2 \left(\int_r^{+\infty} |f| s ds \right) + r^{-1} e^{\frac{r^2}{4}} \left(\int_r^{+\infty} |f| s^4 e^{-\frac{s^2}{4}} ds \right) \\ &\lesssim \sup_{r \geq 1} r^4 |f| \left\{ \left(\int_r^{+\infty} \frac{ds}{s^3} \right) r^2 + r^{-1} e^{\frac{r^2}{4}} \left(\int_r^{+\infty} e^{-\frac{s^2}{4}} ds \right) \right\} \\ &\lesssim \sup_{r \geq 1} r^4 |f|, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} r^5 |\partial_r \tau(f)| &= r^5 \left| \left(\int_r^{+\infty} f u_2 s^6 e^{-\frac{s^2}{4}} ds \right) \partial_r u_1 - \left(\int_r^{+\infty} f u_1 s^6 e^{-\frac{s^2}{4}} ds \right) \partial_r u_2 \right| \\ &\lesssim r^2 \left(\int_r^{+\infty} |f| s ds \right) + (r^{-1} + r) e^{\frac{r^2}{4}} \left(\int_r^{+\infty} |f| s^4 e^{-\frac{s^2}{4}} ds \right) \\ &\lesssim \sup_{r \geq 1} r^4 |f| \left\{ \left(\int_r^{+\infty} \frac{ds}{s^3} \right) r^2 + (r^{-1} + r) e^{\frac{r^2}{4}} \left(\int_r^{+\infty} e^{-\frac{s^2}{4}} ds \right) \right\} \\ &\lesssim \sup_{r \geq 1} r^4 |f|. \end{aligned} \tag{2.20}$$

For $r_0 \leq r \leq 1$, by (2.11) and (2.19), we have

$$\begin{aligned} r^{\frac{5}{2}} |\tau(f)| \leq r^{\frac{5}{2}} &\left| \left(\int_r^1 f u_2 s^6 e^{-\frac{s^2}{4}} ds \right) u_1 - \left(\int_r^1 f u_1 s^6 e^{-\frac{s^2}{4}} ds \right) u_2 \right| \\ &+ r^{\frac{5}{2}} \left| \left(\int_1^{+\infty} f u_2 s^6 e^{-\frac{s^2}{4}} ds \right) u_1 - \left(\int_1^{+\infty} f u_1 s^6 e^{-\frac{s^2}{4}} ds \right) u_2 \right| \end{aligned} \tag{2.21}$$

$$\lesssim \int_{r_0}^{1} |f| s^{\frac{7}{2}} ds + \sup_{r\geq 1} r^4 |f|.$$

Similarly, for $r_0 \leq r \leq 1$, by (2.11), (2.20) and (2.21), we have

$$\begin{aligned} r^{\frac{7}{2}} |\partial_r \tau(f)| &= r^{\frac{7}{2}} \left| \left(\int_r^{+\infty} f u_2 s^6 e^{-\frac{s^2}{4}} ds \right) \partial_r u_1 - \left(\int_r^{+\infty} f u_1 s^6 e^{-\frac{s^2}{4}} ds \right) \partial_r u_2 \right| \\ &\lesssim r^{\frac{7}{2}} \left| \left(\int_r^{1} f u_2 s^6 e^{-\frac{s^2}{4}} ds \right) \partial_r u_1 - \left(\int_r^{1} f u_1 s^6 e^{-\frac{s^2}{4}} ds \right) \partial_r u_2 \right| \\ &\quad + r^{\frac{5}{2}} \left| \left(\int_1^{+\infty} f u_2 s^6 e^{-\frac{s^2}{4}} ds \right) \partial_r u_1 - \left(\int_1^{+\infty} f u_1 s^6 e^{-\frac{s^2}{4}} ds \right) \partial_r u_2 \right| \\ &\lesssim \int_{r_0}^{1} |f| s^{\frac{7}{2}} ds + \sup_{r\geq 1} r^4 |f|. \end{aligned} \tag{2.22}$$

Then (2.13) is obtained by combining (2.19), (2.20), (2.21) and (2.22). □

We construct outer solutions of the self-similar equation in the following.

Proposition 2.3. *Let $0 < r_0 \ll 1$. For any $0 < \varepsilon \ll r_0^{\frac{1}{2}}$, there exists a radial solution to*

$$\Delta \Phi - \frac{1}{2} \Lambda \Phi + 6\Phi^2 + y \cdot \nabla(\Phi^2) = 0, \quad on \ \ [r_0, +\infty) \tag{2.23}$$

with the form

$$\Phi = \Phi_* + \varepsilon u_1 + \varepsilon w,$$

with

$$||w||_{X_{r_0}} \lesssim \varepsilon r_0^{-\frac{1}{2}}, \quad w|_{\varepsilon=0} = 0, \ ||\partial_\varepsilon w||_{X_{r_0}} \lesssim r_0^{-\frac{1}{2}}. \tag{2.24}$$

Proof. Step 1. Fixed point argument. Let $\Phi = \Phi_* + \varepsilon v$ satisfy (2.23) for $r \geq r_0$. Then

$$L(v) = \varepsilon (y \cdot \nabla(v^2) + 6v^2).$$

We set $v = u_1 + w$. Since $L(u_1) = 0$, then w satisfies

$$L(w) = \varepsilon (y \cdot \nabla(u_1 + w)^2 + 6(u_1 + w)^2), \quad \forall\, r \geq r_0.$$

Next, we find the solution of

$$w = \varepsilon\tau(G[u_1]w), \tag{2.25}$$

where $\tau(f)$ is defined in (2.12) and

$$G[u_1]w = r\partial_r(u_1+w)^2 + 6(u_1+w)^2.$$

We claim the following estimates: if $||w_i||_{X_{r_0}} \lesssim 1,\ i = 1, 2$, then

$$\int_{r_0}^{1} |G[u_1]w_i| s^{\frac{7}{2}} ds + \sup_{r\geq 1} r^4 |G[u_1]w_i| \lesssim r_0^{-\frac{1}{2}}, \quad i = 1, 2, \tag{2.26}$$

and

$$\int_{r_0}^{1} |G[u_1]w_1 - G[u_1]w_2| s^{\frac{7}{2}} ds + \sup_{r\geq 1} r^4 |G[u_1]w_1 - G[u_1]w_2| \lesssim r_0^{-\frac{1}{2}} ||w_1 - w_2||_{X_{r_0}}. \tag{2.27}$$

If $\varepsilon r_0^{-\frac{1}{2}} \ll 1$, and (2.26)-(2.27) hold, by the continuity estimate on the resolvent (2.13) and the Banach fixed theorem, there exists a unique solution to (2.25) with $||w||_{X_{r_0}} \lesssim \varepsilon r_0^{-\frac{1}{2}}$. We know from (2.25) that $w|_{\varepsilon=0} = 0$ and $\partial_\varepsilon w = \tau(G[u_1]w)$. Then by (2.13) and (2.26), we get

$$||\partial_\varepsilon w|||_{X_{r_0}} = ||\tau(G[u_1]w)||_{X_{r_0}} \lesssim r_0^{-\frac{1}{2}}.$$

Step 2. Proof of estimates (2.26) and (2.27). By (2.11) and the definition of X_{r_0} in (2.9), for $w \in X_{r_0}$ and $r_0 \leq r \leq 1$, we have

$$|w(r)| + |u_1(r)| + |r\partial_r(w+u_1)| \lesssim r^{-\frac{5}{2}}, \tag{2.28}$$

while for $r \geq 1$,

$$|w(r)| + |u_1(r)| + |r\partial_r(w+u_1)| \lesssim r^{-2}. \tag{2.29}$$

Next, we prove (2.26). For $r_0 \leq r \leq 1$, by (2.28), we have

$$\int_{r_0}^{1} |G[u_1]w| s^{\frac{7}{2}} ds = \int_{r_0}^{1} \left(|s\partial_s(u_1+w)^2| + 6(u_1+w)^2\right) s^{\frac{7}{2}} ds \lesssim \int_{r_0}^{1} s^{-\frac{3}{2}} ds \lesssim r_0^{-\frac{1}{2}}. \tag{2.30}$$

For $r \geq 1$, by (2.29), we have $|G[u_1]w| = r\partial_r(u_1+w)^2 + 6(u_1+w)^2 \lesssim r^{-4}$, and hence

$$\sup_{r\geq 1} r^4 |G[u_1]w| \lesssim 1. \tag{2.31}$$

We conclude the proof of (2.26) by (2.30) and (2.31).

Next, we prove (2.27). For $w_i \in X_{r_0}$ $(i = 1, 2)$, we have

$$G[u_1]w_1 - G[u_1]w_2 = r\partial_r[(2u_1 + w_1 + w_2)(w_1 - w_2)] + 6(2u_1 + w_1 + w_2)(w_1 - w_2).$$

For $r \geq 1$, by (2.10) and the definition of X_{r_0} in (2.9), we get

$$|6(2u_1 + w_1 + w_2)(w_1 - w_2)| \lesssim |w_1 - w_2|, \tag{2.32}$$

and

$$(r\partial_r + 1)(2u_1 + w_1 + w_2) \lesssim 1. \tag{2.33}$$

By (2.33), we obtain

$$\begin{aligned}
&r\partial_r[(2u_1 + w_1 + w_2)(w_1 - w_2)] \\
&= [r\partial_r(2u_1 + w_1 + w_2)](w_1 - w_2) + [r\partial_r(w_1 - w_2)](2u_1 + w_1 + w_2) \\
&\lesssim |w_1 - w_2| + r\partial_r|w_1 - w_2|.
\end{aligned}$$

Then combining (2.32), we have

$$\begin{aligned}
|G[u_1]w_1 - G[u_1]w_2| &\leq |6(2u_1 + w_1 + w_2)(w_1 - w_2)| + |r\partial_r[(2u_1 + w_1 + w_2)(w_1 - w_2)]| \\
&\lesssim r\partial_r|w_1 - w_2| + |w_1 - w_2|,
\end{aligned}$$

and hence

$$\sup_{r\geq 1} r^4|G[u_1]w_1 - G[u_1]w_2| \lesssim ||w_1 - w_2||_{X_{r_0}}. \tag{2.34}$$

For $r_0 \leq r \leq 1$, we have

$$(r\partial_r + 1)|2u_1 + w_1 + w_2| \lesssim r^{-\frac{5}{2}},$$

and hence

$$\begin{aligned}
&r\partial_r[(2u_1 + w_1 + w_2)(w_1 - w_2)] \\
&= [r\partial_r(2u_1 + w_1 + w_2)](w_1 - w_2) + [r\partial_r(w_1 - w_2)](2u_1 + w_1 + w_2) \\
&\lesssim r^{-\frac{5}{2}}(|w_1 - w_2| + r|\partial_r(w_1 - w_2)|).
\end{aligned}$$

Then it follows that

$$
\begin{aligned}
&\int_{r_0}^{1} |G[u_1]w_1 - G[u_1]w_2| s^{\frac{7}{2}} ds \\
&\lesssim \int_{r_0}^{1} \{|s\partial_s[(2u_1 + w_1 + w_2)(w_1 - w_2)]| + 6|(2u_1 + w_1 + w_2)(w_1 - w_2)|\} s^{\frac{7}{2}} ds \\
&\lesssim \int_{r_0}^{1} \left\{ s^{-\frac{5}{2}} (s|\partial_s(w_1 - w_2)| + |w_1 - w_2|) \right\} s^{\frac{7}{2}} ds \\
&\lesssim \sup_{r_0 \le r \le 1} (r^{\frac{5}{2}}|w_1 - w_2| + r^{\frac{7}{2}}|\partial_r(w_1 - w_2)|) \int_{r_0}^{1} s^{-\frac{3}{2}} ds \lesssim r_0^{-\frac{1}{2}} ||w_1 - w_2||_{X_{r_0}}.
\end{aligned}
\tag{2.35}
$$

Combining (2.34) and (2.35), this concludes the proof of (2.27). □

2.3. *Interior profiles*

The purpose of this subsection is to construct a radial solution of (2.5) on $[0, r_0]$, where $0 < r_0 \ll 1$ is given in Proposition 2.3. We define

$$
\bar{Q}(r) = \frac{1}{2r^3} \int_0^r Q(s) s^2 ds. \tag{2.36}
$$

By (1.12), $\bar{Q}$ satisfies

$$
\begin{cases}
\partial_{rr} \bar{Q} + \dfrac{4}{r} \partial_r \bar{Q} + 6\bar{Q}^2 + r\partial_r(\bar{Q}^2) = 0, \\
\bar{Q}(0) = \dfrac{1}{6}, \quad \bar{Q}'(0) = 0.
\end{cases}
\tag{2.37}
$$

We define the linearized operators of (2.37) at Φ_* and $\bar{Q}$, respectively, by the following expressions:

$$
H_\infty := -\partial_{rr} - \frac{4}{r}\partial_r - 12\Phi_* - 2r\partial_r(\Phi_* \cdot), \ H := -\partial_{rr} - \frac{4}{r}\partial_r - 12\bar{Q} - 2r\partial_r(\bar{Q}\cdot). \tag{2.38}
$$

We define Y as the space of continuous functions on $[1, +\infty)$ such that the following norm is finite

$$
||w||_Y = \sup_{r \ge 1} (r^3|w| + r^4|\partial_r w|).
$$

Lemma 2.4. *The equation*

$$
H_\infty(\phi) = 0, \quad \textit{on } (0, +\infty),
$$

has two fundamental solutions

$$\phi_1 = \frac{\sin(\frac{\sqrt{7}}{2}\log(r))}{r^{\frac{5}{2}}}, \quad \phi_2 = \frac{\cos(\frac{\sqrt{7}}{2}\log(r))}{r^{\frac{5}{2}}}. \tag{2.39}$$

In addition, the inverse

$$\psi(f) = \phi_1 \int_r^{+\infty} f\phi_2 \frac{2s^6}{\sqrt{7}} ds - \phi_2 \int_r^{+\infty} f\phi_1 \frac{2s^6}{\sqrt{7}} ds \tag{2.40}$$

satisfies $H_\infty(\psi(f)) = f$ *and*

$$||\psi(f)||_Y \lesssim \sup_{r\geq 1} r^5 |f|. \tag{2.41}$$

Proof. Let $\phi = r^k$. Then by $\Phi_* = \frac{1}{r^2}$, we have

$$H_\infty(\phi) = -r^{k-2}(k^2 + 5k + 8).$$

Since the polynomial $k^2+5k+8=0$ has two complex roots $k = \frac{-5\pm\sqrt{7}i}{2}$, the equation $H_\infty(\phi) = 0$ admits two explicit fundamental solutions

$$\phi_1 = \frac{\sin(\frac{\sqrt{7}}{2}\log(r))}{r^{\frac{5}{2}}}, \quad \phi_2 = \frac{\cos(\frac{\sqrt{7}}{2}\log(r))}{r^{\frac{5}{2}}}, \tag{2.42}$$

and the corresponding Wronskian is given by $W(r) = \phi_1'\phi_2 - \phi_2'\phi_1 = \frac{\sqrt{7}}{2r^6}$. By the variation of constants, the solutions of equation $H_\infty(u) = f$ are given by

$$u = \left(a_{1,0} + \int_r^{+\infty} f\phi_2 \frac{2s^6}{\sqrt{7}} ds\right)\phi_1 + \left(a_{2,0} - \int_r^{+\infty} f\phi_1 \frac{2s^6}{\sqrt{7}} ds\right)\phi_2, \quad a_{1,0},\ a_{2,0} \in \mathbb{R}. \tag{2.43}$$

Hence

$$\psi(f) = \phi_1 \int_r^{+\infty} f\phi_2 \frac{2s^6}{\sqrt{7}} ds - \phi_2 \int_r^{+\infty} f\phi_1 \frac{2s^6}{\sqrt{7}} ds$$

satisfies $H_\infty(\psi(f)) = f$ by choosing $a_{1,0} = a_{2,0} = 0$ in (2.43). For $r \geq 1$, from (2.42), we have

$$\begin{aligned} r^3|\psi(f)| &= r^3 \left| \left(\int_r^{+\infty} f\phi_2 \frac{2s^6}{\sqrt{7}} ds\right)\phi_1 - \left(\int_r^{+\infty} f\phi_1 \frac{2s^6}{\sqrt{7}} ds\right)\phi_2 \right| \\ &\lesssim r^{\frac{1}{2}} \left(\int_r^{+\infty} |f| s^{\frac{7}{2}} ds\right) \lesssim \left(r^{\frac{1}{2}} \int_r^{+\infty} s^{-\frac{3}{2}} ds\right) \sup_{r\geq 1} r^5|f| \lesssim \sup_{r\geq 1} r^5 |f|, \end{aligned} \tag{2.44}$$

and

$$\begin{aligned} r^4|\partial_r\psi(f)| &= r^4\left|\left(\int_r^{+\infty} f\phi_2\frac{2s^6}{\sqrt{7}}ds\right)\partial_r\phi_1 - \left(\int_r^{+\infty} f\phi_1\frac{2s^6}{\sqrt{7}}ds\right)\partial_r\phi_2\right| \\ &\lesssim r^{\frac{1}{2}}\left(\int_r^{+\infty}|f|s^{\frac{7}{2}}ds\right) \lesssim \left(r^{\frac{1}{2}}\int_r^{+\infty}s^{-\frac{3}{2}}ds\right)\sup_{r\geq 1} r^5|f| \lesssim \sup_{r\geq 1} r^5|f|. \end{aligned} \tag{2.45}$$

We conclude the proof of (2.41) by (2.44) and (2.45). □

Lemma 2.5. *The asymptotic profile of $\bar{Q}$ as $r \to +\infty$ is*

$$\bar{Q}(r) = \Phi_* + \frac{c_5\sin(\frac{\sqrt{7}}{2}\log(r))}{r^{\frac{5}{2}}} + O\left(r^{-3}\right), \tag{2.46}$$

where $0 < c_5 \ll 1$ is a constant.

Proof. Assume that

$$\bar{Q} = \Phi_* + \varepsilon v \tag{2.47}$$

solves (2.37) on $[1,\infty)$. Then v satisfies $H_\infty(v) = \varepsilon(6v^2 + r\partial_r v^2)$. Let $v = \phi_1 + w$, by $H_\infty(\phi_1) = 0$, we have $H_\infty(w) = \varepsilon(6(\phi_1+w)^2 + r\partial_r(\phi_1+w)^2)$. We define

$$G[\phi_1](w) = 6(\phi_1+w)^2 + r\partial_r(\phi_1+w)^2.$$

Next, we look for the solution of

$$w = \varepsilon\psi(G[\phi_1](w)), \tag{2.48}$$

where $\psi(f)$ is defined in (2.40). We claim that, if $w \in Y$, then

$$\sup_{r\geq 1} r^5|G[\phi_1](w)| \lesssim 1, \tag{2.49}$$

and for $w_1, w_2 \in Y$, it holds that

$$\sup_{r\geq 1} r^5|G[\phi_1](w_1) - G[\phi_1](w_2)| \lesssim ||w_1 - w_2||_Y. \tag{2.50}$$

If the above claim holds, for $\varepsilon > 0$ small enough, by the resolvent estimate (2.41) and the Banach fixed point theorem, there exists a unique solution $w \in Y$ to (2.48) and hence we find a v for (2.47). Finally we get (2.46) by (2.47).

It remains to show estimates (2.49) and (2.50). By (2.39) and the definition of the space Y, for $r \geq 1$ and $w \in Y$, we have

$$
\begin{aligned}
r^5|G[\phi_1](w)| &= r^5\{6(\phi_1+w)^2 + r\partial_r(\phi_1+w)^2\}\\
&\lesssim r^5[(\phi_1+w+2r\partial_r(\phi_1+w))(\phi_1+w)]\\
&\lesssim r^5(r^{-5}+r^{-6}+r^{-\frac{11}{2}}) \lesssim 1.
\end{aligned}
$$

For $r \geq 1$ and $w_i \in Y$ $(i=1,2)$, by (2.39) and the definition of the space Y, we get

$$
|w_1+w_2+2\phi_1| \lesssim r^{-\frac{5}{2}}, \quad |r\partial_r(w_1+w_2+2\phi_1)| \lesssim r^{-\frac{5}{2}}.
$$

Hence we have

$$
\begin{aligned}
&|G[\phi_1](w_1) - G[\phi_1](w_2)|\\
&= |6(w_1+w_2+2\phi_1)(w_1-w_2) + r\partial_r[(w_1+w_2+2\phi_1)(w_1-w_2)]|\\
&\lesssim r^{-\frac{5}{2}}|w_1-w_2| + |r\partial_r(w_1+w_2+2\phi_1)||w_1-w_2| + |r\partial_r(w_1-w_2)||w_1+w_2+2\phi_1|\\
&\lesssim r^{-\frac{5}{2}}(|w_1-w_2| + |r\partial_r(w_1-w_2)|),
\end{aligned}
$$

and

$$
\begin{aligned}
r^5|G[\phi_1](w_1) - G[\phi_1](w_2)| &\lesssim r^5(r^{-\frac{5}{2}}|w_1-w_2| + r^{-\frac{5}{2}}|r\partial_r(w_1-w_2)|)\\
&= r^{-\frac{1}{2}}(r^3|w_1-w_2| + r^4|\partial_r(w_1-w_2)|)\\
&\leq ||w_1-w_2||_Y.
\end{aligned}
$$

This completes the proof of (2.49) and (2.50). □

Let $r_1 \gg 1$. We define Y_{r_1} as the space of continuous functions on $[0, r_1]$ in which the following norm is finite:

$$
||w||_{Y_{r_1}} = \sup_{0\leq r\leq r_1} (1+r)^{-\frac{1}{2}}(|w| + |r\partial_r w|). \tag{2.51}
$$

Lemma 2.6. *Let H be defined in (2.38). Then the following results hold.*
1. The basis of the fundamental solutions: There holds

$$
H(\Lambda\bar{Q}) = 0, \quad H(\rho) = 0
$$

with the following asymptotic behavior as $r \to +\infty$,

$$
\Lambda\bar{Q} = \frac{c_6\sin(\frac{\sqrt{7}}{2}\log(r))}{r^{\frac{5}{2}}} + O(r^{-3}), \quad \rho = \frac{c_7\cos(\frac{\sqrt{7}}{2}\log(r))}{r^{\frac{5}{2}}} + O(r^{-3}),
$$

where c_6 and c_7 are nonzero constants.
2. The continuity of the resolvent: The inverse

$$S(f)=\left(\int_0^r f\Lambda\bar{Q}\exp\left(\int 2s\bar{Q}(s)ds\right)s^4ds\right)\rho-\left(\int_0^r f\rho\exp\left(\int 2s\bar{Q}(s)ds\right)s^4ds\right)\Lambda\bar{Q},$$

satisfies $H(S(f))=f$ *and*

$$||S(f)||_{Y_{r_1}}\lesssim\sup_{0\le r\le r_1}(1+r)^2|f|. \tag{2.52}$$

Proof. Step 1. Fundamental solutions. Let

$$\bar{Q}_\lambda(r)=\lambda^2\bar{Q}(\lambda r),\quad \lambda>0.$$

Then

$$\partial_{rr}\bar{Q}_\lambda+\frac{4}{r}\partial_r\bar{Q}_\lambda+6\bar{Q}_\lambda^2+r\partial_r(\bar{Q}_\lambda^2)=0,\quad \lambda>0.$$

Differentiating the above equation with λ and evaluating at $\lambda=1$ yields $H(\Lambda\bar{Q})=0$. Let ρ be another solution to $H(\rho)=0$ which is linearly independent of $\Lambda\bar{Q}$. We claim that, all solutions of $H(\phi)=0$ admit an expansion

$$\phi=a_{1,0}\phi_1+a_{2,0}\phi_2+O(r^{-3}),\quad \text{as}\ r\to+\infty, \tag{2.53}$$

where $a_{1,0},a_{2,0}\in\mathbb{R}$ and ϕ_1,ϕ_2 are defined in (2.39).

We rewrite $H(\phi)=0$ in the following form

$$H_\infty(\phi)=-\partial_{rr}\phi-\frac{4}{r}\partial_r\phi-12\Phi_*\phi-2r\partial_r(\Phi_*\phi)=f, \tag{2.54}$$

where

$$f=f(\phi)=12(\bar{Q}-\Phi_*)\phi+2r\partial_r((\bar{Q}-\Phi_*)\phi).$$

Next, we look for the solution of equation (2.54). By (2.43), we shall find a solution in a form

$$\phi=a_{1,0}\phi_1+a_{2,0}\phi_2+\widetilde{\phi}, \tag{2.55}$$

where

$$\widetilde{\phi}=F(\widetilde{\phi})=\left(\int_r^{+\infty}f(\phi)\phi_2\frac{2s^6}{\sqrt{7}}ds\right)\phi_1-\left(\int_r^{+\infty}f(\phi)\phi_1\frac{2s^6}{\sqrt{7}}ds\right)\phi_2:=F_1(\widetilde{\phi})-F_2(\widetilde{\phi}).$$

It follows from (2.39) that

$$|r\partial_r(\phi_1+\phi_2)|\lesssim r^{-\frac{5}{2}}. \tag{2.56}$$

Recall from (2.46) that

$$|\bar{Q}-\Phi_*| \lesssim r^{-\frac{5}{2}}, \quad |r\partial_r(\bar{Q}-\Phi_*)| \lesssim r^{-\frac{5}{2}}, \quad \text{for } r \geq 1. \tag{2.57}$$

For $r \geq 1$, by (2.56) and (2.57), we have

$$\begin{aligned}
F_1(\widetilde{\phi}) \lesssim & \left(\int_r^{+\infty} 12|\bar{Q}-\Phi_*||a_{1,0}\phi_1 + a_{2,0}\phi_2 + \widetilde{\phi}| \frac{2s^6|\phi_2|}{\sqrt{7}} ds \right) |\phi_1| \\
& + \left(\int_r^{+\infty} 2|r\partial_r(\bar{Q}-\Phi_*)||a_{1,0}\phi_1 + a_{2,0}\phi_2 + \widetilde{\phi}| \frac{2s^6|\phi_2|}{\sqrt{7}} ds \right) |\phi_1| \\
& + \left(\int_r^{+\infty} 2|\bar{Q}-\Phi_*||r\partial_r(a_{1,0}\phi_1 + a_{2,0}\phi_2 + \widetilde{\phi})| \frac{2s^6|\phi_2|}{\sqrt{7}} ds \right) |\phi_1| \\
\lesssim & \, r^{-\frac{5}{2}} \left(\int_r^{+\infty} s^{-\frac{3}{2}} + s|\widetilde{\phi}| ds \right) + r^{-\frac{5}{2}} \left(\int_r^{+\infty} s|r\partial_r\widetilde{\phi}| ds \right) \\
\leq & \, r^{-3} + r^{-\frac{5}{2}} \left(\int_r^{+\infty} s(|\widetilde{\phi}| + |r\partial_r\widetilde{\phi}|) ds \right).
\end{aligned}$$

Similarly,

$$F_2(\widetilde{\phi}) \lesssim r^{-3} + r^{-\frac{5}{2}} \left(\int_r^{+\infty} s(|\widetilde{\phi}| + |r\partial_r\widetilde{\phi}|) ds \right).$$

Hence

$$F(\widetilde{\phi}) \lesssim r^{-3} + r^{-\frac{5}{2}} \left(\int_r^{+\infty} s(|\widetilde{\phi}| + |r\partial_r\widetilde{\phi}|) ds \right) \tag{2.58}$$

and

$$F(\widetilde{\phi}_1) - F(\widetilde{\phi}_2) \lesssim r^{-\frac{5}{2}} \left(\int_r^{+\infty} s(|\widetilde{\phi}_1 - \widetilde{\phi}_2| + |r\partial_r(\widetilde{\phi}_1 - \widetilde{\phi}_2)|) ds \right). \tag{2.59}$$

In the same manner, we have

$$r\partial_r F(\widetilde{\phi}) \lesssim r^{-3} + r^{-\frac{5}{2}} \left(\int_r^{+\infty} s(|\widetilde{\phi}| + |r\partial_r\widetilde{\phi}|) ds \right), \tag{2.60}$$

and

$$r\partial_r(F(\widetilde{\phi}_1)-F(\widetilde{\phi}_2))\leq r^{-\frac{5}{2}}\left(\int_r^{+\infty} s(|\widetilde{\phi}_1-\widetilde{\phi}_2|+|r\partial_r(\widetilde{\phi}_1-\widetilde{\phi}_2)|)ds\right). \tag{2.61}$$

For $R\gg 1$, we define Z as the space of continuous functions on $[R,+\infty)$ such that the following norm is finite

$$||\phi||_Z=\sup_{r\geq R} r^3(|\phi|+|r\partial_r\phi|).$$

By (2.58)-(2.61) and the Banach fixed point theorem, there exists a unique solution $\widetilde{\phi}$ that satisfies $F(\widetilde{\phi})=\widetilde{\phi}$ with the bound $||\widetilde{\phi}||_Z\lesssim 1$, and hence we find a solution ϕ in the form (2.55) that solves (2.54). This proves (2.53).

Since $H(\Lambda\bar{Q})=H(\rho)=0$, by (2.39), (2.46) and (2.53), we have

$$\Lambda\bar{Q}=\frac{c_6\sin(\frac{\sqrt{5}}{2}\log(r))}{r^{\frac{5}{2}}}+O(r^{-3}),\quad \rho=\frac{c_7\cos(\frac{\sqrt{5}}{2}\log(r))}{r^{\frac{5}{2}}}+O(r^{-3}),\quad r\to\infty, \tag{2.62}$$

where $c_6, c_7\neq 0$.

Step 2. The estimate of the resolvent. We compute the Wronskian

$$W=\Lambda\bar{Q}'\rho-\Lambda\bar{Q}\rho',\ W'=-\left(\frac{4}{r}+2r\bar{Q}\right)W,\ W=\frac{\exp(-\int 2r\bar{Q}dr)}{r^4}.$$

Take $R_0>0$ small enough. By the definition of W, we have $\frac{W}{(\Lambda\bar{Q})^2}=-\frac{d}{dr}\left(\frac{\rho}{\Lambda\bar{Q}}\right)$, then integrating over $[r, R_0]$ yields

$$\rho(r)=\Lambda\bar{Q}(r)\int_r^{R_0}\frac{\exp(-\int 2s\bar{Q}ds)}{s^4(\Lambda\bar{Q})^2}ds+\frac{\Lambda\bar{Q}(r)\rho(R_0)}{\Lambda\bar{Q}(R_0)}. \tag{2.63}$$

By $\bar{Q}(0)=\frac{1}{6}$ and $\bar{Q}'(0)=0$, we have

$$|\bar{Q}|+|r\partial_r\bar{Q}|\lesssim 1,\ r\in[0,1]. \tag{2.64}$$

Then by (2.63), one has

$$|\rho(r)|\lesssim\frac{1}{r^3},\ |\partial_r\rho(r)|\lesssim\frac{1}{r^4},\quad \text{as } r\to 0. \tag{2.65}$$

If $H(w)=f$, then by the variation of constants, one obtains

$$w=\left(a_3+\int_0^r\frac{f\Lambda\bar{Q}}{W}\right)\rho+\left(a_4-\int_0^r\frac{f\rho}{W}\right)\Lambda\bar{Q},\ \ a_3,\ a_4\in\mathbb{R}. \tag{2.66}$$

Hence,

$$S(f) = \rho \int_0^r \frac{f \Lambda \bar{Q}}{W} ds - \Lambda \bar{Q} \int_0^r \frac{f\rho}{W} ds$$

satisfies $H(S(f)) = f$ by choosing $a_3 = a_4 = 0$ in (2.66). For $0 \le r \le 1$, by (2.64) and (2.65), we get the estimate

$$\begin{aligned}
&(1+r)^{-\frac{1}{2}}|S(f)| \\
&= (1+r)^{-\frac{1}{2}} \left| \left(\int_0^r f \Lambda \bar{Q} \exp\left(\int 2s\bar{Q} ds \right) s^4 ds \right) \rho - \left(\int_0^r f\rho \exp\left(\int 2s\bar{Q} ds \right) s^4 ds \right) \Lambda \bar{Q} \right| \\
&\lesssim \left(\frac{1}{r^3} \int_0^r s^4 ds + \int_0^r s ds \right) \sup_{0 \le r \le 1} |f| \lesssim \sup_{0 \le r \le r_1} (1+r)^2 |f|.
\end{aligned} \tag{2.67}$$

For $1 \le r \le r_1$, we know from (2.46) that

$$|\bar{Q}(r)| \lesssim \frac{1}{r^2}, \quad \exp\left(\int 2s\bar{Q}(s) ds \right) \lesssim r^2.$$

Then combining (2.62) and (2.67), we get

$$\begin{aligned}
&(1+r)^{-\frac{1}{2}}|S(f)| \\
&\lesssim (1+r)^{-\frac{1}{2}} \left| \left(\int_0^1 f\rho \exp\left(\int 2s\bar{Q} ds \right) s^4 ds \right) \Lambda \bar{Q} - \left(\int_0^1 f \Lambda \bar{Q} \exp\left(\int 2s\bar{Q} ds \right) s^4 ds \right) \rho \right| \\
&\quad + (1+r)^{-\frac{1}{2}} \left| \left(\int_1^r f\rho \exp\left(\int 2s\bar{Q} ds \right) s^4 ds \right) \Lambda \bar{Q} - \left(\int_1^r f \Lambda \bar{Q} \exp\left(\int 2s\bar{Q} ds \right) s^4 ds \right) \rho \right| \\
&\lesssim \sup_{0 \le r \le r_1} (1+r)^2 |f| + r^{-3} \int_1^r |f| s^{\frac{7}{2}} ds \lesssim \sup_{0 \le r \le r_1} (1+r)^2 |f|.
\end{aligned} \tag{2.68}$$

Similarly, for $0 \le r \le r_1$, we also have

$$(1+r)^{-\frac{1}{2}} |r\partial_r S(f)| \lesssim \sup_{0 \le r \le r_1} (1+r)^2 |f|. \tag{2.69}$$

We finally get (2.52) by (2.67), (2.68), and (2.69). $\square$

We are now in the position to construct interior solutions for the equation (2.5).

Proposition 2.7. *Let* $0 < r_0 \ll 1$ *and* $0 < \lambda \leq r_0$. *There exists a radial solution* u *to*

$$\Delta\Phi - \frac{1}{2}\Lambda\Phi + 6\Phi^2 + y \cdot \nabla(\Phi^2) = 0, \quad 0 \leq r \leq r_0, \tag{2.70}$$

with the form

$$\Phi = \frac{1}{\lambda^2}(\bar{Q} + \lambda^4 Q_1)\left(\frac{r}{\lambda}\right)$$

with $||Q_1||_{Y_{\frac{r_0}{\lambda}}} \lesssim 1$.

Proof. **Step 1.** Application of the Banach fixed-point theorem. We look for Φ of the form

$$\Phi = \frac{1}{\lambda^2}(\bar{Q} + \lambda^4 Q_1)\left(\frac{r}{\lambda}\right),$$

so that Φ solves (2.70) on $[0, r_0]$. Then,

$$H(Q_1) = J[\bar{Q}, \lambda]Q_1, \quad 0 \leq r \leq r_1, \tag{2.71}$$

where $r_1 = \frac{r_0}{\lambda} \geq 1$ such that $\lambda^2 r_1^2 = r_0^2 \ll 1$, and

$$J[\bar{Q}, \lambda]Q_1 = -\frac{1}{2\lambda^2}\Lambda\bar{Q} - \frac{1}{2}\lambda^2\Lambda Q_1 + \lambda^4(6Q_1^2 + r\partial_r(Q_1^2)).$$

For $w \in Y_{r_1}$, we claim the following estimates:

$$\sup_{0 \leq r \leq r_1} (1+r)^2 |J[\bar{Q}, \lambda]w| \lesssim 1, \tag{2.72}$$

and

$$\sup_{0 \leq r \leq r_1} (1+r)^2 |J[\bar{Q}, \lambda]w_1 - J[\bar{Q}, \lambda]w_2| \lesssim \lambda^2 r_1^2 ||w_1 - w_2||_{Y_{r_1}}. \tag{2.73}$$

If (2.72) and (2.73) hold, by $\lambda^2 r_1^2 \ll 1$, the resolvent estimate (2.52), and the Banach fixed point theorem, there exists a unique solution Q_1 of (2.71) with $||Q_1||_{Y_{\frac{r_0}{\lambda}}} \lesssim 1$.

Step 2. Proof of estimates (2.72) and (2.73). For $0 \leq r \leq r_1$ and $w \in Y_{r_1}$, by the definition of the space Y_{r_1} in (2.51), we have $|\Lambda w| \lesssim 1$. Then, by $|\Lambda\bar{Q}| \lesssim 1$, we get

$$(1+r)^2 |J[\bar{Q}, \lambda]w| \lesssim 1, \text{ on } [0, r_1],$$

which concludes the proof of (2.72).

For $0 \leq r \leq r_1$ and $w_1, w_2 \in Y_{r_1}$, we have

$$|\Lambda(w_1 - w_2)| \lesssim ||w_1 - w_2||_{Y_{r_1}}, \; |w_1 + w_2| \lesssim r\partial_r(w_1 + w_2) \lesssim 1.$$

Then it follows that

$$r\partial_r[(w_1+w_2)(w_1-w_2)] = (w_1-w_2)r\partial_r(w_1+w_2)+(w_1+w_2)r\partial_r(w_1-w_1)$$
$$\lesssim |w_1-w_2|+|r\partial_r(w_1-w_2)| \le ||w_1-w_2||_{Y_{r_1}}.$$

Hence,

$$(1+r)^2|J[\bar{Q},\lambda]w_1 - J[\bar{Q},\lambda]w_2| \lesssim \lambda^2(1+r)^2|\Lambda(w_1-w_2)| + \lambda^4(1+r)^2(w_1+w_2)(w_1-w_2)$$
$$+\lambda^4(1+r)^2 r\partial_r[(w_1+w_2)(w_1-w_1)]$$
$$\lesssim \lambda^2(1+r)^2||w_1-w_2||_{Y_{r_1}} \lesssim \lambda^2 r_1^2||w_1-w_2||_{Y_{r_1}},$$

which concludes the proof of (2.73). □

2.4. *The matching at $r = r_0$*

In this subsection, we prove Proposition 2.1 by matching the value of the exterior solution and interior solution at $r = r_0$ up to the first-order derivative.

Proof of Proposition 2.1. The proof is divided into six steps.
Step 1. Initial setting. From (2.11), we have

$$u_1 = \frac{c_1\sin(\frac{\sqrt{7}}{2}\log(r)+c_3)}{r^{\frac{5}{2}}} + O(r^{-\frac{1}{2}}) \text{ as } r\to 0,\ c_1\neq 0$$

then

$$\Lambda u_1 = c_1\frac{-\frac{1}{2}\sin(\frac{\sqrt{7}}{2}\log(r)+c_3)+\frac{\sqrt{7}}{2}\cos(\frac{\sqrt{7}}{2}\log(r)+c_3)}{r^{\frac{5}{2}}} + O(r^{-\frac{1}{2}}) \text{ as } r\to 0.$$

We choose $0<r_0\ll 1$ such that

$$u_1(r_0) = \frac{c_1}{r_0^{\frac{5}{2}}} + O(r_0^{-\frac{1}{2}}),\quad \Lambda u_1(r_0) = -\frac{c_1}{2r_0^{\frac{5}{2}}} + O(r_0^{-\frac{1}{2}}). \tag{2.74}$$

Then, we choose ε and λ satisfying

$$0<\varepsilon \ll r_0^{\frac{1}{2}},\quad 0<\lambda\le r_0. \tag{2.75}$$

By Proposition 2.3, there exists an radial exterior solution $\Phi_{\text{ext}}[\varepsilon]$ satisfying

$$\Delta\Phi_{\text{ext}} - \frac{1}{2}\Lambda\Phi_{\text{ext}} + 6\Phi_{\text{ext}}^2 + y\cdot\nabla(\Phi_{\text{ext}}^2) = 0,\quad r\ge r_0$$

with the form

$$\Phi_{\text{ext}}[\varepsilon] = \Phi_* + \varepsilon u_1 + \varepsilon w \tag{2.76}$$

and

$$||w||_{X_{r_0}} \lesssim \varepsilon r_0^{-\frac{1}{2}}. \tag{2.77}$$

By Proposition 2.7, there exists an radial interior solution $\Phi_{\text{int}}[\lambda]$ satisfying

$$\Delta\Phi_{\text{int}} - \frac{1}{2}\Lambda\Phi_{\text{int}} + 6\Phi_{\text{int}}^2 + y\cdot\nabla(\Phi_{\text{int}}^2) = 0, \quad 0 \le r \le r_0$$

with the form

$$\Phi_{\text{int}}[\lambda](r) = \frac{1}{\lambda^2}(\bar{Q} + \lambda^4 Q_1)\left(\frac{r}{\lambda}\right), \tag{2.78}$$

with

$$||Q_1||_{Y_{\frac{r_0}{\lambda}}} \lesssim 1. \tag{2.79}$$

Next, we need to match the values of Φ_{ext} with Φ_{int}, and Φ'_{ext} with Φ'_{int} respectively at $r = r_0$, that is,

$$\Phi_{\text{ext}}[\varepsilon](r_0) = \Phi_{\text{int}}[\lambda](r_0), \quad \Phi'_{\text{ext}}[\varepsilon](r_0) = \Phi'_{\text{int}}[\lambda](r_0).$$

Step 2. The matching of Φ_{ext} with Φ_{int} at $r = r_0$. We introduce the map

$$F[r_0](\varepsilon, \lambda) = \Phi_{\text{ext}}[\varepsilon](r_0) - \Phi_{\text{int}}[\lambda](r_0).$$

We compute

$$\partial_\varepsilon F[r_0](\varepsilon, \lambda) = \partial_\varepsilon \Phi_{\text{ext}}[\varepsilon](r_0) = u_1(r_0) + w(r_0) + \varepsilon\partial_\varepsilon w(r_0).$$

By (2.24) and (2.74), we have

$$\partial_\varepsilon F[r_0](0,0) = u_1(r_0) \neq 0. \tag{2.80}$$

For $\lambda \to 0_+$, from the asymptotic behavior of $\bar{Q}$ in (2.46) and the definition of the space Y_{r_1} in (2.51), combining (2.79), we have

$$\left|\frac{1}{\lambda^2}(\bar{Q} - \Phi_* + \lambda^4 Q_1)\left(\frac{r_0}{\lambda}\right)\right| \lesssim \left|\frac{1}{\lambda^2}\left(r^{-\frac{5}{2}} + \lambda^4(1+r)^{\frac{1}{2}}\right)\left(\frac{r_0}{\lambda}\right)\right| = \lambda^{\frac{1}{2}}\left[r_0^{-\frac{5}{2}} + \lambda(\lambda + r_0)^{\frac{1}{2}}\right].$$

Hence

$$\lim_{\lambda\to 0_+}\left|\frac{1}{\lambda^2}(\bar{Q} - \Phi_* + \lambda^4 Q_1)\left(\frac{r_0}{\lambda}\right)\right| = 0.$$

Combining $\Phi_*(r) = \frac{1}{\lambda^2}\Phi_*(\frac{r}{\lambda})$, we have

$$F[r_0](0,0) = \Phi_*(r_0) - \Phi_*(r_0) = 0. \tag{2.81}$$

Combining (2.80) and (2.81), by the implicit function theorem, there exists $0 < \lambda_0 \le r_0$ and a continuous function $\varepsilon(\lambda)$ defined on $[0,\lambda_0)$ such that $\varepsilon(0) = 0$ and

$$F[r_0](\varepsilon(\lambda),\lambda) = 0 \quad \text{for } \lambda \in [0,\lambda_0), \tag{2.82}$$

i.e.,

$$\Phi_{\text{ext}}[\varepsilon(\lambda)](r_0) = \Phi_{\text{int}}[\lambda](r_0) \quad \text{for } \lambda \in [0,\lambda_0).$$

Step 3. Estimate of $\varepsilon(\lambda)$. We claim that for $\lambda \in [0,\lambda_0)$, there holds that

$$\varepsilon(\lambda) = \frac{1}{u_1(r_0)\lambda^2}(\bar{Q} - \Phi_*)\left(\frac{r_0}{\lambda}\right) + O(\lambda(\lambda^{\frac12} r_0^3 + r_0^{-\frac12})). \tag{2.83}$$

In fact, since

$$\Phi_{\text{ext}}[\varepsilon(\lambda)](r_0) = \Phi_{\text{int}}[\lambda](r_0) \quad \text{for } \lambda \in [0,\lambda_0),$$

i.e.,

$$\varepsilon(\lambda)u_1(r_0) + \varepsilon(\lambda)w(r_0) = \frac{1}{\lambda^2}(\bar{Q} - \Phi_* + \lambda^4 Q_1)\left(\frac{r_0}{\lambda}\right), \quad \text{for } \lambda \in [0,\lambda_0).$$

By (2.75), we know that

$$|\varepsilon(\lambda)| \lesssim \lambda^{\frac12}. \tag{2.84}$$

Then by (2.11), (2.77) and (2.79), we have

$$\begin{aligned}\varepsilon(\lambda) &= \frac{1}{\lambda^2 u_1(r_0)}(\bar{Q} - \Phi_* + \lambda^4 Q_1)\left(\frac{r_0}{\lambda}\right) - \frac{\varepsilon(\lambda)w(r_0)}{u_1(r_0)} \\ &= \frac{1}{\lambda^2 u_1(r_0)}(\bar{Q} - \Phi_*)\left(\frac{r_0}{\lambda}\right) + O(\lambda(\lambda^{\frac12} r_0^3 + r_0^{-\frac12})),\end{aligned}$$

which proves our claim.

Step 4. Computation of the spatial derivatives. We consider the difference of the spatial derivatives at r_0

$$\mathcal{F}[r_0](\lambda) = \Phi_{\text{ext}}[\varepsilon(\lambda)]'(r_0) - \Phi_{\text{int}}[\lambda]'(r_0), \quad \lambda \in [0,\lambda_0).$$

We claim that $\mathcal{F}[r_0](\lambda)$ admits the following expansion

$$\mathcal{F}[r_0](\lambda) = \lambda^{\frac12}\left\{\frac{c_1 c_5\sqrt{7}}{2u_1(r_0)r_0^6}\sin\left(-\frac{\sqrt{7}}{2}\log\lambda + c_3\right) + O\left(\lambda^{\frac12} r_0^{-\frac12}\left(r_0^{-\frac72} + \lambda^{\frac32}\right)\right)\right\}. \tag{2.85}$$

From (2.77) and (2.84), it follows that

$$|\varepsilon(\lambda)w'(r_0)| \lesssim \lambda^{\frac{1}{2}}|w'(r_0)| \lesssim \lambda r_0^{-4}.$$

From (2.79), we get $\lambda^2|Q_1'(\frac{r_0}{\lambda})| \lesssim \lambda^{\frac{5}{2}} r_0^{-\frac{1}{2}}$. By (2.83), we have

$$\begin{aligned}
\mathcal{F}[r_0](\lambda) &= \varepsilon(\lambda)u_1'(r_0) - \frac{1}{\lambda^3}(\bar{Q}' - \Phi_*')\left(\frac{r_0}{\lambda}\right) + O\left(\lambda\left(r_0^{-4} + \lambda^{\frac{3}{2}} r_0^{-\frac{1}{2}}\right)\right) \\
&= \frac{1}{u_1(r_0)\lambda^2}(\bar{Q} - \Phi_*)\left(\frac{r_0}{\lambda}\right)u_1'(r_0) - \frac{1}{\lambda^3}(\bar{Q}' - \Phi_*')\left(\frac{r_0}{\lambda}\right) + O\left(\lambda\left(r_0^{-4} + \lambda^{\frac{3}{2}} r_0^{-\frac{1}{2}}\right)\right) \\
&= \frac{\lambda^{\frac{1}{2}}}{u_1(r_0)r_0^{\frac{5}{2}}}\left\{\left(\frac{r_0}{\lambda}\right)^{\frac{5}{2}}(\bar{Q} - \Phi_*)\left(\frac{r_0}{\lambda}\right)u_1'(r_0) - \left(\frac{r_0}{\lambda}\right)^{\frac{7}{2}}(\bar{Q}' - \Phi_*')\left(\frac{r_0}{\lambda}\right)\frac{u_1(r_0)}{r_0}\right\} \\
&\quad + O\left(\lambda\left(r_0^{-4} + \lambda^{\frac{3}{2}} r_0^{-\frac{1}{2}}\right)\right).
\end{aligned} \tag{2.86}$$

Recalling (2.11) and (2.46), by simple calculations, one has

$$u_1(r) = \frac{c_1\sin(\frac{\sqrt{7}}{2}\log(r) + c_3)}{r^{\frac{5}{2}}} + O(r^{-\frac{1}{2}}) \ \text{ as } r \to 0,$$

$$u_1'(r) = \frac{-5c_1\sin(\frac{\sqrt{7}}{2}\log(r) + c_3)}{2r^{\frac{7}{2}}} + \frac{\sqrt{7}c_1\cos(\frac{\sqrt{7}}{2}\log(r) + c_3)}{2r^{\frac{7}{2}}} + O(r^{-\frac{3}{2}}) \ \text{ as } r \to 0,$$

$$\bar{Q}(r) - \Phi_*(r) = \frac{c_5\sin(\frac{\sqrt{7}}{2}\log(r))}{r^{\frac{5}{2}}} + O(r^{-3}) \ \text{ as } r \to +\infty,$$

$$\bar{Q}'(r) - \Phi_*'(r) = \frac{-5c_5\sin(\frac{\sqrt{7}}{2}\log(r)}{2r^{\frac{7}{2}}} + \frac{\sqrt{7}c_5\cos(\frac{\sqrt{7}}{2}\log(r))}{2r^{\frac{7}{2}}} + O(r^{-4}) \ \text{ as } r \to +\infty.$$

Then it follows from the above results that

$$\begin{aligned}
&\left(\frac{r_0}{\lambda}\right)^{\frac{5}{2}}(\bar{Q} - \Phi_*)\left(\frac{r_0}{\lambda}\right)u_1'(r_0) - \left(\frac{r_0}{\lambda}\right)^{\frac{7}{2}}(\bar{Q}' - \Phi_*')\left(\frac{r_0}{\lambda}\right)\frac{u_1(r_0)}{r_0} \\
&= \frac{c_1c_5}{r_0^{\frac{7}{2}}}\sin\left(\frac{\sqrt{7}}{2}(\log r_0 - \log\lambda)\right) \times \left(\frac{\sqrt{7}}{2}\cos\left(\frac{\sqrt{7}}{2}\log r_0 + c_3\right) - \frac{5}{2}\sin\left(\frac{\sqrt{7}}{2}\log r_0 + c_5\right)\right) \\
&\quad - \frac{c_1c_5}{r_0^{\frac{7}{2}}}\left(\frac{\sqrt{7}}{2}\cos\left(\frac{\sqrt{7}}{2}(\log r_0 - \log\lambda)\right) - \frac{5}{2}\sin\left(\frac{\sqrt{7}}{2}(\log r_0 - \log\lambda)\right)\right) \\
&\quad \times \sin\left(\frac{\sqrt{7}}{2}\log(r_0) + c_3\right) + O\left(\lambda^{\frac{1}{2}}\left(r_0^{-4} + \lambda^{\frac{3}{2}} r_0^{-\frac{1}{2}}\right)\right)
\end{aligned}$$

$$= \frac{c_1 c_5 \sqrt{7}}{2 r_0^{\frac{7}{2}}} \sin\left(-\frac{\sqrt{7}}{2}\log\lambda + c_3\right) + O\left(\lambda^{\frac{1}{2}}\left(r_0^{-4} + \lambda^{\frac{3}{2}} r_0^{-\frac{1}{2}}\right)\right).$$

Inserting the above identity into (2.86), we obtain (2.85). This proves our claim.
Step 5. The matching of Φ'_{ext} with Φ'_{int} at $r = r_0$. For $\delta_0 > 0$ small enough, we define

$$\lambda_{k,+} = \exp\left(\frac{2(-k\pi + c_3 - \delta_0)}{\sqrt{7}}\right), \quad \lambda_{k,-} = \exp\left(\frac{2(-k\pi + c_3 + \delta_0)}{\sqrt{7}}\right).$$

Since $\lim_{k\to+\infty} \lambda_{k,\pm} = 0$, we know that there exists $k_0 > 0$ such that for $k \geq k_0$, there holds

$$0 < \cdots < \lambda_{k,+} < \lambda_{k,-} < \cdots < \lambda_{k_0,+} < \lambda_{k_0,-} \leq \lambda_0.$$

For all $k \geq k_0$, we have

$$\sin\left(-\frac{\sqrt{7}}{2}\log\lambda_{k,+} + c_3\right) = (-1)^k \sin(\delta_0),$$
$$\sin\left(-\frac{\sqrt{7}}{2}\log\lambda_{k,-} + c_3\right) = (-1)^{k+1} \sin(\delta_0).$$

By (2.85), we obtain

$$\mathcal{F}[r_0](\lambda_{k,\pm}) = \lambda_{k,\pm}^{\frac{1}{2}} \left\{ \pm(-1)^k \frac{c_1 c_5 \sqrt{7}}{2u_1(r_0) r_0^6} \sin(\delta_0) + O\left(\lambda_{k,\pm}^{\frac{1}{2}}\left(r_0^{-4} + \lambda_{k,\pm}^{\frac{3}{2}} r_0^{-\frac{1}{2}}\right)\right)\right\}.$$

Since $\lim_{k\to+\infty} \lambda_{k,\pm} = 0$, and $\delta_0 > 0$ is small enough, there exists $k_1 \geq k_0$ such that, for any $k \geq k_1$, there holds

$$\mathcal{F}[r_0](\lambda_{k,+})\mathcal{F}[r_0](\lambda_{k,-}) < 0.$$

Due to that fact that the function $\lambda \to \mathcal{F}[r_0](\lambda)$ is continuous, then by the mean value theorem, for any $k \geq k_1$, there exists $\bar{\mu}_k$ such that

$$\mathcal{F}[r_0](\bar{\mu}_k) = 0, \quad \bar{\mu}_k \in (\lambda_{k,+}, \lambda_{k,-}).$$

Combining (2.82), since $0 < \bar{\mu}_k < \lambda_0$, we have $F[r_0](\varepsilon(\bar{\mu}_k), \mu_k) = 0$ and $\mathcal{F}[r_0](\bar{\mu}_k) = 0$, i.e.,

$$\Phi_{\mathrm{ext}}[\varepsilon(\bar{\mu}_k)](r_0) = \Phi_{\mathrm{int}}[\bar{\mu}_k](r_0), \quad \Phi_{\mathrm{ext}}[\varepsilon(\bar{\mu}_k)]'(r_0) = \Phi_{\mathrm{int}}[\bar{\mu}_k]'(r_0).$$

We define $\mu_n := \bar{\mu}_{k+n}$. For $k \geq k_1$ and $n \in \mathbb{N}$, the functions

$$\Phi_n(r) := \begin{cases} \Phi_{\mathrm{int}}[\mu_n](r) & \text{for } 0 \leq r \leq r_0, \\ \Phi_{\mathrm{ext}}[\varepsilon(\mu_n)](r) & \text{for } r > r_0 \end{cases}$$

are smooth radial solutions of (2.5).

Step 6. The asymptotic behavior. Recall from (2.76) that

$$\Phi_n = \Phi_* + \varepsilon(\mu_n)u_1(r) + \varepsilon(\mu_n)w(r), \quad r \geq r_0,$$

where $\lim_{n\to+\infty} \varepsilon(\mu_n) = 0$. By (2.10), (2.11), and (2.24), we have

$$\sup_{r_0\leq r\leq 1} r^{\frac{5}{2}}(r\partial_r + 1)(|u_1| + |w|) + \sup_{r\geq 1} r^2(r\partial_r + 1)(|u_1| + |w|) \lesssim 1.$$

Combining (2.9) and (2.11), we have

$$\begin{aligned}
&\sup_{r\geq r_0}(1+r^2)|(r\partial_r + 1)(\Phi_n - \Phi_*)| \\
&\lesssim \varepsilon(\mu_n)\left(\sup_{r\geq r_0}(r\partial_r + 1)(|u_1| + |w|) + \sup_{r\geq 1} r^2(r\partial_r + 1)(|u_1| + |w|)\right) \lesssim \varepsilon(\mu_n)r_0^{-\frac{5}{2}},
\end{aligned}$$

which implies

$$\lim_{n\to+\infty} \sup_{r\geq r_0}(1+r^2)|(r\partial_r + 1)(\Phi_n - \Phi_*)| = 0. \tag{2.87}$$

Thus, we complete the proof of (2.7).

For the interior part estimate, for $0 \leq r \leq r_0$, we know from (2.78) that

$$\Phi_n = \frac{1}{\mu_n^2}(\bar{Q} + \mu_n^4 Q_1)\left(\frac{r}{\mu_n}\right),$$

where

$$\sup_{0\leq r\leq \frac{r_0}{\mu_n}} (1+r)^{-\frac{1}{2}}(|Q_1| + |r\partial_r Q_1|) \lesssim 1.$$

For $r \leq r_0$, we have

$$(r\partial_r + 1)\left|\Phi_n - \frac{1}{\mu_n^2}\bar{Q}\left(\frac{r}{\mu_n}\right)\right| = \mu_n^2(r\partial_r + 1)\left|Q_1\left(\frac{r}{\mu_n}\right)\right| \lesssim \mu_n^2\left(1 + \frac{r}{\mu_n}\right)^{\frac{1}{2}} = \mu_n^{\frac{3}{2}}(\mu_n + r)^{\frac{1}{2}}.$$

Then by $\lim_{n\to+\infty} \mu_n = 0$, we get

$$\lim_{n\to+\infty} \sup_{r\leq r_0}(r\partial_r + 1)\left|\Phi_n - \frac{1}{\mu_n^2}\bar{Q}\left(\frac{r}{\mu_n}\right)\right| = 0, \tag{2.88}$$

which completes the proof of (2.6). □

3. Self-similar blow-up solutions

We now give the proof of Theorem 1.1 for $d = 3$. As mentioned previously, the proof for $d \in [4, 9]$ is directly extendable.

Proof of Theorem 1.1. Recall from Proposition 2.1 that Φ_n are smooth radially symmetric solutions to equation (2.5). By $\Phi_n = \frac{1}{2r^3}\int_0^r U_n(s)s^2 ds$, we have $6\Phi_n + 2r\partial_r\Phi_n = U_n$. It is clear that $\{U_n\}_{n\ge 1}$ are radially symmetric solutions of (1.5). By (2.87), we get

$$\lim_{n\to+\infty}\sup_{r\ge r_0}(1+r^2)\left|U_n - \frac{2}{r^2}\right| = 0.$$

We know from (2.88) that

$$\lim_{n\to+\infty}\sup_{r\le r_0}\left|U_n - \frac{1}{\mu_n^2}Q\left(\frac{r}{\mu_n}\right)\right| = 0.$$

This completes the proof of (1.13) and (1.14).

For any $0 < T < +\infty$, take $u_0 = T^{-1}U_n\left(T^{-\frac{1}{2}}x\right)$. Since $\{U_n\}_{n\ge 1}$ are self-similar profiles solving (1.5), the corresponding solution u blows up in finite time T with

$$u(x,t) = \frac{1}{T-t}U_n\left(\frac{x}{\sqrt{T-t}}\right). \tag{3.1}$$

Because the functions $\{U_n\}_{n\ge 1}$ are bounded, the blow-up is of type I.

We know from (2.76) that

$$U_n(y) \sim \frac{1}{|y|^2}, \quad \text{as } |y| \to +\infty. \tag{3.2}$$

Assume by contradiction that $B(u_0) \neq 0$. But for any $\delta > 0$ and $|x| \ge \delta$, we have

$$\lim_{t\to T}||u(x,t)||_{L^\infty(\mathbb{R}^3)} = \lim_{t\to T}\left\|\frac{1}{T-t}U_n\left(\frac{x}{\sqrt{T-t}}\right)\right\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{|x|^2} \le \frac{1}{\delta^2} < +\infty, \tag{3.3}$$

which contradicts the assumption $B(u_0) \neq 0$. Therefore, the blow-up point of the solution $u(x,t)$ must be the origin, i.e., $B(u_0) = 0$.

For any $\delta_1 > 0$, by (3.2), (3.3), parabolic regularity and the Arzelà-Ascoli theorem, there exists a function u^* such that

$$\lim_{t\to T} u(x,t) \to u^*, \quad \forall\, |x| \ge \delta_1,$$

where $|u^*(x)| \sim \frac{1}{|x|^2}$. For $p \in [1, \frac{3}{2})$, we get

$$\lim_{t\to T}||u(t) - u^*||^p_{L^p(\mathbb{R}^3)} = \lim_{t\to T}\int_0^{\delta_1}|u(r,t) - u^*(r)|^p r^2 dr \lesssim \int_0^{\delta_1} r^{2-2p}dr \to 0, \text{ as } \delta_1 \to 0,$$

and (1.15) is proved. This completes the proof of Theorem 1.1. □

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Data availability

No data was used for the research described in the article.

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