

Asymptotic and Viscous Stability of Large-Amplitude Solutions of a Hyperbolic System Arising from Biology

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ABSTRACT. In this paper, we study the qualitative behavior of the Cauchy problem of a hyperbolic model

$$\begin{cases} p_t - \nabla \cdot (p\mathbf{q}) = \Delta p, & \mathbf{x} \in \mathbb{R}^d, t > 0, \\ \mathbf{q}_t - \nabla \cdot (\varepsilon|\mathbf{q}|^2 + p) = \varepsilon\Delta\mathbf{q}, \end{cases}$$

which is transformed from a singular chemotaxis system describing the effect of a reinforced random walk in [17, 27]. When $d = 1$ and the initial data are prescribed around a constant ground state $(\bar{p}, 0)$ with $\bar{p} \geq 0$, we prove the global asymptotic stability of constant ground states, and identify the explicit decay rate of solutions under very mild conditions on initial data. Moreover, we study the diffusion (viscous) limit of solutions as $\varepsilon \rightarrow 0$ with convergence rates toward solutions of the non-diffusible (inviscid) problem. While the existence of global large solutions of the system in multi-dimensions remains an outstanding open question, we show that the model exhibits a strong parabolic smoothing effect: namely, solutions are spatially analytic for a short time provided that the initial data belong to $L^q(\mathbb{R}^d)$ for any $q > d \geq 1$. In fact, when $d = 1$, we obtain that the solution remains real analytic for all time.

1. INTRODUCTION

1.1. Background. Chemotaxis involves the cellular detection of a chemical concentration gradient and the subsequent movement up (attractive chemotaxis) or down (repulsive chemotaxis) the gradient. It is a common feature shared by many cells and micro-organisms such as the well-studied bacteria *Escherichia coli* and *Salmonella typhimurium*, the slime mold amoebae *Dictyostelium discoideum*, neutrophils, and so on. The prototype of chemotaxis models, due to Keller and Segel [13, 14], is a system of parabolic PDEs reading as

$$\begin{cases} u_t = \nabla \cdot (D \nabla u - \chi u \nabla \varphi(v)), \\ v_t = \varepsilon \Delta v + g(u, v), \end{cases}$$

where u and v denote the cell density and chemical concentration, respectively. $D > 0$ and $\varepsilon \geq 0$ are cell and chemical diffusion coefficients, respectively. The chemotaxis is called to be attractive if $\chi > 0$ and repulsive if $\chi < 0$ with $|\chi|$ measuring the strength of the chemical signal. The potential function $\varphi(v)$, also called the chemotactic sensitivity function, describes the signal detection mechanism, and $g(u, v)$ characterizes the chemical growth and degradation. Most of the studies on chemotaxis deal with the classical attractive chemotaxis model where $\chi > 0$, $\varphi(v) = v$, and $g(u, v) = u - v$ (see [11]). By contrast, the studies of repulsive chemotaxis were much less. A few results on repulsive chemotaxis have been developed recently (see [2, 30] and the references therein). In this paper, we consider a chemotaxis model with logarithmic sensitivity,

$$(1.1) \quad \begin{cases} u_t = D \Delta u - \nabla \cdot (\chi u \nabla \ln v), \\ v_t = \varepsilon \Delta v + uv - \mu v, \end{cases}$$

which was derived in [17, 27] to model the reinforced random walk. The logarithmic sensitivity $\varphi(v) = \ln v$ indicates that the cell chemotactic response to the chemical signal follows the Weber-Fechner law which has prominent applications in biological modelings (cf. [3, 15]). The term uv entails that the chemical grows exponentially with the rate depending on the cell density. Moreover, $\mu > 0$ is a natural degradation rate of the chemical.

In this paper, we consider the repulsive ($\chi < 0$) case for (1.1). To overcome the possible singularity caused by $\ln v$, by using a Cole-Hopf type transformation (see [17])

$$(1.2) \quad \mathbf{q} = \frac{\nabla v}{v} = \nabla \ln v,$$

and scalings $\tilde{t} = -\chi t/D$, $\tilde{\mathbf{x}} = \mathbf{x}\sqrt{-\chi}/D$, $\tilde{\mathbf{q}} = \mathbf{q}\sqrt{-\chi}$, $p = u$, one can remove the logarithmic singularity and transform (1.1) into a new system of hyperbolic type

PDEs with viscosity

$$(1.3) \quad \begin{cases} p_t - \nabla \cdot (p\mathbf{q}) = \Delta p, \\ \mathbf{q}_t - \nabla \cdot \left(\frac{\varepsilon}{-\chi} |\mathbf{q}|^2 + p \right) = \frac{\varepsilon}{D} \Delta \mathbf{q}. \end{cases}$$

We comment that the characteristic field associated with the flux in (1.3) is hyperbolic. This is in contrast to the case when $\chi > 0$, in which the characteristic field may change type (cf. [17]). The exclusive feature of the transformed model (1.3) is that the parameter $\varepsilon > 0$ is not only a viscosity constant but also a coefficient of nonlinear advection term with quadratic nonlinearity, which distinguishes itself from other (general) hyperbolic systems (e.g., see [1, 9, 28]).

1.2. Literature review and goals. To put things into perspective, we now briefly survey the literature on (1.3) in connection with this work.

When $\varepsilon = 0$, many results concerning the qualitative behavior of solutions to (1.3) have been achieved in a series of recent works:

- one-dimensional explicit and numerical solutions on finite intervals [17]
- local well-posedness and blowup criteria of multi-dimensional and large-amplitude classical solutions on \mathbb{R}^n [5, 19]
- global well-posedness and long-time behavior of multi-dimensional small-amplitude classical solutions on \mathbb{R}^n [8, 19]
- global well-posedness and long-time behavior of one-dimensional small-amplitude classical solutions on finite intervals [34]
- global well-posedness of one-dimensional large-amplitude classical solutions on \mathbb{R} [7]
- global well-posedness of one-dimensional large-amplitude classical solutions on finite intervals [4] through directly working on the original model (1.1)
- long-time behavior of one-dimensional large-amplitude classical solutions on finite intervals [21, 22]
- long-time behavior of one-dimensional large-amplitude classical solutions on \mathbb{R} [20]
- long-time behavior of one-dimensional small-amplitude classical solutions on \mathbb{R} [35]
- local nonlinear stability of one-dimensional traveling wave solutions on \mathbb{R} [12, 23, 24]
- formation of one-dimensional shock waves in \mathbb{R} [31].

Comparing with the magnitude of research conducted on (1.3) with $\varepsilon = 0$, the chemically diffusible model (i.e., (1.3) with $\varepsilon > 0$) has been investigated relatively less. The following results have been recently established:

- Global well-posedness and long-time behavior of one-dimensional large-amplitude classical solutions on finite intervals subject to the Dirichlet-Dirichlet boundary conditions, $p|_{\partial\Omega} = \bar{p}$, $q|_{\partial\Omega} = \bar{q}$ [21]
- Global well-posedness, long-time behavior and diffusion limit of one-dimensional large-amplitude classical solutions on finite intervals subject to the Neumann-Dirichlet boundary conditions, $p_x|_{\partial\Omega} = 0$, $q|_{\partial\Omega} = 0$ [21, 29]
- Global well-posedness, long-time behavior and diffusion limit of small-amplitude classical solutions in \mathbb{R}^N ($N = 2, 3$) [32]
- Existence and nonlinear stability of one-dimensional traveling wave solutions on \mathbb{R} [18, 25, 26].

To the best of our knowledge, the qualitative behavior of classical solutions to the Cauchy problem of (1.3) under *general conditions* on initial data has not been studied in the literature. In this paper, we are interested in the dynamics of (1.3) for fixed values of D and χ . Hence, for simplicity, we take $D = -\chi = 1$, and the resulting equations read as

$$(1.4) \quad \begin{cases} p_t - \nabla \cdot (p\mathbf{q}) = \Delta p, & \mathbf{x} \in \mathbb{R}^d, t > 0, \\ \mathbf{q}_t - \nabla(\varepsilon|\mathbf{q}|^2 + p) = \varepsilon\Delta\mathbf{q}, \\ (p, \mathbf{q})(\mathbf{x}, 0) = (p_0, \mathbf{q}_0)(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^d, \end{cases}$$

where the initial data are prescribed around a constant ground state $(\bar{p}, \mathbf{0})$ where $\bar{p} \geq 0$. The purpose of the paper is to study the qualitative behavior of classical solutions to the Cauchy problem (1.4).

1.3. Motivation. Next, we would like to point out the facts that motivate the present work:

- (1) Recently, the local nonlinear stability of one-dimensional traveling wave solutions to (1.4) was established in [25, 26]. It was shown that traveling wave solutions with large strength are asymptotically stable under sufficiently small perturbations. Numerical simulations in [26] indicated that traveling waves are asymptotically stable even under large perturbations. However, a rigorous justification of the global stability of one-dimensional traveling wave solutions to the model is still missing. As a matter of fact, even the global stability of constant equilibrium states is still completely unknown. In this paper, we investigate the global stability of constant equilibrium states to (1.4) which, we hope, can shed some light on the study of global stability of traveling waves.
- (2) It has been observed in previous works that one major issue encountered in the study of the global well-posedness and long-time behavior of large-amplitude classical solutions to (1.4) is the quadratic non-linearity in the first equation. In [29, 33], such an issue was resolved by employing the

weak Lyapunov functional associated with the system which involves a logarithmic function of the cell density function, and deriving special (Poincaré type) inequalities by taking advantage of the compactness of the domain and conservation of total mass (cf. (3.25) in [29], (2.10) in [33]). However, for the Cauchy problem under consideration, the crucial approaches developed in [29, 33] are no longer valid because of the non-compactness of \mathbb{R} . This raises a significant mathematical challenge.

- (3) It is a common belief that linear diffusion usually has a stabilizing effect. Taking this into consideration, one would expect that any analytical result on the chemically non-diffusible model, that is, (1.1) with $\varepsilon = 0$, could be naturally extended to the chemically diffusible model without spending extra effort. Whether this is true for the case considered in this paper is, however, not clear. In fact, from the preliminary mathematical treatment of (1.1), we see that after applying the Cole-Hopf transformation (cf. (1.2)), in addition to the linear diffusion term $\Delta \mathbf{q}$, a new quadratic nonlinearity $\nabla(|\mathbf{q}|^2)$ is present in the system (cf., (1.3)). From a mathematical point of view, quadratic (or higher-order) nonlinearities usually do not have significant impacts on the long-time dynamics of small-amplitude solutions to nonlinear PDEs. However, the story is completely opposite in the regime of large-amplitude solutions. Therefore, a natural question for (1.3) is, *Can the linear diffusion $\Delta \mathbf{q}$ dominate the quadratic nonlinearity $\nabla(|\mathbf{q}|^2)$ in the regime of large-amplitude solutions?* As mentioned before, such a question has been partially answered in one space dimension when the size of the domain under consideration is finite (cf. [29, 33]). However, the case of infinite domain (Cauchy problem) has largely remained open.
- (4) It has been shown in [29, 33] that on finite one-dimensional intervals subject to the Neumann-Dirichlet boundary conditions, large-amplitude classical solutions of (1.3) with $\varepsilon > 0$ converge to those of (1.3) with $\varepsilon = 0$ as $\varepsilon \rightarrow 0$. The results indicate that the chemically diffusible model is consistent with the non-diffusible model on finite one-dimensional intervals subject to the Neumann-Dirichlet boundary conditions. However, such a phenomenon has not been investigated for the Cauchy problem (1.4).
- (5) It is well documented that solutions to the incompressible Navier-Stokes equations are analytic in space at least for short time (cf. [16] and references therein). In fact, in two dimensions, the radius of spatial analyticity grows like \sqrt{t} . This property can be viewed as a strong expression of the parabolic regularization effect derived from its linear part. It immediately follows that the solutions experience exponential spectral decay at least for a short time. Inspired by such a classic result in mathematical fluid mechanics, and because of the parabolic nature of the equations, in the last part of this paper, we explore the spatial analyticity of solutions to (1.4) under minimal regularity assumptions on the initial data.

To contribute to this contemporary body of knowledge, in the present paper, we derive delicate energy estimates to study the global well-posedness, long-time behavior, and zero chemical diffusion limit of large-amplitude classical solutions to (1.4). The approach developed in the current paper is essentially independent of the size of spatial domain and conservation of total mass, and therefore is more general than that utilized in [29, 33] for the finite domain case. Indeed, it can be readily checked that this approach can be applied directly to the initial-boundary value problems considered in [29, 33]. By using this approach, we can show that positive constant ground states for the Cauchy problem are globally asymptotically stable, which means that classical solutions to the Cauchy problem always converge to positive constant ground states regardless of the strength of initial perturbations. We also show that large-amplitude classical solutions of (1.4) are convergent with respect to the chemical diffusion coefficient ε with a certain convergence rate. Furthermore, we derive explicit algebraic decay rates of solutions under very mild conditions on initial data. The above-mentioned results are derived only in one dimension ($d = 1$). For $d > 1$, by adapting a technique for establishing space-analyticity of solutions to the incompressible Navier-Stokes equations (cf. [6]), we show that the model (1.4) also exhibits this strong parabolic smoothing effect in *any* dimension: namely, its solutions are spatially analytic for a short time provided that the initial data belong to $L^q(\mathbb{R}^d)$ for any $q > d \geq 1$. We underscore that the global existence of large-amplitude solutions of (1.4) in multi-dimensions ($d > 1$) still remains open as of this date.

The rest of this paper is organized as follows. In Section 2, we state our main results for the transformed chemotaxis model (1.4). Section 3 is devoted to the studies of global well-posedness, long-time behavior, and diffusion limits of large-amplitude classical solutions to (1.4). In Section 4, we compute the explicit decay rate of the perturbations under mild conditions on initial data. In Section 5, we study the parabolic smoothing effect of the model. In Section 6, we show numerical simulations to illustrate our theoretical results, and launch some new interesting problems for further study based on some numerical results for a case not proved in this paper.

2. MAIN RESULTS AND IDEAS

We now state the main results derived in the paper. In the sequel, we always assume $d = 1$ (one dimension) unless otherwise stated. We first introduce some notation for convenience.

Notation. Throughout this paper, $\|\cdot\|$, $\|\cdot\|_\infty$, and $\|\cdot\|_{H^s}$ denote the norms of the usual Lebesgue measurable function spaces L^2 , L^∞ , and the Hilbert space H^s , respectively. The functional spaces under consideration are $L^\infty([0, T]; H^s)$ and $L^2([0, T]; H^s)$. Unless otherwise specified, C and C_i denote generic constants which are independent of the unknown functions. The values of the constants may vary line by line according to the context.

In [29, 33], the authors studied the global dynamics of one-dimensional large amplitude classical solutions to (1.4) on finite intervals. However, as mentioned before, the methods developed for the initial-boundary value problems are not directly accessible for the Cauchy problem, because of the non-compactness of the domain. The first theorem addresses the global well-posedness and long-time behavior of large-amplitude classical solutions to the Cauchy problem (1.4) when the initial cell density function is perturbed around a positive constant ground state.

Theorem 2.1. *Consider the one-dimensional version of (1.4). Assume that $p_0(x) \geq 0$ and $(p_0 - \bar{p}, q_0) \in H^2(\mathbb{R})$ for some constant ground state $\bar{p} > 0$. Then, for any fixed $0 < \varepsilon \leq 1$, there exists a unique global classical solution (p, q) to the Cauchy problem (1.4) such that*

$$(p - \bar{p}, q) \in C([0, \infty); H^2(\mathbb{R})), (p_x, q_x) \in L^2([0, \infty); H^2(\mathbb{R})),$$

with the following estimate:

$$\begin{aligned} & \| (p - \bar{p})(t) \|_{H^2}^2 + \| q(t) \|_{H^2}^2 \\ & + \int_0^t (\| p_x(\tau) \|_{H^2}^2 + \| q_x(\tau) \|_{H^1}^2 + \varepsilon \| q_x(\tau) \|_{H^2}^2) d\tau \leq C_0, \end{aligned}$$

where the constant $C_0 > 0$ depends only on p_0, q_0 , and \bar{p} . In addition,

$$\lim_{t \rightarrow \infty} (\| (p - \bar{p})(t) \|_{C^1}^2 + \| q(t) \|_{C^1}^2) = 0.$$

Remark 2.2. From the proof of Theorem 2.1 we will see that the upper bound for ε can be any fixed finite number. We take it to be one for simplicity in the paper.

The next question we shall explore is the diffusion (viscous) limit of (1.4) as $\varepsilon \rightarrow 0$. In [29, 33], the authors studied the diffusion limits of large-amplitude classical solutions to the one-dimensional Neumann-Dirichlet boundary value problem of (1.4) as $\varepsilon \rightarrow 0$, and identified the convergence rates of the diffusible problem toward the non-diffusible problem. Our next theorem addresses these two topics for the Cauchy problem (1.4).

Theorem 2.3. *Let the conditions of Theorem 2.1 hold. Let $(p^\varepsilon, q^\varepsilon)$ be the unique classical solutions to (1.4) with $\varepsilon > 0$. Then, for any fixed $t > 0$, the pair of limiting functions $(p^0, q^0) = \lim_{\varepsilon \rightarrow 0} (p^\varepsilon, q^\varepsilon)$ is a unique classical solution to the non-diffusive problem, that is, (1.4) with $\varepsilon = 0$. Moreover, $(p^\varepsilon, q^\varepsilon)$ approaches (p^0, q^0) with the following convergence rate:*

$$\| (p^\varepsilon - p^0)(t) \|_{H^1}^2 + \| (q^\varepsilon - q^0)(t) \|_{H^1}^2 \leq \alpha_1 e^{\beta_1 t} \varepsilon,$$

where α_1, β_1 are positive constants which depend only on p_0, q_0 , and \bar{p} .

Remark 2.4. Theorem 2.3 and the results in [29, 33] indicate that the fully parabolic system and the parabolic-hyperbolic system are consistent on the entire real line or on one-dimensional finite intervals with Neumann-Dirichlet boundary conditions: $p_x|_{\partial\Omega} = 0$, $q|_{\partial\Omega} = 0$. On the other hand, numerical simulations in [21] showed that the consistency does not hold true on one-dimensional finite intervals with Dirichlet boundary conditions: $p|_{\partial\Omega} = \bar{p}$, $q|_{\partial\Omega} = \bar{q}$ when $\varepsilon > 0$, while $p|_{\partial\Omega} = \bar{p}$ when $\varepsilon = 0$. Indeed, the emergence of boundary layers has been numerically observed in [21] and subsequently rigorously justified in [10]. This is due to the mismatch of the boundary conditions between the diffusible and non-diffusible problems. The scenario is similar to the vanishing viscosity limit of the incompressible Navier-Stokes equations under the no-slip boundary conditions.

Now, we would like to briefly explain the ideas of the proofs of Theorems 2.1–2.3. The results are proved by energy methods. Regarding the proof of Theorem 2.1, since our goal is to prove the convergence of the solution to the positive constant ground state, uniform-in-time estimation of the solution is necessary. It turns out that the most difficult part in building the uniform-in-time estimation comes from the control of the low frequency part (L^2 -norm) of the solution. This is mainly due to the quadratic nonlinearity in the first equation of (1.4). We reach our goal by first employing the weak Lyapunov functional (entropy-entropy flux pair) associated with the equations in (1.4) which involves a logarithmic function of the cell density function p . The Lyapunov functional provides a strong monotonicity formula for the low frequency part of the solution, and lays down a foundation for the global existence of large-amplitude solutions.

Next, in order to control the first-order dissipation $\|\partial_x p\|_{L^2}$, a natural step is to perform an L^2 -type energy estimate on the first equation of (1.4). However, this operation produces the integral of the cubic term, pp_xq , which is difficult to deal with in the regime of large-amplitude solutions. To resolve this issue, we first “complicate” the problem by performing higher-order L^r -type ($r > 2$) energy estimates, and then obtain the control of the dissipation $\|\partial_x p\|_{L^2}$ by coupling the higher-order estimates with the weak Lyapunov functional. After coupling the L^r -estimates together, we discover that some of the resulting non-linear terms cancel each other and compensate the cubic term obtained from the L^2 -estimate. It turns out there is a highly non-trivial intrinsic balance between the higher-order non-linearities. It should be mentioned that a similar idea was applied in [20] to study the global dynamics of large amplitude classical solutions to the chemically non-diffusive problem, that is, (1.4) with $\varepsilon = 0$. In particular, it was shown that the L^2, L^3, L^4 norms of $p - \bar{p}$ are uniformly bounded with respect to time. In this paper, we further generalize this method to incorporate the uniform-in-time estimates of all the L^r norms of $p - \bar{p}$ for r ranging from 2 to $2k$ for any fixed $2 \leq k < \infty$. This requires a significant amount of efforts for manipulating the higher-order non-linearities.

We will see that the energy estimation of the low frequency part of the solution is independent not only of time, but also of the chemical diffusion coefficient ε .

The ε -independent energy estimation plays a key role in proving the diffusion limit and convergence rate of the solution stated in Theorem 2.3.

To proceed with estimating the spatial derivatives of the solution so as to establish the diffusion limit and convergence rate (Theorem 2.3), we follow the spirit of [20] to derive a damping equation for the spatial derivatives of the q -function. However, because of the presence of the non-linear convective-like term $\varepsilon(q^2)_x$, the derivation of the ε -independent energy estimates of the spatial derivatives is complicated, which makes the analysis in this paper much more involved than that in [20]. We achieve the goal by making use of the linear chemical diffusion and the ε -independent energy estimation of the low frequency part of the solution. It turns out that the ε -independent estimates for the spatial derivatives of the q -function play a crucial role in taking the chemical diffusion limit and establishing the convergence rate.

The aforementioned results (Theorems 2.1–2.3) are concerned with the qualitative behavior of large-amplitude classical solutions to (1.4) when the constant ground state is strictly positive. Using a similar idea as in [20], we can prove the global well-posedness of large-amplitude classical solutions to (1.4) when the ground state $\bar{p} = 0$. The idea is to lift the cell density function from the zero ground by a positive distance, and make the entropy expansion of the lifted function around the positive ground state. To be precise, we consider the following Cauchy problem:

$$\begin{cases} \bar{p}_t - (\bar{p}q)_x + q_x = \bar{p}_{xx}, \\ q_t - \bar{p}_x = \varepsilon q_{xx} + \varepsilon(q^2)_x; \\ (\bar{p}, q)(x, 0) = (p_0 + \hat{p}, q_0)(x), \end{cases}$$

where $\bar{p} = p + \hat{p}$, $\hat{p} > 0$ is a constant, and (p, q) denotes the solution to (1.4) when $\bar{p} = 0$, $p_0 \geq 0$, and $(p_0, q_0) \in H^2(\mathbb{R})$. In other words, we lift p to be positive by a positive distance \hat{p} . Such a treatment, together with the arguments on page 2197 of [20], yields that $\|q(t)\|^2 \leq C(t)$, where $C(t)$ can be explicitly expressed as

$$C(t) = e^{t/\hat{p}} \left(2 \int_{\mathbb{R}} [\bar{p}_0 \ln(\bar{p}_0) - \bar{p}_0] - [\hat{p} \ln(\hat{p}) - \hat{p}] - \ln(\hat{p})(\bar{p}_0 - \hat{p}) \, dx + \|q_0\|^2 \right).$$

Then, by combining the proof of Theorem 2.1 in this paper and the arguments on pages 2197–2198 of [20], we can show that the Cauchy problem of (1.4) is globally well posed when $\bar{p} = 0$, $p_0 \geq 0$, and $(p_0, q_0) \in H^2(\mathbb{R})$. Here, we only present the result without going through the technical details. The result is recorded in the following proposition.

Proposition 2.5. *Let the conditions of Theorem 2.1 hold with the exception that $\bar{p} = 0$. Then, for any fixed $0 < \varepsilon \leq 1$, there exists a unique global classical solution (p, q) to the Cauchy problem (1.4) such that*

$$(p, q) \in C([0, \infty); H^2(\mathbb{R})), \quad (p_x, q_x) \in L^2([0, \infty); H^2(\mathbb{R})).$$

Remark 2.6. We point out that in the zero ground state case if one assumes that the initial total mass of the cell population is finite, then the total mass of cell population is finite and conserved for all time (since $p \geq 0$), which can be seen by integrating the first equation of (1.4) over \mathbb{R} . This makes the zero ground state case more biologically meaningful than the positive ground state case. However, we are currently unable to rigorously capture the long-time behavior of the solution in the zero ground state case, because the energy bounds derived in this case are increasing functions of time. We leave the investigation for the future. However, our numerical simulations indicate that (p, q) converge to $(0, 0)$ with a very slow decay rate as t becomes large (see details in Section 6).

So far, the results presented in this section do not assume any smallness condition on the strength of the initial perturbations. However, they (especially Theorem 2.1) also provide no information about the *explicit decay rate* of the perturbations. Our next goal is to compute the explicit decay rate of classical solutions to (1.4) under mild conditions on the initial data. We identify the decay rate by following a standard approach, which is to define anti-derivatives of the perturbations and perform time-weighted energy estimates. To state the result, we introduce the following anti-derivatives:

$$(2.1) \quad \varphi(x, t) = \int_{-\infty}^x (p(y, t) - \bar{p}) \, dy, \quad \psi(x, t) = \int_{-\infty}^x q(y, t) \, dy,$$

for $t \geq 0$, where $\bar{p} > 0$ is any given constant ground state. Then, we have the following result.

Theorem 2.7. *With (2.1), we assume that the initial data satisfy*

$$(\varphi_0, \psi_0)(x) \in H^3(\mathbb{R})$$

and there exists a sufficiently small constant $\eta_0 > 0$ such that $\|\varphi_0\|^2 + \|\psi_0\|^2 \leq \eta_0$. Then, there also exists a unique global solution to the Cauchy problem (1.4) satisfying $(p - \bar{p}, q) \in C([0, \infty); H^2(\mathbb{R})) \cap L^2([0, \infty); H^3(\mathbb{R}))$. Moreover, there exists a constant $\zeta_0 > 0$ which is independent of time, such that for any $t > 0$, it holds that

$$\begin{aligned} & \sum_{k=0}^2 [(t + 1)^{k+1} (\|\partial_x^k (p(t) - \bar{p})\|^2 + \|\partial_x^k q(t)\|^2)] + \\ & + \sum_{m=0}^3 \left[\int_0^t (\tau + 1)^m (\|\partial_x^m p(\tau)\|^2 + \|\partial_x^m q(\tau)\|^2) \, d\tau \right] \leq \zeta_0. \end{aligned}$$

Remark 2.8. It is worth mentioning that in Theorem 2.7, only the L^2 -level energy of the initial anti-derivatives is assumed to be small. This is because of the fact that Theorem 2.1 holds for large energy of p_0 and q_0 . Moreover, in [20], a similar result is obtained for (1.4) when $\varepsilon = 0$ under the assumption that

$\|\varphi_0\|^2 + \|\psi_0\|^2 + \|q_0\|^2$ is small. Comparing the situation to that result, we see that the smallness of $\|q_0\|^2$ is removed when $\varepsilon > 0$. This is due to the parabolic nature of the model.

The last result of this paper is concerned with the parabolic smoothing effect of the model. We show that the solution to (1.4) becomes instantaneously spatially analytic as long as the initial data belong to $L^q(\mathbb{R}^d)$, for any $q > d \geq 1$. The approach we employ is the one developed by Grujic-Kukavica in [6], where spatial analyticity was established for the Navier-Stokes equations. The method relies on approximations to the original system by the heat equation, whose solutions are known to be analytic for such data. By considering a suitable self-map, one can then show that analyticity of the approximation propagates to analyticity of the limiting function for a short time, depending only on the L^q -norm of the initial data.

Theorem 2.9. *Let $d \geq 1$ and $q \in (d, \infty)$. Suppose there is a constant M_q depending on q such that $\|p_0\|_{L^q} + \|q_0\|_{L^q} \leq M_q$. Then, there exist absolute constants $C_1, C_2 > 0$ such that, for $T_0 > 0$ given by*

$$T_0 := \min \left\{ \frac{\varepsilon}{C_1}, \frac{1}{(C_2 M_q)^{2/(1-d/q)}} \right\},$$

the Cauchy problem (1.4) has a solution (p, \mathbf{q}) which satisfies

$$p \in C([0, T_0]; L^q(\mathbb{R}^d)), \quad \mathbf{q} \in C([0, T_0]; L^q(\mathbb{R}^d)^d)$$

with the following property: for every $t \in (0, T_0)$, p, \mathbf{q} are restrictions of the analytic functions $p(\mathbf{x}, \mathbf{y}, t) + i\pi(\mathbf{x}, \mathbf{y}, t)$, and $\mathbf{q}(\mathbf{x}, \mathbf{y}, t) + i\mathbf{u}(\mathbf{x}, \mathbf{y}, t)$, respectively, in the region

$$\mathcal{D}_t := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^d : |\mathbf{y}| \leq C_* t^{1/2} \min\{1, \sqrt{\varepsilon}\}\},$$

for some absolute constant $C_ > 0$, depending only on C_1 . Moreover, we have*

$$(2.2) \quad \begin{aligned} &\|p(\cdot, \mathbf{y}, t)\|_{L^q} + \|\pi(\cdot, \mathbf{y}, t)\|_{L^q} \\ &\quad + \|\mathbf{q}(\cdot, \mathbf{y}, t)\|_{L^q} + \|\mathbf{u}(\cdot, \mathbf{y}, t)\|_{L^q} \leq 4M_q, \end{aligned}$$

for each $t \in (0, T_0)$ and $(\mathbf{x}, \mathbf{y}) \in \mathcal{D}_t$.

Before concluding this section, we point out that all results obtained above for the transformed system (1.4) can be transferred to the original chemotaxis system (1.1) by inverting the Cole-Hopf transformation (1.2). Since this process is quite standard and has been done in many previous works (e.g., see [18, 21, 32, 33]), we omit the details and focus our attention on the transformed system (1.4) only.

3. QUALITATIVE BEHAVIOR OF LARGE SOLUTIONS

In this section, we study the long-time dynamics and diffusion limit of large-amplitude classical solutions to the Cauchy problem (1.4).

3.1. Long-time dynamics of transformed system (proof of Theorem 2.1).
 Now, we consider the global dynamics of the transformed system

$$(3.1) \quad \begin{cases} p_t = p_{xx} + (pq)_x, \\ q_t = \varepsilon q_{xx} + \varepsilon(q^2)_x + p_x, \end{cases}$$

with the initial condition

$$(3.2) \quad \begin{cases} (p, q)(x, 0) = (p_0, q_0)(x), \\ (p_0 - \bar{p}, q_0) \in H^2(\mathbb{R}), \\ p_0(x) \geq 0, \quad x \in \mathbb{R}, \end{cases}$$

where $\bar{p} > 0$ is a constant ground state.

First, by using the arguments in [7], one can show the local existence result as follows.

Proposition 3.1 (Local Existence). *Assume that the initial data satisfy (3.2). Then, there exists a unique local solution (p, q) to (3.1)–(3.2) such that $p \geq 0$, $(p - \bar{p}, q) \in C([0, T_0]; H^2(\mathbb{R}))$ and $(p_x, q_x) \in L^2([0, T_0]; H^2(\mathbb{R}))$ for some finite $T_0 > 0$.*

To get a global solution, we derive *a priori* energy estimates of the local solution.

Proposition 3.2 (A priori Estimates). *Let (p, q) be a solution to (3.1)–(3.2). Then, it holds that*

$$(3.3) \quad \begin{aligned} & \|p(t) - \bar{p}\|_{H^2}^2 + \|q(t)\|_{H^2}^2 + \\ & + \int_0^t (\|p_x(\tau)\|_{H^2}^2 + \|q_x(\tau)\|_{H^1}^2 + \varepsilon \|q_x(\tau)\|_{H^2}^2) \, d\tau \leq C_0, \end{aligned}$$

where the constant $C_0 > 0$ is independent of t and ε .

For the sake of readability, we divide the proof of Proposition 3.2 into four steps which are stated in a sequence of lemmas. We begin with a basic energy estimate based on the weak Lyapunov functional associated with the system (3.1).

Lemma 3.3 (Weak Lyapunov Functional). *Let (p, q) be a solution to (3.1)–(3.2). Then, there is a constant d_1 such that*

$$(3.4) \quad \begin{aligned} & \|q(t)\|^2 + \int_0^t \left(\int_{\mathbb{R}} \frac{(p_x)^2}{p} \, dx + \varepsilon \|q_x\|^2 \right) \, d\tau \\ & \leq C_1 = \frac{1}{2} \|q_0\|^2 + d_1 \|p_0 - \bar{p}\|^2. \end{aligned}$$

Proof. Because of the conservation of total mass, after taking the L^2 inner product of (3.1)₁ with $\ln(p) - \ln(\bar{p})$, we have

$$(3.5) \quad \frac{d}{dt} \left(\int_{\mathbb{R}} \eta(p) - \eta(\bar{p}) - \eta'(\bar{p})(p - \bar{p}) \, dx \right) + \int_{\mathbb{R}} \left(p_x q + \frac{(p_x)^2}{p} \right) \, dx = 0,$$

where $\eta(z) = z \ln(z) - z$ which is a convex function. Taking the L^2 inner product of (3.1)₂ with q , we have

$$(3.6) \quad \frac{1}{2} \frac{d}{dt} \|q\|^2 - \int_{\mathbb{R}} p_x q \, dx + \varepsilon \|q_x\|^2 = 0.$$

Adding (3.6) to (3.5), we get

$$(3.7) \quad \frac{d}{dt} \left(\int_{\mathbb{R}} \eta(p) - \eta(\bar{p}) - \eta'(\bar{p})(p - \bar{p}) \, dx + \frac{1}{2} \|q\|^2 \right) + \int_{\mathbb{R}} \frac{(p_x)^2}{p} \, dx + \varepsilon \|q_x\|^2 = 0.$$

Integrating (3.7) over $[0, t]$, we have

$$(3.8) \quad \left(\int_{\mathbb{R}} \eta(p) - \eta(\bar{p}) - \eta'(\bar{p})(p - \bar{p}) \, dx + \frac{1}{2} \|q\|^2 \right)(t) + \int_0^t \left(\int_{\mathbb{R}} \frac{(p_x)^2}{p} \, dx \right) + \varepsilon \|q_x\|^2 \, d\tau = \left(\int_{\mathbb{R}} \eta(p_0) - \eta(\bar{p}) - \eta'(\bar{p})(p_0 - \bar{p}) \, dx + \frac{1}{2} \|q_0\|^2 \right).$$

By the convexity of $\eta(\cdot)$ and the non-negativity of p , we have

$$\int_{\mathbb{R}} \eta(p) - \eta(\bar{p}) - \eta'(\bar{p})(p - \bar{p}) \, dx \geq 0.$$

On the other hand, since $0 < \bar{p} < +\infty$ and $p_0 \geq 0$, we have

$$(3.9) \quad \int_{\mathbb{R}} \eta(p_0) - \eta(\bar{p}) - \eta'(\bar{p})(p_0 - \bar{p}) \, dx \leq d_1 \|p_0 - \bar{p}\|^2,$$

where the constant d_1 depends only on \bar{p} . Thus, (3.8) and (3.9) complete the proof. □

Although the Lyapunov functional provides a uniform-in-time estimate for $\|q\|^2$, the logarithmic expansion of the p -function is too weak for the subsequent energy estimates. Next, we derive a uniform-in-time estimate for $\|p - \bar{p}\|^2$. It turns out that the standard procedure (L^2 -type energy estimate) is not sufficient to achieve our goal, and we need to employ higher-order estimates. Since the proof of the next lemma is quite lengthy, we divide it into three steps.

Lemma 3.4 (L^2 -Estimate). *Let (p, q) be a solution to (3.1)–(3.2). Then, it follows that*

$$(3.10) \quad \|p(t) - \bar{p}\|^2 + \|q(t)\|^2 + \int_0^t (\|p_x(\tau)\|^2 + \varepsilon \|q_x(\tau)\|^2) \, d\tau \leq C_2,$$

where the constant $C_2 > 0$ is independent of t and ε .

Proof. Step 1. We first reformulate the system (3.1). Let $\tilde{p} = p - \bar{p}$. Substituting \tilde{p} into (3.1), we have

$$(3.11) \quad \begin{cases} \tilde{p}_t - (\tilde{p}q)_x - \tilde{p}q_x = \tilde{p}_{xx}, \\ q_t - \tilde{p}_x = \varepsilon q_{xx} + \varepsilon(q^2)_x. \end{cases}$$

Taking the L^2 inner products of (3.11)₁ with \tilde{p} , (3.11)₂ with $\tilde{p}q$, and integrating by parts, we have

$$(3.12) \quad \frac{d}{dt} \left(\frac{1}{2} \|\tilde{p}\|^2 + \frac{\tilde{p}}{2} \|q\|^2 \right) + \|\tilde{p}_x\|^2 + \varepsilon \tilde{p} \|q_x\|^2 = - \int_{\mathbb{R}} \tilde{p}q\tilde{p}_x \, dx.$$

We discover that the cubic term on the righthand side of (3.12) is hard to deal with in building the desired uniform-in-time estimate for $\|\tilde{p}\|^2$, and so we eliminate such a term. For this purpose, by taking the L^2 inner product of (3.11)₁ with $-\frac{1}{2}(\tilde{p})^2$ and integrating by parts, we have

$$(3.13) \quad \frac{d}{dt} \left(-\frac{1}{6} \int_{\mathbb{R}} (\tilde{p})^3 \, dx \right) - \int_{\mathbb{R}} \tilde{p}(\tilde{p}_x)^2 \, dx = \int_{\mathbb{R}} (\tilde{p})^2 q \tilde{p}_x \, dx + \tilde{p} \int_{\mathbb{R}} \tilde{p}q\tilde{p}_x \, dx.$$

Note that the last term on the righthand side of (3.13) and the cubic term on the righthand side of (3.12) have opposite signs, and they differ by a constant multiple. Therefore, multiplying (3.12) by \tilde{p} , and then adding the resulting equation to (3.13), we get

$$(3.14) \quad \begin{aligned} \frac{d}{dt} \left(\frac{\tilde{p}}{2} \|\tilde{p}\|^2 - \frac{1}{6} \int_{\mathbb{R}} (\tilde{p})^3 \, dx + \frac{(\tilde{p})^2}{2} \|q\|^2 \right) + \tilde{p} \|\tilde{p}_x\|^2 \\ - \int_{\mathbb{R}} \tilde{p}(\tilde{p}_x)^2 \, dx + \varepsilon (\tilde{p})^2 \|q_x\|^2 = \int_{\mathbb{R}} (\tilde{p})^2 q \tilde{p}_x \, dx. \end{aligned}$$

Next, observe that in (3.14) the quantity inside of the temporal derivative is not necessarily positive, since the estimate of the L^∞ -norm of \tilde{p} is unknown at this stage. Therefore, we need to move on to the L^4 -estimate of \tilde{p} in order to compensate the third-order power of \tilde{p} . For this purpose, we take the L^2 inner product of (3.11)₁ with $-\frac{1}{3}(\tilde{p})^3$, multiply (3.14) by \tilde{p} , then add the two resulting equations to get

$$\begin{aligned} \frac{d}{dt} \left(\frac{(\tilde{p})^2}{2} \|\tilde{p}\|^2 - \frac{\tilde{p}}{6} \int_{\mathbb{R}} (\tilde{p})^3 \, dx + \frac{1}{12} \int_{\mathbb{R}} (\tilde{p})^4 \, dx + \frac{(\tilde{p})^3}{2} \|q\|^2 \right) + (\tilde{p})^2 \|\tilde{p}_x\|^2 \\ - \tilde{p} \int_{\mathbb{R}} \tilde{p}(\tilde{p}_x)^2 \, dx + \int_{\mathbb{R}} (\tilde{p})^2 (\tilde{p}_x)^2 \, dx + \varepsilon (\tilde{p})^3 \|q_x\|^2 = - \int_{\mathbb{R}} (\tilde{p})^3 q \tilde{p}_x \, dx. \end{aligned}$$

We would like to note that a similar idea was used in [21] to study the global dynamics of large-amplitude classical solutions to an initial-boundary value problem of (1.4) on finite one-dimensional intervals. However, the proof constructed

in [21] only provides a uniform-in-time estimate of the L^2 , L^3 , and L^4 norms of $p - \bar{p}$. Here, we can show in one stroke that the L^r norm of $p - \bar{p}$ is uniformly bounded with respect to time for any $2 \leq r \leq 2k < \infty$ without appealing to the estimate of spatial derivatives of the perturbation, which cannot be achieved by using the argument in [21]. Indeed, for any fixed integer $2 \leq k < \infty$, by repeating the above procedure, we deduce that

$$\begin{aligned}
 (3.15) \quad & \frac{d}{dt} \left(\sum_{j=2}^{2k} \int_{\mathbb{R}} \frac{(\bar{p})^{2k-j} (-\bar{p})^j}{(j-1)j} dx + \frac{(\bar{p})^{2k-1}}{2} \|q\|^2 \right) + \\
 & + \left(\sum_{j=0}^{2k-2} \int_{\mathbb{R}} (\bar{p})^{2k-2-j} (-\bar{p})^j (\bar{p}_x)^2 dx + \varepsilon (\bar{p})^{2k-1} \|q_x\|^2 \right) \\
 & = - \int_{\mathbb{R}} (\bar{p})^{2k-1} q \bar{p}_x dx.
 \end{aligned}$$

Step 2. This step is to get a proper control of the righthand side of (3.15). However, before doing so we need to examine the two terms on the lefthand side of the equation. First, by using the decomposition

$$\begin{aligned}
 (3.16) \quad & \sum_{j=2}^{2k} \frac{(\bar{p})^{2k-j} (-\bar{p})^j}{(j-1)j} = \frac{1}{4} (\bar{p})^{2k-2} (\bar{p})^2 \\
 & + \frac{1}{2} \sum_{i=1}^{k-1} \left(\frac{(\bar{p})^{2k-2i} (\bar{p})^{2i}}{(2i-1)2i} - \frac{2(\bar{p})^{2k-2i-1} (\bar{p})^{2i+1}}{2i(2i+1)} + \frac{(\bar{p})^{2k-2i-2} (\bar{p})^{2i+2}}{(2i+1)(2i+2)} \right) \\
 & \qquad \qquad \qquad + \frac{1}{2(2k-1)2k} (\bar{p})^{2k},
 \end{aligned}$$

and noticing that, for any integer $1 \leq i \leq k-1$, the middle term on the righthand side of (3.16) can be computed as

$$\begin{aligned}
 & \frac{(\bar{p})^{2k-2i} (\bar{p})^{2i}}{(2i-1)2i} - \frac{2(\bar{p})^{2k-2i-1} (\bar{p})^{2i+1}}{2i(2i+1)} + \frac{(\bar{p})^{2k-2i-2} (\bar{p})^{2i+2}}{(2i+1)(2i+2)} \\
 & = (\bar{p})^{2k-2i-2} (\bar{p})^{2i} \left\{ \frac{1}{(2i-1)2i} \left(\bar{p} - \frac{2i-1}{2i+1} \bar{p} \right)^2 + \frac{2(\bar{p})^2}{2i(2i+1)^2(2i+2)} \right\} > 0,
 \end{aligned}$$

we deduce that

$$\sum_{j=2}^{2k} \frac{(\bar{p})^{2k-j} (-\bar{p})^j}{(j-1)j} > \frac{1}{4} (\bar{p})^{2k-2} (\bar{p})^2 + \frac{1}{2(2k-1)2k} (\bar{p})^{2k}.$$

It follows, by the Young's inequality, that

$$(3.17) \quad \sum_{j=2}^{2k} \int_{\mathbb{R}} \frac{(\bar{p})^{2k-j} (-\bar{p})^j}{(j-1)j} dx > R_1^{\bar{p},k} \sum_{m=1}^k \|\bar{p}\|_{L^{2m}}^{2m},$$

where $R_1^{\bar{p},k}$ is a positive constant depending only on \bar{p} and k .

Second, by a similar argument, we have

$$\sum_{j=0}^{2k-2} (\bar{p})^{2k-2-j} (-\bar{p})^j > \frac{1}{2} (\bar{p})^{2k-2} + \frac{1}{2} (\bar{p})^{2k-2},$$

which implies

$$(3.18) \quad \sum_{j=0}^{2k-2} \int_{\mathbb{R}} (\bar{p})^{2k-2-j} (-\bar{p})^j (\bar{p}_x)^2 dx > R_2^{\bar{p},k} \sum_{n=0}^{k-1} \|(\bar{p})^n \bar{p}_x\|^2,$$

where $R_2^{\bar{p},k}$ is another positive constant depending only on \bar{p} and k .

Step 3. We now go back to and deal with the righthand side of (3.15). By the Cauchy-Schwarz inequality and (3.4), we have

$$(3.19) \quad \left| - \int_{\mathbb{R}} (\bar{p})^{2k-1} q \bar{p}_x dx \right| \leq \frac{R_2^{\bar{p},k}}{2} \|(\bar{p})^{k-1} \bar{p}_x\|^2 + \frac{1}{2R_2^{\bar{p},k}} \|(\bar{p})^k q\|^2 \\ \leq \frac{R_2^{\bar{p},k}}{2} \|(\bar{p})^{k-1} \bar{p}_x\|^2 + \frac{1}{2R_2^{\bar{p},k}} \|\bar{p}\|_{\infty}^{2k} \|q\|^2 \\ \leq \frac{R_2^{\bar{p},k}}{2} \|(\bar{p})^{k-1} \bar{p}_x\|^2 + \frac{C_1}{2R_2^{\bar{p},k}} \|\bar{p}\|_{\infty}^{2k}.$$

Substituting (3.19) into (3.15), we then have

$$(3.20) \quad \frac{d}{dt} \left(\sum_{j=2}^{2k} \int_{\mathbb{R}} \frac{(\bar{p})^{2k-j} (-\bar{p})^j}{(j-1)j} dx + \frac{(\bar{p})^{2k-1}}{2} \|q\|^2 \right) + \\ + \left(\sum_{j=0}^{2k-2} \int_{\mathbb{R}} (\bar{p})^{2k-2-j} (-\bar{p})^j (\bar{p}_x)^2 dx - \frac{R_2^{\bar{p},k}}{2} \|(\bar{p})^{k-1} \bar{p}_x\|^2 + \varepsilon (\bar{p})^{2k-1} \|q_x\|^2 \right) \\ \leq \frac{C_1}{2R_2^{\bar{p},k}} \|\bar{p}\|_{\infty}^{2k}.$$

Note that the quantity inside the second parenthesis on the lefthand side of (3.20) is positive, because of (3.18). To control the L^∞ norm of \bar{p} on the righthand side of (3.20), we observe that

$$\begin{aligned} (\bar{p})^{2k}(x, t) &= 2k \int_{-\infty}^x (\bar{p})^{2k-1} \bar{p}_x \, dx \\ &\leq 2k \left(\int_{\mathbb{R}} (\bar{p})^{4k-2} (\bar{p} + \bar{p}) \, dx \right)^{1/2} \left(\int_{\mathbb{R}} \frac{(\bar{p}_x)^2}{\bar{p} + \bar{p}} \, dx \right)^{1/2} \\ &\leq 2k \|\bar{p}\|_\infty^k \left(\int_{\mathbb{R}} (\bar{p})^{2k-2} (\bar{p} + \bar{p}) \, dx \right)^{1/2} \left(\int_{\mathbb{R}} \frac{(\bar{p}_x)^2}{\bar{p} + \bar{p}} \, dx \right)^{1/2} \\ &\leq k \|\bar{p}\|_\infty^k \left(2 \int_{\mathbb{R}} [(\bar{p})^{2k} + (1 + 2\bar{p})(\bar{p})^{2k-2}] \, dx \right)^{1/2} \left(\int_{\mathbb{R}} \frac{(\bar{p}_x)^2}{\bar{p} + \bar{p}} \, dx \right)^{1/2} \\ &\leq \frac{1}{2} \|\bar{p}\|_\infty^{2k} + k^2 (\|\bar{p}\|_{L^{2k}}^{2k} + (1 + 2\bar{p}) \|\bar{p}\|_{L^{2k-2}}^{2k-2}) \int_{\mathbb{R}} \frac{(\bar{p}_x)^2}{\bar{p} + \bar{p}} \, dx. \end{aligned}$$

Then, we have

$$(3.21) \quad \|\bar{p}\|_\infty^{2k} \leq 2k^2 (\|\bar{p}\|_{L^{2k}}^{2k} + (1 + 2\bar{p}) \|\bar{p}\|_{L^{2k-2}}^{2k-2}) \int_{\mathbb{R}} \frac{(\bar{p}_x)^2}{\bar{p} + \bar{p}} \, dx.$$

Substituting (3.21) into (3.20), we get

$$\begin{aligned} (3.22) \quad \frac{d}{dt} &\left(\sum_{j=2}^{2k} \int_{\mathbb{R}} \frac{(\bar{p})^{2k-j} (-\bar{p})^j}{(j-1)j} \, dx + \frac{(\bar{p})^{2k-1}}{2} \|q\|^2 \right) + \\ &+ \left(\sum_{j=0}^{2k-2} \int_{\mathbb{R}} (\bar{p})^{2k-2-j} (-\bar{p})^j (\bar{p}_x)^2 \, dx \right. \\ &\quad \left. - \frac{R_2^{\bar{p},k}}{2} \|(\bar{p})^{k-1} \bar{p}_x\|^2 + \varepsilon (\bar{p})^{2k-1} \|q_x\|^2 \right) \\ &\leq \frac{C_1 k^2}{R_2^{\bar{p},k}} (\|\bar{p}\|_{L^{2k}}^{2k} + (1 + 2\bar{p}) \|\bar{p}\|_{L^{2k-2}}^{2k-2}) \left(\int_{\mathbb{R}} \frac{(\bar{p}_x)^2}{\bar{p} + \bar{p}} \, dx \right). \end{aligned}$$

By virtue of (3.17), we see that

$$\begin{aligned} &\frac{C_1 k^2}{R_2^{\bar{p},k}} (\|\bar{p}\|_{L^{2k}}^{2k} + (1 + 2\bar{p}) \|\bar{p}\|_{L^{2k-2}}^{2k-2}) \\ &\leq C_3 \left(\sum_{j=2}^{2k} \int_{\mathbb{R}} \frac{(\bar{p})^{2k-j} (-\bar{p})^j}{(j-1)j} \, dx + \frac{(\bar{p})^{2k-1}}{2} \|q\|^2 \right) \end{aligned}$$

for some constant C_3 which is independent of t and ε . Thus, we update (3.22) as

$$\begin{aligned}
 (3.23) \quad & \frac{d}{dt} \left(\sum_{j=2}^{2k} \int_{\mathbb{R}} \frac{(\bar{p})^{2k-j} (-\bar{p})^j}{(j-1)j} dx + \frac{(\bar{p})^{2k-1}}{2} \|q\|^2 \right) + \\
 & + \left(\sum_{j=0}^{2k-2} \int_{\mathbb{R}} (\bar{p})^{2k-2-j} (-\bar{p})^j (\bar{p}_x)^2 dx \right. \\
 & \left. - \frac{R_2^{\bar{p},k}}{2} \|(\bar{p})^{k-1} \bar{p}_x\|^2 + \varepsilon (\bar{p})^{2k-1} \|q_x\|^2 \right) \\
 & \leq C_3 \left(\sum_{j=2}^{2k} \int_{\mathbb{R}} \frac{(\bar{p})^{2k-j} (-\bar{p})^j}{(j-1)j} dx + \frac{(\bar{p})^{2k-1}}{2} \|q\|^2 \right) \int_{\mathbb{R}} \frac{(\bar{p}_x)^2}{\bar{p} + \bar{p}} dx.
 \end{aligned}$$

Note that $\bar{p}_x = p_x$, $\bar{p} + \bar{p} = p$. Then, applying the Gronwall inequality to (3.23), and by (3.4), we have, for any $t \geq 0$,

$$(3.24) \quad \left(\sum_{j=2}^{2k} \int_{\mathbb{R}} \frac{(\bar{p})^{2k-j} (-\bar{p})^j}{(j-1)j} dx + \frac{(\bar{p})^{2k-1}}{2} \|q\|^2 \right) (t) \leq C_4,$$

where the constant $C_4 > 0$ is independent of t and ε . Substituting (3.24) into the righthand side of (3.23), and then integrating over $[0, t]$ for any $t > 0$, we have

$$\begin{aligned}
 & \left(\sum_{j=2}^{2k} \int_{\mathbb{R}} \frac{(\bar{p})^{2k-j} (-\bar{p})^j}{(j-1)j} dx + \frac{(\bar{p})^{2k-1}}{2} \|q\|^2 \right) (t) + \\
 & + \int_0^t \left(\sum_{j=0}^{2k-2} \int_{\mathbb{R}} (\bar{p})^{2k-2-j} (-\bar{p})^j (\bar{p}_x)^2 dx \right. \\
 & \left. - \frac{R_2^{\bar{p},k}}{2} \|(\bar{p})^{k-1} \bar{p}_x\|^2 + \varepsilon (\bar{p})^{2k-1} \|q_x\|^2 \right) d\tau \leq C_5.
 \end{aligned}$$

By (3.17) and (3.18), we have, for any fixed integer $1 \leq k < \infty$,

$$(3.25) \quad \left(\sum_{m=1}^k \|\bar{p}\|_{L^{2m}}^{2m} + \|q\|^2 \right) (t) + \int_0^t \left(\sum_{n=0}^{k-1} \|(\bar{p})^n \bar{p}_x\|^2 + \varepsilon \|q_x\|^2 \right) d\tau \leq C_6,$$

for some constant C_6 independent of t and ε . This completes the proof. □

Now, we move on to estimating spatial derivatives of the solution. The following lemma gives the estimate of the first-order spatial derivatives. Since the proof is lengthy again, we divide it into three steps.

Lemma 3.5 (H^1 -Estimate). *Let (p, q) be a solution to (3.1)–(3.2). Then, it follows that*

$$\|(\bar{p}_x, q_x)(t)\|^2 + \int_0^t (\|\bar{p}_{xx}(\tau)\|^2 + \|q_x(\tau)\|^2 + \varepsilon\|q_{xx}(\tau)\|^2) \, d\tau \leq C_7$$

where the constant $C_7 > 0$ is independent of t and ε .

Proof. For convenience, we spilt our proof into four steps.

Step 1. To control the first-order spatial derivative of solutions, a natural step is to perform the standard L^2 -type energy estimate. However, we find that by doing so one cannot obtain the ε -independent estimate of the second-order spatial derivative of q , because the energy estimation involves the temporal integral of $\|q_x\|^2$ which is *inversely proportional* to ε (cf. (3.10)).

To overcome such a technical barrier, we derive a damping equation for q_x , from which we can establish an estimate such that the temporal integral of $\|q_x\|^2$ is independent of ε . For this purpose, by taking the spatial derivative of (3.11)₂ and using equation (3.11)₁, we have

$$(3.26) \quad q_{xt} = -(\bar{p}q)_x - \bar{p}q_x + \bar{p}_t + \varepsilon q_{xxx} + \varepsilon(q^2)_{xx}.$$

Taking the L^2 inner product of (3.26) with q_x , we have

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|q_x\|^2 \right) + \bar{p} \|q_x\|^2 + \varepsilon \|q_{xx}\|^2 \\ &= - \int_{\mathbb{R}} (\bar{p}q)_x q_x \, dx + \int_{\mathbb{R}} \bar{p}_t q_x \, dx + \varepsilon \int_{\mathbb{R}} (q^2)_{xx} q_x \, dx \\ &= - \int_{\mathbb{R}} (\bar{p}q)_x q_x \, dx + \frac{d}{dt} \int_{\mathbb{R}} \bar{p} q_x \, dx - \int_{\mathbb{R}} \bar{p} q_{xt} \, dx - \varepsilon \int_{\mathbb{R}} (q^2)_x q_{xx} \, dx \\ &= - \int_{\mathbb{R}} (\bar{p}q)_x q_x \, dx + \frac{d}{dt} \int_{\mathbb{R}} \bar{p} q_x \, dx + \|\bar{p}_x\|^2 \\ & \quad + \varepsilon \int_{\mathbb{R}} [q_{xx} + (q^2)_x] \bar{p}_x \, dx - 2\varepsilon \int_{\mathbb{R}} q q_x q_{xx} \, dx, \end{aligned}$$

where we have used the equation

$$q_{xt} = \bar{p}_{xx} + \varepsilon q_{xxx} + \varepsilon(q^2)_{xx}.$$

After rearranging terms, we have

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|q_x\|^2 - \int_{\mathbb{R}} \bar{p} q_x \, dx \right) + \bar{p} \|q_x\|^2 + \varepsilon \|q_{xx}\|^2 \\ &= - \int_{\mathbb{R}} (\bar{p}q)_x q_x \, dx + \varepsilon \int_{\mathbb{R}} [q_{xx} + 2q q_x] \bar{p}_x \, dx - 2\varepsilon \int_{\mathbb{R}} q q_x q_{xx} \, dx + \|\bar{p}_x\|^2. \end{aligned}$$

We estimate the first three terms on the right side of the above equation as follows:

$$\begin{aligned}
 - \int_{\mathbb{R}} (\bar{p}q)_x q_x \, dx &\leq \frac{\bar{p}}{2} \|q_x\|^2 + \frac{1}{\bar{p}} (\|\bar{p}q_x\|^2 + \|q\bar{p}_x\|^2), \\
 \varepsilon \int_{\mathbb{R}} [q_{xx} + 2qq_x] \bar{p}_x \, dx &\leq \frac{\varepsilon}{8} \|q_{xx}\|^2 + 2\varepsilon \|\bar{p}_x\|^2 + \varepsilon \|q\|^2 \|q_x\|_{\infty}^2 + \varepsilon \|\bar{p}_x\|^2 \\
 &\leq \frac{\varepsilon}{8} \|q_{xx}\|^2 + 3\varepsilon \|\bar{p}_x\|^2 + 2C_1 \varepsilon \|q_x\| \|q_{xx}\| \\
 &\leq \frac{\varepsilon}{4} \|q_{xx}\|^2 + 3\varepsilon \|\bar{p}_x\|^2 + 8C_1^2 \varepsilon \|q_x\|^2,
 \end{aligned}$$

and

$$\begin{aligned}
 (3.27) \quad -2\varepsilon \int_{\mathbb{R}} qq_x q_{xx} \, dx &\leq \frac{\varepsilon}{8} \|q_{xx}\|^2 + 8\varepsilon \|q\|^2 \|q_x\|_{\infty}^2 \\
 &\leq \frac{\varepsilon}{4} \|q_{xx}\|^2 + 512C_1^2 \varepsilon \|q_x\|^2,
 \end{aligned}$$

where we have used (3.4) for the estimate of $\|q\|^2$, the inequality $\|q_x\|_{\infty}^2 \leq 2\|q_x\| \|q_{xx}\|$, and the Cauchy-Schwarz inequality at various places. We thus have

$$\begin{aligned}
 (3.28) \quad \frac{d}{dt} \left(\frac{1}{2} \|q_x\|^2 - \int_{\mathbb{R}} \bar{p}q_x \, dx \right) &+ \frac{\bar{p}}{2} \|q_x\|^2 + \frac{\varepsilon}{2} \|q_{xx}\|^2 \\
 &\leq \frac{1}{\bar{p}} (\|\bar{p}q_x\|^2 + \|q\bar{p}_x\|^2) + (3\varepsilon + 1) \|\bar{p}_x\|^2 + 520C_1^2 \varepsilon \|q_x\|^2 \\
 &\leq \frac{1}{\bar{p}} (\|\bar{p}q_x\|^2 + \|q\bar{p}_x\|^2) + 4\|\bar{p}_x\|^2 + 520C_1^2 \varepsilon \|q_x\|^2,
 \end{aligned}$$

where we have used the condition $0 < \varepsilon \leq 1$. To control the first term on the righthand side of (3.28), we deduce

$$\begin{aligned}
 (3.29) \quad \|\bar{p}q_x\|^2 + \|q\bar{p}_x\|^2 &\leq \|\bar{p}\|_{\infty}^2 \|q_x\|^2 + \|q\|_{\infty}^2 \|\bar{p}_x\|^2 \\
 &\leq 2(\|\bar{p}\| \|\bar{p}_x\| \|q_x\|^2 + \|q\| \|q_x\| \|\bar{p}_x\|^2) \\
 &\leq C_8 \|\bar{p}_x\| \|q_x\| (\|q_x\| + \|\bar{p}_x\|) \\
 &\leq C_9 (\|\bar{p}_x\|^2 \|q_x\|^2 + \|\bar{p}_x\|^2) + \frac{\bar{p}^2}{4} \|q_x\|^2,
 \end{aligned}$$

using the inequality $\|f\|_{\infty}^2 \leq 2\|f\| \|f_x\|$ and the uniform estimates of $\|q\|^2$ and $\|\bar{p}\|^2$ due to (3.4) and (3.25), respectively. Substituting (3.29) into (3.28), we get

$$\begin{aligned}
 (3.30) \quad \frac{d}{dt} \left(\frac{1}{2} \|q_x\|^2 - \int_{\mathbb{R}} \bar{p}q_x \, dx \right) &+ \frac{\bar{p}}{4} \|q_x\|^2 + \frac{\varepsilon}{2} \|q_{xx}\|^2 \\
 &\leq C_{10} \|\bar{p}_x\|^2 \|q_x\|^2 + (4 + C_{10}) \|\bar{p}_x\|^2 + 520C_1^2 \varepsilon \|q_x\|^2.
 \end{aligned}$$

Step 2. In this step, we shall make a coupling of the estimates (3.30) and (3.23). For this purpose, we let

$$\rho = \frac{2}{R_1^{\bar{p},k}}.$$

After multiplying (3.23) by ρ and adding the result to (3.30), we have

$$(3.31) \quad \begin{aligned} \frac{d}{dt}L(t) + M(t) &\leq C_{10}\|\bar{p}_x\|^2\|q_x\|^2 \\ &+ \rho C_3\left(\sum_{j=2}^{2k}\int_{\mathbb{R}}\frac{(\bar{p})^{2k-j}(-\bar{p})^j}{(j-1)j}dx + \frac{(\bar{p})^{2k-1}}{2}\|q\|^2\right)\left(\int_{\mathbb{R}}\frac{(\bar{p}_x)^2}{\bar{p} + \bar{p}}dx\right) \\ &+ (4 + C_{10})\|\bar{p}_x\|^2 + 520C_1^2\varepsilon\|q_x\|^2, \end{aligned}$$

where

$$(3.32) \quad \begin{aligned} L(t) &= \\ &= \rho\left(\sum_{j=2}^{2k}\int_{\mathbb{R}}\frac{(\bar{p})^{2k-j}(-\bar{p})^j}{(j-1)j}dx + \frac{(\bar{p})^{2k-1}}{2}\|q\|^2\right) + \frac{1}{2}\|q_x\|^2 - \int_{\mathbb{R}}\bar{p}q_xdx \\ &\geq 2\sum_{m=1}^k\|\bar{p}\|_{L^{2m}}^{2m} + \frac{(\bar{p})^{2k-1}}{R_1^{\bar{p},k}}\|q\|^2 + \frac{1}{2}\|q_x\|^2 - \int_{\mathbb{R}}\bar{p}q_xdx \\ &= 2\sum_{m=2}^k\|\bar{p}\|_{L^{2m}}^{2m} + \|\bar{p}\|^2 + \frac{(\bar{p})^{2k-1}}{R_1^{\bar{p},k}}\|q\|^2 + \frac{1}{4}\|q_x\|^2 + \left\|\bar{p} - \frac{1}{2}q_x\right\|^2, \end{aligned}$$

where we have used (3.17), and

$$\begin{aligned} M(t) &= \rho\left(\sum_{j=0}^{2k-2}\int_{\mathbb{R}}(\bar{p})^{2k-2-j}(-\bar{p})^j(\bar{p}_x)^2dx\right. \\ &\quad \left.- \frac{R_2^{\bar{p},k}}{2}\|(\bar{p})^{k-1}\bar{p}_x\|^2 + \varepsilon(\bar{p})^{2k-1}\|q_x\|^2\right) + \frac{\bar{p}}{4}\|q_x\|^2 + \frac{\varepsilon}{2}\|q_{xx}\|^2. \end{aligned}$$

From (3.32) we can see that

$$L(t) \cong \sum_{m=1}^k\|\bar{p}\|_{L^{2m}}^{2m} + \|q\|^2 + \|q_x\|^2,$$

where \cong stands for the equivalence of quantities up to a multiplication by a constant. Then, there is a constant C_{10} which is independent of t and ε , such that

$$\begin{aligned} &C_9\|\bar{p}_x\|^2\|q_x\|^2 + \\ &+ \rho C_3\left(\sum_{j=2}^{2k}\int_{\mathbb{R}}\frac{(\bar{p})^{2k-j}(-\bar{p})^j}{(j-1)j}dx + \frac{(\bar{p})^{2k-1}}{2}\|q\|^2\right)\left(\int_{\mathbb{R}}\frac{(\bar{p}_x)^2}{\bar{p} + \bar{p}}dx\right) \\ &\leq C_{10}\left(\|\bar{p}_x\|^2 + \int_{\mathbb{R}}\frac{(\bar{p}_x)^2}{\bar{p} + \bar{p}}dx\right)L(t). \end{aligned}$$

It follows from (3.31) that

$$(3.33) \quad \frac{d}{dt}L(t) + M(t) \leq C_{11} \left(\|\tilde{p}_x\|^2 + \int_{\mathbb{R}} \frac{(\tilde{p}_x)^2}{\tilde{p} + \bar{p}} dx \right) L(t) + (4 + C_{10}) \|\tilde{p}_x\|^2 + 520C_1^2 \varepsilon \|q_x\|^2.$$

Applying Gronwall inequality to (3.33) and using the uniform estimates (3.4) and (3.25), we have in particular,

$$(3.34) \quad \|q_x(\cdot, t)\|^2 + \int_0^t \|q_x(\cdot, \tau)\|^2 + \varepsilon (\|q_x(\cdot, \tau)\|^2 + \|q_{xx}(\cdot, \tau)\|^2) d\tau \leq C_{12}.$$

We observe that the constants C_8, \dots, C_{12} are independent of t and ε . This will later allow us to take the zero diffusion limit to obtain the solution to the non-diffusible problem.

Step 3. In this step we derive a uniform-in-time estimate for \tilde{p}_x . By taking the L^2 inner products of the two equations in (3.11) with \tilde{p}_{xx} and $\tilde{p}q_{xx}$, respectively, we have

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|\tilde{p}_x\|^2 + \frac{\bar{p}}{2} \|q_x\|^2 \right) + \frac{1}{2} \|\tilde{p}_{xx}\|^2 + \varepsilon \bar{p} \|q_{xx}\|^2 \\ & \leq \|\tilde{p}q_x\|^2 + \|q\tilde{p}_x\|^2 - 2\varepsilon \bar{p} \int_{\mathbb{R}} q q_x q_{xx} dx \\ & \leq C_{13} (\|\tilde{p}_x\|^2 + \|q_x\|^2) + \frac{\varepsilon \bar{p}}{2} \|q_{xx}\|^2 + 64\bar{p}C_1^2 \varepsilon \|q_x\|^2, \end{aligned}$$

where we have used an argument similar to that used in deriving (3.27) and (3.29), and the uniform estimate of $\|q_x\|^2$ obtained from (3.34). From the above estimate, we have

$$(3.35) \quad \frac{d}{dt} \left(\frac{1}{2} \|\tilde{p}_x\|^2 + \frac{\bar{p}}{2} \|q_x\|^2 \right) + \frac{1}{2} \|\tilde{p}_{xx}\|^2 + \frac{\varepsilon \bar{p}}{2} \|q_{xx}\|^2 \leq C_{13} (\|\tilde{p}_x\|^2 + \|q_x\|^2) + 64\bar{p}C_1^2 \varepsilon \|q_x\|^2.$$

Integrating (3.35) with respect to t and using (3.25) and (3.34), we have in particular

$$(3.36) \quad \|\tilde{p}_x(t)\|^2 + \int_0^t \|\tilde{p}_{xx}(\tau)\|^2 d\tau \leq C_{14},$$

where the constant C_{14} is independent of t and ε . The combination of (3.34) and (3.36) completes the proof. \square

The next lemma gives the uniform-in-time estimate of the second-order spatial derivatives of the solution. The proof is similar to that of Lemma 3.3 by using previously established energy estimates. We shall only give a sketch of the proof.

Lemma 3.6 (H^2 -Estimate). *Let (p, q) be a solution to (3.1)–(3.2). Then, it holds that*

$$(3.37) \quad \|\bar{p}_{xx}(t)\|^2 + \|q_{xx}(t)\|^2 + \int_0^t (\|\bar{p}_{xxx}(\tau)\|^2 + \|q_{xx}(\tau)\|^2 + \varepsilon \|q_{xxx}(\tau)\|^2) d\tau \leq C_{15},$$

where the constant $C_{15} > 0$ is independent of t and ε .

Proof. Using a standard L^2 -based energy method and (3.26), we can show that

$$\frac{d}{dt}V(t) + W(t) \leq C_{16}(\|\bar{p}_{xx}\|^2 + \|\bar{p}_x\|^2 + \|q_x\|^2 + \varepsilon \|q_{xx}\|^2) + C_{17}\|\bar{p}_x\|^2V(t)$$

where the constants C_{16} and $C_{17} > 0$ are independent of t and ε , and

$$\begin{aligned} V(t) &= \frac{1}{2}\|q_{xx}\|^2 - \int_{\mathbb{R}} \bar{p}_x q_{xx} dx + 2\|\bar{p}_{xx}\|^2 \\ &\quad + 2\bar{p}\|q_{xx}\|^2 + 2\|\bar{p}_x\|^2 + 2\bar{p}\|q_x\|^2 \\ &= \frac{1}{4}\|q_{xx}\|^2 + \frac{1}{4}\|q_{xx} - 2\bar{p}_x\|^2 + 2\|\bar{p}_{xx}\|^2 \\ &\quad + 2\bar{p}\|q_{xx}\|^2 + \|\bar{p}_x\|^2 + 2\bar{p}\|q_x\|^2, \\ W(t) &= 2\|\bar{p}_{xxx}\|^2 + \frac{\bar{p}}{2}\|q_{xx}\|^2 + \varepsilon \left(2\bar{p} + \frac{1}{2}\right)\|q_{xxx}\|^2 \\ &\quad + 2\|\bar{p}_{xx}\|^2 + 2\varepsilon\bar{p}\|q_{xx}\|^2. \end{aligned}$$

Applying a Gronwall inequality to the above estimate, and using the uniform-in-time integrability of $\|\bar{p}_{xx}\|^2$, $\|\bar{p}_x\|^2$, $\|q_x\|^2$, and $\varepsilon\|q_{xx}\|^2$, we have obtained (3.37). This completes the proof. \square

Using the uniform-in-time estimates obtained in the preceding lemmas and maximum principle, we know that the function $p(x, t)$ is bounded away from zero for any time. Then, the combination of the L^2 , H^1 , and H^2 estimates derived above proves Proposition 3.2 which, along with the local existence result in Proposition 3.1, yields a global-in-time solution to (3.1)–(3.2). The uniqueness of the solution can be proved by using a standard argument (cf. [22]). Next, we show the asymptotic behavior. Indeed, from the proof of Lemmas 3.5–3.6, we can show that

$$\left| \frac{d}{dt} (\|\bar{p}_x\|_{H^1}^2 + \bar{p}\|q_x\|_{H^1}^2) \right| \leq C_{18} (\|\bar{p}_x\|_{H^2}^2 + \varepsilon\|q_x\|_{H^2}^2)$$

for some t -independent constant C_{18} . Then it holds that, by virtue of (3.3),

$$\int_0^t \left| \frac{d}{d\tau} (\|\bar{p}_x\|_{H^1}^2 + \|q_x\|_{H^1}^2) \right| d\tau \leq C_{18}.$$

Thus, it follows (also by (3.3)) that

$$(\|\tilde{p}_x\|_{H^1}^2 + \tilde{p}\|q_x\|_{H^1}^2)(t) \in W^{1,1}(0, \infty),$$

which implies

$$\lim_{t \rightarrow \infty} (\|\tilde{p}_x\|_{H^1}^2 + \|q_x\|_{H^1}^2)(t) = 0.$$

Since $\|f\|_\infty^2 \leq 2\|f\|\|f_x\|$, and $\|p - \tilde{p}\|^2$ and $\|q\|^2$ are uniformly bounded because of (3.3), it holds that

$$\lim_{t \rightarrow \infty} (\|p - \tilde{p}\|_{C^1}^2 + \|q\|_{C^1}^2)(t) = 0$$

with the help of Sobolev’s embedding theorem.

3.2. Diffusion limit of transformed system (proof of Theorem 2.3). First of all, we observe that by essentially repeating the arguments in Sections 3.1–3.2, one can show there exists a unique solution (p^0, q^0) to the non-diffusible problem (i.e., (1.4) with $\varepsilon = 0$), which satisfies

$$(3.38) \quad \begin{aligned} & \| (p^0 - \tilde{p})(t) \|_{H^2}^2 + \| q^0(t) \|_{H^2}^2 \\ & + \int_0^t (\| p_x^0(\tau) \|_{H^2}^2 + \| q_x^0(\tau) \|_{H^1}^2) d\tau \leq C_{19}, \end{aligned}$$

for some constant $C_{19} > 0$ which is independent of t . (See also [20].) Let p^ε be the solution to the diffusible problem, and let $\tilde{p} = p^0 - \tilde{p}$, $\hat{p} = p^0 - p^\varepsilon$ and $\hat{q} = q^0 - q^\varepsilon$. Then, we have the following Cauchy problem:

$$(3.39) \quad \begin{cases} \hat{p}_t - (\tilde{p}\hat{q})_x - \tilde{p}\hat{q}_x - (\hat{p}q^\varepsilon)_x = \hat{p}_{xx}, \\ \hat{q}_t - \hat{p}_x = -\varepsilon q_{xx}^\varepsilon - \varepsilon[(q^\varepsilon)^2]_x, \\ (\hat{p}, \hat{q})(x, 0) = (0, 0). \end{cases}$$

Taking the L^2 inner products of (3.39)₁ with \hat{p} and (3.39)₂ with $\tilde{p}\hat{q}$, we have

$$(3.40) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\hat{p}\|^2 + \tilde{p}\|\hat{q}\|^2) + \|\hat{p}_x\|^2 \\ & = - \int_{\mathbb{R}} (\tilde{p}\hat{q} + \hat{p}q^\varepsilon)\hat{p}_x dx - \varepsilon\tilde{p} \int_{\mathbb{R}} (q_{xx}^\varepsilon + q^\varepsilon q_x^\varepsilon)\hat{q} dx \\ & \leq \frac{1}{2} \|\hat{p}_x\|^2 + (\|\tilde{p}\|_\infty^2 \|\hat{q}\|^2 + \|q^\varepsilon\|_\infty^2 \|\hat{p}\|^2) \\ & \quad + \frac{\tilde{p}^2}{2} \|\hat{q}\|^2 + \varepsilon^2 (\|q_{xx}^\varepsilon\|^2 + 4\|q^\varepsilon\|_\infty^2 \|q_x^\varepsilon\|^2) \\ & \leq \frac{1}{2} \|\hat{p}_x\|^2 + C_{20} (\|\hat{q}\|^2 + \|\hat{p}\|^2) + C_{21} \varepsilon^2 (\|q_{xx}^\varepsilon\|^2 + \|q_x^\varepsilon\|^2) \end{aligned}$$

where we have used the Cauchy-Schwarz inequality, the Sobolev inequality $\|f\|_\infty^2 \leq 2\|f\| \|f_x\|$, and the uniform-in-time estimates of \bar{p} and q^ε , due to (3.38) and (3.3), respectively.

Taking the spatial derivatives of the two equations in (3.39), we get

$$(3.41) \quad \begin{cases} \hat{p}_{xt} - (\bar{p}\hat{q})_{xx} - \bar{p}\hat{q}_{xx} - (\hat{p}q^\varepsilon)_{xx} = \hat{p}_{xxx}, \\ \hat{q}_{xt} - \hat{p}_{xx} = -\varepsilon q_{xxx}^\varepsilon - \varepsilon[(q^\varepsilon)^2]_{xx}. \end{cases}$$

Taking the L^2 inner products of (3.41)₁ with \hat{p}_x and (3.41)₂ with \hat{q}_x , and using arguments similar to those used in deriving (3.40), we have

$$(3.42) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\hat{p}_x\|^2 + \bar{p}\|\hat{q}_x\|^2) + \|\hat{p}_{xx}\|^2 \\ &= - \int_{\mathbb{R}} [(\bar{p}\hat{q})_x + (\hat{p}q^\varepsilon)_x] \hat{p}_{xx} \, dx + \varepsilon \int_{\mathbb{R}} [q_{xxx}^\varepsilon + 2(q^\varepsilon q_x^\varepsilon)_x] \hat{q}_x \, dx \\ &\leq \frac{1}{2} \|\hat{p}_{xx}\| + C_{22}(\|\hat{p}_x\| + \|\hat{q}_x\| + \|\hat{p}\| + \|\hat{q}\|) \\ &\quad + C_{23}\varepsilon^2(\|q_{xxx}^\varepsilon\|^2 + \|q_{xx}^\varepsilon\|^2 + \|q_x^\varepsilon\|^2). \end{aligned}$$

By coupling (3.40) and (3.42) together, we have

$$(3.43) \quad \begin{aligned} & \frac{d}{dt} (\|\hat{p}\|_{H^1}^2 + \bar{p}\|\hat{q}\|_{H^1}^2) + \|\hat{p}_x\|_{H^1}^2 \\ &\leq C_{24}(\|\hat{p}\|_{H^1}^2 + \bar{p}\|\hat{q}\|_{H^1}^2) + C_{25}\varepsilon^2\|q_x^\varepsilon\|_{H^2}^2. \end{aligned}$$

The Gronwall inequality then implies that

$$\|\hat{p}(t)\|_{H^1}^2 + \bar{p}\|\hat{q}(t)\|_{H^1}^2 \leq (e^{C_{24}t} C_{25}) \varepsilon \left(\int_0^t \varepsilon \|q_x^\varepsilon\|_{H^2}^2 \, d\tau \right)$$

where we have used the fact that $(\hat{p}, \hat{q})(x, 0) = (0, 0)$. Using the uniform temporal integrability of $\varepsilon\|q_x^\varepsilon\|_{H^2}^2$ due to (3.3), we get

$$(3.44) \quad \|\hat{p}(t)\|_{H^1}^2 + \|\hat{q}(t)\|_{H^1}^2 \leq C_{26}e^{C_{24}t} \varepsilon, \quad \forall t > 0.$$

Furthermore, by plugging (3.44) into (3.43), we have

$$\int_0^t \|\hat{p}_x(\tau)\|_{H^1}^2 \, d\tau \leq C_{27}(e^{C_{24}t} + 1)\varepsilon.$$

This completes the proof of Theorem 2.3.

4. ALGEBRAIC DECAY RATE (PROOF OF THEOREM 2.7)

This section is devoted to further investigation of the qualitative behavior of the solution obtained in Theorem 2.1. Although Theorem 2.1 gives a definite answer to the question of global well-posedness and long-time behavior of classical solutions to (1.4), it provides no information about the *explicit decay rate* of the perturbations, which is physically important and mathematically challenging. In this section, we compute the explicit decay rate of the solution with respect to time under mild conditions on initial data.

The proof relies heavily on the energy framework developed in Section 3.1. Let $(\bar{p}, 0)$ be any given constant state satisfying $\bar{p} > 0$. Without loss of generality, we assume $\bar{p} = 1$. Upon integrating the perturbed system with respect to x from $-\infty$ to x , we have the following initial value problem:

$$(4.1) \quad \begin{cases} \varphi_t - \varphi_x \psi_x - \psi_x = \varphi_{xx}, \\ \psi_t - \varphi_x = \varepsilon \psi_{xx} + \varepsilon (\psi_x)^2, \\ (\varphi, \psi)(x, 0) = (\varphi_0, \psi_0)(x), \end{cases}$$

where

$$\varphi(x, t) = \int_{-\infty}^x (p(y, t) - 1) dy, \quad \psi(x, t) = \int_{-\infty}^x q(y, t) dy$$

denote the anti-derivatives of the perturbed functions $p - 1$ and $q - 0$, respectively. Following a standard procedure, we carry out energy estimates under the *a priori* assumption:

$$(4.2) \quad \sup_{0 \leq t \leq T} (\|\varphi(t)\|^2 + \|\psi(t)\|^2) \leq \eta,$$

for some small constant $\eta > 0$ which will be determined later.

Remark 4.1. Since we are concerned with the explicit decay rate of the solution for fixed ε , throughout this section we use D_i to denote generic constants which are independent of t and the unknown functions, but may depend on ε , in order to distinguish such constants from those in the previous sections.

Proof of Theorem 2.7. For convenience, we divide the proof into four steps.

Step 1. Taking the L^2 inner products of the equations in (4.1) with φ and ψ , respectively, then adding the results, we have

$$(4.3) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\varphi\|^2 + \|\psi\|^2) + \|\varphi_x\|^2 + \varepsilon \|\psi_x\|^2 \\ &= \int_{\mathbb{R}} \varphi_x \psi_x \varphi dx + \varepsilon \int_{\mathbb{R}} (\psi_x)^2 \psi dx \\ &\leq \frac{1}{2} \|\varphi\|_{\infty} (\|\varphi_x\|^2 + \|\psi_x\|^2) + \varepsilon \|\psi\|_{\infty} \|\psi_x\|^2 \\ &\leq \|\varphi\|^{1/2} \|\varphi_x\|^{1/2} (\|\varphi_x\|^2 + \|\psi_x\|^2) + 2\varepsilon \|\psi\|^{1/2} \|\psi_x\|^{1/2} \|\psi_x\|^2. \end{aligned}$$

By definition and Theorem 2.1, we know that

$$\|\varphi_x\|^{1/2} = \|p - 1\|^{1/2} \leq D_1 \quad \text{and} \quad \|\psi_x\|^{1/2} = \|q\|^{1/2} \leq D_2$$

for some constants D_1 and D_2 which are independent of t and ε . Thus, we update (4.3) as

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\varphi\|^2 + \|\psi\|^2) + \|\varphi_x\|^2 + \varepsilon \|\psi_x\|^2 \\ \leq D_1 \eta^{1/4} (\|\varphi_x\|^2 + \|\psi_x\|^2) + \varepsilon D_2 \eta^{1/4} \|\psi_x\|^2, \end{aligned}$$

where we have used (4.2). We observe that when

$$\eta \leq \min \left\{ \left(\frac{1}{2D_1} \right)^4, \left(\frac{\varepsilon}{2(D_1 + \varepsilon D_2)} \right)^4 \right\},$$

it holds that

$$\frac{1}{2} \frac{d}{dt} (\|\varphi\|^2 + \|\psi\|^2) + \frac{1}{2} \|\varphi_x\|^2 + \frac{\varepsilon}{2} \|\psi_x\|^2 \leq 0,$$

which implies

$$(4.4) \quad \|\varphi(t)\|^2 + \|\psi(t)\|^2 + \int_0^t (\|\varphi_x(\tau)\|^2 + \varepsilon \|\psi_x(\tau)\|^2) d\tau \leq \|\varphi_0\|^2 + \|\psi_0\|^2.$$

From a standard continuation argument, we know that (4.2) holds true for all time, provided that

$$\|\varphi_0\|^2 + \|\psi_0\|^2 \leq \frac{1}{2} \min \left\{ \left(\frac{1}{2D_1} \right)^4, \left(\frac{\varepsilon}{2(D_1 + \varepsilon D_2)} \right)^4 \right\}.$$

Next, we carry out weighted-in-time energy estimates and identify the explicit decay rate by a bootstrap argument.

Step 2. Taking ∂_x to the equations in (4.1), we have

$$(4.5) \quad \begin{cases} \varphi_{xt} - \varphi_{xx}\psi_x - \varphi_x\psi_{xx} - \psi_{xx} = \varphi_{xxx}, \\ \psi_{xt} - \varphi_{xx} = \varepsilon\psi_{xxx} + 2\varepsilon\psi_x\psi_{xx}. \end{cases}$$

The operation

$$\int_{\mathbb{R}} [(4.5)_1 \times (t + 1)\varphi_x + (4.5)_2 \times (t + 1)\psi_x] dx$$

then yields

$$(4.6) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} [(t+1)(\|\varphi_x\|^2 + \|\psi_x\|^2)] + (t+1)\|\varphi_{xx}\|^2 + \varepsilon(t+1)\|\psi_{xx}\|^2 \\ &= \frac{1}{2}(\|\varphi_x\|^2 + \|\psi_x\|^2) + (t+1) \int_{\mathbb{R}} (\varphi_{xx}\psi_x + \varphi_x\psi_{xx})\varphi_x \, dx. \end{aligned}$$

We estimate the nonlinear term on the right-hand side of (4.6) as follows:

$$\begin{aligned} & t+1 \int_{\mathbb{R}} (\varphi_{xx}\psi_x + \varphi_x\psi_{xx})\varphi_x \, dx \\ &= \frac{(t+1)}{2} \int_{\mathbb{R}} \varphi_x^2 \psi_{xx} \, dx \leq \frac{t+1}{4\delta} \|\varphi_x b\|_{L^4}^4 + \frac{\delta(t+1)}{4} \|\psi_{xx}\|^2, \end{aligned}$$

where $\delta > 0$ is a constant to be determined. With the help of the Gagliardo-Nirenberg inequality

$$\|\nabla f\|_{L^4} \lesssim \|f\|_{\infty}^{1/2} \|\nabla^2 f\|^{1/2}, \quad \forall f \in W^{2,2}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n),$$

we update (4.6) (by choosing $\delta = 2\varepsilon$) as

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [(t+1)(\|\varphi_x\|^2 + \|\psi_x\|^2)] + (t+1)\|\varphi_{xx}\|^2 + \varepsilon(t+1)\|\psi_{xx}\|^2 \\ & \leq \frac{1}{2}(\|\varphi_x\|^2 + \|\psi_x\|^2) + \frac{D_3(t+1)}{8\varepsilon} \|\varphi\|_{\infty}^2 \|\varphi_{xx}\|^2 + \frac{\varepsilon(t+1)}{2} \|\psi_{xx}\|^2 \\ & \leq \frac{1}{2}(\|\varphi_x\|^2 + \|\psi_x\|^2) + \frac{D_3(t+1)}{4\varepsilon} \|\varphi\| \|\varphi_x\| \|\varphi_{xx}\|^2 + \frac{\varepsilon(t+1)}{2} \|\psi_{xx}\|^2 \\ & \leq \frac{1}{2}(\|\varphi_x\|^2 + \|\psi_x\|^2) + \frac{D_4(t+1)}{4\varepsilon} \eta^{1/2} \|\varphi_{xx}\|^2 + \frac{\varepsilon(t+1)}{2} \|\psi_{xx}\|^2, \end{aligned}$$

where we have applied (4.2) and the uniform estimate of $\|\varphi_x\| = \|p-1\|$ obtained from Theorem 2.1. It then follows that, when $\eta \leq \left(\frac{2\varepsilon}{D_4}\right)^2$, it holds that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [(t+1)(\|\varphi_x\|^2 + \|\psi_x\|^2)] + \frac{(t+1)}{2} \|\varphi_{xx}\|^2 + \frac{\varepsilon}{2} (t+1) \|\psi_{xx}\|^2 \\ & \leq \frac{1}{2}(\|\varphi_x\|^2 + \|\psi_x\|^2). \end{aligned}$$

Integrating the above inequality over time and using (4.4), we get for $\forall t > 0$

$$(4.7) \quad \begin{aligned} & (t+1)(\|\varphi_x\|^2 + \|\psi_x\|^2) + \int_0^t (\tau+1)(\|\varphi_{xx}(\tau)\|^2 + \|\psi_{xx}(\tau)\|^2) \, d\tau \\ & \leq D_5(\|\varphi_0\|_{H^1}^2 + \|\psi_0\|_{H^1}^2). \end{aligned}$$

This gives the first-order algebraic decay rate of the perturbations.

Step 3. By repeating the above procedure, and using the uniform-in-time estimates of $\|\varphi_x\|_\infty^2 = \|p - 1\|_\infty^2$ and $\|\psi_x\|_\infty^2 = \|q\|_\infty^2$ obtained from Theorem 2.1, we can show that

$$\begin{aligned}
 (4.8) \quad & \frac{1}{2} \frac{d}{dt} [(t + 1)(\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2)] \\
 & + \frac{(t + 1)}{2} \|\varphi_{xxx}\|^2 + \frac{\varepsilon(t + 1)}{2} \|\psi_{xxx}\|^2 \\
 & \leq \frac{1}{2} (\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2) \\
 & \quad + D_6(t + 1)(\|\varphi_x\|_\infty^2 + \|\psi_x\|_\infty^2)(\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2) \\
 & \leq \frac{1}{2} (\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2) + D_7(t + 1)(\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2),
 \end{aligned}$$

which, together with (4.7) and Theorem 2.1, implies that

$$\begin{aligned}
 (4.9) \quad & (t + 1)(\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2) + \\
 & + \int_0^t (\tau + 1)(\|\varphi_{xxx}(\tau)\|^2 + \|\psi_{xxx}(\tau)\|^2) d\tau \leq D_8, \quad \forall t > 0.
 \end{aligned}$$

Next, we push the decay rate of the second-order derivatives further.

Step 4. Indeed, as a consequence of (4.7), (4.9), and the inequality $\|f\|_\infty^2 \leq 2\|f\| \|f_x\|$, we have

$$(4.10) \quad \|\varphi_x\|_\infty^2 \lesssim (t + 1)^{-1} \quad \text{and} \quad \|\psi_x\|_\infty^2 \lesssim (t + 1)^{-1}.$$

Then, multiplying (4.8) by $(t + 1)$, we infer that

$$\begin{aligned}
 (4.11) \quad & \frac{1}{2} \frac{d}{dt} [(t + 1)^2(\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2)] + \\
 & + \frac{(t + 1)^2}{2} \|\varphi_{xxx}\|^2 + \frac{\varepsilon(t + 1)^2}{2} \|\psi_{xxx}\|^2 \\
 & \leq (t + 1)(\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2) \\
 & \quad + D_6(t + 1)^2(\|\varphi_x\|_\infty^2 + \|\psi_x\|_\infty^2)(\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2) \\
 & \leq (t + 1)(\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2) + D_7(t + 1)(\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2)
 \end{aligned}$$

where we have used (4.10). Integrating (4.11), we get

$$\begin{aligned}
 & (t + 1)^2(\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2) \\
 & + \int_0^t (\tau + 1)^2(\|\varphi_{xxx}(\tau)\|^2 + \|\psi_{xxx}(\tau)\|^2) d\tau \leq D_9, \quad \forall t > 0.
 \end{aligned}$$

The decay rate of the third-order derivatives can be proved in a completely similar fashion, and we omit further details. This completes the proof of Theorem 2.7. \square

5. PARABOLIC SMOOTHING EFFECT (PROOF OF THEOREM 2.9)

5.1. Setup. Consider the following approximation to (1.4): let $p^{(0)} \equiv 0$ and $\mathbf{q}^{(0)} \equiv 0$, for $n = 0$, and

$$\begin{aligned} p_t^{(n)} - \Delta p^{(n)} &= \nabla \cdot (p^{(n-1)} \mathbf{q}^{(n-1)}), & p^{(n)}(\mathbf{x}, 0) &= p_0(\mathbf{x}), \\ \mathbf{q}_t^{(n)} - \varepsilon \Delta \mathbf{q}^{(n)} &= \nabla(\varepsilon |\mathbf{q}^{(n-1)}|^2 + p^{(n-1)}), & \mathbf{q}^{(n)}(\mathbf{x}, 0) &= \mathbf{q}_0(\mathbf{x}), \end{aligned}$$

for $n > 0$. By properties of the heat equation, we know that for all $n \geq 0$, $(p^{(n)}, \mathbf{q}^{(n)})$ is a real analytic pair of functions on \mathbb{R}^d for all $t > 0$. To show that the limiting function (p, \mathbf{q}) is real analytic and solves (1.4), we consider the complex extension of (1.4) and obtain uniform estimates in L^q . Indeed, let us consider

$$p \mapsto p + i\pi, \quad \mathbf{q} \mapsto \mathbf{q} + i\mathbf{u}, \quad \mathbf{x} \mapsto \mathbf{x} + i\mathbf{y} \in \mathbb{C}^d.$$

Then (1.4) becomes the system of inhomogeneous heat equations:

$$(5.1) \quad \begin{cases} p_t^{(n)} - \Delta p^{(n)} = \nabla \cdot (p^{(n-1)} \mathbf{q}^{(n-1)} - \pi^{(n-1)} \mathbf{u}^{(n-1)}), & p^{(n)}(\mathbf{x}, 0) = p_0(\mathbf{x}), \\ \pi_t^{(n)} - \Delta \pi^{(n)} = \nabla \cdot (p^{(n-1)} \mathbf{u}^{(n-1)} + \pi^{(n-1)} \mathbf{q}^{(n-1)}), & \pi(\mathbf{x}, 0) = 0, \\ \mathbf{q}_t^{(n)} - \varepsilon \Delta \mathbf{q}^{(n)} = \varepsilon \nabla(|\mathbf{q}^{(n-1)}|^2 + |\mathbf{u}^{(n-1)}|^2) + \nabla p^{(n-1)}, & \mathbf{q}(\mathbf{x}, 0) = \mathbf{q}_0(\mathbf{x}), \\ \mathbf{u}_t^{(n)} - \varepsilon \Delta \mathbf{u}^{(n)} = \nabla \pi^{(n-1)}, & \mathbf{u}(\mathbf{x}, 0) = \mathbf{0}. \end{cases}$$

For $\tilde{\alpha} \in \mathbb{R}^d$, let us make the following change of variables:

$$(5.2) \quad \begin{aligned} P_{\tilde{\alpha}}^{(n)}(\mathbf{x}, t) &:= p^{(n)}(\mathbf{x}, \tilde{\alpha}t, t), & \Pi_{\tilde{\alpha}}^{(n)}(\mathbf{x}, t) &:= \pi^{(n)}(\mathbf{x}, \tilde{\alpha}t, t), \\ Q_{\tilde{\alpha}}^{(n)}(\mathbf{x}, t) &:= \mathbf{q}^{(n)}(\mathbf{x}, \tilde{\alpha}t, t), & U_{\tilde{\alpha}}^{(n)}(\mathbf{x}, t) &:= \mathbf{u}^{(n)}(\mathbf{x}, \tilde{\alpha}t, t). \end{aligned}$$

Observe that given $f(\mathbf{x}, \tilde{\alpha}t, t) + ig(\mathbf{x}, \tilde{\alpha}t, t)$ analytic, then by the chain rule and Cauchy-Riemann equations, we have

$$\begin{aligned} \partial_t(f(\mathbf{x}, \tilde{\alpha}t, t)) &= \tilde{\alpha} \cdot (\nabla_y f)(\mathbf{x}, \tilde{\alpha}t, t) + (\partial_t f)(\mathbf{x}, \tilde{\alpha}t, t) \\ &= -\tilde{\alpha} \cdot (\nabla_x g)(\mathbf{x}, \tilde{\alpha}t, t) + (\partial_t f)(\mathbf{x}, \tilde{\alpha}t, t), \\ \partial_t(g(\mathbf{x}, \tilde{\alpha}t, t)) &= \tilde{\alpha} \cdot (\nabla_y g)(\mathbf{x}, \tilde{\alpha}t, t) + (\partial_t g)(\mathbf{x}, \tilde{\alpha}t, t) \\ &= \tilde{\alpha} \cdot (\nabla_x f)(\mathbf{x}, \tilde{\alpha}t, t) + (\partial_t g)(\mathbf{x}, \tilde{\alpha}t, t). \end{aligned}$$

Thus, upon integrating by parts and applying Duhamel’s formula, the solution of the corresponding system can be expressed in the following way:

$$(5.3a) \quad P_\alpha^{(n)}(\mathbf{x}, t) = e^{t\Delta} p_0(\mathbf{x}) - \int_0^t \tilde{\alpha} \cdot \nabla e^{(t-s)\Delta} \Pi_\alpha^{(n)} ds + \int_0^t \nabla e^{(t-s)\Delta} \cdot (P_\alpha^{(n-1)} Q_\alpha^{(n-1)} - \Pi_\alpha^{(n-1)} U_\alpha^{(n-1)}) ds,$$

$$(5.3b) \quad \Pi_\alpha^{(n)}(\mathbf{x}, t) = \int_0^t \tilde{\alpha} \cdot \nabla e^{(t-s)\Delta} P_\alpha^{(n)} ds - \int_0^t \nabla e^{(t-s)\Delta} \cdot (P_\alpha^{(n-1)} U_\alpha^{(n-1)} + \Pi_\alpha^{(n-1)} Q_\alpha^{(n-1)}) ds$$

$$(5.3c) \quad Q_\alpha^{(n)}(\mathbf{x}, t) = e^{t\varepsilon\Delta} \mathbf{q}_0(\mathbf{x}) - \int_0^t \tilde{\alpha} \cdot \nabla e^{(t-s)\varepsilon\Delta} U_\alpha^{(n)} ds, - \varepsilon \int_0^t \nabla e^{(t-s)\varepsilon\Delta} (|Q_\alpha^{(n-1)}|^2 + |U_\alpha^{(n-1)}|^2) ds - \int_0^t \nabla e^{(t-s)\varepsilon\Delta} P_\alpha^{(n-1)} ds,$$

$$(5.3d) \quad U_\alpha^{(n)}(\mathbf{x}, t) = \int_0^t \tilde{\alpha} \cdot \nabla e^{(t-s)\varepsilon\Delta} Q_\alpha^{(n)} ds - \int_0^t \nabla e^{(t-s)\varepsilon\Delta} \Pi_\alpha^{(n-1)} ds.$$

Let $d < q < \infty$. We define the functional $\varphi^{(n)}$ by

$$\varphi^{(n)}(t) := \|P_\alpha^{(n)}(\cdot, t)\|_{L_x^q} + \|\Pi_\alpha^{(n)}(\cdot, t)\|_{L_x^q} + \|Q_\alpha^{(n)}(\cdot, t)\|_{L_x^q} + \|U_\alpha^{(n)}(\cdot, t)\|_{L_x^q}.$$

Given $T > 0$, define

$$\Phi_T^{(n)} := \sup_{0 \leq t \leq T} \varphi^{(n)}(t).$$

To prove Theorem 2.9, we show the following:

- (1) $\Phi_T^{(n)} < \infty$, for some $|\tilde{\alpha}|$, T sufficiently small.
- (2) $(p^{(n)}, \pi^{(n)}, \mathbf{q}^{(n)}, \mathbf{u}^{(n)}) \rightarrow (p, \pi, \mathbf{q}, \mathbf{u})$ uniformly on compact subsets of a domain in $\mathbb{C}^d \times (0, T)$.
- (3) (p, \mathbf{q}) are classical solutions to (1.4), which are real-analytic for short time.

5.2. A priori estimates for approximate solutions. We will establish the following uniform bounds for $\Phi_T^{(n)}$.

Lemma 5.1. *Let $d < q < \infty$ and $T > 0$. Suppose that*

$$(5.4) \quad \|p_0\|_{L^q} + \|\mathbf{q}_0\|_{L^q} \leq M_q.$$

Then, there exists an absolute constant $C_1 > 0$ such that if $\tilde{\alpha} \in \mathbb{R}^d$ satisfies

$$(5.5) \quad 2C_1|\tilde{\alpha}|T^{1/2} < \min\{1, \varepsilon^{1/2}\},$$

it holds that

$$\Phi_T^{(n)} \leq 2M_q + 2C_1\varepsilon^{-1/2}T^{1/2}\Phi_T^{(n-1)} + 2C_2T^{(1-d/q)/2}(\Phi_T^{(n-1)})^2, \quad n \geq 1,$$

for some absolute constant $C_2 > 0$ (given by (5.9) below). In particular, if

$$(5.6) \quad T \leq \min \left\{ \frac{\varepsilon}{64C_1^2}, \frac{1}{(32C_2M_q)^{2/(1-d/q)}} \right\},$$

then

$$(5.7) \quad \Phi_T^{(n)} \leq 4M_q, \quad n \geq 1.$$

Since $(p^{(n)}, \pi^{(n)}, \mathbf{q}^{(n)}, \mathbf{u}^{(n)})$ is real analytic on \mathbb{R}^d , for all $t > 0$, we immediately have the following result from Lemma 5.1 and the definition (5.2).

Corollary 5.2. *Let $d < q < \infty$ and $T > 0$. Suppose that (5.4), (5.5), and (5.6) hold. Then, for*

$$(5.8) \quad |\mathbf{y}| \leq \frac{1}{2}C_1^{-1}T^{1/2} \min\{1, \varepsilon^{1/2}\},$$

we have

$$\begin{aligned} & \|p^{(n)}(\cdot, \mathbf{y}, t)\|_{L_x^q} + \|\pi^{(n)}(\cdot, \mathbf{y}, t)\|_{L_x^q} \\ & + \|\mathbf{q}^{(n)}(\cdot, \mathbf{y}, t)\|_{L_x^q} + \|\mathbf{u}^{(n)}(\cdot, \mathbf{y}, t)\|_{L_x^q} \leq 4M_q, \quad n \geq 1, \end{aligned}$$

for each $t \in (0, T)$.

To prove Lemma 5.1, we will make use of the following elementary lemma regarding estimates for the heat kernel. Note that we have rescaled the heat kernel by a factor of γ (cf. [6]).

Lemma 5.3. *Let $T > 0$. Then, the heat kernel $e^{\gamma t \Delta}$ satisfies*

$$\sup_{t>0} \|e^{\gamma t \Delta}\|_{L^1(\mathbb{R}^d)} \leq 1,$$

and there exists an absolute constant $C = C(r)$ such that

$$\gamma \int_0^T \|\nabla e^{\gamma t \Delta}\|_{L^r(\mathbb{R}^d)} dt \leq C(r)(\gamma T)^{(r+d-dr)/(2r)}, \quad 1 \leq r < \frac{d}{d-1}.$$

Next, we prove Lemma 5.1 by using Lemma 5.3.

Proof of Lemma 5.1. Let $1/q + 1/q' = 1$ with $q > d$, and fix $T > 0$. We estimate (5.3) by applying Young's convolution inequality (with $1 + 1/q = 1/q' + 2/q$), Lemma 5.3, and the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} \|P_\alpha^{(n)}(\cdot, t)\|_{L_x^q} &\leq \|p_0\|_{L_x^q} + C(1)|\tilde{\alpha}|T^{1/2}\|\Pi_\alpha^{(n)}\|_{L_T^\infty L_x^q} \\ &\quad + 2C(q')T^{1/2-d/2+d(q-1)/(2q)} \\ &\quad \times (\|P_\alpha^{(n-1)}\|_{L_T^\infty L_x^q} \|Q_\alpha^{(n-1)}\|_{L_T^\infty L_x^q} \\ &\quad + \|\Pi_\alpha^{(n-1)}\|_{L_T^\infty L_x^q} \|U_\alpha^{(n-1)}\|_{L_T^\infty L_x^q}), \\ \|P_\alpha^{(n)}(\cdot, t)\|_{L_x^q} &\leq C(1)|\tilde{\alpha}|T^{1/2}\|P_\alpha^{(n)}\|_{L_T^\infty L_x^q} \\ &\quad + 2C(q')T^{1/2-d/2+d(q-1)/(2q)} \\ &\quad \times (\|P_\alpha^{(n-1)}\|_{L_T^\infty L_x^q} \|U_\alpha^{(n-1)}\|_{L_T^\infty L_x^q} \\ &\quad + \|\Pi_\alpha^{(n-1)}\|_{L_T^\infty L_x^q} \|Q_\alpha^{(n-1)}\|_{L_T^\infty L_x^q}). \end{aligned}$$

We also estimate $Q_\alpha^{(n)}, U_\alpha^{(n)}$ similarly:

$$\begin{aligned} \|Q_\alpha^{(n)}(\cdot, t)\|_{L_x^q} &\leq \|q_0\|_{L_x^q} + C(1)|\tilde{\alpha}|\varepsilon^{-1/2}T^{1/2}\|U_\alpha^{(n)}\|_{L_T^\infty L_x^q} \\ &\quad + 2C(q')T^{1/2-d/2+d(q-1)/(2q)} \\ &\quad \times (\|Q_\alpha^{(n-1)}\|_{L_T^\infty L_x^q}^2 + \|U_\alpha^{(n-1)}\|_{L_T^\infty L_x^q}^2) \\ &\quad + C(1)\varepsilon^{-1/2}T^{1/2}\|P_\alpha^{(n-1)}\|_{L_T^\infty L_x^q}, \\ \|U_\alpha^{(n)}(\cdot, t)\|_{L_x^q} &\leq C(1)|\tilde{\alpha}|\varepsilon^{-1/2}T^{1/2}\|Q_\alpha^{(n)}\|_{L_T^\infty L_x^q} \\ &\quad + C(1)\varepsilon^{-1/2}T^{1/2}\|\Pi_\alpha^{(n-1)}\|_{L_T^\infty L_x^q}. \end{aligned}$$

Therefore, by adding the above inequalities, using the fact that $\tilde{\alpha}$ satisfies (5.5) with C_0 satisfying (5.9), and taking the supremum over $t \in [0, T]$, we obtain

$$\frac{1}{2}\Phi_T^{(n)} \leq M_q + C_1\varepsilon^{-1/2}T^{1/2}\Phi_T^{(n-1)} + C_2T^{(1-d/q)/2}(\Phi_T^{(n-1)})^2,$$

where

$$(5.9) \quad C_1 = C(1), \quad C_2 = 6C(q').$$

By induction, we suppose that $\Phi_T^{(n-1)} \leq 4M_q$. Then, $\Phi_T^{(n)} \leq 4M_q$, provided that (5.6) holds. \square

5.3. Contraction. In this section, we show that the map, \mathcal{T} , induced by Duhamel’s formula applied to $(\mathbf{p}^{(n)}, \boldsymbol{\pi}^{(n)}, \mathbf{q}^{(n)}, \mathbf{u}^{(n)})$ of (5.1), is a contraction in the ball Z defined by

$$Z = \{z \in C([0, T]; L^q(\mathbb{R}^d)^{2+2d}) : \|z - e^{t\Delta}(\mathbf{p}_0, 0, \mathbf{q}_0, 0)\|_{L_T^\infty L_x^q} \leq 4M_q\},$$

for $T > 0$ sufficiently small.

Lemma 5.4. *Let C_1, C_2 be absolute constants given by (5.9). Suppose that $T > 0$ satisfies*

$$(5.10) \quad T \leq \min \left\{ \frac{\varepsilon}{256C_1^2}, \frac{1}{(64C_2M_q)^{2/(1-d/q)}} \right\}.$$

Then, $\mathcal{T} : Z \rightarrow Z$ defines a self-map and is a contraction.

Proof. First, we show that $\mathcal{T}^{(n)}$ is a self-map. Indeed, observe that by (5.10) and Corollary 5.2, we have

$$\begin{aligned} \|\mathbf{p}^{(n)} - e^{t\Delta}\mathbf{p}_0\|_{L_T^\infty L_x^q} &\leq 2C(q')T^{1/2-d/2+d(q-1)/(2q)} \\ &\quad \times (\|\mathbf{p}^{(n-1)}\|_{L_T^\infty L_x^q} \|\mathbf{q}^{(n-1)}\|_{L_T^\infty L_x^q} \\ &\quad + \|\boldsymbol{\pi}^{(n-1)}\|_{L_T^\infty L_x^q} \|\mathbf{u}^{(n-1)}\|_{L_T^\infty L_x^q}) \\ &\leq 4C(q')T^{1/2-d/(2q)}M_q^2 \leq M_q. \end{aligned}$$

We have similar estimates for $\boldsymbol{\pi}^{(n)}, \mathbf{q}^{(n)}, \mathbf{u}^{(n)}$, which imply that \mathcal{T} is a self-map.

To show that \mathcal{T} is a contraction, observe that

$$\begin{aligned} \mathbf{p}^{(n)} - \mathbf{p}^{(n-1)} &= \int_0^t \nabla e^{(t-s)\Delta}(\mathbf{p}_\alpha^{(n-1)} - \mathbf{p}^{(n-2)})\mathbf{q}^{(n-1)} \, ds \\ &\quad + \int_0^t \nabla e^{(t-s)\Delta} \nabla \mathbf{p}^{(n-2)}(\mathbf{q}^{(n-1)} - \mathbf{q}^{(n-2)}) \, ds \\ &\quad - \int_0^t \nabla e^{(t-s)\Delta} \cdot (\boldsymbol{\pi}^{(n-1)} - \boldsymbol{\pi}^{(n-2)})\mathbf{u}^{(n-1)} \, ds \\ &\quad - \int_0^t \nabla e^{(t-s)\Delta} \cdot \boldsymbol{\pi}^{(n-2)}(\mathbf{u}^{(n-1)} - \mathbf{u}^{(n-2)}) \, ds. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} &\|\mathbf{p}^{(n)} - \mathbf{p}^{(n-1)}\|_{L_T^\infty L_x^q} \\ &\leq 4C(q')M_qT^{(q-d)/(2q)}\|\mathbf{p}^{(n-1)} - \mathbf{p}^{(n-2)}\|_{L_T^\infty L_x^q} \\ &\quad + 4C(q')M_qT^{(q-d)/(2q)}(\|\mathbf{q}^{(n-1)} - \mathbf{q}^{(n-2)}\|_{L_T^\infty L_x^q} \\ &\quad + \|\boldsymbol{\pi}^{(n-1)} - \boldsymbol{\pi}^{(n-2)}\|_{L_T^\infty L_x^q} + \|\mathbf{u}^{(n-1)} - \mathbf{u}^{(n-2)}\|_{L_T^\infty L_x^q}). \end{aligned}$$

Similarly,

$$\begin{aligned} & \|\pi_\alpha^{(n)} - \pi_\alpha^{(n-1)}\|_{L_T^\infty L_x^q} \\ & \leq 4C(q')M_q T^{(q-d)/(2q)} \|p^{(n-1)} - p^{(n-2)}\|_{L_T^\infty L_x^q} \\ & \quad + 4C(q')M_q T^{(q-d)/(2q)} (\|Q_\alpha^{(n-1)} - Q_\alpha^{(n-2)}\|_{L_T^\infty L_x^q} \\ & \quad + \|\pi_\alpha^{(n-1)} - \pi_\alpha^{(n-2)}\|_{L_T^\infty L_x^q} + \|\mathbf{u}^{(n-1)} - \mathbf{u}^{(n-2)}\|_{L_T^\infty L_x^q}). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \|\mathbf{q}^{(n)} - \mathbf{q}^{(n-1)}\|_{L_T^\infty L_x^q} \\ & \leq C(1)\varepsilon^{-1/2}T^{1/2}\|p^{(n-1)} - p^{(n-2)}\|_{L_T^\infty L_x^q} + 8C(q')M_q T^{(q-d)/(2q)} \\ & \quad \times (\|\mathbf{q}^{(n-1)} - \mathbf{q}^{(n-2)}\|_{L_T^\infty L_x^q} + \|\mathbf{u}^{(n-1)} - \mathbf{u}^{(n-2)}\|_{L_T^\infty L_x^q}), \end{aligned}$$

and

$$\|\mathbf{u}^{(n)} - \mathbf{u}^{(n-1)}\|_{L_T^\infty L_x^q} \leq C(1)\varepsilon^{-1/2}T^{1/2}\|\pi_\alpha^{(n-1)} - \pi_\alpha^{(n-2)}\|_{L_T^\infty L_x^q}.$$

Therefore, by summing the estimates, we have

$$\begin{aligned} & \|\mathcal{T}^{(n)} - \mathcal{T}^{(n-1)}\|_{L_T^\infty L_x^q} \\ & \leq 4(C(1)\varepsilon^{-1/2}T^{1/2} + 8C(q')M_q T^{(1-d/q)/2})\|\mathcal{T}^{(n-1)} - \mathcal{T}^{(n-2)}\|_{L_T^\infty L_x^q}. \end{aligned}$$

Observe that by (5.9) and (5.10), we have

$$4(C_1\varepsilon^{-1/2}T^{1/2} + 4C_2M_q T^{(1-d/q)/2}) \leq \frac{1}{2},$$

as desired. This completes the proof. □

5.4. Proof of Theorem 2.9. Finally, we are ready to prove Theorem 2.9. We will require the following lemma, which guarantees that the limiting function from the contraction mapping theorem is analytic. It can be found in [6].

Lemma 5.5. *Let \mathcal{F} be the set of all functions f which are analytic in an open set $\Omega \subset \mathbb{C}^d$ and for which*

$$\int_\Omega |f(x, y)|^q dx dy \leq M_0 < \infty.$$

Then, \mathcal{F} is a normal family.

Proof. First, Lemma 5.4 implies that the sequence

$$\mathcal{T}^{(n)} = (p^{(n)}, \pi^{(n)}, \mathbf{q}^{(n)}, \mathbf{u}^{(n)})$$

converges to a unique point $(p, \pi, \mathbf{q}, \mathbf{u}) \in C([0, T_0]; L^q(\mathbb{R}^d)^{2+2d})$. On the other hand, $(p, \pi, \mathbf{q}, \mathbf{u})$ is a classical solution of (1.4). Indeed, we may argue exactly as in [6]. By Lemma 5.5, one can extract a subsequence, which converges uniformly to $(\bar{p}, \bar{\pi}, \bar{\mathbf{q}}, \bar{\mathbf{u}})$ on compact sets of the domain defined by

$$\mathcal{D} := \{(\mathbf{x}, \mathbf{y}, t) \in \mathbb{C}^d \times (0, T_0) \mid \mathbf{y} \text{ satisfies (5.8) and } T_0 \text{ satisfies (5.10)}\}.$$

By uniqueness of the limits, we must have $(\bar{p}, \bar{\pi}, \bar{\mathbf{q}}, \bar{\mathbf{u}}) = (p, \pi, \mathbf{q}, \mathbf{u})$. Since the family is normal, all derivatives $\partial_t^k \partial_x^\ell$ exist, and they are uniformly bounded because of the inhomogeneous heat equations (5.1). We again may show that these derivatives converge to the derivatives of the limiting function from the contraction mapping theorem. Therefore, (p, \mathbf{q}) is a classical solution of (1.4). Since the $(p^{(n)}, \pi^{(n)}, \mathbf{q}^{(n)}, \mathbf{u}^{(n)})$ are analytic and form a normal family over \mathcal{D} , it follows that $(p, \mathbf{q}, \pi, \mathbf{u})$ is analytic over \mathcal{D} . By applying Fatou’s lemma to (5.7), we obtain (2.2). This completes the proof of Theorem 2.9 with $C_1 = 256C_1^2$, $C_2 = 64C_2$, and $C_* = 1/(2C_1)$. □

6. NUMERICAL ILLUSTRATIONS

The chemotaxis model (1.1) is generally difficult to solve by using routine numerical schemes, because of the singularity term $\nabla \ln(v) = \nabla v/v$. The Cole-Hopf transformation (1.2) converts the original chemotaxis model (1.1) to a parabolic system (1.4) where the cell density $u = p$ remains the same but the logarithmic singularity is removed. We solve system (1.4) to obtain the numerical value of u which is of the most interest in the model.

It is important to note that the long-time behavior results obtained in this paper have a prominent assumption $\bar{p} > 0$. It is unknown if the results still hold true when $\bar{p} = 0$. Hence, we have two goals in this section:

- numerical illustration of the long-time behavior result obtained in Theorem 2.1
- numerical simulation of the zero ground state case, and comparison with the $\bar{p} > 0$ case.

Since the domain under consideration is infinite, we choose a large domain $\Omega = (-100, 100)$ to mimic the infinite domain with appropriate initial conditions. Since $(p_0 - \bar{p}, q_0) \in H^2(\mathbb{R})$, which implies that $p_0 = \bar{p}$ and $q_0 = 0$ at the infinite boundary, we impose compatible boundary conditions on the initial data such that $p_0|_{\partial\Omega} \approx \bar{p}$ and $q_0|_{\partial\Omega} \approx 0$. We employ the Matlab PDE solver, which is based on the finite difference scheme, to simulate the model (1.4). Here, we set the time step size $\Delta t = 1$ and spatial step size $\Delta x = 0.1$. In Figure 6.1, we choose $\bar{p} = 1$ and $p_0(x) = 1 + \exp(-x - \exp(-x))$ and $q_0(x) = \exp(-x - \exp(-x))$,

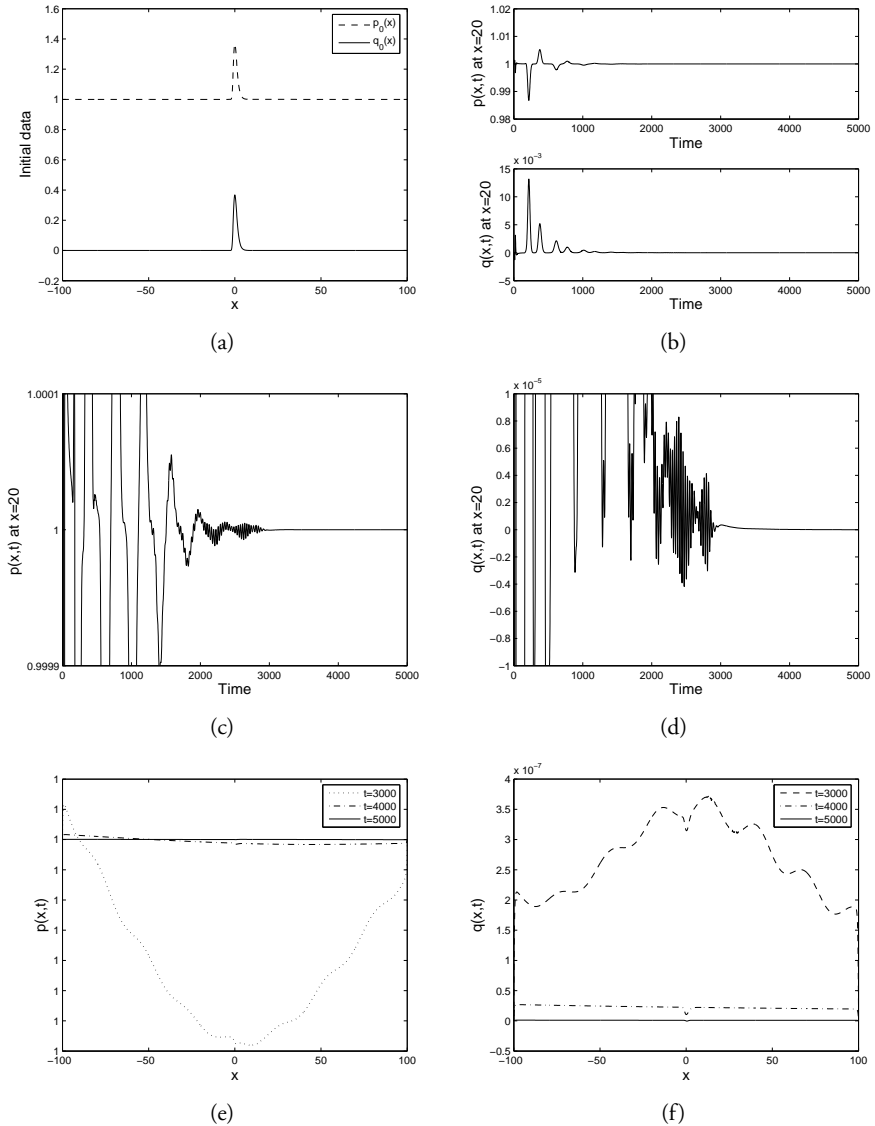


FIGURE 6.1. Numerical solutions to system (1.4) with initial data $p_0(x) = 1 + \exp(-x - \exp(-x))$, $q_0(x) = \exp(-x - \exp(-x))$, $\varepsilon = 0.1$, and $\bar{p} = 1$. Figure (a) plots the initial distribution $(p_0(x), q_0(x))$, (b) plots the time evolution of the solution $(p(x, t), q(x, t))$ at $x = 20$, (c) and (d) are the magnified visualizations of $p(20, t)$ and $q(20, t)$, respectively, and (e) and (f) plot the solution profiles of $p(x, t)$ and $q(x, t)$ at several large time steps.

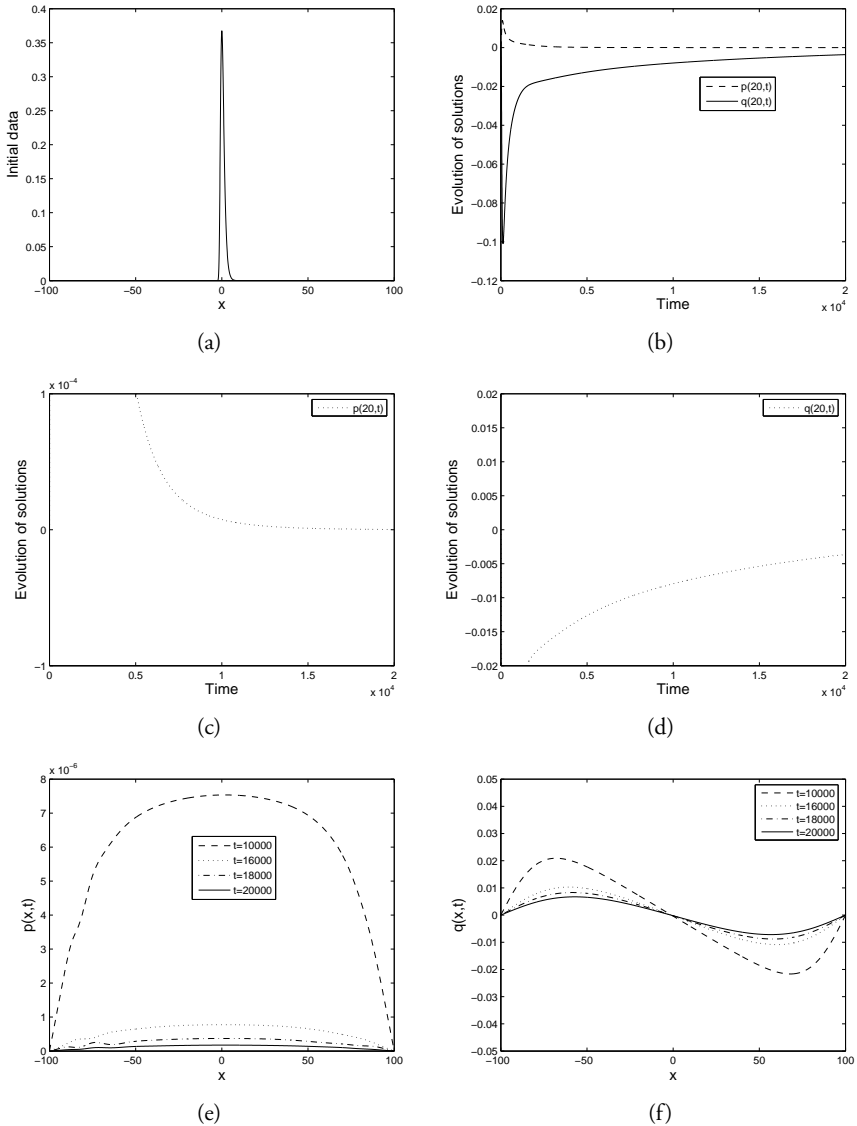


FIGURE 6.2. Numerical solutions to system (1.4) with initial data $p_0(x) = q_0(x) = \exp(-x - \exp(-x))$ and $\varepsilon = 0.1$, $\bar{p} = 0$. Figure (a) plots the initial distribution $(p_0(x), q_0(x))$, (b) plots the evolution of the solution $(p(x, t), q(x, t))$ at spatial point $x = 20$, (c) and (d) are the magnified visualizations of $p(20, t)$ and $q(20, t)$, respectively, and (e) and (f) plot the solution profiles of $p(x, t)$ and $q(x, t)$ at several large time steps.

as plotted in Figure 6.1 (a), such that $(p_0 - 1, q_0)$ satisfies the conditions in Theorem 2.1. To illustrate the fact that the solution converges to the ground state $(1, 0)$ as time tends to infinity, we arbitrarily choose one spatial point and visualize the time evolution of the solution at that point. Figure 6.1 (b) plots the evolution of the functions $p(x, t)$ and $q(x, t)$ at $x = 20$, respectively. To better visualize the converging process of the solution, we present magnified views of $p(20, t)$ and $q(20, t)$, which are plotted in Figure 6.1 (c) and (d), respectively, from which we see that the solution approaches the ground state oscillatorily. Moreover, the evolution of the functions over the whole region at several time steps is plotted in Figure 6.1 (e) and (f).

In Figure 6.2, we choose a new set of initial data, as plotted in Figure 6.2 (a), with the ground state being $(0, 0)$. We point out that, since the total mass of the cell population is conserved and finite when the ground state is zero, this case is more biologically relevant and meaningful than the positive ground state case. However, analytical results for this case largely remain open, especially regarding the long-time asymptotic behavior of the solution. Therefore, it is worthwhile to explore the long-time behavior of the solution numerically first, and predict some qualitative behavior of the model for future investigation. From the simulations shown in Figures 6.2 (b), (c), and (d) we see that the solution gradually approaches the zero ground state, which is similar to the scenarios presented in Figure 6.1. However, we have several observations which are distinct from those for the positive ground state case, as follows:

- (1) Although the solution seems to approach the ground state $(0, 0)$ as time proceeds, the converging time (≈ 20000) is much longer than that for the positive ground state case (≈ 3000). Indeed, from the simulations we see that even after $t \approx 20000$, the solution still stays away from zero. This suggests that, in the ideal case ($\Omega = \mathbb{R}$), the solution might become homogeneously distributed over the whole region with conserved total mass. The analysis of this case is considerably more difficult than that of the positive ground state case.
- (2) From the simulations we see that the solution approaches $(0, 0)$ monotonically at each spatial point, while the convergence in the case of positive ground state case is in an oscillatory fashion. This is also difficult to prove based on the method in this paper.

The aforementioned observations motivate us to make the following conjecture: *When initial data are perturbations of the ground state $(0, 0)$, global solutions to the Cauchy problem (1.4) monotonically approach the ground state as time tends to infinity with certain decay rate.* This is an interesting question to examine, but the method in this paper does not apply directly. However, the numerical simulations in Figure 6.2 provide some useful information about the dynamics of solutions for this case, which leaves a new problem for the future. Another challenging open question we would like to mention is the global well-posedness of (1.4) for large

data in higher dimensions when $\varepsilon > 0$. The main difficulty is that the weak Lyapunov functional (3.7) is valid only in one-dimensional space, and hence some *a priori* estimates cannot be established in higher dimensions.

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